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Uniqueness in inverse elastic scattering with finitely many incident waves

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Abstract

We consider the third and fourth exterior boundary value problems of linear isotropic elasticity and present uniqueness results for the corresponding inverse scattering problems with polyhedral-type obstacles and a finite number of incident plane elastic waves. Our approach is based on a reflection principle for the Navier equation.

1 Introduction and main results

The inverse scattering problem of determining a bounded obstacle by its far field pattern is fundamental for exploring bodies by acoustic, electromagnetic or elastic waves. Establishing the uniqueness in this inverse problem by using the far field data from only one or, at most, finitely many incident plane waves remains a challenging open problem; see, e.g., [4], [8]. Recent progress in this direction was obtained in inverse acoustic and electromagnetic scattering by polyhedral scatterers which, in $\mathbb{R}^3$, are composed of finitely many solid polyhedra and subsets of two-dimensional planes; see [3], [2], [10], [6] for the Helmholtz equation and [11] for the Maxwell system. To date, there is no corresponding result for elastic wave scattering, but we refer to [7] for uniqueness results with an infinite number of incident elastic waves in the case of smooth obstacles.

It is the purpose of this paper to derive uniqueness results for polyhedral elastic scatterers with finitely many incoming plane waves. We will focus on the fourth and third exterior boundary value problems for the Navier equation where normal stress and tangential displacement (resp. normal displacement and tangential stress) vanish on the boundary of the obstacle; see [9, Chap. 1.14]. Our approach exploits the above-mentioned developments in inverse acoustic and electromagnetic scattering for polyhedral scatterers and combines a reflection principle for the Navier equation with a path argument. This technique was first employed in [2] for acoustic scattering by sound-soft polyhedral scatterers and then modified and extended to the sound-hard case [10], [6] and to electromagnetic scattering [11].

We will now state the direct and inverse elastic scattering problems. Let $D \subset \mathbb{R}^3$ be a scatterer, i.e., a compact set such that its exterior $D^c = \mathbb{R}^3 \backslash D$ is connected, and let ν denote the unit normal vector on its boundary ∂D directed into D^c. The parameters λ, μ and ω are constants such that

$$\mu > 0\,, \quad \lambda + 2\mu/3 > 0\,, \quad \omega > 0\,. \tag{1.1}$$

As usual, $a \cdot b$ denotes the scalar product and $a \times b$ denotes the vector product of a, $b \in \mathbb{R}^3$. The propagation of time harmonic elastic waves in D^c is governed by the Navier equation (or system)

$$(\Delta^* + \omega^2)u = 0 \quad \text{in} \quad D^c\,, \quad \Delta^* := \mu\Delta + (\lambda + \mu)\,\text{grad div}\,, \tag{1.2}$$

where u denotes the displacement field. Any solution u of (1.2) can be decomposed as

$$u = u_p + u_s\,, \quad u_p := (-1/k_p^2)\,\text{grad div}\ u\,, \quad u_s := (1/k_s^2)\,\text{curl curl}\ u\,, \tag{1.3}$$

where $k_p := \omega/\sqrt{2\mu+\lambda}$, $k_s := \omega/\sqrt{\mu}$ are the compressional and shear wave numbers respectively. Moreover, u_p and u_s satisfy the vector Helmholtz equations

$$(\Delta + k_p^2)u_p = 0 \quad \text{and} \quad (\Delta + k_s^2)u_s = 0 \quad \text{in} \quad D^c \tag{1.4}$$

respectively; note that curl curl $= -\Delta +$ grad div . The traction (or stress) operator on ∂D is defined by

$$Tu := 2\mu\, \partial_\nu u + \lambda\, (\text{div}\ u)\, \nu + \mu\, \nu \times \text{curl}\ u\,. \tag{1.5}$$

We assume that a time harmonic plane elastic wave u^{in} is incident on the scatterer D, which takes the general form

$$u^{in}(x) = Ad\, \exp(ik_p d\cdot x) + A_1 f_1\, \exp(ik_s d \cdot x) + A_2 f_2\, \exp(ik_s d\cdot x)\,, \tag{1.6}$$

where $A,\ A_1,\ A_2 \in \mathbb{C},\ d,\ f_1,\ f_2 \in \mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ and $f_1 \cdot d = f_2 \cdot d = f_1 \cdot f_2 = 0$. In particular,

$$u_p^{in}(x) = d\, \exp(ik_p d\cdot x) \tag{1.7}$$

is called an incident pressure wave, and

$$u_s^{j,in}(x) = f_j\, \exp(ik_s d\cdot x)\,,\ j = 1, 2\,, \quad f_1 \cdot d = f_2 \cdot d = f_1 \cdot f_2 = 0\,, \tag{1.8}$$

are incident shear waves propagating in direction d.

A solution u to (1.2) is called radiating if it satisfies Kupradze's radiation condition, i.e.

$$\begin{gathered} \partial_r u_p - ik_p u_p = o(r^{-1})\,, \quad \partial_r u_s - ik_s u_s = o(r^{-1}) \\ \text{uniformly in} \quad \hat{x} = x/r\,, \quad \text{as} \quad r = |x| \to \infty\,, \end{gathered} \tag{1.9}$$

which coincides with Sommerfeld's condition for the compressional and shear parts u_p and u_s of u and the corresponding vector Helmholtz equations (1.4); see [9, Chap. 3.2].

Direct scattering problem (DP): Given a scatterer $D \subset \mathbb{R}^3$ and an incident field u^{in} of the form (1.6), find the total field $u = u^{in} + u^{sc}$ in D^c, where the scattered field u^{sc} satisfies (1.2) and condition (1.9), and the total field u satisfies the boundary conditions of the fourth kind,

$$\nu \times u = 0\,, \quad \nu \cdot Tu = 0 \quad \text{on} \quad \partial D\,. \tag{1.10}$$

Direct problem (DP'): Replace the boundary conditions in (DP) by the boundary conditions of the third kind,

$$\nu \cdot u = 0\,, \quad \nu \times Tu = 0 \quad \text{on} \quad \partial D\,. \tag{1.11}$$

We refer to the monograph [9] for a comprehensive treatment of the basic boundary value problems of elasticity, including the boundary conditions of the third and fourth kinds. It is well known [9, Chap. 3.6] that the problems (DP) and (DP') admit at most one solution $u \in H^1_{loc}(D)^3$, and a standard method to prove existence for scatterers with C^2 boundaries is the integral equation method (see [9, Chap. 7.3]). Using the method of limiting absorption, the existence of solutions can be proved if the exterior domain D^c satisfies the cone condition; see [12, Chap. 4] for the case of the exterior Neumann problem for the Helmholtz equation, and we refer to [1, Chap. 4] for a nice account of the cone condition and its relation to other geometric properties of domains. Thus, in particular, the unique solvability of the scattering problems (DP) and (DP') holds within the class of polyhedral scatterers which are defined as follows.

Definition 1 *A scatterer $D \subset \mathbb{R}^3$ is called a polyhedral scatterer if its boundary ∂D is a finite union of cells. Here a cell is defined as the closure of an open connected subset of a two-dimensional plane.*

Note that in general a polyhedral scatterer consists of finitely many solid polyhedra and planar sets.

To state our inverse problems, we have to introduce the far field pattern of the scattered field u^{sc} appearing in (DP) or (DP'). It is well known [9], [7] that the radiating solution u^{sc} to the Navier equation has an asymptotic behaviour of the form

$$u^{sc}(x) = r^{-1}\,\exp(ik_p r)\,u_p^\infty(\hat{x})\,\hat{x} + r^{-1}\,\exp(ik_s r)\,u_s^\infty(\hat{x}) + o(r^{-1}) \qquad \text{as} \quad r \to \infty\,, \quad \text{with} \quad \hat{x}\cdot u_s^\infty(\hat{x}) = 0 \quad \forall \hat{x} \in \mathbb{S}^2\,, \tag{1.12}$$

uniformly in all directions $\hat{x}$. Here u_p^∞ is a uniquely determined scalar function, and u_s^∞ is a uniquely determined vector function. Note that, in analogy to (1.3), we have $u^{sc} = u_p^{sc} + u_s^{sc}$ where u_s^{sc} is a divergence free and radiating solution to a vector Helmholtz equation, while u_p^{sc} is the gradient of a radiating solution of a scalar Helmholtz equation.

Definition 2 *The far field pattern of the scattered field u^{sc} is given by*

$$u^\infty(\hat{x}) := u_p^\infty(\hat{x})\,\hat{x} + u_s^\infty(\hat{x})\,, \quad \hat{x} \in \mathbb{S}^2\,. \tag{1.13}$$

Inverse scattering problem (IP) resp. (IP'): From the knowledge of the far field pattern $u^\infty(\hat{x})$, $\hat{x} \in \mathbb{S}^2$, of $u^{sc} = u - u^{in}$ for one or several incident waves u^{in} of fixed incidence direction d, determine the shape of the scatterer.

The aim of this paper is to prove the following uniqueness results within the class of polyhedral scatterers.

Theorem 1 *Let the parameters λ, μ, ω and the incidence direction d be fixed. Then, in the inverse problem (IP), we have uniqueness within the class of polyhedral scatterers with a single incident pressure wave of direction d. For the inverse problem (IP'), a polyhedral scatterer is uniquely determined by the far field patterns for two linearly independent incident shear waves of direction d.*

In the final section 3 we will mainly deal with the proof for (IP) and then sketch the corresponding modifications in the case of (IP'). Our proof is essentially based on a reflection principle for the Navier equation that will be established in section 2. Unfortunately, such an approach seems to be impossible in the case of the physically more relevant boundary conditions of the first or second kind, which correspond to a clamped or free boundary of the elastic obstacle.

2 Reflection principle for the Navier equation

Let $D \subset \mathbb{R}^3$ be a scatterer, and let u be a solution to (1.2) which is real-analytic in D^c. For the proof of our uniqueness result, the notion of a flat set of the displacement field u is of importance (cf. also [10], [6], [11] for acoustic and electromagnetic scattering).

Definition 3 *Let Π be a two-dimensional plane in $\mathbb{R}^3$ with unit normal ν_Π. A non-void open connected component π of $\Pi \cap D^c$ will be called a flat set of u if*

$$\nu_\Pi \times u = 0 \quad and \quad \nu_\Pi \cdot Tu = 0 \quad on \quad \pi\,, \tag{2.1}$$

where T is the traction operator defined in (1.5).

Let $R = R_\Pi$ denote the reflection with respect to Π in $\mathbb{R}^3$. We are now ready to state the reflection principle. Note that this result is already implicitly contained in [9, Chap. 14, Thm. 3.2] (where a continuation formula at a planar boundary is given), but we prefer to present an independent proof here.

Theorem 2 *Let π_1 be a flat set of u different from $\pi \subset \Pi$, and let $G \subset D^c$ be a domain such that $R(G) = G$ and $\pi \cup \pi_1 \subset G$. Then the reflection $R(\pi_1)$ of π_1 with respect to Π is also a flat set of u.*

Proof. Step 1. Let first $\Pi = \{x_3 = 0\}$, and let $\pi \subset \Pi$ be a flat set of u, i.e., its components u_j $(j = 1, 2, 3)$ satisfy

$$u_1 = u_2 = 0\,, \quad \partial_3 u_3 = 0 \quad \text{on} \quad \pi\,, \tag{2.2}$$

since $\nu = \nu_\Pi = (0, 0, 1)$, $\nu \times u = (-u_2, u_1, 0)$, and

$$\nu \cdot Tu = 2\mu\, \nu \cdot \partial_\nu u + \lambda \operatorname{div} u = 2\mu\, \partial_3 u_3 + \lambda(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) = 0\,,$$

implying $(\lambda + 2\mu)\,\partial_3 u_3 = 0$, hence $\partial_3 u_3 = 0$ on π (compare (1.1)). Then, in the domain G which is symmetric with respect to $\{x_3 = 0\}$, u_1 and u_2 must be odd symmetric in x_3, and u_3 must be even symmetric in x_3. Indeed, the function v with the components

$$v_j(x) := -u_j(x_1, x_2, -x_3) = -u_j(Rx)\,,\ j = 1, 2\,, \quad v_3 := u_3(x_1, x_2, -x_3) = u_3(Rx)$$

satisfies equation (1.2) in G, and by (2.2) it has the same Cauchy data on π as the function u, so that $u = v$ in G.

Now let $\pi_1 \subset \Pi_1$ be another flat set of u in G, and let ν_1 be a unit normal of the plane Π_1. Since $R\nu_1 = \nu_{R(\Pi_1)}$ (by choosing the directions of the normals suitably) and $(x_1, x_2, -x_3) \in R(\pi_1)$ for any $(x_1, x_2, x_3) \in \pi_1$, it is easy to check that the relations

$$\nu_1 \times u = 0\,, \quad \nu_1 \cdot Tu = 2\mu\nu_1 \cdot \partial_{\nu_1} u + \lambda \operatorname{div} u = 0 \quad \text{on} \quad \pi_1$$

imply, on using the symmetry properties of u_j $(j = 1, 2, 3)$,

$$\nu_{R(\Pi_1)} \times u|_{R(\Pi_1)} = -\nu_1 \times u|_{\pi_1} = 0\,, \quad \nu_{R(\Pi_1)} \cdot Tu|_{R(\Pi_1)} = -\nu_1 \cdot Tu|_{\pi_1} = 0\,,$$

hence $R(\Pi_1)$ is a flat set of u.

Step 2. Assume now that the plane Π with normal ν contains the origin. Let $U \in \mathbb{R}^{3\times 3}$ be a rotation matrix such that $x^* = U^{-1}x = U^t x\,,\ x \in \mathbb{R}^3$, maps the plane Π onto the plane $\Pi^* = \{x_3^* = 0\}$ with normal $\nu^* = U^{-1}\nu = (0, 0, 1)$. Then we can reduce the proof to the case considered in step 1. With the solution u of the Navier equation (1.2) in G satisfying the boundary conditions (2.1) on $\pi \subset \Pi$ and $\pi_1 \subset \Pi_1$, we associate the function

$$u^*(x^*) := U^{-1}u(Ux^*)\,, \quad x^* \in \pi^* = U^{-1}(\pi) \subset \Pi^*\,.$$

From $x = Ux^*$ and the relations

$$\nabla_{x^*} = U^{-1}\nabla_x\,, \quad \nabla_{x^*}\cdot u^*(x^*) = U^{-1}\nabla_x \cdot U^{-1}u(x) = \nabla_x \cdot u(x)$$

we obtain

$$(\Delta^*_{x^*} + \omega^2)\, u^*(x^*) = U^{-1}(\Delta^*_x + \omega^2)\, u(x)\,, \tag{2.3}$$

$$\begin{aligned} \nu^* \times u^*(x^*) &= U^{-1}\nu \times U^{-1}u(x) = U^{-1}(\nu \times u(x))\,, \\ \nu^* \cdot T_{x^*}u^*(x^*) &= U^{-1}\nu \cdot U^{-1}T_x u(x) = \nu \cdot T_x u(x) \end{aligned} \tag{2.4}$$

since

$$\begin{aligned} \Delta_{x^*} = \nabla_{x^*}\cdot\nabla_{x^*} &= \Delta_x\,, \quad \nabla_{x^*}(\nabla_{x^*}\cdot u^*(x^*)) = U^{-1}\nabla_x(\nabla_x \cdot u(x)) \\ T_{x^*}u^*(x^*) &= 2\mu\,(\nu^*\cdot\nabla_{x^*})\,u^*(x^*) + \lambda\,\nu^*\,\nabla_{x^*}\cdot u^*(x^*) + \mu\,\nu^*\times\nabla_{x^*}\times u^*(x^*) \\ &= 2\mu\, U^{-1}(\nu\cdot\nabla_x)\,u(x) + \lambda\, U^{-1}\nu\,\nabla_x\cdot u(x) + \mu\, U^{-1}\nu \times U^{-1}\,(\nabla_x \times u(x)) \\ &= U^{-1}T_x u(x)\,. \end{aligned}$$

Here we have also used the relation $U(a\times b) = Ua \times Ub$ for arbitrary vectors a, b and any orthogonal matrix U. Then, by (2.3) and (2.4), u^* satisfies the homogeneous Navier equation in $G^* = U^{-1}(G)$ and the boundary conditions (2.1) on π^* and $\pi_1^* = U^{-1}(\pi_1) \subset \Pi_1^* = U^{-1}(\Pi_1)$. By step 1, the reflection $R^*(\pi_1^*)$ of π_1^* with respect to Π^* is also a flat set of u^*, i.e.,

$$\nu_{R^*(\Pi_1^*)} \times u^*(x^*) = 0\,, \quad \nu_{R^*(\Pi_1^*)} \cdot T_{x^*}u^*(x^*) = 0\,, \quad x^* \in R^*(\pi_1^*)\,. \tag{2.5}$$

Finally, noting that

$$R^*(\Pi_1^*) = R^*(U^{-1}(\Pi_1)) = U^{-1}(R(\Pi_1))\,, \quad \nu_{U^{-1}(R(\Pi_1))} = U^{-1}\nu_{R(\Pi_1)}$$

and using the relations (2.4), we see from (2.5) that

$$U^{-1}(\nu_{R(\Pi_1)} \times u) = 0\,, \quad \nu_{R(\Pi_1)} \cdot Tu = 0 \quad \text{on} \quad R(\pi_1)\,,$$

i.e., $R(\pi_1)$ is also a flat set of u.

Step 3. In the case of a plane Π not passing the origin, we obtain the result from step 2 by using the fact that the operators Δ^* and T are invariant with respect to translations in $\mathbb{R}^3$. This finishes the proof in the general case. $\square$

Remark 1 *Theorem 2 carries over to the boundary conditions* (1.11) *of the third kind if we replace the conditions* (2.1) *in the definition of a flat set* $\pi \subset \Pi$ *by*

$$\nu_\Pi \cdot u = 0 \quad \textit{and} \quad \nu_\Pi \times Tu = 0 \quad \textit{on} \quad \pi\,. \tag{2.6}$$

The proof is analogous to that of Theorem 2; see also [9, Chap. 14, Thm. 3.3]. Note that, in the special case $\Pi = \{x_3 = 0\}$, we have to apply even reflection (with respect to Π) for the first two components of u and odd reflection for the third component.

3 Proof of Theorem 1

Let D be a polyhedral scatterer in the sense of Definition 1, and assume that a pressure wave u_p^{in} of direction $d \in \mathbb{S}^2$ (see (1.7)) is incident on D. Let $u = u_p^{in} + u^{sc} \in H^1_{loc}(D^c)^3$ be the solution of the direct problem (DP), and let u^∞ be the far field pattern of the scattered field u^{sc} (see (1.13)). To prove the Theorem 1 in this case, we use path and reflection arguments first developed in [2] for the Helmholtz equation and later modified in [10], [6]. Here we follow [6, Sect. 3.1] in spirit, but employ the reflection principle of Theorem 2.

Step 1: existence of a flat set of the displacement field
Assume contrarily that there is another polyhedral obstacle $D_1 \neq D$ such that the far field pattern of $u^1 - u_p^{in}$ coincides with u^∞ on $\mathbb{S}^2$, where u^1 is the solution of problem (DP) for D_1. Then we have (see, e.g., [7])

$$u^1 = u \quad \text{in the unbounded connected component } \Omega \text{ of } \mathbb{R}^n \backslash (D \cup D_1)\,. \tag{3.1}$$

Moreover, since D^c and D_1^c are connected, we obtain (see [5], [10] for the details) $\partial\Omega \not\subset D \cap D_1$ and can assume without loss of generality that

$$S := (\partial D_1 \backslash D) \cap \partial\Omega \neq \emptyset\,.. \tag{3.2}$$

It follows from (3.1) and (3.2) that there exists a cell $F \subset S$ such that condition (1.10) is satisfied on F. Denoting by Π the plane (with normal ν) containing F and by $\mathrm{int}(F)$ the interior of the set F, we find a flat set π of u such that $\mathrm{int}(F) \subset \pi \subset \Pi$.

Next we verify that π must be bounded. Note that the scattered field u^{sc} satisfies

$$\lim_{|x|\to\infty} \{|u^{sc}(x)| + |\nabla u^{sc}(x)|\} = 0\,. \tag{3.3}$$

This follows from the decomposition $u^{sc} = u_p^{sc} + u_s^{sc}$ (see (1.3)) and the fact that both terms u_p^{sc} and u_s^{sc} satisfy relation (3.3) as radiating solutions of the corresponding Helmholtz equations (1.4) (see [3], [10]). If π were unbounded, then condition (1.10) on π and (3.3) would imply that

$$\nu \times u_p^{in} \to 0\,, \quad \nu \cdot T u_p^{in} \to 0\,, \quad \text{as} \quad |x| \to \infty \quad \text{on} \quad \Pi\,. \tag{3.4}$$

From the first relation of (3.4) and (1.7), we obtain $\nu \times d = 0$, hence $\nu = \pm d$. From the second relation, we then have $(2\mu + \lambda)k_p = 0$ since

$$\begin{aligned} \nu \cdot T u_p^{in} &= 2\mu\, \nu \cdot \partial_\nu u_p^{in} + \lambda \,\mathrm{div}\, u_p^{in} + \mu\, \nu \cdot (\nu \times \mathrm{curl}\, u_p^{in}) \\ &= \{2\mu\, (\nu \cdot d)^2 + \lambda |d|^2\}\, i k_p \exp(i k_p d \cdot x)\,. \end{aligned}$$

Hence $k_p = 0$ which is a contradiction proving the boundedness of any flat set of u.

Step 2: path argument
Choose a point $P \in \mathrm{int}(F)$ and a continuous and injective path $\gamma(t)$, $t \geq 0$, starting at $P = \gamma(0)$ and leading to infinity in the connected set Ω. Let $\mathcal{M}$ be the set of intersection points of γ with all flat sets of u. By Step 1, $\mathcal{M} \neq \emptyset$. Furthermore, $\mathcal{M}$ is bounded since a flat set of u outside a sufficiently large ball would be unbounded. (Note that ∂D is bounded, and u is real-analytic in D^c.)

Moreover, the set $\mathcal{M}$ is also closed, hence compact. Let $\{x_n\}$ be a sequence of intersection points of flat sets π_n, $x_n \in \pi_n$, with the path γ, and such that x_n converges to a point $x_0 \in \gamma$. Choosing a unit normal ν_n to π_n and passing to a convergent subsequence $\nu_n \to \nu_0$, one can now prove that the plane Π_0 through x_0 with unit normal ν_0 contains a flat set π_0 of u such that $x_0 \in \pi_0$; see the arguments in the proof of [10, Lemma 2].

Thus there exists $t^* \geq 0$ such that no flat set of u can intersect $\gamma(t)$ for $t > t^*$. Let $\pi^* \subset \Pi^*$ be a flat set passing $\gamma(t^*)$ and lying on a plane Π^*.

Step 3: reflection argument and final contradiction
Following the argument of [2, Lemma 3.7], we now apply the reflection principle of Theorem 2 to prove the existence of a flat set π' intersecting $\gamma(t)$ at some $t' > t^*$ which is a contradiction.

Let R denote the reflection with respect to the plane Π^*, and choose $x^+ = \gamma(t^* + \varepsilon)$ for $\varepsilon > 0$ sufficiently small and $x^- = R(x^+)$. Let $G^\pm$ be the connected component of $D^c \backslash \pi^*$ containing $x^\pm$, and denote by $E^\pm$ the connected component of $G^\pm \cap R(G^\mp)$ containing $x^\pm$. Setting $E = E^+ \cup \pi^* \cup E^-$, we observe that E is a connected open set whose boundary consists of cells of ∂D and $R(\partial D)$. Then, by Theorem 2, u satisfies the boundary conditions (1.10) on ∂E and $E \cap \Pi^*$. Moreover, E is bounded since otherwise Π^* would contain an unbounded flat set of u.

Hence, $\gamma(t)$ must intersect ∂E at some $t' > t^*$, so that there exists a flat set π' passing $\gamma(t')$. This contradiction finishes the proof of the theorem in the case of the inverse problem (IP). $\square$

We now discuss the necessary modifications in the above proof for the inverse problem (IP') corresponding to the third exterior boundary value problem (DP') for the Navier equation. Assume that the two (linearly independent) shear waves $u_s^{j,in}$, $j = 1, 2$, of direction d (see (1.8)) are incident on a polyhedral scatterer D. Let $u^j \in H^1_{loc}(D^c)^3$ be the solution of the direct problem (DP') with incident plane wave $u_s^{j,in}$.

Definition 4 *Let Π be a two-dimensional plane in $\mathbb{R}^3$ with unit normal ν. A non-void open connected component π of $\Pi \cap D^c$ is called a flat set of (u^1, u^2) if*

$$\nu \cdot u^j = 0 \quad and \quad \nu \times Tu^j = 0 \quad on \quad \pi\,, \quad j = 1\,, 2\,. \tag{3.5}$$

Then the reflection principle of Remark 1 carries over to this case. To prove Theorem 1 for problem (IP') by repeating the above arguments in steps 1-3, it is sufficient to prove the boundedness of a flat set of (u^1, u^2). To do so, it is enough to verify that, for an unbounded flat set π, the relations

$$\nu \cdot u_s^{j,in} \to 0\,, \quad \nu \times Tu_s^{j,in} \to 0\,, \quad \text{as} \quad |x| \to \infty \quad \text{on} \quad \pi\,, \quad j = 1\,, 2\,, \tag{3.6}$$

lead to a contradiction. Note that (3.6) follows from (3.3) and (3.5), and (1.8) and the first relations of (3.6) imply $\nu \cdot f_j = 0$, $j = 1, 2$, hence $\nu = \pm d$. Together with the second relations of (3.6), we then obtain $\mu\, k_s = 0$ which is impossible. Note that, on π,

$$\nu \times Tu = 2\mu\, \nu \times \partial_\nu u + \mu\, \nu \times (\nu \times \text{curl}\; u)\,,$$

and thus, using $d \cdot f_j = 0$, $j = 1, 2$,

$$\nu \times \partial_\nu u_s^{j,in} = ik_s\, (\nu \times f_j)\, (\nu \cdot d)\, \exp(ik_s d \cdot x) = ik_s\, (d \times f_j)\, \exp(ik_s d \cdot x)\,,$$

$$\nu \times (\nu \times \text{curl}\; u_s^{j,in}) = ik_s\, d \times (d \times (d \times f_j))\, \exp(ik_s d \cdot x) = -ik_s\, (d \times f_j)\, \exp(ik_s d \cdot x)\,.$$

Remark 2 *Unfortunately, our approach does not give the uniqueness in (IP') with only one incident shear wave. Note that then the boundedness of a corresponding flat set of the displacement field u cannot be proved in general. Of course, the uniqueness in (IP') with one incident shear wave holds in the case of two-dimensional elasticity.*

Remark 3 *In both inverse problems (IP) and (IP'), we can prove the uniqueness within the more restrictive class of polyhedral obstacles with a single incident plane elastic wave of the general form* (1.6)*. Here a polyhedral scatterer is called a polyhedral obstacle if it consists of finitely many solid polyhedra only. In this case, following the approach of [6], we have to allow unbounded flat sets of the displacement field and need to show that their number must be finite; see [6, Lemma 2] for acoustic scattering. Then the arguments of [6, Sect. 3.2] imply the desired uniqueness results. The details will be presented in a future paper.*

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