Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

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submitted: 29. July 2009

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No. 1436
Berlin 2009

2000 *Mathematics Subject Classification.* 34K06, 34K20, 34K26.

Key words and phrases. Spectrum, Delay Differential Equation, Large Delay.

This work has been supported by DFG Research Center Matheon, 'Mathematics for key technologies' in Berlin.

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THE SPECTRUM OF DELAY DIFFERENTIAL EQUATIONS WITH LARGE DELAY

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Abstract. We show that the spectrum of linear delay differential equations with large delay splits into two different parts. One part, called the strong spectrum, converges to isolated points when the delay parameter tends to infinity. The other part, called the pseudocontinuous spectrum, accumulates near criticality and converges after rescaling to a set of spectral curves, called the asymptotic continuous spectrum. We show that the spectral curves and strong spectral points provide a complete description of the spectrum for sufficiently large delay and can be comparatively easily calculated by approximating expressions.

1. Introduction

Delay differential equations with large delay play an important role for modeling of many real world systems, e.g. for optoelectronic systems with optical feedback or coupling [8], neural systems [10], and others [1, 11]. One of the basic questions in such systems concerns the stability of steady states and, in particular, spectral properties of the corresponding linearized systems. In this paper we investigate the spectrum of linear delay differential equations (DDEs) of the form

$$\frac{du}{dt}(t) = Au(t) + Bu(t-\tau), \tag{1.1}$$

where $u \in \mathbb{R}^N$, $A, B \in \mathbb{R}^{N\times N}$, $B \neq 0$ in the limit of large delay Note that the limit of large delay can be represented in the form of a singularly perturbation

$$\epsilon\frac{d\bar{u}}{d\bar{t}}(\bar{t}) = A\bar{u}(\bar{t}) + B\bar{u}(\bar{t}-1) \tag{1.2}$$

after the change of variables $t = \bar{t}\tau$, $\bar{u}(\bar{t}) = u(\bar{t}\tau)$, where $\epsilon = 1/\tau$.

The limit of large delay has been studied previously by many authors. In particular, Hale, Huang, Mallet-Paret, Nussbaum, and others have studied in detail the appearance of periodic solutions for certain types of scalar equations. In the work of Ivanov and Sharkovsky [5], the closeness of the DDE (1.2) to the corresponding difference equation, formally obtained by putting $\varepsilon = 0$, has been exploited. Using formal asymptotics, Kashchenko et al. [6, 7] provided a derivation of the Ginzburg-Landau equation as a local normal form. Some aspects of the spectrum have been previously investigated in [2, 3, 4, 14, 15, 16, 17] by Politi and Giacomelli and the authors of this paper.

In our paper, we provide a complete description of the spectrum for a general system of DDEs (1.1) with a single large delay. We show that the spectrum splits into two parts with different scaling behavior: the *strong spectrum*, which converges to a finite number of isolated points when the delay parameter tends to infinity, and the

pseudocontinuous spectrum, which accumulates near criticality and converges after a suitable rescaling to a set of spectral curves, called the asymptotic continuous spectrum. The results allow for a detailed description of the location of the spectrum and possible instability mechanisms. Moreover, they provide comparatively simple approximating asymptotic expressions for the location of the spectrum. The main results are formulated in Sec. 2, the corresponding proofs in Sec. 4, and illustrative examples are given in Sec. 3.

2. Main results

Our main goal is to investigate the set of solutions of the characteristic function

$$\chi(\lambda,\epsilon) := \det\left(-\lambda I + A + Be^{-\frac{\lambda}{\epsilon}}\right) \tag{2.1}$$

of equation (1.1) and to describe the spectrum

$$\Sigma^\epsilon := \{\lambda \in \mathbb{C} \mid \chi(\lambda,\epsilon) = 0\}$$

as $\varepsilon = \frac{1}{\tau} \longrightarrow 0$. We will pay special attention to the case when the matrix B is noninvertible, which can be found in many practical examples [1, 9, 12]. However, in order to avoid technical difficulties, we assume that $\operatorname{Ker} B = \operatorname{Ker} B^2$, such that we have after a suitable change of coordinates

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B} \end{pmatrix}, \tag{2.2}$$

where $\tilde{B} \in \mathbb{R}^{d\times d}$ is invertible with $1 \le d \le N$. Correspondingly we decompose

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{2.3}$$

where $A_4 \in \mathbb{R}^{d\times d}$ and $A_1 \in \mathbb{R}^{(N-d)\times(N-d)}$. Based on this, we can define the following spectral sets for (1.1):

Definition 1. The set

$$\mathcal{A}_+ := \{\lambda \in \mathbb{C} \mid \det(\lambda I - A) = 0, \quad \mathfrak{Re}\,\lambda > 0\}$$

is called the *asymptotic strong unstable spectrum*,

$$\mathcal{A}_- := \{\lambda \in \mathbb{C} \mid \det(\lambda I - A_1) = 0, \quad \mathfrak{Re}\,\lambda < 0\}$$

is called the *asymptotic strong stable spectrum* and

$$\mathcal{A}_s := \mathcal{A}_+ \cup \mathcal{A}_-$$

is called the *asymptotic strong spectrum*.

We call

$$\mathcal{A}_c := \left\{\gamma + i\omega \in \mathbb{C} \mid \exists \varphi \in \mathbb{R} : \, p_\omega\left(e^{-\gamma}e^{-i\varphi}\right) = 0\right\},$$

the *asymptotic continuous spectrum*, where

$$p_\omega(Y) := \det(-i\omega I + A + BY)$$

and we assume that $p_\omega(Y) \not\equiv 0$ for all $\omega \in \mathbb{R}$.[1]

These asymptotic spectral sets will turn out to determine the asymptotic location of the full spectrum Σ^ϵ. We define now subsets of Σ^ϵ corresponding to $\mathcal{A}_+$, $\mathcal{A}_-$, $\mathcal{A}_s$, and $\mathcal{A}_c$, respectively:

[1] the assumption is made here only to avoid technifical difficulties, it will be removed later.

Definition 2. The set

$$\Sigma_s^\epsilon := \Sigma_+^\epsilon \cup \Sigma_-^\epsilon,$$

where

$$\Sigma_+^\epsilon := \Sigma^\epsilon \cap B_r(\mathcal{A}_+),\ \Sigma_-^\epsilon := \Sigma^\epsilon \cap B_r(\mathcal{A}_-)$$

and $r := \frac{1}{3}\min\{r_0, \mathrm{dist}\,(\mathcal{A}_s, i\mathbb{R})\}$, $r_0 := \min\{|\lambda-\mu| \mid \lambda,\mu \in \mathcal{A}_s, \lambda \neq \mu\}$ is called the *strong spectrum*. The set

$$\Sigma_c^\epsilon := \Sigma^\epsilon \setminus \Sigma_s^\epsilon$$

is called the *pseudo-continuous spectrum*. Finally

$$\Pi_c^\epsilon := S_\epsilon(\Sigma_c^\epsilon),$$

where $S_\epsilon : \mathbb{C} \to \mathbb{C}$ is the rescaling

$$S_\epsilon(a+ib) := a\frac{1}{\epsilon} + ib \quad \text{for} \quad a,b \in \mathbb{R},$$

is called the *rescaled pseudo-continuous spectrum*.

Here $B_r(M) := \cup_{x\in M}\{z \in \mathbb{C} \mid |x-z| < r\}$ denotes the set of balls around a set $M \subset \mathbb{C}$.

The following main theorem establishes that for $\epsilon \downarrow 0$ the strong spectrum Σ_s^ϵ converges to $\mathcal{A}_s$, the pseudo-continuous spectrum Σ_c^ϵ converges to the imaginary axis $i\mathbb{R}$ and the rescaled pseudo-continuous spectrum Π_c^ϵ converges to the set of curves given by the asymptotic continuous spectrum $\mathcal{A}_c$. The stability of (1.1) for sufficiently long delay is given by the set of curves $\mathcal{A}_c$ (see Corollaries 5 and 6). Note that for $\det B \neq 0$ there is no strong stable spectrum, i.e the strong spectrum contains only eigenvalues with positive real part.

Theorem 3. *(i) Let $\lambda \in \mathcal{A}_+$. Then for $0 < \delta \leq r$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the number of eigenvalues of (1.1) in $B_\delta(\lambda)$ equals the multiplicity of λ as an eigenvalue of A.*
(ii) Let $\lambda \in \mathcal{A}_-$. Then for $0 < \delta \leq r$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the number of eigenvalues of (1.1) in $B_\delta(\lambda)$ (counting multiplicities) equals the multiplicity of λ as an eigenvalue of A_1.
(iii) For $\mu \in \mathcal{A}_c$ and $\delta > 0$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ there exists $\lambda \in \Sigma_c^\epsilon$ with $|S_\epsilon(\lambda) - \mu| < \delta$.
(iv) Let $R > 0$. For $0 < \delta \leq r$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and $\lambda \in \Sigma_c^\epsilon$ with $|\mathfrak{Im}\,\lambda| < R$ we have $|\mathfrak{Re}\,\lambda| < \delta$ and there exists $\mu \in \mathcal{A}_c$ with $|S_\epsilon(\lambda) - \mu| < \delta$.

In the next theorem, we state some properties of the asymptotic continuous spectrum $\mathcal{A}_c$. We claim that it consists of $d = \mathrm{rank}B$ curves that may have a finite number of singularities. The singularities mediate the transition of eigenvalues from the strong spectrum to the pseudo-continous spectrum. The regular parts of the curve, relevant for bifurcation analysis, are straightforward to compute numerically and in many cases analytically.

Theorem 4. *(i) There exist $d = \mathrm{rank}B$ continuous functions $\gamma_1, \cdots, \gamma_d : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ such that $\mathcal{A}_c = \bigcup_{l=1}^d \{\gamma_l(\omega) + i\omega \mid \omega \in \mathbb{R},\ \gamma_l(\omega) \notin \{-\infty,\infty\}\}$, which are called the* spectral curves of $\mathcal{A}_c$.

(ii) Let $i\omega \notin \sigma(A_1)$. Then there exist $l \in \{1,\dots,d\}$ such that $\gamma_l(\omega) = \infty$ if and only if $i\omega \in \sigma(A)$.
(iii) Let $i\omega \notin \sigma(A)$ and $d < N$. Then there exist $l \in \{1,\dots,d\}$ such that $\gamma_l(\omega) = -\infty$ if and only if $i\omega \in \sigma(A_1)$.

Based on these results, the conditions for the asymptotic stability of the DDE (1.1) can be formulated in terms of the spectral curves as follows.

Corollary 5. *(i) If the spectral curves are in the negative half-plane, i.e. $\gamma_l(\omega) < 0$ for all $\omega \in \mathbb{R}$, $1 \le l \le d$, and $\mathcal{A}_+ = \emptyset$, then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the delay differential equation (1.1) is exponentially asymptotically stable.*
(ii) If some spectral curve admits positive values, i.e. $\gamma_l(\omega) > 0$ for some $\omega \in \mathbb{R}$ and $l \in \{1,\dots,d\}$, or $\mathcal{A}_+ \neq \emptyset$, then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the delay differential equation (1.1) is exponentially asymptotically unstable.

In particular, it is evident that the onset of instability is always given by a spectral curve touching the imaginary axis.

Corollary 6. *There exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the following assertion is true: Assume that $i\omega$ is the eigenvalue in Σ^ε with largest real part, and that the polynomial $p_\omega(Y)$ is not identically zero. Then the eigenvalue $i\omega$ belongs to the pseudo-continous spectrum.*

Here, we need the nondegeneracy condition $p_\omega(Y) \not\equiv 0$, which is always satisfied if $\det B \neq 0$. This condition is here necessary to avoid trivial situations, for example where equation (1.1) can be decomposed into a skew-product of a DDE with an ODE, where pure ODE spectrum independent on the delay appears.

The theorems above still hold true when the polynomial $p_\omega(Y)$ becomes trivial for some $\omega \in \mathbb{R}$. In this case, we have to adapt our definitions slightly and include the additional set of of *strong critical spectrum*

$$\mathcal{A}_0 := \{i\omega \in \mathbb{C} \mid i\omega \in \sigma(A),\, p_\omega(Y) = 0 \text{ for all } Y \in \mathbb{C}\} \subset \Sigma^\epsilon \quad \text{for all } \epsilon{>}0. \tag{2.4}$$

into the strong spectrum $\mathcal{A}_s$. Note that these eigenvalues are independent on ε, i.e they are present for all values of the delay τ. If $\mathcal{A}_0 \neq \emptyset$ then the pseudocontinuous spectrum has to be defined as

$$\Sigma^\epsilon_c := \Sigma^\epsilon \setminus (\Sigma^\epsilon_s \cup \mathcal{A}_0)\,. \tag{2.5}$$

The definition of the asymptotic continuous spectrum has to be replaced by

$$\mathcal{A}_c := Cl\left\{\gamma + i\omega \in \mathbb{C} \mid i\omega \notin \mathcal{A}_0 \text{ and } \exists \varphi \in \mathbb{R} :\, p_\omega\left(e^{-\gamma}e^{-i\varphi}\right) = 0\right\}, \tag{2.6}$$

where Cl denotes the topological closure.

Using these definitions the results in Theorems 3 and 4 as well as Corollaries 6 and 5(ii) literally hold true. In Corollary 5(i), a nonempty $\mathcal{A}_0$ clearly prevents asymptotic exponential stability. Before we prove the above results in Sec. 4, we will illustrate them by some examples.

3. Examples

3.1. **Example 1: pseudo-continuous and strong unstable spectrum.** With the first example, we illustrate the case $\det B \neq 0$, where only pseudo-continuous

and strong unstable spectrum is present. We insert into system (1.1) the matrices

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where α, β are real parameters. The strong spectrum Σ_s^ϵ consists only of the strong unstable part Σ_+^ϵ, which is approximated by the strong spectrum $\mathcal{A}_+$ (see Theorem 3). For the asymptotic strong spectrum we obtain from $\sigma(A)=\{\alpha \pm i\beta\}$ that

$$\mathcal{A}_+ = \begin{cases} \sigma(A), & \text{if } \alpha > 0, \\ \emptyset, & \text{if } \alpha \le 0. \end{cases} \tag{3.1}$$

This means that for sufficiently large delay and $\alpha > 0$, there exist two unstable eigenvalues approaching $\alpha \pm i\beta$ as $\tau \to \infty$.

When $\tau \to \infty$, the rescaled pseudo-continuous spectrum Π_c^ϵ approaches the asymptotic continuous spectrum $\mathcal{A}_c$. It is determined by the condition $p_\omega\left(e^{-\gamma}e^{-i\varphi}\right) = 0$ that reads here as

$$\left(\alpha - i\omega + e^{-\gamma}e^{-i\varphi}\right)^2 + \beta^2 = 0. \tag{3.2}$$

This equation can be solved with respect to γ and φ. As a result, we obtain two curves of asymptotic continuous spectrum

$$\gamma_\pm(\omega) = -\frac{1}{2}\ln\left(\alpha^2 + (\beta \pm \omega)^2\right), \quad \omega \in \mathbb{R}.$$

Using Theorem 4(i) we have

$$\mathcal{A}_c = \{\gamma_+(\omega) + i\omega | \omega \in \mathbb{R}, \beta \neq -\omega \vee \alpha \neq 0\} \cup \{\gamma_-(\omega) + i\omega | \omega \in \mathbb{R}, \beta \neq \omega \vee \alpha \neq 0\}.$$

It is easy to see that the asymptotic continuous spectrum is stable if $|\alpha| > 1$ and unstable if $|\alpha| < 1$. Similarly, for sufficiently large τ, the pseudo-continuous spectrum Σ_c^ϵ is stable if $|\alpha| > 1$ and unstable for $|\alpha| < 1$. At $\alpha = 0$, the spectral curves have singularities $\gamma_\pm(\mp\beta) = \infty$ (cf. Theorem 4(ii)).

According to Theorem 3, for large delay the two above mentioned spectra Π_c^ϵ and Σ_s^ϵ completely describe the structure of the whole spectrum. Figure 3.1 shows the spectrum for different values of the parameter α. In particular,
– for $\alpha < -1$, the pseudocontinuous spectrum is stable and there is no strong unstable spectrum (Fig. 3.1(a));
– at $\alpha = -1$, a critical situation occurs when the asymptotic continuous spectrum touches the imaginary axis;
– for $-1 < \alpha < 0$, the pseudocontinuous spectrum is unstable (Fig. 3.1(b));
– for $0 < \alpha < 1$, two strong unstable spectral points are present (Fig. 3.1(d)) appearing via the two singularities $\gamma_\pm(\mp\beta) = \infty$ of the spectral curve at $\alpha = 0$ (Fig. 3.1(c));
– for $\alpha > 1$, the pseudocontinuous spectrum becomes stable again, but the strong unstable spectrum is still present (Fig. 3.1(e)).

3.2. **Example 2: pseudo-continuous and strong stable spectrum.** With the following example, we illustrate the strong stable spectrum and its interaction with the pseudo-continuous spectrum as some parameter is varied. We consider the linear system with

$$A = \begin{pmatrix} \alpha & 1/2 \\ 2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

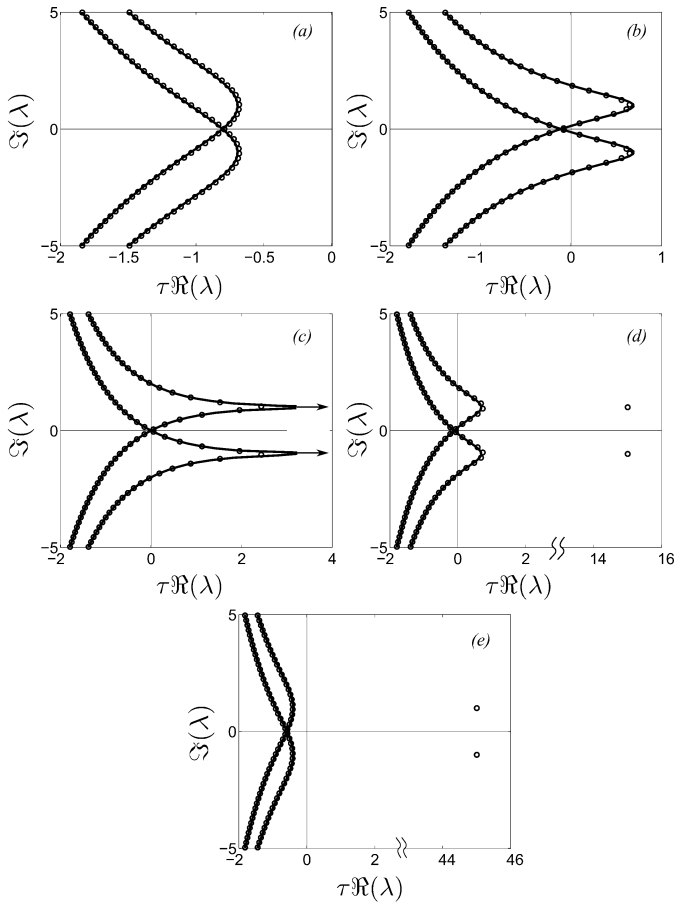

FIGURE 3.1. Spectra of example 1 for large delay. Asymptotic continuous spectrum (solid lines) is shown together with numerically computed eigenvalues for fixed $\tau = 30$. Parameter values: (a) $\alpha = -2$; (b) $\alpha = -0.5$; (c) $\alpha = 0$; (d) $\alpha = 0.5$: (e) $\alpha = 1.5$. For all figures $\beta = 1$.

For this system $A_1 = \alpha$ and we get asymptotic strong stable spectrum $\mathcal{A}_- = \{\alpha\}$ if $\alpha < 0$. Moreover, there is asymptotic strong unstable spectrum $\mathcal{A}_+ = \left\{\frac{\alpha}{2} + \sqrt{1 + \frac{\alpha^2}{4}}\right\}$ for all α. Since $\mathrm{rank}(B) = 1$, the asymptotic continuous spectrum consists here only of one spectral curve

$$\gamma(\omega) = -\frac{1}{2} \ln \left[\frac{\left(1 + \omega^2\right)^2 + \alpha^2 \omega^2}{(\alpha^2 + \omega^2)} \right],$$

As it follows from Theorem 4, the corresponding spectral curve develops a singularity $\gamma(0) = -\infty$ if $\alpha = 0$. Figure 3.2 illustrates the strongly stable spectrum and the singularity of the asymptotic continuous spectrum.

3.3. **Example 3: strong stable spectrum in the Lang-Kobayashi model.** The following example is the Lang-Kobayashi model, which describes the dynamics

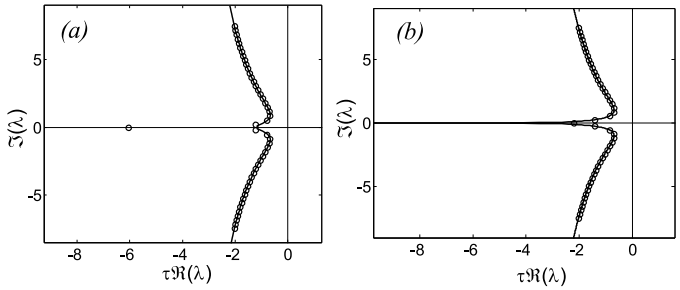

FIGURE 3.2. Spectrum of system (3.3) with large delay. Asymptotic continuous spectrum (solid lines) is shown together with numerically computed eigenvalues for fixed $\tau = 20$. Parameter values: (a) $\alpha = -0.3$, (b) $\alpha = 0$. Strong unstable spectrum is not shown.

of a semiconductor laser with optical feedback [9]

$$\begin{aligned} E'(t) &= (1+i\alpha)n(t)E(t) + \eta e^{-i\varphi}E(t-\tau), \\ n'(t) &= \nu[J - n(t) - (2n(t)+1)|E(t)|^2]. \end{aligned} \tag{3.4}$$

Here the variables $E \in \mathbb{C}$ and $n \in \mathbb{R}$ denote the electrical field and carrier density respectively. Additionally, there are real parameters $\alpha > 0, \nu > 0, J > 0, \eta > 0$, and φ. The system has rotating wave solutions (external cavity modes) of the form $E(t) = ae^{i\Omega t}, n(t) = N$. Their stability is determined by a linear system of the form (1.1) in three real variables with

$$A = \begin{bmatrix} N & \beta & a \\ -\beta & N & \alpha a \\ -2a\nu(2N+1) & 0 & -\nu(1+2a^2) \end{bmatrix}, \quad B = \begin{bmatrix} -N & -\beta & 0 \\ \beta & -N & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\beta = \Omega - \alpha N$. Note that $\det B = 0$ and

$$A_1 = -\nu(1+2a^2)$$

is a negative scalar value. Hence, the strong stable spectrum always exists $\mathcal{A}_- = \{-\nu(1+2a^2)\}$. Figure 3.3 illustrates the spectrum of some external cavity mode, which contains two spectral curves and one strong stable eigenvalue. A detailed study of the stability properties of all external cavity modes, using the results presented here can be found in [13].

4. Proofs of the main results

In this section we prove the main results, which are formulated in Sec. 2. First we prove Theorem 4. We start with the following lemma.

Lemma 7. *The degree of p_ω equals $d = \mathrm{rank}B$ if and only if $i\omega \notin \sigma(A_1)$.*

Proof. Use Leibniz formula

$$\det(-i\omega I + A + BY) = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_j (-i\omega I + A + BY)_{j,\sigma(j)},$$

where S_N denotes the set of all permutations on $\{1, \ldots, N\}$ and the subindex $(j, \sigma(j))$ denotes the element in the j-th row and $\sigma(j)$-th column. Assume without loss of generality that $\tilde{B}$ is in Jordan canonical form containing the eigenvalues on

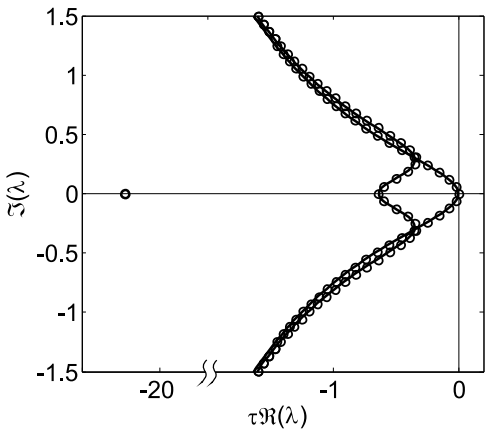

FIGURE 3.3. Spectrum of the external cavity mode with $N = -0.3$, $\Omega = -0.6$, $a = 1$ for the Lang-Kobayashi system (3.4). Asymptotic continuous spectrum(solid lines) and circles numerically computed eigenvalues (circles) for $\tau = 100$ and parameters: $\eta = 0.3$, $\varphi = 0$, $\alpha = 2$, $\nu = 0.07$, $J = 0.1$.

its diagonal. The leading order coefficient for Y^d is then calculated by summing over all permutations σ which satisfy $\sigma(k) = k$ for $N - d < k \leq N$. Hence the coefficient for the monomial Y^d is $\det(-i\omega + A_1) \prod_{j=1}^{d} \tilde{\lambda}_j$, which is not zero, since the eigenvalues $\tilde{\lambda}_j$ of $\tilde{B}$ are not zero by assumption. □

Using Lemma 7 we can conclude there exist d continuous functions $Y_l : \mathbb{R} \setminus \{\omega \in \mathbb{R} \mid i\omega \notin \sigma(A_1)\} \to \mathbb{C}$ such that $p_\omega(Y_l(\omega)) = 0$ for $1 \leq l \leq d$ and $i\omega \notin \sigma(A_1)$. Define $\gamma_l(\omega) := -\log|Y_l(\omega)|$ and extend γ_l continuously onto $\mathbb{R}$ with values in $\mathbb{R} \cup \{-\infty, \infty\}$. This proves assertion (i) of the theorem.

To prove assertion (ii), we have to study the case where, for a specific choice of ω, zero is a root of $p_\omega(Y)$. Indeed, we have for all $i\omega \notin \sigma(A_1)$ that

$$\gamma_l(\omega) = \infty \text{ for some } 1 \leq l \leq d \Longleftrightarrow p_\omega(0) = 0.$$

But $p_\omega(0) = 0$ is equivalent to $i\omega \in \sigma(A)$, finishing the proof of statement (ii).

To prove assertion (iii), we have to study the case where, for varying ω, a root of $p_\omega(Y)$ tends to infinity. To this end it will be useful to define also the polynomial

$$q_\omega(Z) := \det\left(Z\left(-i\omega I + A\right) + B\right). \tag{4.1}$$

Note that for $Z \neq 0$ we have $q_\omega(Z) = 0$ exactly if $p_\omega(\frac{1}{Z}) = 0$. Hence, the spectral curves $\gamma_l(\omega)$, $1 \leq l \leq d$, satisfy $p_\omega\left(e^{-\gamma_l(\omega)} e^{-i\varphi}\right) = 0$ as well as $q_\omega\left(e^{\gamma_l(\omega)} e^{i\varphi}\right) = 0$ for $\gamma_l(\omega) \notin \{-\infty, \infty\}$ with some $\varphi \in \mathbb{R}$. We study now roots of $q_\omega(Z)$ that tend to zero. Recalling that $\mathrm{rank} B = d < N$ it is obvious from (4.1) that $Z = 0$ is already a root of q_ω with multiplicity $N - d$. By this reason, we need the following lemma that separates a nontrivial component $\tilde{q}_\omega(Z)$ of $q_\omega(Z)$ for Z close to zero.

Lemma 8. *For all $\omega \in \mathbb{R}$ there exists $\delta > 0$ such that for $|Z| < \delta$ we have*

$$q_\omega(Z) = Z^{N-d} \det C(Z)\, \tilde{q}_\omega(Z), \tag{4.2}$$

where

$$\tilde{q}_\omega(Z) = \det\left(-i\omega I + A_1 - ZA_2C(Z)^{-1}A_3\right) \tag{4.3}$$

and $C(Z) = \tilde{B} + Z(-i\omega I + A_4)$.

Proof. We use the block structure (2.2) and (2.3) of A and B. For $|Z|$ sufficiently small, invertibility of the block $-i\omega I + A_4 + \tilde{B}Z^{-1}$ follows from invertibility of $\tilde{B}$ and we can apply the formula for the determinant of block matices:

$$\begin{aligned} q_\omega(Z) =& \det\left(Z\begin{pmatrix} -i\omega I + A_1 & A_2 \\ A_3 & -i\omega I + A_4 + \tilde{B}Z^{-1}\end{pmatrix}\right) \\ =& Z^N \det\left(-i\omega I + A_4 + \tilde{B}Z^{-1}\right) \times \\ & \times \det\left(-i\omega I + A_1 - A_2(-i\omega I + A_4 + \tilde{B}Z^{-1})^{-1}A_3\right). \end{aligned}$$

The representation (4.2) then follows immediately. □

Using this lemma, we can argue now for $\tilde{q}_\omega$ in the same way as we did above for p_ω: First, we observe that for $\tilde{q}_\omega(Z) \not\equiv 0$ we have

$$\gamma_l(\omega) = -\infty \text{ for some } 1 \le l \le d \Longleftrightarrow \tilde{q}_\omega(0) = 0.$$

Then, recalling (4.3), we note that $\tilde{q}_\omega(0) = 0$ is equivalent to $i\omega \in \sigma(A_1)$. This finishes the proof, under the assumption that $i\omega \notin \sigma(A)$ implies $\tilde{q}_\omega(Z) \not\equiv 0$. But q_ω and $\tilde{q}_\omega$ are identically zero exactly for the same values of ω where $p_\omega$is identically zero. For these values, we have by definition that $i\omega \in \mathcal{A}_0$. Hence we are finished when we show the statement $\mathcal{A}_0 \subset \sigma(A)$.This follows from that fact that if p_ω is identically zero, then in particular $p_\omega(0) = \det(-i\omega I + A)$ has to be zero. We have proved Theorem 4 and remark that Lemma 8 implies

$$\mathcal{A}_0 \subset \sigma(A) \cap \sigma(A_1).$$

We continue now with Theorem3. At first, we prove statement *(i)*, that for $0 < \delta \le r$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and $\lambda \in \mathcal{A}_+$, the sum of the eigenvalues in $B_\delta(\lambda)$ counting multiplicities equals the multiplicity of λ as an eigenvalue of A: Let $\lambda \in \mathcal{A}_+$. For $z \in B_{2\delta}(\lambda)$ and $\epsilon \downarrow 0$ the holomorphic function $\chi(z,\epsilon)$ converges uniformly to $-zI + A$. Hence, Hurwitz theorem implies that there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the functions $-zI + A$ and $\chi(z,\epsilon)$ have the same number of zeros in $B_\delta(\lambda)$. Since the set $\mathcal{A}_+$ is finite, this proves statement (i) of Theorem 3.

For statement (ii), we have to show that for $0 < \delta \le r$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and $\lambda \in \mathcal{A}_-$ we have that the number of the eigenvalues in $B_\delta(\lambda)$ counting multiplicities equals the multiplicity of λ as an eigenvalue of A_1. Let $\lambda \in \mathcal{A}_-$. We define $f(z,\epsilon) := \chi(z+\lambda,\epsilon)$ and use again the block structure (2.2) and (2.3) to obtain

$$f(z,\epsilon) = \det\left[\begin{pmatrix} -(z+\lambda)I_1 + A_1 & A_2 \\ A_3 & -(z+\lambda)I_4 + A_4 + \tilde{B}e^{-\frac{z+\lambda}{\epsilon}}\end{pmatrix}\right],$$

where I_1 and I_4 denote identity matrices in $\mathbb{R}^{N-d}$ and $\mathbb{R}^d$, respectively. Then

$$f(z,\epsilon) = \left(e^{-\frac{z+\lambda}{\epsilon}}\right)^d \det(M)\det(N),$$

where

$$M := \tilde{B} + e^{\frac{z+\lambda}{\epsilon}}\left(-(z+\lambda)I_4 + A_4\right), \quad N := -(z+\lambda)I_1 + A_1 - e^{\frac{z+\lambda}{\epsilon}} A_2 M^{-1} A_3.$$

Note that M is invertible, because

$$\det M = \det(\tilde{B}) + O(|e^{\frac{z+\lambda}{\epsilon}}|).$$

Since $M^{-1} = \tilde{B}^{-1} + O(|e^{\frac{z+\lambda}{\epsilon}}|)$ we have

$$\det(N) = \det(-(z+\lambda)I_1 + A_1) + O(|e^{\frac{z+\lambda}{\epsilon}}|).$$

Hence

$$f(z,\epsilon)\left(e^{\frac{z+\lambda}{\epsilon}}\right)^d = \det(\tilde{B})\det(-(z+\lambda)I_1 + A_1) + O(|e^{\frac{z+\lambda}{\epsilon}}|).$$

Because $\mathfrak{Re}\,\lambda < 0$ it follows that for given $0 < \delta \leq r$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ $f(z,\epsilon)$ has the same number of zeros as $\det(-(z+\lambda)I_1 + A_1)$ in $B_\delta(\lambda)$. This proves statement (ii) of Theorem 3.

Now we come to statement (iii). We must show that for $\mu \in \mathcal{A}_c$ and $\delta > 0$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ there exists $\lambda \in \Sigma_c^\epsilon$ with $|S_\epsilon(\lambda) - \mu| < \delta$. To this end, we take $\mu = \gamma_0 + i\omega_0 \in \mathcal{A}_c$. Without loss of generality we assume that $i\omega_0 \notin \mathcal{A}_0$. For $\epsilon > 0$ let

$$f_\epsilon(z) := \det\left(-\epsilon z - i\epsilon 2\pi k(\epsilon) + A + Be^{-z}\right),$$

where $k(\epsilon) := \left[\frac{\omega_0}{\epsilon 2\pi}\right]$ is the largest integer smaller than $\frac{\omega_0}{\epsilon 2\pi}$. For $\epsilon \downarrow 0$ we have $f_\epsilon(z) \to f_0(z) := \det\left(-i\omega_0 I + A + Be^{-z}\right)$ locally uniformly on $\mathbb{C}$. Because $i\omega_0 \notin \mathcal{A}_0$, the function f_0 is nontrivial. By assumption there exists $\varphi_0 \in \mathbb{R}$ such that $f_0(\gamma_0 + i\varphi_0) = 0$. Hence for $\eta > 0$, chosen such that f_0 has only $\gamma_0 + i\varphi_0$ as a zero on the closed η disk around $\gamma_0 + i\varphi_0$, there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ f_0 and f_ϵ have the same number of zeros in the open η disk of $\gamma_0 + i\varphi_0$. If z_ϵ is such a zero, then $\lambda_\epsilon = \epsilon z_\epsilon + i\epsilon 2\pi k(\epsilon) \in \Sigma^\epsilon$. Given $\delta > 0$ we choose $\eta > 0$ and $\epsilon_0 > 0$ sufficiently small such that $|S_\epsilon(\lambda_\epsilon) - \mu| < \delta$. This proves statement (iii) of Theorem 3.

To prove the remaining statement (iv) of Theorem 3 we formulate the following Lemmas 9 and 12.

Lemma 9. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers converging to $i\omega_0 \in \mathbb{C}$, where $\omega_0 \in \mathbb{R}$, and let $(\epsilon_n)_{n\in\mathbb{N}}$, $\epsilon_n > 0$, be a sequence of positive numbers converging to 0 such that $\chi(\lambda_n, \epsilon_n) = 0$. Then there exists a subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ such that one of the following holds:*
(i) $\lim_{k\to\infty} S_{\epsilon_{n_k}}(\lambda_{n_k}) \in \mathcal{A}_c$
(ii) $\lim_{k\to\infty} S_{\epsilon_{n_k}}(\lambda_{n_k}) = \infty$ *and there exists a spectral curve γ_l, $l \in \{1,\dots,d\}$, with $\gamma_l(\omega_0) = \infty$.*
(iii) $\lim_{k\to\infty} S_{\epsilon_{n_k}}(\lambda_{n_k}) = -\infty$ *and there exists a spectral curve γ_l, $l \in \{1,\dots,d\}$, with $\gamma_l(\omega_0) = -\infty$.*
(iv) $\lambda_{n_k} = \lambda_{n_{k+1}} = i\omega_0 \in \mathcal{A}_0$ for all $k \in \mathbb{N}$.

To prove Lemma 9 in the general case including nonempty $\mathcal{A}_0$ we need the following

Lemma 10. *Let $i\omega_0 \in \mathcal{A}_0$. There exist $l > 0$, $l \in \mathbb{N}$, and a polynomial $\tilde{p}(Y, X)$ in X and Y, which is nontrivial in Y for all X belonging to an open neighborhood $U \subset \mathbb{C}$ of $i\omega_0$, such that*

$$\det(-XI + A + BY) = (X - i\omega_0)^l \tilde{p}(Y, X).$$

In particular

$$p_\omega(Y) = (\omega - \omega_0)^l i^l \tilde{p}(Y, i\omega),$$

where $\tilde{p}(Y, i\omega)$ is a nontrivial polynomial in Y for ω in some open neighborhood Ω of ω_0.

There exist $m > 0$, $m \in \mathbb{N}$, and a polynomial $\tilde{q}(Z, X)$ in Z and X, which is nontrivial in Z for X belonging to an open neighborhood $V \subset \mathbb{C}$ of $i\omega_0$, such that

$$\det(Z(-XI + A) + B) = (X - i\omega_0)^m \tilde{q}(Z, X).$$

In particular

$$q_\omega(Z) = (\omega - \omega_0)^m i^m \tilde{q}(Z, i\omega),$$

where $\tilde{q}(Z, i\omega)$ is a nontrivial polynomial in Z for ω in some open neighborhood $\tilde{\Omega}$ of ω_0.

Proof. Write

$$\det(-XI + A + BY) = \det(-(X - i\omega_0)I - i\omega_0 I + A + BY)$$

and consider it as a polynomial in $X - i\omega_0$ and Y. Because $i\omega_0 \in \mathcal{A}_0$ a factor $(X - i\omega_0)^l$, $l > 0$, $l \in \mathbb{N}$ can be split off. By choosing l maximal the polynomial

$$\tilde{p}(Y, X) := \frac{\det(-XI + A + BY)}{(X - i\omega_0)^l}$$

in X and Y becomes nontrivial in Y for all X sufficiently close to $i\omega_0$.Similarly write

$$\det(Z(-XI + A) + B) = \det(-Z(X - i\omega_0)I + Z(-i\omega_0 I + A) + B)$$

and consider it as a polynomial in $X - i\omega_0$ and Z. □

From Lemma 10 it is not difficult to see the following Remark for the definition (2.6) of the asymptotic continuous spectrum in the case $\mathcal{A}_0 \neq \emptyset$:

Remark 11. Let $i\omega_0 \in \mathcal{A}_0$. Then $\gamma + i\omega_0 \in \mathcal{A}_c$ if and only if there exists $\varphi \in \mathbb{R}$ such that $\tilde{p}(e^{-\gamma} e^{-i\varphi}, i\omega_0) = 0$. Hence we have

$$\mathcal{A}_c = \left\{ \gamma + i\omega \in \mathbb{C} \mid \exists \varphi \in \mathbb{R} : \tilde{p}(e^{-\gamma} e^{-i\varphi}, i\omega) = 0 \right\},$$

which is compatible to Definition 1 in the case $\mathcal{A}_0 = \emptyset$.

Proof. [Proof of Lemma 9] Write

$$\lambda_n = \epsilon_n \gamma_n + i\epsilon_n \varphi_n + i\epsilon_n 2\pi m_n,$$

where $\varphi_n \in [0, 2\pi[$ and $m_n \in \mathbb{Z}$. By assumption we have $\lim_{n\to\infty}(i\epsilon_n 2\pi m_n) = i\omega_0$ and $\lim_{n\to\infty}(\epsilon_n \gamma_n) = 0$. By passing to a subsequence we can assume that $\lim_{n\to\infty} \varphi_n = \varphi_0 \in \mathbb{R}$. Define

$$\rho_n(y) := \det\left(-\epsilon_n y - i\epsilon_n 2\pi m_n + A + Be^{-y}\right), \quad y \in \mathbb{C}.$$

The sequence $\rho_n(y)$ of holomorphic functions converges uniformly on bounded sets of $\mathbb{C}$ to $p_{\omega_0}(e^{-y})$ and we have

$$\chi(\lambda_n, \epsilon_n) = \rho_n(\gamma_n + i\varphi_n) = 0. \tag{4.4}$$

Suppose γ_n is bounded. Then there exists a subsequence $(\gamma_{n_k})_{k\in\mathbb{N}}$ converging to some $\gamma_0 \in \mathbb{R}$. From (4.4) by letting $k \to \infty$ we get $p_{\omega_0}(e^{-\gamma_0 - i\varphi_0}) = 0$. If $i\omega_0 \notin \mathcal{A}_0$ it follows by definition of $\mathcal{A}_c$ that $\lim_{k\to\infty} S_{\epsilon_{n_k}}(\lambda_{n_k}) = \gamma_0 + i\omega_0 \in \mathcal{A}_c$, so we have (i). Suppose $i\omega_0 \in \mathcal{A}_0$. By applying Lemma 10 we get

$$\lambda_{n_k} = i\omega_0 \quad \text{or} \quad \tilde{p}(e^{-\gamma_{n_k} - i\varphi_{n_k}}, \lambda_{n_k}) = 0. \tag{4.5}$$

If the set $\{l \in \mathbb{N} \mid \lambda_{n_l} = i\omega_0\}$ is infinite, then we have shown (iv). Otherwise we can assume that $\lambda_{n_k} \neq i\omega_0$ for all k. From (4.5)

$$\tilde{p}(e^{-\gamma_0 - i\varphi_0}, i\omega_0) = \lim_{k\to\infty} \tilde{p}(e^{-\gamma_{n_k} - i\varphi_{n_k}}, \lambda_{n_k}) = 0.$$

Since $\tilde{p}(Y, i\omega)$ is nontrivial for ω near ω_0 it follows that $\tilde{p}(Y, i\omega)$ and hence $p_\omega(Y)$ has a root $y_l(\omega)$ which converges to $e^{-\gamma_0 - i\varphi_0}$ for $\omega \to \omega_0$. Hence the corresponding spectral curve $\gamma_l(\omega) = -\log|y_l(\omega)|$ satisfies $\gamma_l(\omega_0) = \gamma_0$. Hence $\gamma_0 + i\omega_0 \in \mathcal{A}_c$ (compare with Remark 11).

Suppose γ_n is unbounded. We consider four different cases. First let $\lim_{k\to\infty} \gamma_{n_k} = \infty$ and $i\omega_0 \notin \mathcal{A}_0$. Letting $k \to \infty$ and using (4.4) we get $p_{\omega_0}(0) = 0$. Since $i\omega_0 \notin \mathcal{A}_0$ it follows that there exists $l \in \{1, \dots, d\}$ such that $\gamma_l(\omega_0) = \infty$. Hence we have (ii). Now treat the case $\lim_{k\to\infty} \gamma_{n_k} = \infty$ and $i\omega_0 \in \mathcal{A}_0$. Again using Lemma 10 we have (4.5). Hence either we have (iv) or we can assume that $\lambda_{n_k} \neq i\omega_0$ and $\tilde{p}(e^{-\gamma_{n_k} - i\varphi_{n_k}}, \lambda_{n_k}) = 0$ for all k. In the latter case we have $\tilde{p}(0, i\omega_0) = 0$ which implies (ii).

Next consider the case $\lim_{k\to\infty} \gamma_{n_k} = -\infty$ and $i\omega_0 \notin \mathcal{A}_0$. Define

$$\sigma_n(y) := \det\left(e^y\left(-\epsilon_n y - i\epsilon_n\varphi_n - i\epsilon_n 2\pi m_n + A\right) + B\right), \quad y \in \mathbb{C}.$$

The sequence $\sigma_n(y)$ of holomorphic functions converges uniformly on bounded sets of $\mathbb{C}$ to $q_{\omega_0}(e^y)$ and we have

$$\sigma_n(\gamma_n + i\varphi_n) = \left(e^{\gamma_n + i\varphi_n}\right)^N \chi(\lambda_n, \epsilon_n) = 0. \tag{4.6}$$

Letting $k \to \infty$ we get $q_{\omega_0}(0) = 0$. Since $q_\omega(Z)$ is nontrivial for ω near ω_0 we see (iii).

Finally let $\lim_{k\to\infty} \gamma_{n_k} = -\infty$ and $i\omega_0 \in \mathcal{A}_0$. Applying the second part of Lemma 10 we get

$$\lambda_{n_k} = i\omega_0 \quad \text{or} \quad \tilde{q}(e^{\gamma_{n_k} + i\varphi_{n_k}}, \lambda_{n_k}) = 0.$$

This implies that either (iv) or (iii) holds. □

Lemma 12. *Let $R > 0$. For $n \in \mathbb{N}$ let $\epsilon_n > 0$ be such that $\lim_{n\to\infty} \epsilon_n = 0$. Consider $\lambda_n \in \Sigma_c^{\epsilon_n}$ with $|\mathfrak{Im}\,\lambda_n| \le R$. Then λ_n is bounded and for any convergent subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ we have $\mathfrak{Re}\left(\lim_{k\to\infty} \lambda_{n_k}\right) = 0$.*

Proof. Definition (2.1) implies that λ_n is bounded. Indeed, suppose λ_n was unbounded. Then there would exist a subsequence $(\lambda_{m_k})_{k\in\mathbb{N}}$ such that either

$$\lim_{k\to\infty} \mathfrak{Re}\,(\lambda_{m_k})_{k\in\mathbb{N}} = \infty \tag{4.7}$$

or

$$\lim_{k\to\infty} \mathfrak{Re}\,(\lambda_{m_k})_{k\in\mathbb{N}} = -\infty \tag{4.8}$$

In the case (4.7) we have a contradiction to the fact that the spectral radius of $A + B\exp(-\lambda_{m_k}/\epsilon_{m_k})$ is bounded. Let us consider the case (4.8). We apply Leibniz formula for the determinant of $-\lambda_{m_k} I + A + B\exp(-\lambda_{m_k}/\epsilon_{m_k})$ (compare with

Lemma 7) and note that the leading order summand corresponds to the identity permutation which contains d factors of order $\exp(-\lambda_{m_k}/\epsilon_{m_k})$ and $n-d$ factors of oder λ_{m_k}. This gives us a contradiction to the assumption $\lambda_n \in \Sigma_c^{\epsilon_n}$. Hence λ_n is bounded.

Let $(\lambda_{n_k})_{k\in\mathbb{N}}$ be a subsequence converging to λ_0. Suppose $\mathfrak{Re}\,\lambda_0 > 0$. Then one can pass to the limit in (2.1) and gets $\det(-\lambda_0 I + A) = 0$. This contradicts that $\lambda_{n_k} \notin \Sigma_+^{\epsilon}$. Suppose $\mathfrak{Re}\,\lambda_0 < 0$. There exist converging subsequences $(\lambda_{n_k})_{k\in\mathbb{N}}$ and $v_k \in \mathbb{C}^N$, $\|v_k\| = 1$, such that

$$(-\lambda_{n_k} I + A + B\exp(-\lambda_{n_k}/\epsilon_{n_k}))\, v_k = 0. \tag{4.9}$$

Multiplying with $\exp(\lambda_{n_k}/\epsilon_{n_k})$ and passing to the limit we get $Bv_0 = 0$, where $v_0 := \lim_{k\to\infty} v_k$. Next passing to the limit in the first $N-d$ lines of (4.9) we get that $\lambda_0 \in \mathcal{A}_-$. This contradicts $\lambda_{n_k} \notin \Sigma_-^{\epsilon_{n_k}}$. □

Lemmas 9 and 12 prove statement *(iv)* of Theorem 3. Indeed, assume statement *(iv)* was false. Then there would exist $R_0 > 0$ and $\delta_0 > 0$ and for $n \in \mathbb{N}$ $\lambda_n \in \Sigma_c^{\epsilon_n}$, $|\mathfrak{Im}\,\lambda_n| \le R_0$, $\epsilon_n > 0$, $\lim \epsilon_n = 0$, such that

$$|\mathfrak{Re}\lambda_n| \ge \delta_0 \text{ for all } n \in \mathbb{N} \tag{4.10}$$

or

$$|S_{\epsilon_n}(\lambda_n) - \mu| \ge \delta_0 \text{ for all } \mu \in \mathcal{A}_c \text{ and } n \in \mathbb{N}. \tag{4.11}$$

Case (4.10) would contradict Lemma 12. Case (4.11) together with our definition (2.5) of the pseudocontinuous spectrum would contradict Lemma 9. Hence the proof of Theorem 3 is complete.

Finally, we should comment that Corollary 5 follows directly from the main Theorem 3 and the following

Remark 13. For all $C > 0$ there exists $R > 0$ such that for all $\epsilon > 0$ we have that if $\lambda \in \Sigma^{\epsilon}$ - and $|\mathfrak{Im}\lambda| \ge R$ then $\mathfrak{Re}\lambda \le -C\epsilon$.

Proof. If the assertion was false then there would exist $C_0 > 0$ and sequences $(\lambda_n)_{n\in\mathbb{N}}$, $(\epsilon_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$, $\lambda_n \in \mathbb{C}$, $\epsilon_n > 0$, $v_n \in \mathbb{R}^{\mathbb{N}}$, $\|v_n\| = 1$, $\lim_{n\to\infty} v_n = v$, with the properties

$$-\mathfrak{Re}\frac{\lambda_n}{\epsilon_n} < C_0, \quad \lim_{n\to\infty} |\lambda_n| = \infty, \quad \left(-\lambda_n I + A + Be^{\frac{-\lambda_n}{\epsilon_n}}\right) v_n = 0.$$

Deviding the last equation by λ_n and passing $n \to \infty$ yields the contradiction $v = 0$. □

The authors acknowledge the support of DFG Research Center Matheon "Mathematics for key technologies".

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