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# Variational approach to scattering of plane elastic waves by diffraction gratings


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**Abstract**

The scattering of a time-harmonic plane elastic wave by a two-dimensional periodic structure is studied. The grating profile is given by a Lipschitz curve on which the displacement vanishes. Using a variational formulation in a bounded periodic cell involving a nonlocal boundary operator, existence of solutions in quasi-periodic Sobolev spaces is investigated by establishing the Fredholmness of the operator generated by the corresponding sesquilinear form. Moreover, by a Rellich identity, uniqueness is proved under the assumption that the grating profile is given by a Lipschitz graph. The direct scattering problem for transmission gratings is also investigated. In this case, uniqueness is proved except for a discrete set of frequencies.

# 1 Introduction

This paper is concerned with the scattering of a time-harmonic plane elastic wave by an unbounded periodic structure. Such structures are also called diffraction gratings and have many important applications in diffractive optics, radar imaging and non-destructive testing. We refer to the monograph [8] for historical remarks and details of these applications.

During the last twenty years, significant progress has been made concerning the mathematical analysis and the numerical approximation of grating diffraction problems for the case of incident acoustic or electromagnetic waves, using integral equation methods (e.g., [27], [16], [22], [25], [28]) and variational methods (e.g., [22], [15], [9], [6], [18], [19], [29], [7]). In particular, the variational approach appeared to be well adapted to the analytical and numerical treatment of rather general two-dimensional and three-dimensional periodic diffractive structures involving complex materials and non-smooth interfaces.

In this paper we assume that a periodic surface divides the three-dimensional space into two non-locally perturbed half-spaces filled with homogeneous and isotropic elastic media. Moreover, this surface is assumed to be invariant in the $x_3$-direction, and its cross-section in the $(x_1, x_2)$-plane is to be represented by a curve $\Lambda$ which is periodic in $x_1$. All elastic waves are assumed to be propagating perpendicular to the $x_3$-axis, so that the problem can be treated as a problem of plane elasticity. The special case of an inpenetrable surface on which all displacement vanishes leads to the Dirichlet (or first boundary value) problem for the Navier system in the unbounded domain above the grating profile $\Lambda$, while the scattering by a transmission grating is modeled by a corresponding transmission problem on the whole $(x_1, x_2)$-plane.

The first attempt to rigorously prove existence and uniqueness of solutions for the scattering of elastic waves by unbounded surfaces is due to T. Arens; see [2], [3] for two-dimensional diffraction gratings and [4], [5] for more general rough surfaces. In particular, in [2] existence and uniqueness of quasi-periodic solutions to the Dirichlet problem was established in the case that the grating profile $\Lambda$ is given by the graph of a smooth ($C^2$) periodic function.

The existence proof is based on the boundary integral equation method where the solution is sought as a superposition of single and double layer potentials.

Our main aim in this paper is to study the same problem, but via a variational approach in general Lipschitz domains, which is broad enough to cover most cases that arise in applications of diffraction gratings. We reduce the Navier system with Dirichlet boundary condition in the unbounded domain to an equivalent strongly elliptic variational problem in a bounded periodic cell with a non-local boundary condition. An explicit representation of the Dirichlet-to-Neumann (DtN) map on the artificial boundary is worked out, and a detailed analysis of this DtN map is employed to prove the strong ellipticity of the sesquilinear form. Applying the Fredholm alternative, we then prove that there always exists a quasiperiodic solution for either an incident pressure wave or an incident shear wave.

To extend the uniqueness result of [2] to grating profiles given by a Lipschitz graph, we use a Rellich identity and adapt an approach by Nečas [26, Chap. 5] to deal with the Lipschitz boundary. This generalizes the result of [20] for the scalar quasi-periodic Helmholtz equation to the case of the Navier system. More general Rellich identities for the Navier equation (on bounded domains) can be found in [14].

Moreover, the variational approach is extended to the case of transmission gratings where a Lipschitz interface separates two homogenous elastic media characterized by constant elastic parameters. This allows us to obtain general existence results, and uniqueness is proved except for a discrete set of frequencies. Note that this approach also applies to the case of several Lipschitz interfaces.

The paper is organized as follows. In Section 2 we give the mathematical formulation of the scattering problem in the case of an inpenetrable surface. Following [2], a radiation condition at infinity based on Rayleigh expansions is used. In Section 3 we formulate the variational problem in a bounded periodic cell which is equivalent to the boundary value problem. Using Korn's inequality and the Fourier series representation of the DtN map, we prove the strong ellipticity of the variational equation over the energy space. In Section 4 we present our solvability results for the Dirichlet case. The well-posedness for the boundary value problem with mixed Dirichlet and impedance boundary conditions is also established. In Section 5 we prove existence and uniqueness results for the transmission problem.

The problem of scattering by a diffraction grating can be seen as a special case of scattering by a rough surface. Note that the periodicity considerably simplifies the mathematical argument, because the compact imbedding of Sobolev spaces can be applied to a single period of the unbounded domain. For a rigorous mathematical analysis of rough surface scattering problems for the Helmholtz equation via variational methods, we refer to [11], [12], [10]. The variational approach to scattering by a rough surface in an elastic medium will be the task of future work.

# 2 Formulation of the Dirichlet problem

Let the profile of the diffraction grating be given by a Lipschitz curve $\Lambda \subset \mathbb{R}^2$ which is $2\pi$-periodic in $x_1$, and let $D$ be the unbounded domain above $\Lambda$. We assume the region $D$ is filled with an isotropic, homogenous elastic medium characterized by the Lamé constants

$\lambda, \mu$ satisfying $\mu > 0,\ \lambda + \mu > 0$. Let

$$k_p := \omega/\sqrt{2\mu + \lambda}\,, \quad k_s := \omega/\sqrt{\mu}$$

be the compressional and shear wave numbers respectively. We assume that a time harmonic plane elastic wave $u^{in}$ with incident angle $\theta \in (-\pi/2, \pi/2)$ is incident on $\Lambda$ from above, which is either an incident pressure wave taking the form

$$u^{in} = u_p^{in}(x) = \hat{\theta} \exp(ik_p \hat{\theta} \cdot x) \quad \text{with} \quad \hat{\theta} := (\sin\theta, -\cos\theta) \tag{2.1}$$

or an incident shear wave of the form

$$u^{in} = u_s^{in}(x) = \hat{\theta}^{\perp} \exp(ik_s \hat{\theta} \cdot x) \quad \text{with} \quad \hat{\theta}^{\perp} := (\cos\theta, \sin\theta)\,. \tag{2.2}$$

The propagation of time harmonic elastic waves in $D$ is governed by the Navier equation (or system)

$$(\Delta^* + \omega^2)u = 0 \quad \text{in} \quad D\,, \quad \Delta^* := \mu\Delta + (\lambda + \mu)\,\text{grad div}\,, \tag{2.3}$$

where $u = u^{in} + u^{sc}$ is the total displacement field and $u^{sc}$ denotes the scattered field. Here $\omega > 0$ stands for the angular frequency of the harmonic motion, and we assume for simplicity that the mass density of the elastic medium is equal to one. Moreover, we require that the total field satisfies the boundary condition

$$u = 0 \quad \text{on} \quad \Lambda\,. \tag{2.4}$$

The periodicity of the structure, together with the form of the incident waves, implies that the solution $u$ must be quasiperiodic with phase-shift $\alpha$ (or $\alpha$-quasiperiodic), i.e.

$$u(x_1 + 2\pi, x_2) = \exp(2i\alpha\pi x_1)\, u(x_1, x_2)\,, \quad (x_1, x_2) \in D\,, \tag{2.5}$$

where either $\alpha := k_p \sin\theta$ for the incident pressure wave (2.1), or $\alpha := k_s \sin\theta$ for the incident shear wave (2.2).

To ensure well-posedness of the boundary value problem (2.3)–(2.5), a radiation condition must be imposed as $x_2 \to +\infty$. First we note that the scattered field $u^{sc}$, which also satisfies the Navier equation (2.3), can be decomposed in $D$ as

$$u^{sc} = \frac{1}{i}\,(\text{grad}\ \varphi + \overrightarrow{\text{curl}}\,\psi) \quad \text{with} \quad \varphi := -\frac{i}{k_p^2}\,\text{div}\ u^{sc}\,, \quad \psi := \frac{i}{k_s^2}\,\text{curl}\ u^{sc}\,, \tag{2.6}$$

where the two curl operators in $\mathbb{R}^2$ are defined by

$$\text{curl}\ u := \partial_1 u_2 - \partial_2 u_1\,, \quad u = (u_1, u_2)^{\top} \quad \text{and} \quad \overrightarrow{\text{curl}}\, v := (\partial_2 v, -\partial_1 v)^{\top}\,,$$

and the scalar functions $\varphi$, $\psi$ satisfy the homogeneous Helmholtz equations

$$(\Delta + k_p^2)\,\varphi = 0 \quad \text{and} \quad (\Delta + k_s^2)\,\psi = 0 \quad \text{in} \quad D\,. \tag{2.7}$$

Here and in the following the notation $\partial_j v = \partial v/\partial x_j$ is used. Note that the relations (2.6) and (2.7) follow from the well known decomposition [23] of the scattered field $u^{sc}$ into its compressional and shear parts,

$$u^{sc} = u_p + u_s\,, \quad u_p := -\frac{1}{k_p^2}\,\text{grad div}\ u^{sc}\,, \quad u_s := \frac{1}{k_s^2}\,\overrightarrow{\text{curl}}\ \text{curl}\ u^{sc}\,,$$

and the fact that $u^{sc}$ satisfies equation (2.3).

Now, as $\varphi$ and $\psi$ are $\alpha$-quasiperiodic solutions to the Helmholtz equations (2.7) in the unbounded domain $D$, we impose the usual outgoing wave condition on them (see, e.g., [22]). For $x_2 > \Lambda^+$, we assume that $\varphi$, $\psi$ have Rayleigh expansions of the form

$$\varphi(x) = \sum_{n\in\mathbb{Z}} A_{p,n} \exp(i\alpha_n x_1 + i\beta_n x_2)\,, \quad \psi(x) = \sum_{n\in\mathbb{Z}} A_{s,n} \exp(i\alpha_n x_1 + i\gamma_n x_2)\,, \tag{2.8}$$

where the constants $A_{p,n}$, $A_{s,n} \in \mathbb{C}$ are called Rayleigh coefficients and

$$\Lambda^+ := \max_{(x_1,x_2)\in\Lambda} x_2\,, \quad \alpha_n := \alpha + n\,, \quad \beta_n := \begin{cases} \sqrt{k_p^2 - \alpha_n^2} & \text{if} \quad |\alpha_n| \le k_p \\ i\sqrt{\alpha_n^2 - k_p^2} & \text{if} \quad |\alpha_n| > k_p\,, \end{cases} \tag{2.9}$$

and $\gamma_n$ is defined analogously as $\beta_n$ with $k_p$ replaced by $k_s$. It follows from (2.6) that the two components of the scattered field $u^{sc}$ in $D$ can be represented as

$$u_1^{sc} = \frac{1}{i}\left(\partial_1\varphi + \partial_2\psi\right), \quad u_2^{sc} = \frac{1}{i}\left(\partial_2\varphi - \partial_1\psi\right). \tag{2.10}$$

Therefore, we finally obtain a corresponding expansion of $u^{sc}$ into outgoing plane elastic waves:

$$\begin{aligned} u^{sc}(x) = \sum_{n\in\mathbb{Z}} \Bigg\{ & A_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \exp(i\alpha_n x_1 + i\beta_n x_2) \\ & + A_{s,n} \begin{pmatrix} \gamma_n \\ -\alpha_n \end{pmatrix} \exp(i\alpha_n x_1 + i\gamma_n x_2) \Bigg\}\,, \end{aligned} \tag{2.11}$$

for $x_2 > \Lambda^+$. This is the radiation condition we are going to use in the following; see also [2]. Since $\beta_n$ and $\gamma_n$ are real for at most a finite number of indices, only a finite number of plane waves in (2.11) propagate into the far field, with the remaining evanescent waves (or surface waves) decaying exponentially as $x_2 \to +\infty$. The above expansion converges uniformly with all derivatives in the half-plane $\{x \in \mathbb{R}^2 : x_2 \ge b\}$, for any $b > \Lambda^+$, and the Rayleigh coefficients are uniquely determined by the Fourier coefficients $\hat{u}_n$ of the function $\exp(-i\alpha x_1)u^{sc}(x_1, b)$:

$$\hat{u}_n = D_n \begin{pmatrix} A_{p,n}\exp(i\beta_n b) \\ A_{s,n}\exp(i\gamma_n b) \end{pmatrix}, \quad D_n := \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & -\alpha_n \end{pmatrix}. \tag{2.12}$$

Note here that $\det D_n = -(\alpha_n^2 + \beta_n\gamma_n) \neq 0$ for all $n \in \mathbb{Z}$. Our diffraction problem can now be formulated as the following boundary value problem.

**Dirichlet problem (DP):** Given a grating profile curve $\Lambda \subset \mathbb{R}^2$ (which is $2\pi$-periodic in $x_1$) and an incident field $u^{in}$ of the form (2.1) or (2.2), find a vector function $u = u^{in} + u^{sc} \in H^1_{loc}(D)^2$ that satisfies (2.3)–(2.5) and the radiation condition (2.11).

# 3 Variational formulation of (DP)

Following the approach of [22] in the case of the scalar Helmholtz equation, we propose an equivalent variational formulation of the boundary value problem (DP), which is posed in a

bounded periodic cell in $\mathbb{R}^2$ and is enforcing the radiation condition. Introduce an artificial boundary

$$\Gamma_b := \{(x_1, b) : \ 0 \le x_1 \le 2\pi\}, \quad b > \Lambda^+,$$

and the bounded domain

$$\Omega_b = \Omega_{\Lambda,b} := \{(x_1, x_2) \in D : \ 0 < x_1 < 2\pi, \ x_2 < b\},$$

lying between the segment $\Gamma_b$ and one period of the grating profile curve which we denote by $\Lambda$ again. We assume that $\Lambda$ is a Lipschitz curve, so that $\Omega_b$ is a bounded Lipschitz domain.

Let $H^1_\alpha(\Omega_b)$ denote the Sobolev space of scalar functions on $\Omega_b$ which are $\alpha$-quasiperiodic with respect to $x_1$. We introduce the space

$$V_\alpha = V_\alpha(\Omega_b) := \{u \in H^1_\alpha(\Omega_b)^2 : u|_\Lambda = 0\},$$

which is the energy space for our variational problem. In the following $V_\alpha$ is equipped with the norm in the usual Sobolev space $H^1(\Omega_b)^2$ of vector functions.

By the first Betti formula, it follows that for $u, \varphi \in V_\alpha$

$$-\int_{\Omega_b} (\Delta^* + \omega^2) u \cdot \overline{\varphi}\, dx = \int_{\Omega_b} (a_L(u, \overline{\varphi}) - \omega^2 u \cdot \overline{\varphi})\, dx - \int_{\Gamma_b} \overline{\varphi} \cdot Tu\, ds \tag{3.1}$$

where the bar indicates the complex conjugate, and

$$\begin{aligned} a_L(u, \varphi) \ &= \ (2\mu + \lambda)\,(\partial_1 u_1\, \partial_1 \varphi_1 + \partial_2 u_2\, \partial_2 \varphi_2) + \mu\,(\partial_2 u_1\, \partial_2 \varphi_1 + \partial_1 u_2\, \partial_1 \varphi_2) \\ &\quad + \mu\,(\partial_1 u_1\, \partial_2 \varphi_2 + \partial_2 u_2\, \partial_1 \varphi_1) + \lambda\,(\partial_2 u_1\, \partial_1 \varphi_2 + \partial_1 u_2\, \partial_2 \varphi_1), \end{aligned} \tag{3.2}$$

and $Tu$ stands for the *stress vector* or *traction* having the form:

$$Tu = 2\mu\, \partial_{\mathbf{n}} u + \lambda\, \mathbf{n}\, \mathrm{div}\ u + \mu \begin{pmatrix} n_2\,(\partial_1 u_2 - \partial_2 u_1) \\ n_1\,(\partial_2 u_1 - \partial_1 u_2) \end{pmatrix}, \tag{3.3}$$

where $\mathbf{n} = (n_1, n_2)^\top$ denotes the exterior unit normal on the boundary of $\Omega_b$. Moreover, we have

$$\begin{aligned} Tu = T(\mu, \lambda)u := 2\mu\, \partial_2 u + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\partial_1 u_1 + \partial_2 u_2) \\ + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\partial_1 u_2 - \partial_2 u_1) \quad \text{on} \quad \Gamma_b. \end{aligned} \tag{3.4}$$

Now we introduce the DtN map $\mathcal{T}$ on the artificial boundary $\Gamma_b$. For any $u \in H^1_\alpha(\Omega_b)^2$, we have

$$v := u|_{\Gamma_b} \in H^{1/2}_\alpha(\Gamma_b)^2, \quad \exp(-i\alpha x_1)\, v \in H^{1/2}_{per}(\Gamma_b)^2$$

from the trace theorem, where $H^s_\alpha(\Gamma_b)$ and $H^s_{per}(\Gamma_b)$ denote the Sobolev spaces of order $s \in \mathbb{R}$ of functions on $\Gamma_b$ that are $\alpha$-quasiperiodic and periodic respectively. Note that an equivalent norm on $H^s_\alpha(\Gamma_b)^2$ is given by

$$\|v\|_{H^s_\alpha(\Gamma_b)^2} = \Big( \sum_{n \in \mathbb{Z}} (1 + |n|)^{2s}\, |\hat{v}_n|^2 \Big)^{1/2},$$

where $\hat{v}_n \in \mathbb{C}^2$ are the Fourier coefficients of $\exp(-i\alpha x_1)\, v(x_1, b)$. For any $v \in H^{1/2}_\alpha(\Gamma_b)^2$, we define $\mathcal{T}v$ as the traction $Tu^{sc}$ on $\Gamma_b$ where $u^{sc}$ is the unique $\alpha$-quasiperiodic solution of the homogenous Navier equation in $\{x_2 > b\}$ which satisfies (2.11) and $u^{sc} = v$ on $\Gamma_b$. The next lemma shows an explicit representation of $\mathcal{T}$.

**Lemma 1** *With the notation introduced in (2.9), we have*

$$\mathcal{T}v = \mathcal{T}(\omega, \alpha)v = -\sum_{n\in\mathbb{Z}} W_n \hat{v}_n \exp(i\alpha_n x_1) \quad \text{for} \quad v = \sum_{n\in\mathbb{Z}} \hat{v}_n \exp(i\alpha_n x_1) \in H^{1/2}_\alpha(\Gamma_b)^2\,, \quad (3.5)$$

*where*

$$W_n = W_n(\omega, \alpha) := \frac{1}{i} \begin{pmatrix} \omega^2\beta_n/d_n & 2\mu\alpha_n - \omega^2\alpha_n/d_n \\ -2\mu\alpha_n + \omega^2\alpha_n/d_n & \omega^2\gamma_n/d_n \end{pmatrix}, \quad d_n := \alpha_n^2 + \beta_n\gamma_n. \quad (3.6)$$

*Proof.* Let $u^{sc}$ be the radiating solution of (2.3) in $\{x_2 > 0\}$ such that $u^{sc} = v$ on $\Gamma_b$. Then $u^{sc}$ takes the form (2.11), where the corresponding Rayleigh coefficients $A_{p,n}$, $A_{s,n}$ are given by

$$\begin{pmatrix} A_{p,n}\, \exp(i\beta_n b) \\ A_{s,n}\, \exp(i\gamma_n b) \end{pmatrix} = D_n^{-1}\, \hat{v}_n \quad ; \quad (3.7)$$

see (2.12). Moreover, from the representation (2.10) of $u^{sc}$, the Rayleigh expansions (2.8), (2.11) and the relations (2.7), (3.4), we obtain

$$\begin{aligned} Tu^{sc} &= 2\mu\, \partial_2 u^{sc} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\partial_1 u_1^{sc} + \partial_2 u_2^{sc}) + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\partial_1 u_2^{sc} - \partial_2 u_1^{sc}) \\ &= 2\mu\, \partial_2 u^{sc} - i\lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Delta\varphi + i\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Delta\psi \\ &= 2\mu \sum_{n\in\mathbb{Z}} \left\{ i\beta_n A_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \exp(i\alpha_n x_1 + i\beta_n x_2) + i\gamma_n A_{s,n} \begin{pmatrix} \gamma_n \\ -\alpha_n \end{pmatrix} \exp(i\alpha_n x_1 + i\gamma_n x_2) \right\} \\ &\quad + \sum_{n\in\mathbb{Z}} \left\{ i\lambda k_p^2 A_{p,n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\alpha_n x_1 + i\beta_n x_2) - i\mu k_s^2 A_{s,n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(i\alpha_n x_1 + i\gamma_n x_2) \right\}. \end{aligned}$$

Together with (3.7), this implies

$$\begin{aligned} \widehat{(\mathcal{T}v)}_n &= \begin{pmatrix} 2i\mu\beta_n\alpha_n & 2i\mu\gamma_n^2 - i\mu k_s^2 \\ 2i\mu\beta_n^2 + \lambda i k_p^2 & -2i\mu\gamma_n\alpha_n \end{pmatrix} \begin{pmatrix} A_{p,n}\exp(i\beta_n b) \\ A_{s,n}\exp(i\gamma_n b) \end{pmatrix} \\ &= i \begin{pmatrix} 2\mu\alpha_n\beta_n & \omega^2 - 2\mu\alpha_n^2 \\ \omega^2 - 2\mu\alpha_n^2 & -2\mu\alpha_n\gamma_n \end{pmatrix} D_n^{-1}\, \hat{v}_n \\ &= \frac{i}{d_n} \begin{pmatrix} 2\mu\alpha_n\beta_n & \omega^2 - 2\mu\alpha_n^2 \\ \omega^2 - 2\mu\alpha_n^2 & -2\mu\alpha_n\gamma_n \end{pmatrix} \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & -\alpha_n \end{pmatrix} \hat{v}_n \\ &= i \begin{pmatrix} \omega^2\beta_n/d_n & 2\mu\alpha_n - \omega^2\alpha_n/d_n \\ -2\mu\alpha_n + \omega^2\alpha_n/d_n & \omega^2\gamma_n/d_n \end{pmatrix} \hat{v}_n \\ &= -W_n \hat{v}_n\,, \end{aligned}$$

where $\widehat{(\mathcal{T}v)}_n$ denotes the $n$-th Fourier coefficient of $\exp(-i\alpha x_1)\,\mathcal{T}v = \exp(-i\alpha x_1)\,Tu^{sc}$. This completes the proof by recalling the definitions of $W_n$ and $d_n$ in (3.6). $\square$

Next we introduce the sesquilinear form $B(u,\varphi)$ defined by

$$B(u,\varphi) := \int_{\Omega_b} \big(a_L(u,\overline{\varphi}) - \omega^2 u\cdot\overline{\varphi}\big)\,dx - \int_{\Gamma_b} \overline{\varphi}\cdot\mathcal{T}u\,ds\,,\ \forall\ u,\phi\in V_\alpha\,, \tag{3.8}$$

with $\mathcal{T}u := \mathcal{T}(u|_{\Gamma_b})$. Note that, by Lemma 1, $\mathcal{T}u$ takes the form

$$\mathcal{T}u = -\sum_{n\in\mathbb{Z}} W_n\hat{u}_n \exp(i\alpha_n x_1)\,, \tag{3.9}$$

where $\hat{u}_n$ are the Fourier coefficients of $\exp(-i\alpha x_1)\,u(x_1,b)$. Applying Betti's identity (3.1) to a solution $u = u^{sc}+u^{in}$ of (DP) and using the fact that

$$Tu = T(u^{sc}+u^{in}) = \mathcal{T}u^{sc} + Tu^{in} = \mathcal{T}u + f_0\,,\quad \text{with}\quad f_0 := Tu^{in} - \mathcal{T}u^{in}\,,$$

we obtain the following variational formulation of (DP): Find $u\in V_\alpha$ such that

$$B(u,\varphi) = \int_{\Gamma_b} f_0\cdot\overline{\varphi}\,ds\,,\quad \forall\ \varphi\in V_\alpha\,. \tag{3.10}$$

Here

$$f_0 = f_{p,0} := \frac{2i\beta_0 k_p(\lambda+2\mu)}{d_0}\begin{pmatrix} -\alpha \\ \gamma_0\end{pmatrix}\exp(i\alpha x_1 - i\beta_0 b) \tag{3.11}$$

for an incident pressure wave of the form (2.1), and

$$f_0 = f_{s,0} := -\frac{2i\gamma_0 k_s\mu}{d_0}\begin{pmatrix} \beta_0 \\ \alpha\end{pmatrix}\exp(i\alpha x_1 - i\gamma_0 b) \tag{3.12}$$

for an incident shear wave of the form (2.2). The problems (DP) and (3.10) are equivalent in the following sense.

**Remark 1** *If $u\in H^1_{loc}(D)^2$ is a solution of the boundary value problem (DP), then $u|_{\Omega_b}$ satisfies the variational problem (3.10). Conversely, a solution $u\in V_\alpha(\Omega_b)$ of (3.10) can be extended to a solution $u = u^{in}+u^{sc}$ of the Navier equation (2.3) for $x_2\geq b$, where $u^{sc}$ is defined by the relations (2.11), (2.12) via the Fourier coefficients $\hat{u}_n$ of $\exp(-i\alpha x_1)\,(u-u^{in})(x_1,b)$.*

To study the form $B$, the following lemma is needed. For a matrix $M\in\mathbb{C}^{2\times 2}$, let $\operatorname{Re} M := (M+M^*)/2$, and we shall write $\operatorname{Re} M > 0$ if $\operatorname{Re} M$ is positive-definite. Here $M^*$ is the adjoint of $M$ with respect to the scalar product $(\cdot,\cdot)_{\mathbb{C}^2}$ in $\mathbb{C}^2$.

**Lemma 2** *Let $W_n = W_n(\omega,\alpha)$ be defined as in Lemma 1. Then*
*(i) Given a fixed frequency $\omega>0$, we have $\operatorname{Re} W_n > 0$ for all sufficiently large $|n|$.*
*(ii) There exists a sufficiently small frequency $\omega_0>0$ such that*

$$(\operatorname{Re} W_n\, z, z)_{C^2} \geq C\,|n|\,|z|^2\,,\quad \forall\ z\in\mathbb{C}^2\,,\quad \forall\ \omega\in(0,\omega_0]\,,\quad \forall\ n\neq 0 \tag{3.13}$$

*with some constant $C>0$ independent of $\omega$ and $n$.*

**Proof.** We can write the matrix $W_n$ as

$$W_n = \begin{pmatrix} a_n & ic_n \\ -ic_n & b_n \end{pmatrix}, \quad a_n := -i\,\frac{\omega^2\beta_n}{d_n}, \quad b_n := -i\,\frac{\omega^2\gamma_n}{d_n}, \quad c_n := \frac{\alpha_n}{d_n}(\omega^2 - 2\mu d_n). \tag{3.14}$$

Let first $\omega > 0$ be fixed. We have

$$\beta_n = i\sqrt{(n+\alpha)^2 - k_p^2} \sim i\,|n|, \quad \gamma_n = i\sqrt{(n+\alpha)^2 - k_s^2} \sim i\,|n| \quad \text{as} \quad |n| \to \infty, \tag{3.15}$$

and, on using Taylor expansions,

$$\begin{aligned} d_n = \alpha_n^2 + \beta_n\gamma_n &= (n+\alpha)^2\left\{1 - \sqrt{1 - \frac{k_p^2}{(n+\alpha)^2}}\sqrt{1 - \frac{k_s^2}{(n+\alpha)^2}}\right\} \\ &\sim \frac{k_p^2 + k_s^2}{2}, \quad \text{as } |n| \to \infty. \end{aligned} \tag{3.16}$$

Moreover, from (3.14)–(3.16) we have, for sufficiently large $|n|$,

$$a_n > 0, \quad b_n > 0 \qquad c_n \in \mathbb{R}, \quad \operatorname{Re} W_n = \begin{pmatrix} a_n & i\,c_n \\ -i\,c_n & b_n \end{pmatrix}. \tag{3.17}$$

Note that the relation $\operatorname{Re}.W_n > 0$ holds if and only if

$$a_n > 0 \quad \text{and} \quad \det(\operatorname{Re} W_n) = a_n\,b_n - c_n^2 > 0. \tag{3.18}$$

It is easily seen that

$$\begin{aligned} \det(\operatorname{Re} W_n) &= \frac{1}{d_n^2}\left(-\omega^4\,\beta_n\,\gamma_n - \alpha_n^2\,(\omega^2 - \mu\,d_n)^2\right) \\ &= \frac{1}{d_n}\left(-\omega^4 + 4\alpha_n^2\,\mu\,(\omega^2 - \mu d_n)\right), \end{aligned} \tag{3.19}$$

which together with (3.16) and the relations $k_p = \omega/\sqrt{2\mu+\lambda}$, $k_s = \omega/\sqrt{\mu}$ implies that

$$\omega^2 - \mu\,d_n \sim \omega^2 - \frac{\mu}{2}\left(\frac{\omega^2}{2\mu+\lambda} + \frac{\omega^2}{\mu}\right) = \omega^2\,\frac{\mu+\lambda}{2(2\mu+\lambda)} > 0, \quad \text{as} \quad |n| \to \infty. \tag{3.20}$$

From (3.19) and (3.20) we now obtain the second inequality of (3.18) for all $|n|$ sufficiently large, which completes the proof of assertion (i).

To prove assertion (ii), we also need to analyze the behavior of $\operatorname{Re} W_n$ as $\omega \to 0$. Notice that, for all sufficiently small $\omega > 0$ and $n \neq 0$, we have the relations

$$\beta_n/i \geq C\,|n|, \quad \gamma_n/i \geq C\,|n|, \tag{3.21}$$

with a positive constant $C$ independent of $\omega$ and $n$. Moreover, by arguing as in (3.16),

$$d_n = \omega^2\,\frac{3\mu+\lambda}{2\mu\,(2\mu+\lambda)} + \mathcal{O}(\omega^4), \quad \text{as} \quad \omega \to 0,$$

which yields

$$\omega^2 - \mu\,d_n = \omega^2\,\frac{\mu+\lambda}{2(2\mu+\lambda)} + \mathcal{O}(\omega^4), \quad \text{as} \quad \omega \to 0, \tag{3.22}$$

uniformly in $n \neq 0$. Thus, combining (3.19), (3.21) and (3.22), we find that there exists a sufficiently small frequency $\omega_0 > 0$ such that, for all $\omega \in (0, \omega_0]$ and $n \neq 0$,

$$|n|^{-1}\, a_n \geq c > 0\,, \quad |n|^{-2}\, \det(\mathrm{Re}\, W_n) \geq c > 0\,,$$

which means that the matrices $|n|^{-1}\, \mathrm{Re}\, W_n$ are uniformly positive-definite; compare (3.18). This implies estimate (3.13) and finishes the proof of assertion (ii). $\square$

It follows from Lemma 1 and the relations (3.14)–(3.16) that the DtN operator $\mathcal{T}$ maps the Sobolev space $H_\alpha^{1/2}(\Gamma_b)^2$ continuously into $H_\alpha^{-1/2}(\Gamma_b)^2$. Therefore, the sesquilinear form $B(u, \varphi)$ defined in (3.8) is bounded on the energy space $V_\alpha$. Setting

$$B(u, \varphi) = (\mathcal{B}u, \varphi)_{\Omega_b} \quad \forall\, u\,,\, \varphi \in V_\alpha\,, \tag{3.23}$$

the form $B$ obviously generates a continuous linear operator $\mathcal{B} : V_\alpha \to V_\alpha'$. Here $V_\alpha'$ denotes the dual of the space $V_\alpha$ with respect to the duality $(\cdot, \cdot)_{\Omega_b}$ extending the scalar product in $L^2(\Omega_b)^2$.

We call a bounded sesquilinear form $B(\cdot, \cdot)$ given on some Hilbert space $X$ strongly elliptic if there exists a compact form $q(\cdot, \cdot)$ such that

$$|\mathrm{Re}\, B(u, u)| \geq c\, ||u||_X^2 - q(u, u)\,, \quad \forall\, u \in X.$$

To establish the strong ellipticity of the sesquilinear form $B$ defined in (3.8), we need the following auxiliary results on the bilinear form $a_L$ defined in (3.2), which can be written as

$$a_L(u, v) = \lambda\, \mathrm{div}\; u\, \mathrm{div}\; v + 2\mu \sum_{i,j=1}^{2} \varepsilon_{ij}(u)\, \varepsilon_{ij}(v) \quad \varepsilon_{ij}(u) := (\partial_j u_i + \partial_i u_j)/2\,.$$

Under our assumptions on the Lamé constants, $\mu > 0$, $\lambda + \mu > 0$, we have the estimate (e.g., [21, Chap. 5.4])

$$\int_G a_L(u, \overline{u})\, dx \geq C(G) \sum_{i,j=1}^{2} ||\varepsilon_{ij}(u)||_{L^2(G)}^2\,, \quad \forall\, u \in H^1(G)^2\,, \tag{3.24}$$

with a positive constant $C(G)$, for each bounded Lipschitz domain $G \subset \mathbb{R}^2$. To obtain a lower bound for the second term in (3.24), the well known Korn's inequality can be used; see e.g. [24, Chapter 10], [17, Chapter 3] for a proof.

**Lemma 3** *For each bounded Lipschitz domain $G \subset \mathbb{R}^2$, we have the inequality*

$$\sum_{i.j=1}^{2} ||\varepsilon_{ij}(v)||_{L^2(G)}^2 + \sum_{i=1}^{2} ||v_i||_{L^2(G)}^2 \geq C(G)\, ||v||_{H^1(G)^2}^2\,, \quad \forall\, v \in H^1(G)^2\,. \tag{3.25}$$

**Remark 2** *Let $G$ be a bounded Lipschitz domain in $\mathbb{R}^2$, and suppose that $\Gamma_0 \subset \partial G$ has positive Lebesgue measure. Then, using (3.25) and the arguments in the proof of [17, Chapter 3,Theorem 3.3], one can prove that*

$$||v||_{H^1(G)^2}^2 \leq C(G) \left( ||v||_{L^2(\Gamma_0)^2}^2 + \sum_{i,j=1}^{2} ||\varepsilon_{ij}(v)||_{L^2(G)}^2 \right)\,, \quad \forall\, v \in H^1(G)^2\,.$$

*In particular, if $v \in H^1(G)^2$ satisfies $v|_{\Gamma_0} = 0$, we see that*

$$|v| := \left( \sum_{i,j=1}^{2} ||\varepsilon_{ij}(v)||^2_{L^2(G)} \right)^{1/2}$$

*is an equivalent norm of $v$ in $H^1(G)^2$, and from (3.24) we then have the estimate*

$$\int_G a_L(v, \overline{v})\, dx \geq C(G)\, ||v||^2_{H^1(G)^2}\,,$$

*with $C(G) > 0$ not depending on $v$.*

We are now ready to prove the main result of this section.

**Theorem 1** *Assume that the grating profile $\Lambda$ is a Lipschitz curve. Then the sesquilinear form $B$ defined in (3.8) is strongly elliptic over $V_\alpha$. Moreover, the operator $\mathcal{B}$ defined by (3.23) is always a Fredholm operator with index zero.*

**Proof.** Since $u$ vanishes on $\Lambda$, it follows from Korn's inequality (see Lemma 3 and Remark 2) that there exists a positive constant $C$ such that

$$\int_{\Omega_b} a_L(u, \bar{u})\, dx \geq C\, ||u||^2_{H^1(\Omega_b)^2} = C\, ||u||^2_{V_\alpha}\,, \quad \forall\, u \in V_\alpha\,. \tag{3.26}$$

Moreover, the operator $\mathcal{K} : V_\alpha \to V_\alpha'$ defined by

$$(\mathcal{K}u, \varphi)_{\Omega_b} = -\omega^2 \int_{\Omega_b} u \cdot \bar{\varphi}\, dx\,, \quad \forall\, u, \varphi \in V_\alpha \tag{3.27}$$

is compact. To prove the strong ellipticity of the form $B$ defined in (3.8), it is now sufficient to verify that $\mathcal{T}$ is the sum of a finite dimensional operator and an operator $\mathcal{T}_1$ with

$$\mathrm{Re}\left\{ -\int_{\Gamma_b} \overline{u} \cdot \mathcal{T}_1 u\, ds \right\} \geq 0\,, \quad \forall\, u \in H^1_\alpha(\Omega_b)^2\,. \tag{3.28}$$

To do so, we apply (3.9) and set

$$\mathcal{T}_1 u := -\sum_{|n| \geq n_0} W_n\, \hat{u}_n\,, \quad \mathcal{T}_0 := \mathcal{T} - \mathcal{T}_1\,.$$

where $n_0 \in \mathbb{N}$ is sufficiently large, so that

$$\mathrm{Re}\,(W_n z, z)_{\mathbb{C}^2} \geq 0\,, \quad \forall\, z \in \mathbb{C}^2\,, \quad \forall\, |n| \geq n_0 \tag{3.29}$$

by Lemma 2 (i). Then the operator $\mathcal{T}_0$ is finite dimensional, and (3.28) is a consequence of (3.29). This finishes the proof of the strong ellipticity of the form $B$ over $V_\alpha$, and the Fredholm property of $\mathcal{B}$ follows in a standard way. □

# 4 Existence and uniqueness results

In this section, we establish existence and uniqueness theorems for the boundary value problem (DP), or equivalently, the variational problem (3.10) in the case of arbitrary frequencies. Problem (3.10) can also be written in the form

$$\mathcal{B}u = \mathcal{F}_0\,, \quad \mathcal{F}_0 \in V_\alpha'\,, \tag{4.1}$$

where $\mathcal{F}_0$ is given by the right hand side of (3.10), and the operator $\mathcal{B} : V_\alpha \to V_\alpha'$ is defined by (3.23) via the sesquilinear form (3.8).

Let $u \in V_\alpha$ be a solution of the homogeneous equation $\mathcal{B}u = 0$. Then $u$ can be extended to a radiating solution of (2.3) in $D$ by setting $u(x) = u^{sc}(x)$ for $x_2 \geq b$, where $u^{sc}$ is defined by the expansion (2.11) with the Rayleigh coefficients $A_{p,n}$, $A_{s,n}$, which are uniquely determined by the Fourier coefficients $\hat{u}_n$ of $\exp(-i\alpha x_1)\, u(x_1, b)$ via the relation (2.12). We will need the following technical result, which has already been proved in [2]. Here we prefer to give a more direct proof that is based on the Fourier series representation of the DtN operator.

**Lemma 4** *If $u \in V_\alpha$ satisfies $\mathcal{B}u = 0$, then*

$$A_{p,n} = 0 \quad \textit{for} \quad |\alpha_n| < k_p \quad \textit{and} \quad A_{s,n} = 0 \quad \textit{for} \quad |\alpha_n| < k_s\,. \tag{4.2}$$

**Proof.** Taking imaginary parts in the variational equation (3.10) with $\varphi = u$ and $f_0 = 0$, we are going to prove that

$$\mathrm{Im}\, B(u,u) = -\mathrm{Im} \int_{\Gamma_b} \overline{u} \cdot \mathcal{T}u \, ds = -2\pi\omega^2 \left( \sum_{|\alpha_n|<k_p} \beta_n \, |A_{p,n}|^2 + \sum_{|\alpha_n|<k_s} \gamma_n \, |A_{s,n}|^2 \right), \tag{4.3}$$

which implies (4.2) since the left hand side of (4.3) is zero. Rewrite the relation (2.12) in the form

$$\hat{u}_n = D_n \, A_n\,, \quad A_n := \begin{pmatrix} A_{p,n} \, \exp(i\beta_n b) \\ A_{s,n} \, \exp(i\gamma_n b) \end{pmatrix}.$$

Then the Fourier coefficients $\hat{w}_n$ of $\exp(-i\alpha x_1)\, \mathcal{T}u(x_1, b)$ can be written

$$\hat{w}_n = i \, G_n \, A_n\,, \quad G_n := \begin{pmatrix} 2\mu\alpha_n\beta_n & \omega^2 - 2\mu\alpha_n^2 \\ \omega^2 - 2\mu\alpha_n^2 & -2\mu\alpha_n\gamma_n \end{pmatrix};$$

see the proof of Lemma 1. Hence we get

$$\int_{\Gamma_b} \overline{u} \cdot \mathcal{T}u \, ds = 2\pi \sum_{n\in\mathbb{Z}} (iG_nA_n, D_nA_n)_{\mathbb{C}^2} == 2\pi \sum_{n\in\mathbb{Z}} (iL_nA_n, A_n)_{\mathbb{C}^2}\,,$$

where $L_n := D_n^* G_n$ can be written as

$$\begin{aligned} L_n &= \begin{pmatrix} \alpha_n & \overline{\beta}_n \\ \overline{\gamma}_n & -\alpha_n \end{pmatrix} \begin{pmatrix} 2\mu\alpha_n\beta_n & \omega^2 - 2\mu\alpha_n^2 \\ \omega^2 - 2\mu\alpha_n^2 & -2\mu\alpha_n\gamma_n \end{pmatrix} \\ &= \begin{pmatrix} 4\mu\alpha_n^2(\mathrm{Im}\beta_n)i + \omega^2\overline{\beta}_n & (\omega^2 - 2\mu\alpha_n^2)\alpha_n - 2\mu\alpha_n\overline{\beta}_n\gamma_n \\ 2\mu\alpha_n\beta_n\overline{\gamma}_n - (\omega^2 - 2\mu\alpha_n^2)\alpha_n & 4\mu\alpha_n^2(\mathrm{Im}\gamma_n)i + \omega^2\overline{\gamma}_n \end{pmatrix}. \end{aligned}$$

Thus we obtain

$$\operatorname{Im}\int_{\Gamma_b} \overline{u}\cdot\mathcal{T}u\,ds = 2\pi\operatorname{Im}\sum_{n\in\mathbb{Z}}(iL_nA_n,A_n)_{\mathbb{C}^2} = 2\pi\sum_{n\in\mathbb{Z}}(\operatorname{Re}L_n\,A_n,A_n)_{\mathbb{C}^2}\,.$$

Finally, we note that the matrix $\operatorname{Re}L_n$ is diagonal,

$$\operatorname{Re}L_n = \operatorname{diag}\{e_n,f_n\}\quad\text{with}\quad e_n := \begin{cases}\omega^2\beta_n & \text{for } \alpha_n^2<k_p^2\,,\\ 0 & \text{for } \alpha_n^2\geq k_p^2\,,\end{cases}\qquad f_n := \begin{cases}\omega^2\gamma_n & \text{for } \alpha_n^2<k_s^2\,,\\ 0 & \text{for } \alpha_n^2\geq k_s^2\,.\end{cases}$$

This completes the proof of (4.3). □

The above lemma shows that a solution to the homogenous equation $\mathcal{B}u=0$ can only consist of exponentially decaying modes. Obviously this does not imply the uniqueness in problem (4.1); however, a solvability result can be proved by combining Theorem 1 and Lemma 4.

**Theorem 2** *Assume that the grating profile $\Lambda$ is a Lipschitz curve. Then, for all incident waves of the form (2.1) or (2.2), there exists a solution to the variational problem (3.10) and hence to problem (DP).*

**Proof.** By Theorem 1, equation (4.1) is solvable if its right hand side $\mathcal{F}_0$ is orthogonal (with respect to the duality $(\cdot,\cdot)_{\Omega_b}$) to all solutions $v$ of the homogenous adjoint equation $\mathcal{B}^*v=0$. Here the adjoint operator $\mathcal{B}^*: V_\alpha\to V_\alpha'$ of $\mathcal{B}$ satisfies (cf. (3.8) and (3.23))

$$(\mathcal{B}^*v,\psi)_{\Omega_b} = (v,\mathcal{B}\psi)_{\Omega_b} = \overline{B(\psi,v)} = \int_{\Omega_b}(a_L(v,\overline{\psi})-\omega^2\,v\cdot\overline{\psi})\,dx - \int_{\Gamma_b}\overline{\psi}\cdot\mathcal{T}^*v\,ds\,,\quad\forall\,\psi\in V_\alpha\,,$$

where the adjoint $\mathcal{T}^*$ of $\mathcal{T}$ takes the form (cf. Lemma 1 and (3.9))

$$\mathcal{T}^*v = -\sum_{n\in\mathbb{Z}} W_n^*\,\hat{v}_n\,\exp(i\alpha_n x_1)\quad\text{for}\quad v|_{\Gamma_b} = \sum_{n\in\mathbb{Z}}\hat{v}_n\,\exp(i\alpha_n x_1)\,.$$

Let $v\in V_\alpha$ be an arbitrary solution of the equation $\mathcal{B}^*v=0$, i.e.,

$$B(\psi,v)=0,\quad\forall\,\psi\in V_\alpha\,. \tag{4.4}$$

Then we can extend $v$ to a solution of (2.3) in the unbounded domain $D$ by setting

$$\begin{aligned} v(x) = \sum_{n\in\mathbb{Z}}\Bigg\{ & A_{p,n}\begin{pmatrix}\alpha_n\\ -\overline{\beta}_n\end{pmatrix}\exp(i\,\alpha_n x_1 i\,-\overline{\beta}_n x_2)\\ & + A_{s,n}\begin{pmatrix}-\overline{\gamma}_n\\ -\alpha_n\end{pmatrix}\exp(i\,\alpha_n x_1 - i\,\overline{\gamma}_n x_2)\Bigg\}\,, \end{aligned} \tag{4.5}$$

for $x_2\geq b$, where the Rayleigh coefficients $A_{p,n}$, $A_{s,n}\in\mathbb{C}$ of $v$ are determined by the Fourier coefficients $\hat{v}_n$ of $\exp(-i\alpha x_1)\,v|_{\Gamma_b}$ via the relation

$$\hat{v}_n = \begin{pmatrix}\alpha_n & -\bar{\gamma}_n\\ -\bar{\beta}_n & -\alpha_n\end{pmatrix}\begin{pmatrix}A_{p,n}\,\exp(-i\bar{\beta}_n b)\\ A_{s,n}\,\exp(-i\bar{\gamma}_n b)\end{pmatrix}\,; \tag{4.6}$$

compare (2.12). Note that (4.5) is an expansion into incoming plane elastic waves, and as in the proof of Lemma 1, it can be verified that $Tv=\mathcal{T}^*v$ on $\Gamma_b$. Moreover, arguing as in the

proof of Lemma 4, it follows that each solution $v$ of (4.4) has vanishing Rayleigh coefficients of the incoming modes,

$$A_{p,n} = 0 \quad \text{for} \quad |\alpha_n| < k_p \quad \text{and} \quad A_{s,n} = 0 \quad \text{for} \quad |\alpha_n| < k_s \,. \tag{4.7}$$

Consider first equation (4.1) in the case of an incident pressure wave (2.1) where the right hand side is given by (3.10), (3.11). Then (4.7) implies that $A_{p,0} = A_{s,0} = 0$, hence

$$(\mathcal{F}_0, v)_{\Omega_b} = \int_{\Gamma_b} f_{p,0} \cdot \overline{v}\, ds = 0 \quad \text{for each solution } v \text{ of (4.4)}\,;$$

note that $k_p < k_s$ and $\alpha = \alpha_0 = k_p \sin\theta$ . For an incident shear wave (2.2), where the right hand side of (4.1) is given by (3.10), (3.12), with $\alpha = k_s \sin\theta$, from (4.7) we only obtain $A_{s,0} = 0$ in general. However, this is enough to imply, together with (3.12) and (4.5), that

$$(\mathcal{F}_0, v)_{\Omega_b} = \int_{\Gamma_b} f_{s,0} \cdot \overline{v}\, ds = 2\pi\, \overline{A_{p,0}}\, f_{s,0} \cdot \begin{pmatrix} \alpha \\ -\beta_0 \end{pmatrix} = 0\,,$$

for each solution $v$ of (4.4). Thus the right hand side of equation (4.1) is orthogonal to each solution of (4.4), which finishes the proof of the theorem. □

We next give the main theorem of this section. Supposing the grating surface is given by a Lipschitz graph, we establish the uniqueness in the Dirichlet problem for arbitrary frequencies. Such a uniqueness result has already been obtained in [2] for smooth profile functions; see also [22] in the case of the scalar Helmholtz equation. Our uniqueness proof is essentially based on a (periodic) Rellich identity and follows the approach of [20] in the scalar case. To deal with the Lipschitz boundary, we adapt Nečas' method [26, Chap. 5] of approximating the grating profile by smooth curves.

**Theorem 3** *If $\Lambda$ is a Lipschitz graph, then the operator $\mathcal{B} : V_\alpha \to V_\alpha'$ is invertible. In particular, the variational problem (3.10) and hence problem (DP) have a unique solution for all incident waves of the form (2.1) or (2.2).*

**Proof.** By Theorem 1, we only need to prove the uniqueness. Let $u \in V_\alpha$ be a solution of the homogeneous equation $\mathcal{B}u = 0$, and let $A_{p,n}$, $A_{s,n}$ be its Rayleigh coefficients which are determined by the Fourier coefficients $\hat{u}_n$ of $\exp(-i\alpha x_1)\, u|_{\Gamma_b}$ via the relation (2.12).

**Step 1.** We first prove that the theorem holds for periodic $C^2$ graphs. In this case, $u \in H^2(\Omega_b)^2 \cap V_\alpha$, and using integration by parts, we obtain

$$2\,\mathrm{Re} \int_{\Omega_b} (\Delta^* + \omega^2)\, u \cdot \partial_2 \overline{u}\, dx = \int_{\partial\Omega_b} (\partial_{\mathbf{n}} u \cdot \partial_2 \overline{u} + \partial_{\mathbf{t}} u \cdot \partial_1 \overline{u} + n_2\, \omega^2\, |u|^2)\, ds\,, \tag{4.8}$$

where $\partial_{\mathbf{t}}$ denotes the tangential derivative on the boundary. Analogously, using integration by parts again, we get

$$\begin{aligned} 2\,\mathrm{Re} \int_{\Omega_b} \mathrm{grad\ div}\ u \cdot \partial_2 \overline{u}\, dx = {} & 2\,\mathrm{Re} \int_{\partial\Omega_b} n_1\, \partial_2 u_1\, \mathrm{div}\ \overline{u}\, ds \\ & + \int_{\partial\Omega_b} n_2\, (|\partial_2 u_2|^2 - |\partial_1 u_1|^2)\, ds\,. \end{aligned} \tag{4.9}$$

Then it follows from (4.8) and (4.9) that

$$\begin{aligned} 2\,\mathrm{Re}\int_{\Omega_b}(\Delta^*+\omega^2)u\cdot\partial_2\overline{u}\,dx &= \int_{\partial\Omega_b}\left(\mu(\partial_{\mathbf{n}}u\cdot\partial_2\overline{u}+\partial_{\mathbf{t}}u\cdot\partial_1\overline{u})+n_2\,\omega^2\,|u|^2\right)ds \\ &+(\lambda+\mu)\left\{2\,\mathrm{Re}\int_{\partial\Omega_b}n_1\,\partial_2u_1\mathrm{div}\;\overline{u}\,ds+\int_{\partial\Omega_b}n_2\,(|\partial_2u_2|^2-|\partial_1u_1|^2)\,ds\right\}. \end{aligned} \tag{4.10}$$

Note that (4.10 is a special case of the Rellich identity for the Navier equation proved in [14, Proposition 2]. Since $u$ vanishes on $\Lambda$, we have $\partial_{\mathbf{t}}u=-n_2\,\partial_1u+n_1\,\partial_2u=0$ on $\Lambda$, which implies that

$$n_1\,\partial_2u=n_2\,\partial_1u\,,\quad \partial_1u=n_1\,\partial_{\mathbf{n}}u\quad\text{and}\quad\partial_2u=n_2\,\partial_{\mathbf{n}}u\quad\text{on}\;\;\Lambda\,.$$

Thus the integral over $\Lambda$ on the right hand side of (4.10) takes the form

$$\int_\Lambda\left(\mu\,|\partial_{\mathbf{n}}u|^2\,+(\lambda+\mu)\,|\mathrm{div}\;u|^2\right)n_2\;ds\,. \tag{4.11}$$

Moreover, using the Rayleigh expansion (2.11) of $u$ for $x_2\geq b$, one can verify by careful calculations that the integral over $\Gamma_b$ in (4.10) takes the form

$$\begin{aligned} &\int_{\Gamma_b}\left(\mu\,(|\partial_2u_1|^2-|\partial_1u_2|^2)+(\lambda+2\mu)\,(|\partial_2u_2|^2-|\partial_1u_1|^2)+\omega^2\,|u|^2\right)ds \\ &=4\pi\omega^2\left(\sum_{|\alpha_n|<k_p^2}\beta_n^2\,|A_{p,n}|^2+\sum_{|\alpha_n|<k_s^2}\gamma_n^2\,|A_{s,n}|^2\right), \end{aligned} \tag{4.12}$$

and combining (4.10)–(4.12) gives

$$\begin{aligned} 2\,\mathrm{Re}\int_{\Omega_b}(\Delta^*+\omega^2)u\cdot\partial_2\overline{u}\,dx &= \int_\Lambda\left(\mu\,|\partial_{\mathbf{n}}u|^2\,+(\lambda+\mu)\,|\mathrm{div}\;u|^2\right)n_2\,ds \\ &+4\pi\omega^2\left(\sum_{|\alpha_n|<k_p^2}\beta_n^2\,|A_{p,n}|^2+\sum_{|\alpha_n|<k_s^2}\gamma_n^2\,|A_{s,n}|^2\right). \end{aligned} \tag{4.13}$$

This is just the quasiperiodic version of the Rellich identity (4.10) for our variational problem (4.1). Now we observe that the left hand side of (4.13) vanishes, and by Lemma 4 the boundary term (4.12) vanishes, too. Therefore, (4.13) implies that $\partial_{\mathbf{n}}u=0$ on $\Lambda$, using the fact that $-n_2\geq C>0$ on $\Lambda$. Note that $\Lambda$ is assumed to be the graph of a $C^2$ function. Finally, as a consequence of Holmgren's uniqueness theorem and the unique continuation principle, $u$ must vanish in all of $\Omega$.

**Step 2.** Now we consider the general case that the profile of the diffraction grating is given by the graph

$$\Lambda=\Lambda_f:=\left\{(t,f(t))\in\mathbb{R}^2:t\in[0,2\pi]\right\}$$

where $f$ is a periodic Lipschitz function of period $2\pi$. Again we have to show that a solution $u\in V_\alpha$ to the homogeneous problem (4.1) vanishes in $\Omega_b=\Omega_{\Lambda,b}$; recall that $b>\max\{f(t):t\in[0,2\pi]\}$. Consider the inhomogeneous boundary value problem

$$\begin{aligned} &(\Delta^*+\omega^2+i)\,v=g:=iu\quad\text{in}\;\;\Omega_b\,, \\ &v|_\Lambda=0\,,\quad Tv-\mathcal{T}(\omega,\alpha)\,v=0\quad\text{on}\;\;\Gamma_b\,. \end{aligned} \tag{4.14}$$

One easily verifies that the operator $\mathcal{B}_1 : V_\alpha \to V_\alpha'$ generated by the sesquilinear form

$$B_1(v,\varphi) := \int_{\Omega_b} \big(a_L(v,\overline{\varphi}) - (\omega^2 + i)v \cdot \overline{\varphi}\big)\, dx - \int_{\Gamma_b} \overline{\varphi} \cdot \mathcal{T}(\omega,\alpha)v\, ds$$

is invertible. Indeed, as in Theorem 1 it follows that $\mathcal{B}_1$ is Fredholm with index zero, and arguing as in the proof of Lemma 4 we obtain that $\operatorname{Im} B_1(w,w) = 0$, $w \in V_\alpha$, implies that

$$4\pi\omega^2 \left( \sum_{|\alpha_n|<k_p^2} \beta_n^2\, |\tilde{A}_{p,n}|^2 + \sum_{|\alpha_n|<k_s^2} \gamma_n^2\, |\tilde{A}_{s,n}|^2 \right) + \int_{\Omega_b} |w|^2\, dx = 0\,,$$

where $\tilde{A}_{p,n}$, $\tilde{A}_{s,n}$ are the Rayleigh coefficients of a solution $w$ to the homogeneous problem (4.14) (with $g = 0$). Hence $w$ must vanish in $\Omega_b$.

Therefore $v = u$ is the unique solution of the inhomogeneous problem (4.14) in $V_\alpha = V_\alpha(\Omega_b)$. Following the proof of [26, Thm. 5.1.1], we choose $C^\infty$ profiles $\Lambda_j = \Lambda_{f_j}$ such that the Lipschitz constants of $f_j$ are uniformly bounded in $j$, and

$$\Omega_b^j = \Omega_{\Lambda_j,b} \subset \Omega_b\,, \quad \max\{|f_j(t) - f(t)| : t \in [0,2\pi]\} \to 0\,, \quad \text{as} \quad j \to \infty\,. \tag{4.15}$$

Let $u^j \in V_\alpha(\Omega_b^j)$ be the solution of the problem (4.14) for $\Omega_b^j$, which is unique by step 1. Extending $u^j$ by zero to $\Omega_b \setminus \Omega_b^j$, we regard $u^j \in V_\alpha(\Omega_b)$ as a solution of the problem (4.14) with the right hand side $g^j \in L^2(\Omega_b)$ where $g^j$ denotes the extension of $-iu|_{\Omega_b^j}$ by zero. Then, from (4.15) we have $g^j \to g$ in $L^2(\Omega_b)$, and the invertibility of $\mathcal{B}_1$ implies

$$u^j \to u \quad \text{in} \quad V_\alpha(\Omega_b)\,, \quad j \to \infty\,. \tag{4.16}$$

We rewrite the boundary value problem for $u^j$ as

$$\begin{aligned} &(\Delta^* + \omega^2)\, u^j = h^j := i(u - u^j) \quad \text{in} \quad \Omega_b^j\,, \\ &v|_{\Lambda_j} = 0\,, \quad Tu^j - \mathcal{T}(\omega,\alpha)\, u^j = 0 \quad \text{on} \quad \Gamma_b\,. \end{aligned} \tag{4.17}$$

Note that $u^j \in V_\alpha(\Omega_b^j) \subset V_\alpha(\Omega_b)$ can be extended to a radiating solution of the Navier equation in the unbounded domain $D$, using the expansion (2.11) with the Rayleigh coefficients $A_{p,n}^j$, $A_{s,n}^j$ determined by the Fourier coefficients $\hat{u}_n^j$ of $\exp(-i\alpha x_1)\, u^j(x_1,b)$ via the relation (2.12). Applying the periodic Rellich identity (4.13) to problem (4.17), we obtain

$$\begin{aligned} 2\operatorname{Re} \int_{\Omega_b} h^j \cdot \partial_2 \overline{u^j}\, dx &= \int_{\Lambda_j} \left(\mu\, |\partial_{\mathbf{n}} u^j|^2 \, + (\lambda+\mu)\, |\operatorname{div} u^j|^2\right) n_2\, ds \, + \, I_j\,, \\ I_j &:= 4\pi\omega^2 \left( \sum_{|\alpha_n|<k_p^2} \beta_n^2\, |A_{p,n}^j|^2 + \sum_{|\alpha_n|<k_s^2} \gamma_n^2\, |A_{s,n}^j|^2 \right)\,. \end{aligned} \tag{4.18}$$

Moreover, setting $\varphi = u^j$ in the variational formulation of (4.17),

$$\begin{aligned} B(u^j,\varphi) &:= \int_{\Omega_b} \big(a_L(u^j,\overline{\varphi}) - \omega^2 u^j \cdot \overline{\varphi}\big)\, dx - \int_{\Gamma_b} \overline{\varphi} \cdot \mathcal{T}(\omega,\alpha) u^j\, ds \\ &= -\int_{\Omega_b} h^j \cdot \overline{u^j}\, dx\,, \ \varphi \in V_\alpha(\Omega_b)\,, \end{aligned}$$

and taking imaginary parts, we get (cf. (4.3))

$$I_j = -\mathrm{Im}\, B(u^j, u^j) = \mathrm{Im} \left( \int_{\Omega_b} h^j \cdot \overline{u^j}\, dx \right) ,$$

which implies $I_j \to 0$ as $j \to \infty$ in view of (4.16) and the definition of $h^j$ in (4.17). From (4.18) we then have, on using the uniform estimate $-n_2 \geq C > 0$ on $\Lambda_j$ for all $j \in \mathbb{N}$,

$$\int_{\Lambda_j} |\partial_{\mathbf{n}} u^j|^2\, ds \to 0\,, \quad j \to \infty\,. \tag{4.19}$$

We may identify the spaces $L^2(\Lambda_j)$ and $L^2(\Lambda)$ with $L^2(0, 2\pi)$ via the norm

$$\|v \circ f_j\|_{L^2(0,2\pi)} = \left( \int_0^{2\pi} |v(t, f_j(t))|^2\, dt \right)^{1/2} , \quad v \in L^2(\Lambda_j)\,,$$

with $\Lambda_0 = \Lambda$, $f_0 = f$, which is a uniformly equivalent norm with respect to $j$. From (4.19) we get $\partial_{\mathbf{n}} u^j|_{\Lambda_j} \to 0$ in $L^2(0, 2\pi)^2$, which together with $u^j|_{\Lambda_j} = 0\,, j \in \mathbb{N}$, implies that $Tu^j|_{\Lambda_j} \to 0$ in $L^2(0, 2\pi)^2$. Here $T$ denotes the traction operator defined in (3.3). Moreover, then it follows from (4.16) and the relation $\varphi|_{\Lambda_j} \to \varphi|_{\Lambda}$ in $L^2(0, 2\pi)^2$ (cf. Lemma 2.4.5 in [26]) that, by passing to the limit in Betti's identity,

$$\int_{\Lambda_j} \overline{\varphi} \cdot Tu^j\, ds = B(u^j, \varphi) + \int_{\Omega_b} h^j \cdot \overline{u^j}\, dx\,, \quad \forall\ \varphi \in H^1_\alpha(\Omega_b)^2\,,$$

we obtain that $B(u, \varphi) = 0$ for all $\varphi \in H^1_\alpha(\Omega_b)^2$, hence $Tu|_\Lambda = 0$. Note that the trace $Tu|_\Lambda$ in the sense of $H^{-1/2}$ is defined by

$$\int_{\Lambda} \overline{\varphi} \cdot Tu\, ds = B(u, \varphi)\,, \quad \forall\ \varphi \in H^1_\alpha(\Omega_b)^2\,.$$

Finally, since the Dirichlet and Neumann data of $u$ vanish on $\Lambda$, we obtain $u = 0$ in $\Omega_b$ by the unique continuation principle. □

**Remark 3** *(i) Assume that $\Lambda$ is given by a piecewise smooth graph having only a finite number of corner points (with non-zero angles). Then the uniqueness already follows from the arguments in step 1 of the above proof. In that case each solution to problem (3.10) satisfies $u \in H^{3/2+\epsilon}(\Omega_b)^2$ for some $\epsilon > 0$, so that the integration by parts in the Rellich identity (4.10) is justified. Moreover, then the uniqueness result extends to the case that the $x_2$-component of the normal, $-n_2$, vanishes on a subset of $\Lambda$ and has a positive lower bound on the other parts, e.g, in the case of rectangular groove gratings where the profile consists of a finite number of horizontal and vertical segments only.*

*(ii) If the grating profile $\Lambda$ is given by a general Lipschitz curve, we can only prove the uniqueness for all sufficiently small frequencies $\omega$. To see this, we decompose the operator $\mathcal{B}$ into the sum $\mathcal{A} + \mathcal{K}$, where $\mathcal{K}$ is the operator defined in (3.27) and $\mathcal{A}$ is defined by*

$$(\mathcal{A}v, \varphi)_{\Omega_b} = \int_{\Omega_b} a_L(v, \overline{\varphi})\, dx - \int_{\Gamma_b} \overline{\varphi} \cdot \mathcal{T} v\, ds\,, \quad \forall\ v, \varphi \in V_\alpha\,. \tag{4.20}$$

*From Lemma 1 we get, for any $v \in V_\alpha$,*

$$\mathrm{Re} \left\{ - \int_{\Gamma_b} \overline{v} \cdot \mathcal{T} v\ ds \right\} = 2\pi \sum_{n \neq 0} \mathrm{Re}\, (W_n \hat{v}_n, \hat{v}_n)_{\mathbb{C}^2} + 2\pi\, \mathrm{Re}\, (W_0 \hat{v}_0, \hat{v}_0)_{\mathbb{C}^2}\,, \tag{4.21}$$

*where $\hat{v}_n$ are the Fourier coefficients of* $\exp(-i\alpha x_1)\, v(x_1, b)$. *For the last term in (4.21) we have*

$$|(W_0\hat{u}_0, \hat{v}_0)_{\mathbb{C}^2}| = \mathcal{O}(\omega)|\hat{v}_0|^2 \quad as \quad \omega \to 0\,;$$

*see the definition of $W_n$ in (3.6). Then it follows from Lemma 2 (ii) applied to the second term in (4.21) and from estimate (3.26) that the operator $\mathcal{A}$ defined in (4.20) is coercive, i.e.,*

$$|\mathrm{Re}\,(\mathcal{A}v, v)_{\Omega_b}| \geq C\,\|v\|^2_{V_\alpha}\,, \quad \forall\ v \in V_\alpha\,,$$

*if $\omega$ is sufficiently small. Here the constant $C > 0$ does not depend on $\omega$. Finally, we have*

$$\|\mathcal{K}\|_{V_\alpha \to V'_\alpha} = \mathcal{O}(\omega^2) \quad as \quad \omega \to 0\,,$$

*which implies that the operator $\mathcal{B} = \mathcal{A} + \mathcal{K}$ is always invertible if $\omega$ is sufficiently small.*

*(iii) Relying on the above uniqueness result for small frequencies, it is possible to prove the invertibility of the operator $\mathcal{B}$ for all frequencies $\omega > 0$ with the possible exception of a discrete set in $(0, \infty)$; see Theorem 6 below in the case of the transmission problem.*

To conclude this section, we present an existence and uniqueness result in the case where the Dirichlet condition (2.4) in the diffraction problem (DP) is replaced by the mixed Dirichlet and Robin boundary conditions:

$$u = 0 \ \text{ on } \ \Lambda_D\,, \quad Tu - i\eta u = 0 \ \text{ on } \ \Lambda_I\,. \tag{4.22}$$

We assume that $\Lambda$ has a Lipschitz dissection $\Lambda = \Lambda_D \cup \Sigma \cup \Lambda_I$, where $\Lambda_D$ and $\Lambda_I$ are two disjoint and relative open subsets of $\Lambda$ having $\Sigma$ as their common boundary (see [24, p. 99]). On $\Lambda_I$, $\eta \in \mathbb{C}$ is assumed to be a constant with $\mathrm{Re}\,\eta > 0$. In this case, the proof of uniqueness becomes easy because of the impedance coefficient $\eta$ on $\Lambda_I$. The boundary conditions (4.22) lead to the following variational problem in the bounded periodic cell $\Omega_b$: Find $u \in E_\alpha := \{v \in H^1_\alpha(\Omega_b)^2 : v = 0 \text{ on } \Lambda_D\}$ such that

$$\begin{aligned} \int_{\Omega_b} \big(a_L(u, \overline{\varphi}) - \omega^2 u \cdot \overline{\varphi}\big)\, dx - i\eta \int_{\Lambda_I} u \cdot \overline{\varphi}\, ds - \int_{\Gamma_b} \overline{\varphi} \cdot \mathcal{T}u\, ds \\ = \int_{\Gamma_b} f_0 \cdot \overline{\varphi}\, ds\,, \quad \forall\ \varphi \in E_\alpha\,, \end{aligned} \tag{4.23}$$

where $f_0$ is defined by (3.11) for an incident pressure wave, and by (3.12) for an incident shear wave.

**Theorem 4** *If $\Lambda_I \neq \emptyset$ , then there always exists a unique solution $u \in E_\alpha$ to the variational problem (4.23).*

**Proof.** It follows from the proof of Theorem 1 that the operator generated by the sesquilinear form of (4.23) is a Fredholm operator with index zero. Thus it is enough to prove the uniqueness. Letting $f_0 = 0$, $u = \varphi$ and taking imaginary parts in (4.23), we have (cf. (4.3))

$$-\mathrm{Re}\,\eta \int_{\Lambda_I} |u|^2\, ds = \mathrm{Im} \int_{\Gamma_b} \overline{u} \cdot \mathcal{T}u\, ds \geq 0\,,$$

which implies that $u = 0$ on $\Lambda_I$. This means that $u$ has vanishing Dirichlet and Neumann data on $\Lambda_I$, and as a consequence of the unique continuation principle, $u = 0$ on $\Omega_b$. □

# 5 Solvability results for transmission gratings

The aim of this section is to provide a solvability theory of quasiperiodic transmission problems for the two-dimensional Navier system. Suppose the whole $(x_1, x_2)$-plane is filled with elastic materials which are homogenous above and below a certain periodic interface $\Lambda$. We assume throughout this section that $\Lambda$ is a $2\pi$-periodic Lipschitz curve. Let $D^\pm$ be the unbounded domains above and below $\Lambda$ respectively. We assume that the Lamé coefficients $\mu^\pm, \lambda^\pm$ in $D^\pm$ are constants satisfying $\mu^\pm > 0$, $\lambda^\pm + \mu^\pm > 0$, and that the mass densities $\rho^\pm$ are positive constants in these subdomains. Let

$$k_p^\pm := \omega \sqrt{\rho^\pm/(2\mu^\pm + \lambda^\pm)}\,, \quad k_s^\pm := \omega \sqrt{\rho^\pm/\mu^\pm} \tag{5.1}$$

be the corresponding compressional and shear wave numbers respectively. As in Section 2 we assume that a time harmonic plane elastic wave $u^{in}$ with incident angle $\theta$ is incident on $\Lambda$ from $D^+$, which is either an incident pressure wave of the form (2.1), or an incident shear wave of the form (2.2), with $k_p$, $k_s$ replaced by $k_p^+$, $k_s^+$. Then we are looking for the total displacenent field $u$,

$$u = u^{in} + u^+ \quad \text{in} \quad D^+\,, \quad u = u^- \quad \text{in} \quad D^-\,, \tag{5.2}$$

where the scattered fields $u^\pm$ satisfy the corresponding $\alpha$-quasiperiodic Navier equations

$$(\Delta^* + \omega^2\rho^\pm)\, u^\pm = 0 \quad \text{in} \quad D^\pm\,, \quad \text{with} \quad u^\pm(x_1 + 2\pi, x_2) = \exp(2i\alpha\pi x_1)\, u^\pm(x_1, x_2)\,, \tag{5.3}$$

and either $\alpha := k_p^+ \sin\theta$ for an incident pressure wave, or $\alpha := k_s^+ \sin\theta$ for an incident shear wave. On the interface the continuity of the displacement and the stress lead to the transmission conditions

$$u^{in} + u^+ = u^-\,, \quad T^+(u^{in} + u^+) = T^- u^- \quad \text{on} \quad \Lambda\,, \tag{5.4}$$

where the corresponding stress operators are defined as in (3.3), with $\mu$, $\lambda$ replaced by $\mu^\pm$, $\lambda^\pm$. Finally, we need to impose appropriate radiation conditions on the scattered fields as $x_2 \to \pm\infty$. Introduce the notation

$$\Lambda^+ := \max_{(x_1,x_2)\in\Lambda} x_2\,, \quad \Lambda^- := \min_{(x_1,x_2)\in\Lambda} x_2\,,$$

let $\alpha_n := \alpha + n$, and define $\beta_n^\pm$ and $\gamma_n^\pm$ as in (2.9) with $k_p$, $k_s$ replaced by $k_p^\pm$, $k_s^\pm$. Then we insist that the scattered fields $u^\pm$ admit the following Rayleigh expansions (cf. (2.11)), for $x_2 \gtrless \Lambda^\pm$:

$$\begin{aligned} u^\pm(x) = \sum_{n\in\mathbb{Z}} \Bigg\{ & A_{p,n}^\pm \begin{pmatrix} \alpha_n \\ \pm\beta_n^\pm \end{pmatrix} \exp(i\alpha_n x_1 \pm i\beta_n^\pm x_2) \\ & + A_{s,n}^\pm \begin{pmatrix} \pm\gamma_n^\pm \\ -\alpha_n \end{pmatrix} \exp(i\alpha_n x_1 \pm i\gamma_n^\pm x_2) \Bigg\}\,, \end{aligned} \tag{5.5}$$

where for any $b^+ > \Lambda^+$, $b^- < \Lambda^-$, the Rayleigh coefficients are related with the Fourier coefficients $\hat{u}_n^\pm$ of $\exp(-i\alpha x_1)\, u^\pm(x_1, \pm b)$ by the relations (cf. (2.12))

$$\hat{u}_n^\pm = D_n^\pm\, A_n^\pm\,, \quad D_n^\pm := \begin{pmatrix} \alpha_n & \pm\gamma_n^\pm \\ \pm\beta_n^\pm & -\alpha_n \end{pmatrix}\,, \quad A_n^\pm := \begin{pmatrix} A_{p,n}^\pm \exp(\pm i\beta_n^\pm b^\pm) \\ A_{s,n}^\pm \exp(\pm i\gamma_n^\pm b^\pm) \end{pmatrix}\,. \tag{5.6}$$

Note that $\det D_n^{\pm} \neq 0$ for all $n \in \mathbb{Z}$. The diffractition problem for transmission gratings can now be formulated as the following boundary value problem.

**Transmission problem (TP):** Given a grating profile curve $\Lambda \subset \mathbb{R}^2$ (which is $2\pi$-periodic in $x_1$) and an incident plane pressure or shear wave $u^{in}$, find a vector function $u \in H^1_{loc}(\mathbb{R}^2)^2$ that satisfies (5.2)–(5.5).

Following the approach of Section 3, we reduce the problem (TP) to a variational problem in a bounded periodic cell in $\mathbb{R}^2$, enforcing the transmission and radiation conditions. Introduce artificial boundaries

$$\Gamma^{\pm} := \{(x_1, b^{\pm}) : \ 0 \leq x_1 \leq 2\pi\}\,, \quad b^+ > \Lambda^+\,, \quad b^- < \Lambda^-$$

and the bounded domains

$$\Omega = \Omega_{b^-,b^+} := (0, 2\pi) \times (b^-, b^+)\,, \quad \Omega^{\pm} := D^{\pm} \cap \Omega\,.$$

The DtN maps $\mathcal{T}^{\pm}$ on the artificial boundaries $\Gamma^{\pm}$ have the Fourier series representations (cf. (5.6) and Lemma 1)

$$\mathcal{T}^{\pm} u^{\pm} := -\sum_{n\in\mathbb{Z}} W_n^{\pm} \hat{u}_n^{\pm} \exp(i\alpha_n x_1)\ , \quad u^{\pm} = \sum_{n\in\mathbb{Z}} \hat{u}_n^{\pm} \exp(i\alpha_n x_1) \in H^{1/2}_{\alpha}(\Gamma^{\pm})^2\,, \tag{5.7}$$

where the matrices $W_n^{\pm} = W_n^{\pm}(\omega, \alpha)$ take the form (cf. (3.6))

$$W_n^{\pm} := \frac{1}{i} \begin{pmatrix} \omega^2\rho^{\pm}\beta_n^{\pm}/d_n^{\pm} & 2\mu^{\pm}\alpha_n - \omega^2\rho^{\pm}\alpha_n/d_n^{\pm} \\ -2\mu^{\pm}\alpha_n + \omega^2\rho^{\pm}\alpha_n/d_n^{\pm} & \omega^2\rho^{\pm}\gamma_n^{\pm}/d_n^{\pm} \end{pmatrix}\,, \quad d_n^{\pm} := \alpha_n^2 + \beta_n^{\pm}\gamma_n^{\pm}\,. \tag{5.8}$$

Applying the first Betti formula on each subdomain $\Omega^{\pm}$ to a solution of (TP), and using the transmission conditions (5.4) at the interface and the DtN operators (5.7), we obtain the following variational formulation of (TP) on the bounded domain $\Omega$: Find $u \in H^1_{\alpha}(\Omega)^2$ such that

$$\begin{aligned} B(u,\varphi) &:= \int_{\Omega} \left(a_L(u,\overline{\varphi}) - \omega^2\rho\, u\cdot\overline{\varphi}\right) dx - \int_{\Gamma^+} \overline{\varphi}\cdot\mathcal{T}^+ u\, ds - \int_{\Gamma^-} \overline{\varphi}\cdot\mathcal{T}^- u\, ds \\ &= \int_{\Gamma^+} f_0\cdot\overline{\varphi}\, ds\,, \quad \forall\ \varphi \in H^1_{\alpha}(\Omega)^2\,. \end{aligned} \tag{5.9}$$

Here the domain integral is understood as the sum of the integrals

$$\int_{\Omega^{\pm}} \left(a_L^{\pm}(u,\overline{\varphi}) - \omega^2\rho^{\pm}\, u\cdot\overline{\varphi}\right) dx$$

where the bilinear forms $a_L^{\pm}$ are defined as in (3.2), with $\mu$, $\lambda$ replaced by $\mu^{\pm}$, $\lambda^{\pm}$, and the right hand side is given by (cf. (3.10)–(3.12))

$$f_0 = f_{p,0} := \frac{2i\beta_0^+ k_p^+(\lambda^+ + 2\mu^+)}{d_0^+} \begin{pmatrix} -\alpha \\ \gamma_0^+ \end{pmatrix} \exp(i\alpha x_1 - i\beta_0^+ b^+) \tag{5.10}$$

for an incident pressure wave, and

$$f_0 = f_{s,0} := -\frac{2i\gamma_0^+ k_s^+ \mu^+}{d_0^+} \begin{pmatrix} \beta_0^+ \\ \alpha \end{pmatrix} \exp(i\alpha x_1 - i\gamma_0^+ b^+) \tag{5.11}$$

for an incident shear wave. As in (3.23), the sesquilinear form $B$ defined in (5.9) generates a continuous linear operator $\mathcal{B}$ from $H^1_\alpha(\Omega)^2$ into its dual $(H^1_\alpha(\Omega)^2)'$, with respect to the pairing $(u,\varphi)_\Omega = \int_\Omega u\cdot\bar\varphi$, via

$$B(u,\varphi) = (\mathcal{B}u,\varphi)_\Omega\,, \quad \forall\, u\,,\ \varphi \in H^1_\alpha(\Omega)^2\,. \tag{5.12}$$

The following lemma extends Lemma 4 to the transmission case.

**Lemma 5** *Let $\mathcal{B}$ be the operator defined in (5.12). If $u \in H^1_\alpha(\Omega)^2$ satisfies $\mathcal{B}u = 0$, then*

$$A^\pm_{p,n} = 0 \ \ \textit{for}\ \ |\alpha_n| < k_p^\pm \ \ \textit{and}\ \ A^\pm_{s,n} = 0 \ \ \textit{for}\ \ |\alpha_n| < k_s^\pm\,, \tag{5.13}$$

*where $A^\pm_{p,n}$, $A^\pm_{s,n}$ are the Rayleigh coefficients of $u$ defined via (5.6) with the Fourier coefficients $\hat u_n^\pm$ of $\exp(-i\alpha x_1)\,u(x_1,b^\pm)$.*

**Proof.** As in the proof of Lemma 4, we can verify the identity

$$\begin{aligned} \operatorname{Im} B(u,u) &= -\operatorname{Im}\int_{\Gamma^+} \overline{u}\cdot\mathcal{T}^+u\,ds - \operatorname{Im}\int_{\Gamma^-}\overline{u}\cdot\mathcal{T}^-u\,ds \qquad (5.14)\\ &= -2\pi\omega^2\left(\sum_{|\alpha_n|<k_p^+}\beta_n^+\,|A^+_{p,n}|^2 + \sum_{|\alpha_n|<k_s^+}\gamma_n^+\,|A^+_{s,n}|^2 + \sum_{|\alpha_n|<k_p^-}\beta_n^-\,|A^-_{p,n}|^2 + \sum_{|\alpha_n|<k_s^-}\gamma_n^-\,|A^-_{s,n}|^2\right), \end{aligned}$$

and taking imaginary parts in the variational equation (5.9) with $\varphi = u$ and $f_0 = 0$, we then obtain the relation (5.13). □

The following result extends Theorems 1 and 2 to the transmission problem.

**Theorem 5** *(i) The sesquilinear form $B$ defined by (5.9) is strongly elliptic over $H^1_\alpha(\Omega)^2$, and the operator $\mathcal{B}$ defined in (5.12) is Fredholm with index zero.*
*(ii) For all incident plane pressure or shear waves, there exists a solution to the variational problem (5.9) and hence to problem (TP).*

**Proof.** (i) It follows from the estimate (3.24) applied to the subdomains $\Omega^\pm$ and from Korn's inequality (see Lemma 3) on $\Omega$ that there exist positive constants $c$, $C$ such that

$$\int_\Omega \big(a_L(u,\bar u) + c\,|u|^2\big)\,dx \ge C\,||u||^2_{H^1(\Omega)^2}\,, \quad \forall\ u\in H^1_\alpha(\Omega)^2\,. \tag{5.15}$$

As in the proof of Theorem 1, from Lemma 2 (i) we obtain

$$\operatorname{Re}\left\{-\int_{\Gamma^\pm}\overline{u}\cdot\mathcal{T}_1^\pm u\,ds\right\} \ge 0\,, \quad \forall\ u\in H^1_\alpha(\Omega)^2 \tag{5.16}$$

by setting (cf. (5.7), (5.8))

$$\mathcal{T}_1^\pm u := -\sum_{|n|\ge n_0} W_n^\pm\,\hat u_n^\pm\,, \quad \mathcal{T}_0^\pm := \mathcal{T}^\pm - \mathcal{T}_1^\pm\,,$$

where $\hat u_n^\pm$ are the Fourier coefficients of $\exp(-i\alpha x_1)\,u(x_1,b^\pm)$ and $n_0$ is sufficiently large. Note that the operators $\mathcal{T}_0^\pm$ are finite dimensional. Moreover, the operator $\mathcal{K} : H^1_\alpha(\Omega)^2 \to (H^1_\alpha(\Omega)^2)'$ defined by

$$(\mathcal{K}u,\varphi)_\Omega = -(\omega^2+c)\int_\Omega u\cdot\bar\varphi\,dx\,, \quad \forall\ u,\varphi\in H^1_\alpha(\Omega)^2$$

is compact. Now the strong ellipticity of the form $B$ defined in (5.9) follows from (5.15) and (5.16).

(ii) To ensure existence of solutions, we only need to prove that the relation

$$\int_{\Gamma^+} f_0 \cdot \overline{v}\, ds = 0 \tag{5.17}$$

holds for all $v \in H^1_\alpha(\Omega)^2$ in the null space of the adjoint operator, i.e., $\mathcal{B}^* v = 0$, where $f_0$ is the right hand side defined in (5.10) and (5.11) respectively; see the proof of Theorem 2. Here the adjoint $\mathcal{B}^*$ of $\mathcal{B}$ satisfies (cf. (5.9) and (5.12)), for all $\psi \in H^1_\alpha(\Omega)^2$,

$$(\mathcal{B}^* v, \psi)_\Omega = \overline{B(\psi, v)} = \int_\Omega (a_L(v, \overline{\psi}) - \omega^2\, v \cdot \overline{\psi})\, dx - \int_{\Gamma^+} \overline{\psi} \cdot (\mathcal{T}^+)^* v\, ds - \int_{\Gamma^-} \overline{\psi} \cdot (\mathcal{T}^-)^* v\, ds\,,$$

where the adjoints $(\mathcal{T}^\pm)^*$ take the form (cf. (5.7) and (5.8))

$$(\mathcal{T}^\pm)^* v = -\sum_{n\in\mathbb{Z}} (W_n^\pm)^*\, \hat{v}_n^\pm\, \exp(i\alpha_n x_1) \quad \text{for} \quad v|_{\Gamma^\pm} = \sum_{n\in\mathbb{Z}} \hat{v}_n^\pm\, \exp(i\alpha_n x_1)\,.$$

Let $v \in H^1_\alpha(\Omega)^2$ be an arbitrary solution of the equation $\mathcal{B}^* v = 0$, i.e.,

$$B(\psi, v) = 0\,, \quad \forall\ \psi \in H^1_\alpha(\Omega)^2\,. \tag{5.18}$$

We can extend $v$ to a solution of (2.3) in $\mathbb{R}^2$ by using the Rayleigh expansions (5.5) for $x_2 \geq b^+$ and $x_2 \leq b^-$ respectively, with $\beta_n^\pm$, $\gamma_n^\pm$ replaced by $-\overline{\beta_n^\pm}$, $-\overline{\gamma_n^\pm}$. Here the Rayleigh coefficients $A_{p,n}^\pm$, $A_{s,n}^\pm$ of $v$ are determined by the Fourier coefficients $\hat{v}_n^\pm$ of $\exp(-i\alpha x_1)\, v|_{\Gamma^\pm}$ via the relations (5.6), again with $\beta_n^\pm$, $\gamma_n^\pm$ replaced by $-\overline{\beta_n^\pm}$, $-\overline{\gamma_n^\pm}$; compare (4.5) and (4.6).

Arguing as in the proof of Lemma 5, we now obtain that each solution $v$ of (5.18) has vanishing Rayleigh coefficients of the incoming modes in $D^+$,

$$A_{p,n}^+ = 0 \ \text{ for } \ |\alpha_n| < k_p^+ \ \text{ and } \ A_{s,n}^+ = 0 \ \text{ for } \ |\alpha_n| < k_s^+\,. \tag{5.19}$$

Finally, recalling the definition of $f_0$ (see (5.10) or (5.11)), the relation (5.17) follows from (5.19) as in the proof of Theorem 2. □

Following the approach in [18], [19], [29] in the case of electromagnetic diffraction gratings, we finally establish some uniqueness results for the variational problem (5.9) and hence for the boundary value problem (TP).

**Theorem 6** *If $u^{in}$ is an incident pressure wave of the form (2.1) (with $k_p = k_p^+$), then*
*(i) There exists $\omega_0 > 0$ such that the variational problem (5.9) admits a unique solution $u \in H^1_\alpha(\Omega)^2$ for all incident angles and for all frequencies $\omega \in (0, \omega_0]$.*
*(ii) For all but a sequence of countable frequencies $\omega_j$, $\omega_j \to \infty$, the variational problem (5.9) (with fixed incidence angle $\theta$) admits a unique solution $u \in H^1_\alpha(\Omega)^2$.*

**Proof.** (i) Assuming there exists a solution $u \in H^1_\alpha(\Omega)^2$ to the homogenous problem (5.9), so that $B(u,u) = 0$, we shall prove that $u = 0$ in $\Omega$. Applying Lemma 2 (ii) to the DtN operators (5.7), we obtain that, for all $\omega \in (0, \omega_0]$ with $\omega_0$ sufficiently small,

$$\begin{aligned} I \ &:= \ \mathrm{Re}\left( -\int_{\Gamma^+} \overline{u} \cdot \mathcal{T}^+ u\, ds - \int_{\Gamma^-} \overline{u} \cdot \mathcal{T}^- u\, ds \right) \\ &= \ 2\pi \sum_{n\in\mathbb{Z}} \left( \mathrm{Re}\,(W_n^+ \hat{u}_n^+, \hat{u}_n^+)_{\mathbb{C}^2} + \mathrm{Re}\,(W_n^- \hat{u}_n^-, \hat{u}_n^-)_{\mathbb{C}^2} \right) \qquad (5.20) \\ &\geq \ C \sum_{n\neq 0} \left( |n| (|\hat{u}_n^+|^2 + |\hat{u}_n^-|^2) \right) + \mathrm{Re}\,(W_0^+ \hat{u}_0^+, \hat{u}_0^+)_{\mathbb{C}^2} + \mathrm{Re}\,(W_0^- \hat{u}_0^-, \hat{u}_0^-)_{\mathbb{C}^2}\,, \end{aligned}$$

where $\hat{u}_n^{\pm}$ are the Fourier coefficients of $\exp(-i\alpha x_1)\, u(x_1, b^{\pm})$. Here and in the following $C$ denotes various positive constants not depending on $u$ and $\omega$. Let $A_{p,n}^{\pm}$, $A_{p,n}^{\pm}$ be the Rayleigh coefficients of $u$ which are defined via the relations (5.6).

Since $k_s^+ > k_p^+$, it follows from Lemma 5 that $A_{p,0}^+ = A_{s,0}^+ = 0$, which implies $\hat{u}_0^+ = 0$. Recall that

$$\beta_n^{\pm} = \sqrt{(k_p^{\pm})^2 - \alpha_n^2}\,, \quad \gamma_n^{\pm} = \sqrt{(k_s^{\pm})^2 - \alpha_n^2}\,, \quad \alpha_n = n + k_p^+ \sin\theta\,, \quad \alpha = \alpha_0 \tag{5.21}$$

in the case of an incident pressure wave with incidence angle $\theta$, where the square roots are chosen such that their imaginary parts are non-negative. Therefore, the estimate (5.20) can be written as

$$I \geq C \left( ||u||^2_{H_\alpha^{1/2}(\Gamma^+)^2} + \sum_{n \neq 0} |n| |\hat{u}_n^-|^2 \right) + \mathrm{Re}\,(W_0^- \hat{u}_0^-, \hat{u}_0^-)_{\mathbb{C}^2}\,, \quad \forall\ \omega \in (0, \omega_0] \tag{5.22}$$

Furthermore, from the definition of $W_0^-$ in (5.8), we have the bound

$$|(W_0^- \hat{u}_0^-, \hat{u}_0^-)_{\mathbb{C}^2}| \leq C\,\omega\, |\hat{u}_0^-|^2 \leq C\,\omega ||u||^2_{H^1(\Omega)^2}\,. \tag{5.23}$$

Combining the estimates (5.20), (5.22) and (5.23) and using the definition of the sesquilinear form $B$ in (5.9), we obtain for $\omega \in (0, \omega_0]$

$$0 = \mathrm{Re}\, B(u,u) \geq \int_\Omega a_L(u, \overline{u})\, dx + C\, ||u||^2_{H_\alpha^{1/2}(\Gamma^+)^2} - C\,\omega\, ||u||^2_{H^1(\Omega)^2}\,,$$

which leads to

$$\int_\Omega a_L(u, \overline{u})\, dx + C\, ||u||^2_{H_\alpha^{1/2}(\Gamma^+)^2} \leq C\,\omega\, ||u||^2_{H^1(\Omega)^2}\,. \tag{5.24}$$

Now it follows from the estimate (3.24) applied to the subdomains $\Omega^{\pm}$ and from Remark 3 applied to $\Omega$ that the square root of the left hand side of the inequality (5.24) is an equivalent norm on $H_\alpha^1(\Omega)^2$. Therefore, it follows that $u = 0$ in $\Omega$ if the frequency $\omega$ is sufficiently small.

(ii) To study the uniqueness for arbitrary frequencies $\omega$ using analytic Fredholm theory, it is necessary to replace equation (5.9) on the $\omega$-dependent space $H_\alpha^1(\Omega)^2$ by an equivalent variational problem acting on the same energy space,

$$V = H^1_{per}(\Omega)^2 := \{u \in H^1(\Omega)^2 :\ u \ \text{ is } 2\pi - \text{periodic in } x_1\}\,,$$

for each $\omega$. Recall that (cf. (5.1) and (5.21))

$$\alpha = k_p^+ \sin\theta = \omega \sin\theta \sqrt{\rho^+/(2\mu^+ + \lambda^+)}\,. \tag{5.25}$$

So, instead of the operator $\mathcal{B} : H_\alpha^1(\Omega)^2 \to (H_\alpha^1(\Omega)^2)'$ defined by (5.12), we consider the operator

$$\mathcal{B}_\alpha : V \to V'\,, \quad \mathcal{B}_\alpha u := \exp(-i\alpha x_1)\, \mathcal{B}(\exp(i\alpha x_1)\, u)\,, \quad u \in V\,, \tag{5.26}$$

where $V'$ is the dual of $V$ with respect to the pairing $(\cdot,\cdot)_\Omega$. Note that $\mathcal{B}_\alpha$ is then generated by the sesquilinear form

$$B_\alpha(u, \varphi) := B(\exp(i\alpha x_1)\, u, \exp(i\alpha x_1)\, \varphi)\,, \quad u\,,\, \varphi \in V\,,$$

which can be written as (cf. (5.9))

$$B_\alpha(u,\varphi) = \int_\Omega \left(a_{L,\alpha}(u,\overline{\varphi}) - \omega^2 \rho\, u\cdot\overline{\varphi}\right) dx - \int_{\Gamma^+} \overline{\varphi}\cdot \mathcal{T}_\alpha^+ u\, ds - \int_{\Gamma^-} \overline{\varphi}\cdot \mathcal{T}_\alpha^- u\, ds\,, \qquad (5.27)$$

where the bilinear form $a_{L,\alpha}$ on $\Omega^\pm$ is defined as in (3.2), with $\mu$, $\lambda$ replaced by $\mu^\pm$, $\lambda^\pm$, and $\partial_1$ replaced by the differential operator $\partial_{1,\alpha} = \partial_1 + i\alpha$, and where (cf. (5.7), (5.8))

$$\mathcal{T}_\alpha^\pm u := -\sum_{n\in\mathbb{Z}} W_n^\pm(\omega,\alpha)\hat{u}_n^\pm \exp(inx_1)\,, \quad u|_{\Gamma^\pm} = \sum_{n\in\mathbb{Z}} \hat{u}_n^\pm \exp(inx_1) \in H^{1/2}_{per}(\Gamma^\pm)^2\,. \qquad (5.28)$$

To indicate the dependence on the frequency $\omega$, we shall write $\mathcal{B}_\alpha = \mathcal{B}(\omega)$ and $\mathcal{T}_\alpha^\pm = \mathcal{T}^\pm(\omega)$ in the following. Note that the operator generated by the first term of the form (5.27) depends analytically on $\omega\in\mathbb{C}$, while for the DtN operators (5.28) this is only valid if one avoids the set of exceptional values (the Rayleigh frequencies) where one of the numbers $\beta_n^\pm$, $\gamma_n^\pm$ vanishes (cf. (5.1), (5.21)):

$$\mathcal{R} = \left\{\omega : \exists\, n\in\mathbb{Z} \ \text{ such that } \ \alpha_n^2 = \omega^2\rho^\pm/(2\mu^\pm+\lambda^\pm) \ \text{ or } \ \alpha_n^2 = \omega^2\rho^\pm/\mu^\pm\right\}. \qquad (5.29)$$

It follows immediately from Theorem 5 and (5.26) that $\mathcal{B}(\omega): V\to V'$ is a Fredholm operator with index zero for all $\omega>0$. Moreover, by assertion (i), we can choose $\omega_0>0$ sufficiently small so that $\mathcal{B}(\omega_0)$ is invertible. Then $\mathcal{B}(\omega)$ is invertible if and only if the operator

$$\mathcal{A}(\omega) := I + \mathcal{K}(\omega) : V\to V\,, \quad \mathcal{K}(\omega) := \mathcal{B}(\omega_0)^{-1}\left(\mathcal{B}(\omega)-\mathcal{B}(\omega_0)\right) \qquad (5.30)$$

is invertible, where $I$ denotes the identity operator. To prove that the operator $\mathcal{K}(\omega)$ defined in (5.30) is compact on $V$, we note that

$$\begin{aligned}\left((\mathcal{B}(\omega)-\mathcal{B}(\omega_0))\,u,\varphi\right)_\Omega \;&=\; -\int_{\Gamma^+}\overline{\varphi}\,(\mathcal{T}^+(\omega)-\mathcal{T}^+(\omega_0))u\,ds - \int_{\Gamma^-}\overline{\varphi}\,(\mathcal{T}^-(\omega)-\mathcal{T}^-(\omega_0))\,u\,ds \\ &\quad -(\omega-\omega_0)\int_\Omega \rho\, u\,\overline{\varphi}\,dx\,, \quad u\,,\,\varphi\in V\,, \end{aligned} \qquad (5.31)$$

and (cf. (5.28))

$$(\mathcal{T}^\pm(\omega)-\mathcal{T}^\pm(\omega_0))\,u = -\sum_{n\in\mathbb{Z}}(W_n^\pm(\omega,\alpha)-W_n^\pm(\omega_0,\alpha))\,\hat{u}_n^\pm\exp(inx_1)\,.$$

Then the uniform estimates

$$\|W_n^\pm(\omega,\alpha)-W_n^\pm(\omega_0,\alpha)\|_{\mathbb{C}^2\to\mathbb{C}^2} \le c(\omega,\omega_0)\,, \quad \forall\, n\in\mathbb{Z}\,,$$

together with the trace and imbedding theorems for periodic Sobolev spaces, imply the compactness of the form (5.31) and hence that of $\mathcal{K}(\omega)$.

Since $\mathcal{K}(\omega)$ is a compact operator function depending analytically on $\omega$ if $\omega\notin\mathcal{R}$ (cf. (5.29), (5.21)) and $\mathcal{A}(\omega_0)$ is invertible, it follows from the analytic Fredholm theory (e.g., [13, Theorem 8.26]) that $\mathcal{A}(\omega)$ is invertible for all $\omega\in\mathcal{U} := (0,\infty)\setminus\mathcal{R}$, with the possible exception of some discrete subset, say $\mathcal{D}$, of $\mathcal{U}$. Thus assertion (ii) is proved if we show that a point $\omega_*\in\mathcal{R}$ cannot be an accumulation point of $\mathcal{D}$. It follows from the definition of $\beta_n^\pm$, $\gamma_n^\pm$ (cf. (5.21)) that, in some neighbourhood of $\omega_*$, the operator functions $\mathcal{T}^\pm(\omega)$, and hence $\mathcal{B}(\omega)$, $\mathcal{K}(\omega)$, $\mathcal{A}(\omega)$, are analytic in $z := (\omega-\omega_*)^{1/2}$, where the branch of the root is chosen such that its imaginary part is non-negative. Then, applying [13, Theorem 8.26] to the operator function $\mathcal{A}(z) = I+\mathcal{K}(z)$ in a neighbourhood of $z=0$, gives the desired result. □

**Remark 4** *(i) For an incident shear wave $u^{in}$, Theorem 6 holds under the additional assumption that $k_p^- > k_s^+$, or equivalently, $\rho^-/(2\mu^- + \lambda^-) > \rho^+/\mu^+$. Note that the relations (5.21) hold with $\alpha_n = n + k_s^+ \sin\theta$, so that in the proof of the corresponding assertion (i) one obtains $\hat{u}_0^- = 0$ and thus estimate (5.24) with the corresponding boundary term on $\Gamma^-$. We do not know whether this condition can be removed.*

*(ii) Assume that the elastic material is homogeneous above a periodic Lipschitz interface $\Lambda^+$ and below another periodic Lipschitz interface $\Lambda^-$, whereas the elastic medium between $\Lambda^+$ and $\Lambda^-$ may be inhomogeneous with piecewise constant Lamé parameters $\lambda$, $\mu$ and density $\rho$ having jumps at certain (finitely many) disjoint periodic Lipschitz interfaces. Then Theorems 5 and 6 can easily be extended to these more general periodic diffractive structures.*

*(iii) The uniqueness result of Theorem 3 does not hold for the transmission problem (TP). Even in the special case of two half-planes with certain elastic parameters $\lambda^\pm$, $\mu^\pm$, $\rho^\pm$ and the transmission conditions (5.4) on the line $\{x_2 = 0\}$, there may exist non-trivial solutions of the homogeneous problem (Rayleigh surface waves) that decay exponentially as $x_2 \to \pm\infty$; see [1]. Hence additional conditions must be imposed on the elastic parameters to guarantee the uniqueness. However, so far we do not know of any general result in this direction.*

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