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Absorbing Boundary Condition for Hyperbolic Systems

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Abstract

This paper deals with absorbing boundary conditions for hyperbolic systems in one and two space dimensions. We prove the strict well–posedness of the resulting initial boundary value problem in 1D. Afterwards we establish the GKS–stability of the corresponding Lax-Wendroff–type finite difference scheme. Hereby, we have to extend the classical proofs, since the (discretized) absorbing boundary conditions do not fit the standard form of boundary conditions for hyperbolic systems.

1. Introduction

This article is concerned with the numerical approximation of hyperbolic partial differential equations that are posed on an unbounded spatial domain (usually $\mathbb{R}^N$). When solving this *whole space problem* numerically one is facing the problem that one has to confine the computational domain. Typical examples for first order hyperbolic equations are the Maxwell equations, the (linearized) shallow water equations and the classical hydrodynamic equation in semiconductor simulation [30](without heat conduction term).

Sometimes it is useful to apply a coordinate transformation with *conformal mappings* in order to transfer the original whole space problem to a new problem defined on a bounded domain. Unfortunately, the differential equation often becomes quite complicated [35] and this transformation technique of conformal mappings fails, if the solution is oscillating at infinity and turns out to be not suitable for many physical problems [16].

Another strategy is to confine the domain by introducing artificial boundaries without making any changes to the considered differential equation. At these artificial boundaries one defines so–called *absorbing boundary conditions* (ABCs), which are designed such that the solution of the bounded domain approximates well the solution on the original unbounded domain. The quality of approximation will be higher, if the components leaving the interior of the bounded domain (*outflow components*) have only small reflections at the artificial boundary. Especially the amplitudes of the waves that are reflected from the artificial boundaries should be as small as possible [10].

There exist a couple of applications of absorbing boundary conditions in the literature, e.g. in electro dynamics [6], in fluid dynamics [5] and in geology [7]. In meteorology ABCs are used in local area weather forecasts [9], since the original domain (earth surface) would require a too high computational effort to solve the simulation in the given time frame and coarsening the grid would lead to unsatisfactory results.

This work consists of two parts: an analytic part and a numerical (discrete) part. In the first analytic section we will use the technique of pseudodifferential operators [36] to construct a *hierarchy of absorbing boundary conditions* for linear first order hyperbolic systems. Our procedure closely follows the classical work of Engquist and Majda [10]. Afterwards we will investigate the one–dimensional case and prove that the resulting initial boundary value problems (IBVP) are *well-posed* in the strict sense of Kreiss and Lorenz [24]. For hyperbolic systems in two spatial dimensions Engquist and Majda already showed that ABCs may give rise to not well–posed problems [10].

In the second, numerical part of this article the derived absorbing boundary conditions are discretized adequately and we show that the resulting *Lax-Wendroff difference scheme* for the IBVP in 1D is *GKS–stable* (stable in the sense of Gustafsson, Kreiss and Sundström). The technique of our proof can be generalized to other finite difference schemes and other discretizations of the ABCs. We will present several numerical examples and focus on the numerical stability and the *discrete absorptions qualities* of the discretized ABCs.

2. Absorbing Boundary Conditions for Linear Hyperbolic Systems

In this section we will first introduce the most important definitions for first order hyperbolic systems. Notice that differential equations of higher order can always be rewritten as first order systems [39].

In general boundary conditions for hyperbolic equations cannot be chosen arbitrarily. For first order systems in 1D (see Section 2.1) one can interpret each component of the solution as a *propagating wave* and it will turn out that each acceptable boundary condition on the one side prescribe the behaviour of inflowing parts and on the other side must not impose any condition on outflowing parts. These 'parts' correspond to the characteristics of the systems. This situation gets more complicated in two spatial dimensions: it is not easy to distinguish between inflowing and outflowing parts of the solution. Especially, waves can propagate tangentially to the boundary and it is a-priori unclear if and how to impose boundary conditions for these components.

2.1. Systems in one Spatial Dimension

We consider systems of the form

$$U_t + A(x,t)U_x + C(x,t)U = F(x,t) \tag{2.1a}$$

on the strip $0 \le x \le L$, $t \ge 0$ with the initial condition

$$U(x,0) = f(x), \qquad 0 \le x \le L, \tag{2.1b}$$

with boundary conditions at $x = 0$, $x = L$. The coefficients $A(x,t)$, $C(x,t) \in \mathbb{R}^{N\times N}$, the source term $F(x,t) \in \mathbb{R}^N$ and the initial data $f(x) \in \mathbb{R}^N$ are assumed to be C^∞–smooth.

Next, we assume that the system (2.1a) is *hyperbolic*, i.e. $A(x,t)$ has for $x \in [0,L]$, $t \ge 0$ real eigenvalues $\lambda_j(x,t)$ and a complete set of real eigenvectors. This property allows the simplification of the system (2.1a), since then $A = A(x,t)$ is *diagonalizable*, i.e. there exists a regular matrix $T = T(x,t)$ (containing the eigenvectors), such that

$$T^{-1}AT = \Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_N), \qquad \lambda_j = \lambda_j(x,t), \qquad j = 1,2,\dots,N. \tag{2.2}$$

Furthermore, for systems with *variable coefficients* we have to assume that the matrix norms of $T(x,t)$, $T^{-1}(x,t)$ remain bounded for $x \in [0,L]$, $t \ge 0$. A simple calculation yields

$$(T^{-1}U)_t = -\Lambda(T^{-1}U)_x - (T^{-1}C - T_t^{-1} - T^{-1}AT\,T_x^{-1})U + T^{-1}F.$$

Using the characteristic variables $V(x,t) := T^{-1}(x,t)U(x,t)$ the system (2.1a) transforms into its characteristic form

$$V_t + \Lambda(x,t)V_x + \tilde{C}(x,t)V = \tilde{F}(x,t), \tag{2.3}$$

with $\tilde{C} = (T^{-1}C - T_t^{-1} - T^{-1}AT\,T_x^{-1})T = T^{-1}CT - T_t^{-1}T + T^{-1}AT_x$ and $\tilde{F} = T^{-1}F$.

Remark 2.1. *Without loss of generality we assume in the sequel that the system* (2.1a) *is already formulated in characteristic variables, i.e. the system matrix A is diagonal:* $A = \Lambda$.

Furthermore, we make the following assumption

Assumption 2.1 (Constant Partition at the Boundary [24])**.**

$$\lambda_j(0,t) \quad \textit{and} \quad \lambda_j(L,t), \qquad j = 1,2,\dots,N \tag{2.4}$$

as functions of time do not change their sign, i.e. each function (2.4) *is either* > 0 *for all* t, $= 0$ *for all* t *or* < 0 *for all times* t.

The case of *N* scalar equations. In order to investigate the system (2.1a), we use the *method of characteristics* and start with the *decoupled case* $C = 0$. Then the system reduces to N independent scalar hyperbolic equations. Furthermore we assume $F = 0$, i.e. we have

$$\frac{\partial U_j}{\partial t} + \lambda_j \frac{\partial U_j}{\partial x} = 0, \qquad j = 1,2,\dots,N. \tag{2.5}$$

This equations correspond to ordinary differential equations along the *characteristic curve*

$$\frac{d}{dt}\Big(U_j(x(t),t)\Big) = 0 \qquad \text{for} \quad \frac{dx}{dt} = \lambda_j(x,t).$$

Hence, the components U_j are constant on the characteristic curve $(x(t),t)$ and propagate with the *characteristic velocities* $\lambda_j = dx/dt$. First, we assume that the velocities λ_j are constant, i.e. $\lambda_j(x,t) = \lambda_j = \text{const.}$, $j = 1,2,\dots,N$ and introduce the following notation:

Notation . U^+, U^0, U^- consist of the variables $U_j = U_j(x,t)$ with index j, for which $\lambda_j > 0$, $\lambda_j = 0$, $\lambda_j < 0$ holds. This *indexing concept* of components will be transfered later to other quantities, e.g. f, or sub matrices, etc..

Obviously, U^0 is determined solely by the initial data: $U^0(x,t) = f^0(x)$, $0 \le x \le L$, $t \ge 0$, i.e. boundary conditions for characteristic variables U^0 with velocities $\lambda_j = 0$ are neither necessary nor feasible.

For $\lambda_j > 0$ the characteristics are from left to right and besides initial data one need boundary data at the left boundary $x = 0$, the so–called *inflow boundary*. It is not permitted to impose any boundary conditions at the left boundary $x = L$, *outflow boundary*, since they could contradict the initial condition and prevent the existence of a solution. For $\lambda_j < 0$ the meaning and the notation of the two boundaries is interchanged. Hence we formulate

$$U^+(0,t) = g_0(t), \qquad U^-(L,t) = g_L(t), \qquad t > 0,$$

i.e. we impose a condition for the incoming characteristic variables at each boundary. These boundary condition can be generalized to

$$\begin{aligned} U^+(0,t) &= S_0(t)U^-(0,t) + g_0(t), \qquad t > 0, \\ U^-(L,t) &= S_L(t)U^+(L,t) + g_L(t), \qquad t > 0, \end{aligned} \tag{2.6}$$

where $S_0(t)$, $S_L(t)$ are matrices of appropriate dimensions.

Along the characteristics, starting in the corner points $(x,t)=(0,0)$, $(x,t)=(L,0)$, the solution will have a jump discontinuity, if the initial function f and the boundary data g_0, g_L are *not compatible*, i.e. if $f(0)\neq g_0(0)$, $f(L)\neq g_L(0)$. This is a fundamental property of *linear* hyperbolic equations: singularities only propagates along characteristics. Thus, we assume that the initial data is *compatible* with the boundary data, meaning not only the coincidence of the function values but also the values of derivatives or arbitrary order:

Assumption 2.2 (Compatibility of the Data)**.** *The boundary data g_0, g_L are assumed to be C^∞-smooth and compatible with the initial function f, i.e. $f(0)=g_0(0)$, $f(L)=g_L(0)$, and also compatible in derivatives of arbitrary high order.*

Notice that this assumption is e.g. fulfilled, if the data f, g_0 and g_L vanish identically close to the corner points $(x,t)=(0,0)$, $(x,t)=(L,0)$.

We turn now to the case $\lambda_j=\lambda_j(x,t)$. Then, in general, the characteristics are no lines any more. We assume that all eigenvalues at the boundary are non–zero, i.e. the boundary is *not characteristic*. If an eigenvalue $\lambda_j(x,t)$ changes its sign on $0<x<L$, then the component U_j may belong e.g. at $x=0$ to a positive and at $x=L$ to a negative characteristic velocity. Also in this case we will use the notation from page 4, to reference the components U_j at the boundary points.

Remark 2.2 (coupled case)**.** *For $C\neq 0$ the resulting system* (2.1a) *is in general* coupled*, but only in the low order terms. These terms do not influence the* partition *into inflow and outflow parts, [21, 37], i.e. the boundary conditions must fulfill the same criteria as before.*

For a hyperbolic problem with boundary conditions of the form (2.6) the well-posedness is well–known. For convenience we repeat here the Theorem 7.6.4 on the existence of a C^∞-solution and its estimate from [24]:

Theorem 2.1. *Assume that the boundary is not characteristic and the data F, f, g_0, g_L are compatible at $t=0$. Then the hyperbolic IBVP* (2.1)*,* (2.6) *has a unique solution U that is a C^∞-function. For each finite time interval $0\le t\le T$ there exists a constant c_T, such that*

$$\|U(.,t)\|_2^2+\int_0^t\left(|U(0,\tau)|^2+|U(L,\tau)|^2\right)d\tau \le c_T\left[\left\|f\right\|_2^2+\int_0^t\left(\left|g_0(\tau)\right|^2+\left|g_L(\tau)\right|^2+\|F(.,t)\|_2^2\right)d\tau\right] \quad (2.7)$$

for $0\le t\le T$. The constant c_T is independent of the data F, f, g_0, g_L.

Note that the C^∞-smoothness of the data is assumed only for simplicity reasons. The proof also works with much less regularity of the data (cf. Remark 2.5).

Thus this problem is *well–posed*. However, the ABCs that we will formulate in Section 2.2 do not have the form (2.6): instead of the matrices S_0, S_L integral operators

appear. Hence we need for showing the well–posedness of the problem (2.1) with ABCs in 1D another proof that we will present in Section 2.3 using Theorem 2.1.

Next we consider *systems in two spatial dimensions* of the form

$$U_t + A(x,y,t)U_x + B(x,y,t)U_y + C(x,y,t)U = F(x,y,t) \tag{2.8a}$$

on the strip $0 \le x \le L$, $-\infty < y < \infty$, $t \ge 0$ with the initial condition

$$U(x,y,0) = f(x,y), \qquad 0 \le x \le L,\ -\infty < y < \infty \tag{2.8b}$$

and supplied with boundary conditions at $x = 0$, $x = L$. The coefficients and the initial function are again assumed to be C^∞–smooth.

In general, it is not possible in 2D to fully investigate the solutions using solely the method of characteristics and additional problems will appear if waves propagate tangentially to the boundary. Here we will only introduce two important definitions:

Definition 2.1 (strictly hyperbolic system [21])**.** *The system* (2.8a) *is called* strictly hyperbolic*, if the matrices $k_1A + k_2B$ have distinct real eigenvalues for all $(k_1,k_2) \neq (0,0)$.*

Definition 2.2 (symmetrizable hyperbolic systems [21])**.** *The system* (2.8a) *is called* symmetrizable hyperbolic*, if the matrices A, B are symmetric for all arguments $(x,y,t) \in [0,L] \times \mathbb{R} \times \mathbb{R}_0^+$.*

Definition 2.2 can be generalized to systems where the coefficient of the time derivative is not the identity: in this case this coefficient must be positive definite.

2.2. Derivation of the Absorbing Boundary Conditions

We sketch briefly for convenience the derivation of *absorbing boundary conditions (ABCs)* at $x = 0$. For this purpose we review the classical construction from Engquist and Majda [10, Section 2]. We consider *strict hyperbolic* systems of first order with variable coefficients

$$U_t + \Lambda(x,y,t)U_x + B(x,y)U_y + C(x,y,t)U = 0 \tag{2.9}$$

posed on the half space $x \ge 0$, $-\infty < y < \infty$, $t \ge 0$, where Λ, B are symmetric $N \times N$ matrices. We assume that Λ is regular for all $(x,y,t) \in \mathbb{R}_0^+ \times \mathbb{R} \times \mathbb{R}_0^+$ and assume without loss of generalization that (2.9) is already in characteristic form, i.e.

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_N),$$

with $\lambda_j > 0$, $1 \le j \le m$ and $\lambda_j < 0$, $m+1 \le j \le N$. According to our convention we write

$$\Lambda^+ = \operatorname{diag}(\lambda_1, \ldots, \lambda_m), \quad \Lambda^- = \operatorname{diag}(\lambda_{m+1}, \ldots, \lambda_N).$$

Note that due to the strict hyperbolicity of (2.9) we have $\lambda_j \neq \lambda_k$ for $j \neq k$.

We reformulate the system (2.9) as

$$U_x = -\Lambda^{-1}U_t - \Lambda^{-1}BU_y + \tilde{C}U,$$

with $\tilde{C} = -\Lambda^{-1}C$ and define $M = M(\xi,\omega)$ as $M(\xi,\omega) := -i\Lambda^{-1}\xi - i\Lambda^{-1}B\omega$. Then

$$U_x = M\left(\frac{\partial_t}{i}, \frac{\partial_y}{i}\right)U + \tilde{C}U. \tag{2.10}$$

In the essential step we decouple the positive and negative λ. Following the *construction of Taylor* [38, 39] there exist a smooth matrix $V = V(\xi,\omega,x,y,t)$, invertible for all (ξ,ω) with $|\omega/\xi| + |\omega| < c_0$ for one $c_0 > 0$, such that (2.10) can be transformed using

$$W = V\left(\frac{\partial_t}{i}, \frac{\partial_y}{i}, x, y, t\right)U \tag{2.11}$$

to

$$W_x = \begin{pmatrix} \Omega_{11} & 0 \\ \Omega_{21} & \Omega_{22} \end{pmatrix} W. \tag{2.12}$$

Hereby, $\Omega_{11} = \Omega_{11}\left(\partial_t/i, \partial_y/i, x, y, t\right)$ denotes a $m \times m$ pseudodifferential operator of first order, i.e. $\Omega_{11} \in \mathrm{PS}(1)$. Ω_{21} and Ω_{22} are $(N-m)\times m$, $(N-m)\times(N-m)$ pseudodifferential operators of first order, respectively.

Once the differential equation is in *decoupled form* (2.12), it is an easy task to formulate the ABC, since Ω_{11} contains exactly the positive eigenvalues, that corresponds to the inflow components at $x = 0$. Therefore, the ABC must eliminate exactly this components at $x = 0$:

$$W^+\Big|_{x=0} = (VU)^+\Big|_{x=0} = 0 \qquad \text{with} \quad W^+ = (W_1, \ldots, W_m). \tag{2.13}$$

In other words, the reflected part W^+ at $x = 0$ is set to zero.

Taylor's construction further state that for $V(\xi,\omega,x,y,t)$ exists an *asymptotic expansion*

$$\begin{aligned} V(\xi,\omega,x,y,t) \cong\ & V_0(\xi,\omega,x,y,t) + \xi^{-1}V_{-1}(\xi,\omega,x,y,t) \\ & + \xi^{-2}V_{-2}(\xi,\omega,x,y,t) + \ldots, \end{aligned} \tag{2.14}$$

where each V_j is homogeneous of degree zero in (ξ,ω).

Due to the strict hyperbolicity $M(1,0)$ is a diagonal matrix with pairwise different eigenvalues, and hence there exist a constant c_0, such that $M(\xi,\omega)$ possess pairwise different eigenvalues in a conical neighborhood around $(1,0)$, also for $|\omega/\xi| + |\omega| < c_0$. Now V_0 is chosen in such a way that

$$V_0 M V_0^{-1} = \begin{pmatrix} {}_1\Omega_{11} & 0 \\ {}_1\Omega_{21} & {}_1\Omega_{22} \end{pmatrix}. \tag{2.15}$$

Note that V_0 is not uniquely determined. In 1D ($B = 0$) or if B is diagonal, then $M(\xi,\omega)$ is a diagonal matrix and one can choose $V_0(\xi,\omega) = I$.

By terminating the asymptotic expansion (2.14) after a finite number of terms we obtain a *hierarchy of ABCs*. We use the approximation $V \cong V_0 + \xi^{-1}V_{-1} + \mathrm{O}(\xi^{-2}) = (1 + K_1)V_0 + \mathrm{O}(\xi^{-2})$ with $K_1V_0 = \xi^{-1}V_{-1}$, $(1 + K_1) \in \mathrm{PS}(-1)$, $V_0 \in \mathrm{PS}(0)$, to achieve an

equation of the form (2.12) for $\widetilde{W} := (1+K_1)V_0U$ up to errors of order $\mathrm{O}(\xi^{-1})$ on symbol level. Afterwards we proceed as in (2.13). With $V^{-1} \cong V_0^{-1}(1-K_1)+\mathrm{O}(\xi^{-2})$ we obtain

$$W_x = V_xU + VU_x = V_xU + V(MU+\tilde{C}U) = V_xV^{-1}W + V(M+\tilde{C})V^{-1}W.$$

The *composition formula for pseudodifferential operators* [36, Theorem 3.4] gives

$$\begin{aligned} V_{0_x}V_0^{-1} &+ (1+K_1)V_0(M+\tilde{C})V_0^{-1}(1-K_1) \\ &= V_{0_x}V_0^{-1} + V_0MV_0^{-1} + K_1V_0MV_0^{-1} - V_0MV_0^{-1}K_1 + V_0\tilde{C}V_0^{-1} + \mathrm{O}(\xi^{-1}). \end{aligned}$$

Now one has to determine K_1 in such a way that

$$D(x,y,t) := K_1V_0MV_0^{-1} - V_0MV_0^{-1}K_1 + V_0\left[V_0^{-1}V_{0_x} - \Lambda^{-1}C(x,y,t)\right]V_0^{-1}$$

becomes a lower block diagonal matrix. Here equation (2.15) was used. For $B = \text{const.}$ we have $D(x,y,t) = -\Lambda^{-1}C(x,y,t)$, and in this case a lower block diagonal matrix can be obtained if K_1 has the form

$$K_1(x,y,t) = \begin{pmatrix} 0 & K(x,y,t) \\ 0 & 0 \end{pmatrix}$$

and fulfills the equation

$$-K(x,y,t)(\Lambda^-)^{-1} + (\Lambda^+)^{-1}K(x,y,t) - (\Lambda^+)^{-1}C^{+-}(x,y,t) = 0.$$

It is ${}_1\Omega_{11} = -(\Lambda^+)^{-1}$ and ${}_1\Omega_{22} = -(\Lambda^-)^{-1}$ and thus we have

$$K(x,y,t) = (k_{j\ell})_{\substack{1\le j\le m \\ m+1\le \ell\le N}} \quad \text{with} \quad k_{j\ell}(x,y,t) = \frac{\lambda_\ell}{\lambda_\ell - \lambda_j}\, c_{j\ell}(x,y,t). \tag{2.16}$$

Let us remark that there exists a unique K, if $(\Lambda^+)^{-1}$ and $(\Lambda^-)^{-1}$ have disjoint spectra [38, Section 1]. This property is obviously fulfilled here.

From a *perturbation calculation* in [10, Section 2C] follows

$$V_0(1,0) = I \quad \text{and hence} \quad \xi^{-1}V_{-1}(1,0,x,y,t) = K_1(x,y,t), \tag{2.17a}$$

$$V_0(\xi,\omega) = V_0(1,\frac{\omega}{\xi}) = I + \frac{\omega}{\xi}\left(\frac{\partial}{\partial\omega}V_0\right)(1,0) + \mathrm{O}\left(\left|\frac{\omega}{\xi}\right|^2\right). \tag{2.17b}$$

Remark 2.3. *Since it is $\omega/\xi = \sin\theta$, with θ being the impact angle of the waves, the expansion* (2.17b) *is around an orthogonal impact angle.*

Furthermore one obtains for the ansatz

$$\frac{\partial}{\partial\omega}V_0(1,0,x,y) = \begin{pmatrix} 0 & X(x,y) \\ 0 & 0 \end{pmatrix}$$

the condition

$$-X(x,y)(\Lambda^-)^{-1}+(\Lambda^+)^{-1}X(x,y)-(\Lambda^+)^{-1}B^{+-}(x,y)=0,$$

such that finally

$$X(x,y)=(\chi_{j\ell})_{\substack{1\le j\le m\\ m+1\le \ell\le N}} \quad \text{with} \quad \chi_{j\ell}(x,y)=\frac{\lambda_\ell}{\lambda_\ell-\lambda_j}\,b_{j\ell}(x,y). \tag{2.18}$$

From the asymptotic expansion (2.14) and the Taylor expansion (2.17b) around the *orthogonal impact angle* θ (it is $\omega/\xi=\sin\theta$) one finally obtains (after a multiplication with ξ) as in [10] the ABCs:

$$\underline{\text{zero order ABC:}}\ \text{Error}\ \mathrm{O}(|\omega/\xi|+1/|\xi|): U^+(0,y,t)=0 \tag{2.19a}$$

$$\underline{\text{`}\tfrac{1}{2}\text{ order' ABC:}}\ \text{Error}\ \mathrm{O}(|\omega/\xi|+1/|\xi|^2): U_t^+(0,y,t)+K(0,y,t)U^-(0,y,t)=0 \tag{2.19b}$$

$$\underline{\text{first order ABC:}}\ \text{Error}\ \mathrm{O}(|\omega/\xi|^2+1/|\xi|^2):$$
$$U_t^+(0,y,t)+X(0,y)U_y^-(0,y,t)+K(0,y,t)U^-(0,y,t)=0 \tag{2.19c}$$

Remark 2.4. *If B is a diagonal matrix, then* (2.19b) *is identical to* (2.19c).

Until now we assumed that the initial data vanishes outside the computational domain. If this is not the case one has to use *inhomogeneous boundary conditions*. The *inhomogeneous first order ABC* in 1D reads:

$$U_t^+(0,t)+K(0,t)U^-(0,t)=g_t(t),\quad t\ge 0, \tag{2.20}$$

$$\text{or}\quad U^+(0,t)=U^+(0,0)-\int_0^t K(0,\tau)U^-(0,\tau)\,d\tau+g(t)-g(0),\quad t\ge 0. \tag{2.21}$$

In the sequel we set $K_0(\tau):=K(0,\tau)$.

2.3. Well–posedness in One Space Dimension

We consider the problem (2.1) with variable coefficients and inhomogeneous first order absorbing boundary conditions on the strip $0\le x\le L$:

$$U_t+\Lambda(x,t)U_x+C(x,t)U=F(x,t),\quad 0\le x\le L,\quad t\ge 0, \tag{2.22a}$$
$$\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_N)\quad\text{with}\quad \lambda_1\ge\lambda_2\ge\cdots\ge\lambda_m>0>\lambda_{m+1}\ge\cdots\ge\lambda_N,$$
$$U(x,0)=f(x),\quad 0\le x\le L, \tag{2.22b}$$
$$\begin{aligned} U^+(0,t)&=U^+(0,0)-\int_0^t K_0(\tau)U^-(0,\tau)\,d\tau+g_0(t)-g_0(0),\quad t\ge 0,\\ U^-(L,t)&=U^-(L,0)-\int_0^t K_L(\tau)U^+(L,\tau)\,d\tau+g_L(t)-g_L(0),\quad t\ge 0. \end{aligned} \tag{2.22c}$$

The coefficients $\Lambda(x,t)$, $C(x,t) \in \mathbb{R}^{N\times N}$ and $F(x,t)$, $f(x)$, $(g_0(t), g_L(t))^\top \in \mathbb{R}^N$ are C^∞-smooth, and especially, due to the *compatibility Assumption* 2.2, we have in the ABCs (2.22c) $U^+(0,0) = g_0(0)$ and $U^-(L,0) = g_L(0)$. Roughly speaking, the IBVP (2.22) is *well–posed*, if for all smooth compatible data F, f, g_0, g_L there exists a unique smooth solution U, that can be bounded by terms of the data in every finite time interval $0 \le t \le T$.

Let us further remark that it is not necessary, to further precise the *smoothness assumptions* for F, f, g_0 and g_L. One can always substitute 'smooth' by 'C^∞'. Once an estimate for this case is derived one can approximate the *less smooth data*, as long the used norms are defined for this data, cf. [24, p. 223]. This holds also for Theorem 2.1.

Besides the L^2-norm $\|.\|_2$ on $[0,L]$ we will use in the sequel the following notation for *norms at the boundary*, i.e. for $x = 0$, $x = L$:

$$\begin{aligned} &\underline{\text{Euclidean norm:}} \quad |.|_\pm \\ \text{inflow part:} \quad &|U(t)|_-^2 := |U^+(0,t)|^2 + |U^-(L,t)|^2 \\ \text{outflow part:} \quad &|U(t)|_+^2 := |U^+(L,t)|^2 + |U^-(0,t)|^2 \end{aligned}$$

$$\begin{aligned} &\underline{\text{weighted norm:}} \quad \|.\|_\pm \\ \text{inflow part:} \quad &\|U(t)\|_-^2 := \sum_{\lambda_j(0,t)>0} \lambda_j(0,t) \left|U_j(0,t)\right|^2 - \sum_{\lambda_j(L,t)<0} \lambda_j(L,t) \left|U_j(L,t)\right|^2 \\ \text{outflow part:} \quad &\|U(t)\|_+^2 := \sum_{\lambda_j(L,t)>0} \lambda_j(L,t) \left|U_j(L,t)\right|^2 - \sum_{\lambda_j(0,t)<0} \lambda_j(0,t) \left|U_j(0,t)\right|^2 \end{aligned}$$

For matrices $|.|$ denotes a matrix norm that is *compatible* with the Euclidean vector norm.

Definition 2.3 (strong well-posedness [24, p. 224])**.** *The IBVP* (2.22) *is called* strongly well-posed*, if for all smooth data F, f, g_0, g_L exists a unique smooth solution U and for every finite time interval $0 \le t \le T$ exists a constant c_T, such that*

$$\begin{aligned} \|U(.,t)\|_2^2 + \int_0^t \left(\|U(\tau)\|_-^2 + \|U(\tau)\|_+^2\right) d\tau \\ \le c_T \left[\left\|f\right\|_2^2 + \int_0^t \left(\left|g_0(\tau)\right|^2 + \left|g_L(\tau)\right|^2 + \|F(.,\tau)\|_2^2\right) d\tau\right] \end{aligned} \tag{2.23}$$

for $0 \le t \le T$. The constant c_T must be independent of F, f, g_0, g_L.

Remark 2.5 (Smoothness of data)**.** *The well–posedness of* (2.22) *can be proven under much less strict assumptions on the smoothness of the data; obviously it is sufficient to require: $F \in L^2((0,T), L^2((0,L), \mathbb{R}^N))$, $f \in L^2((0,L), \mathbb{R}^N)$ and $(g_0, g_L)^\top \in L^2((0,T), \mathbb{R}^N)$.*

We should address now the question what happens in case of *zero crossings* of one eigenvalue $\lambda_j(x,t)$ in the interior of the strip, i.e. what happens if $\Lambda(x,t)$ gets *singular* on $0 < x < L$. In Assumption 2.3 it was sufficient to require the uniform boundedness

of Λ only at the boundaries $x = 0$ and $x = L$. Therefor one has (besides the uniform boundedness of Λ_x) the following assumption to the eigenvalues λ_j :

$$\text{const.} \geq |\lambda_j(0,t)|,\ |\lambda_j(L,t)| \geq \text{const.} > 0, \quad j = 1,\ldots,N.$$

Hence, an eigenvalue λ_j may become zero on $0 < x < L$, as long as Λ_x remains uniformly bounded, e.g. $\lambda_j(x,t) = \alpha(x - x_0) + a(t)$, with $a(t)$ suitable chosen.

2.3.1. The Estimate of the Solution

We obtain for the constant coefficient system as the difference of outflow and inflow parts:

$$\begin{aligned}\|U(t)\|_+^2 - \|U(t)\|_-^2 &= \sum_{m=1}^{N} \lambda_j \left|U_j(L,t)\right|^2 - \sum_{m=1}^{N} \lambda_j \left|U_j(0,t)\right|^2 \\ &= \langle \Lambda U(x,t), U(x,t)\rangle \Big|_{x=0}^{x=L} = 2\int_0^L \left\langle \Lambda U_x(x,t), U(x,t)\right\rangle \, dx.\end{aligned}$$

We multiply (2.22a) with $2U$ and integrate w.r.t. x :

$$\begin{aligned}\partial_t \|U(.,t)\|_2^2 + \|U(t)\|_+^2 - \|U(t)\|_-^2 + 2\int_0^L \langle CU(x,t), U(x,t)\rangle \, dx \\ = 2\int_0^L \langle F(x,t), U(x,t)\rangle \, dx. \quad (2.24)\end{aligned}$$

An integration w.r.t. time yields the following *inflow–outflow balance* :

$$\begin{aligned}\|U(.,t)\|_2^2 = \left\|f\right\|_2^2 + \int_0^t \left(\|U(\tau)\|_-^2 - \|U(\tau)\|_+^2\right) d\tau - 2\int_0^t \int_0^L \langle CU(x,\tau), U(x,\tau)\rangle \, dx\, d\tau \\ + 2\int_0^t \int_0^L \langle F(x,\tau), U(x,\tau)\rangle \, dx\, d\tau.\end{aligned}$$

Note that if C is skew-symmetric, then $\langle CU(x,\tau), U(x,\tau)\rangle = 0$. Our goal is to estimate the inflow data by terms of the outflow data. We have

$$\|U(t)\|_-^2 \leq \rho_1 \left(|U^+(0,t)|^2 + |U^-(L,t)|^2\right), \quad \rho_1 := \rho(\Lambda) = \max_{p=1,\ldots,N} \left|\lambda_p\right|. \quad (2.25)$$

From the boundary condition (2.22c) one obtains with the Cauchy-Schwarz inequality

$$\left|\int_0^t K_0 U^-(0,\tau)\, d\tau\right| \leq \left|K_0\right| \sqrt{t} \left(\int_0^t |U^-(0,\tau)|^2 \, d\tau\right)^{\frac{1}{2}}$$

and thus

$$\begin{aligned} |U^+(0,t)|^2 &\le 2\left|g_0(t)\right|^2 + 2\left|\int_0^t K_0 U^-(0,\tau)\,d\tau\right|^2 \\ &\le 2\left|g_0(t)\right|^2 + 2\left|K_0\right|^2 t\int_0^t |U^-(0,\tau)|^2\,d\tau. \end{aligned} \tag{2.26}$$

Analogously, one gets from the boundary condition at $x = L$ the estimate

$$|U^-(L,t)|^2 \le 2\left|g_L(t)\right|^2 + 2\left|K_L\right|^2 t\int_0^t |U^+(L,\tau)|^2\,d\tau. \tag{2.27}$$

Hence, we have, with $\kappa := \max(\left|K_0\right|, \left|K_L\right|)$, the estimate

$$\begin{aligned} \|U(t)\|_-^2 &\le 2\rho_1\left(\left|g_0(t)\right|^2 + \left|g_L(t)\right|^2 + \kappa^2 t\int_0^t \left(|U^-(0,\tau)|^2 + |U^+(L,\tau)|^2\right)d\tau\right) \\ &\le 2\rho_1\left(\left|g_0(t)\right|^2 + \left|g_L(t)\right|^2 + \kappa^2 t\int_0^t |U(\tau)|_+^2\,d\tau\right). \end{aligned} \tag{2.28}$$

If we consider the function

$$z(t) := \|U(.,t)\|_2^2 + \int_0^t \left(\|U(\tau)\|_-^2 + \|U(\tau)\|_+^2\right)d\tau, \tag{2.29}$$

then for $0 \le t \le T$ holds:

$$\begin{aligned} z(t) &= \left\|f\right\|_2^2 + 2\int_0^t \|U(\tau)\|_-^2\,d\tau - 2\int_0^t\int_0^L \left(\langle CU(x,\tau), U(x,\tau)\rangle - \langle F(x,\tau), U(x,\tau)\rangle\right)dx d\tau \\ &\le \left\|f\right\|_2^2 + 4\rho_1\left(\int_0^t\left(\left|g_0(\tau)\right|^2 + \left|g_L(\tau)\right|^2\right)d\tau + \kappa^2 t\int_0^t\int_0^\tau |U(s)|_+^2\,ds\,d\tau\right) \\ &\quad + 2|C|\int_0^t\int_0^L |U(x,\tau)|^2\,dx\,d\tau + \int_0^t\int_0^L \left(|F(x,\tau)|^2 + |U(x,\tau)|^2\right)dx d\tau \\ &\le \left\|f\right\|_2^2 + \left(4\rho_1\rho_2\kappa^2 t + 2|C| + 1\right)\int_0^t\left(\|U(.,\tau)\|_2^2 + \int_0^\tau \|U(s)\|_+^2\,ds\right)d\tau \\ &\quad + 4\rho_1\int_0^t\left(\left|g_0(\tau)\right|^2 + \left|g_L(\tau)\right|^2\right)d\tau + \int_0^t \|F(.,\tau)\|_2^2\,d\tau \\ &\le z(0) + c_1\int_0^t z(\tau)\,d\tau + c_2\int_0^t\left(\left|g_0(\tau)\right|^2 + \left|g_L(\tau)\right|^2 + \|F(.,\tau)\|_2^2\right)d\tau \end{aligned}$$

with the constants $c_1 := 4\rho_1\rho_2\kappa^2 T + 2|C| + 1$, $c_2 := 4\rho_1 + 1$ and $\rho_2 := \rho\left(\Lambda^{-1}\right)$.

Now an application of *Gronwall's inequality* [24, Lemma 3.1.1] gives

$$z(t) \le e^{c_1 t} \left\| f \right\|_2^2 + c_2 e^{c_1 t} \int_0^t \left(\left| g_0(\tau) \right|^2 + \left| g_L(\tau) \right|^2 + \| F(.,\tau) \|_2^2 \right) d\tau .$$

Finally, we obtain with $c_T := (1 + c_2) e^{c_1 T}$ an estimate of the form:

$$\begin{aligned} \| U(.,t) \|_2^2 + \int_0^t \left(\| U(\tau) \|_-^2 + \| U(\tau) \|_+^2 \right) d\tau \\ \le c_T \left[\left\| f \right\|_2^2 + \int_0^t \left(\left| g_0(\tau) \right|^2 + \left| g_L(\tau) \right|^2 + \| F(.,\tau) \|_2^2 \right) d\tau \right] . \end{aligned} \tag{2.30}$$

This estimation can also be done for variable coefficients, if the following assumption holds.

Assumption 2.3. *The coefficient functions Λ_x and the matrices appearing in the boundary conditions K_0, K_L are* uniformly bounded*, i.e. $|\Lambda(x,t)|$, $|C(x,t)|$, $|\Lambda_x(x,t)| \le$ const. and $|K_0(t)| \le \kappa_0$, $|K_L(t)| \le \kappa_L$ for $0 \le x \le L$ and all times $t \in [0,T]$. Furthermore, for all $t \in [0,T]$ must hold $|\lambda_j(0,t)|, |\lambda_j(L,t)| \ge$ const. > 0, $j = 1, \ldots, N$, hence the norms defined on page 10 are equivalent.*

This assumption holds from now on.

2.3.2. The Existence and Uniqueness of the Solution

We will only sketch the proof based on a fix point argument for the following iteration in $L^2((0,T), \mathbb{R}^N)$: For given inflow data the equation with variable coefficients is solved and the solution is evaluated at the outflow boundary. Using the ABCs one gets back to inflow data and the iteration is finished. More precisely, the *fixpoint operator* $\boldsymbol{F}$ is defined as follows: for $V = (V^+, V^-)^\top \in L^2((0,T), \mathbb{R}^N)$ and fixed, finite time T solve the problem

$$Y_t + \Lambda(x,t) Y_x + C(x,t) Y = F(x,t), \quad 0 \le x \le L, \quad t \ge 0, \tag{2.31a}$$

$$Y(x,0) = f(x), \quad 0 \le x \le L, \tag{2.31b}$$

$$\begin{aligned} Y^+(0,t) &= V^+(t), \quad t \ge 0, \\ Y^-(L,t) &= V^-(t), \quad t \ge 0. \end{aligned} \tag{2.31c}$$

Then, $\boldsymbol{F}V = ((\boldsymbol{F}V)^+, (\boldsymbol{F}V)^-)^\top$ is defined through the integral equations

$$\begin{aligned} (\boldsymbol{F}V)^+(t) &:= g_0(t) - \int_0^t K_0(\tau) Y^-(0,\tau)\, d\tau, \quad t \ge 0, \\ (\boldsymbol{F}V)^-(t) &:= g_L(t) - \int_0^t K_L(\tau) Y^+(L,\tau)\, d\tau, \quad t \ge 0. \end{aligned} \tag{2.32}$$

To prove that this iteration is well–defined in $L^2((0,T), \mathbb{R}^N)$ we show

Lemma 2.1. *The fixpoint operator $\boldsymbol{F}$ is a self mapping on $L^2((0,T),\mathbb{R}^N)$.*

Proof. After solving (2.31) we have with Theorem 2.1 :

$$\|(Y(.,t)\|_2^2+\int_0^t \left(|Y(0,\tau)|^2+|Y(L,\tau)|^2\right)\,d\tau \le c_T'\left[\left\|f\right\|_2^2+\int_0^t \left(|V(\tau)|^2+\|F(.,\tau)\|_2^2\right)\,d\tau\right]$$

for $0\le t\le T$ and since $V\in L^2((0,T),\mathbb{R}^N)$ we obtain

$$\int_0^T \left(|Y^-(0,\tau)|^2+|Y^+(L,\tau)|^2\right)\,d\tau \le \text{const.}$$

Hence $(Y^-(0,.),Y^+(L,.))^\top \in L^2((0,T),\mathbb{R}^N)$ and with (2.32) we get $\boldsymbol{F}V\in L^2((0,T),\mathbb{R}^N)$. □

Furthermore we show

Lemma 2.2. *$\boldsymbol{F}$ is contractive at least on some subinterval $(0,T_1)$ of $(0,T)$.*

Proof. For two given inflow data V_1, V_2, one can estimate after solving (2.31) the difference of the corresponding outflow data using Theorem 2.1:

$$\begin{aligned}\left\|(Y_1-Y_2)(.,t)\right\|_2^2+\int_0^t \left(\left|(Y_1-Y_2)(0,\tau)\right|^2+\left|(Y_1-Y_2)(L,\tau)\right|^2\right)\,d\tau&\\ \le \tilde{c}_T\int_0^t \left(\left|(V_1^+-V_2^+)(\tau)\right|^2+\left|(V_1^--V_2^-)(\tau)\right|^2\right)\,d\tau&\end{aligned}$$

and thus

$$\int_0^t \left|Y_1(\tau)-Y_2(\tau)\right|_+^2 \le \tilde{c}_T\int_0^t \left|V_1(\tau)-V_2(\tau)\right|^2\,d\tau. \tag{2.33}$$

Analogously to (2.26) we obtain from the boundary conditions (2.32)

$$\begin{aligned}\left|\boldsymbol{F}V_1^+(t)-\boldsymbol{F}V_2^+(t)\right|^2 &\le \kappa_0^2 t\int_0^t \left|Y_1^-(0,\tau)-Y_2^-(0,\tau)\right|^2\,d\tau,\\ \left|\boldsymbol{F}V_1^-(t)-\boldsymbol{F}V_2^-(t)\right|^2 &\le \kappa_L^2 t\int_0^t \left|Y_1^+(L,\tau)-Y_2^+(L,\tau)\right|^2\,d\tau,\end{aligned}$$

and this yields with (2.33) and $\kappa := \max(\kappa_0,\kappa_L)$

$$\begin{aligned}\left|\boldsymbol{F}V_1(t)-\boldsymbol{F}V_2(t)\right|^2 &\le \kappa^2 t\int_0^t \left|Y_1(\tau)-Y_2(\tau)\right|_+^2\,d\tau\\ &\le \tilde{c}_T\kappa^2 t\int_0^t \left|V_1(\tau)-V_2(\tau)\right|^2\,d\tau.\end{aligned} \tag{2.34}$$

With an integration by parts w.r.t. time one obtains finally from (2.34)

$$\int_0^T \left|\boldsymbol{F}V_1(t)-\boldsymbol{F}V_2(t)\right|^2 dt \le \frac{\tilde{c}_T\kappa^2}{2}\left(t^2\int_0^t \left|V_1(\tau)-V_2(\tau)\right|^2 d\tau\Big|_0^T - \int_0^T t^2\left|V_1(t)-V_2(t)\right|^2 dt\right)$$
$$\le \frac{\tilde{c}_T}{2}\kappa^2T^2\int_0^T \left|V_1(t)-V_2(t)\right|^2 dt,$$

i.e. $\boldsymbol{F}$ is contractive for $T_1 < \sqrt{2}/(\kappa\sqrt{\tilde{c}_T})$. □

Let us point out that the interval of contraction of $\boldsymbol{F}$ depends only on $\kappa\sqrt{\tilde{c}_T}$, thus the iteration can be applied at least on a subinterval $(0,T_1)$ of $(0,T)$. Since this depends only on the length of the considered time interval, the resulting local solution can be extended on these subintervals of same length until the final time T is reached.

3. Numerical Analysis of the Linear Case in 1D

We consider finite difference methods for linear hyperbolic systems in 1D

$$U_t + AU_x + CU = F(x,t), \qquad 0 \le x \le L,\ t \ge 0, \tag{3.1}$$

where A, C are constant $N\times N$ matrices. The case of space- and time dependent coefficients can be treated analogously. With a *spatial step size* $h := \Delta x$ and a *temporal step size* $k := \Delta t$ we discretize the (x,t)–strip $[0,L]\times\mathbb{R}_0^+$ using the following *grid points* $x_j = jh$, $j = 0,1,2,\ldots,J$, $t_n = nk$, $n = 0,1,2,\ldots$. We further assume, that the hyperbolic *mesh ratio* $\lambda := k/h$ is constant and hence the choice of a time step $k > 0$ defines a uniform grid. The finite difference methods yield *approximations* $U_j^n \in \mathbb{R}^N$ of the analytic solution $U(x_j,t_n)$ at the discrete grid points (x_j,t_n). Here we consider the classical *Lax–Wendroff method*

$$\begin{aligned} U_j^{n+1} = U_j^n &- \frac{1}{2}\lambda A(U_{j+1}^n - U_{j-1}^n) - kCU_j^n + \frac{1}{2}(\lambda A)^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)\\ &+ \frac{1}{4}\lambda k(AC+CA)(U_{j+1}^n - U_{j-1}^n) + \frac{1}{2}(kC)^2U_j^n\\ &+ \frac{1}{2}k(F_j^{n+1}+F_j^n) - \frac{1}{4}\lambda kA(F_{j+1}^n - F_{j-1}^n) - \frac{1}{2}k^2CF_j^n. \end{aligned} \tag{3.2}$$

Besides the discretization of the analytic inflow condition one needs additional *numerical boundary conditions* at the outflow boundary to close the scheme. For this purpose we will apply a *first order horizontal extrapolation*

$$(U^-)_0^{n+1} = 2(U^-)_1^{n+1} - (U^-)_2^{n+1}, \tag{3.3a}$$
$$(U^+)_J^{n+1} = 2(U^+)_{J-1}^{n+1} - (U^+)_{J-2}^{n+1}. \tag{3.3b}$$

In order to identify inflow and outflow components for the proper formulation of the boundary conditions one possibly has to transform the solution at the boundary with the matrix that diagonalize A (if A is not diagonal).

3.1. The Stability of the Difference Scheme in one dimension

In this section we will perform a *stability analysis* of the considered difference scheme to solve the IBVP with inhomogeneous first order ABCs. We will investigate concisely the stability of the interior discretization in combination with the chosen boundary conditions. In this context we will refer to the stability theory of Gustafsson, Kreiss and Sundström [18], [23] (briefly: *GKS stability theory*), that provides sufficient and necessary conditions for the stability of a discrete IBVP in one space dimension. The complicated algebraic conditions of the GKS theory were simplified in the succeeding works of Goldberg und Tadmor [13, 15]. As for the well–posedness in Section 2.3 the problem appears, that the ABCs do not fit into the standard form of the boundary conditions in the GKS theory and hence this stability theory is not directly applicable.

The *discrete IBVP with two boundaries*, that is posed on the bounded index range $0 \le j \le J$, is stable, if the corresponding IBVPs defined on the semi–unbounded index ranges $-\infty < j \le J$ and $0 \le j < \infty$ are stable [18, Theorem 5.4]. This is a consequence of the *finite speed of propagation* for hyperbolic equations. Hence, it is sufficient to consider the problem on the positive half line $x \ge 0$:

$$\begin{gathered} U_t + \Lambda U_x + CU = F(x,t), \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N), \quad x \ge 0,\ t \ge 0 \\ \text{with} \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m > 0 > \lambda_{m+1} \ge \cdots \ge \lambda_N, \\ \text{i.e.} \quad \Lambda^+ = \operatorname{diag}(\lambda_1, \dots, \lambda_m), \quad \Lambda^- = \operatorname{diag}(\lambda_{m+1}, \dots, \lambda_N) \end{gathered} \tag{3.4a}$$

$$U(x,0) = f(x), \quad x \ge 0 \tag{3.4b}$$

$$U^+(0,t) = U^+(0,0) - \int_0^t K_0 U^-(0,\tau)\, d\tau + g(t) - g(0), \quad t \ge 0. \tag{3.4c}$$

Due to the *compatibility Assumption* 2.2 we have in the ABC (3.4c) $U^+(0,0) = g(0)$.

Remark 3.1. *The matrix $K_0 \in \mathbb{R}^{m \times (N-m)}$ does not necessarily stem from an ABC.*

We want to solve problem (3.4) with a consistent difference approximation of the form

$$\begin{gathered} U_j^{n+1} = QU_j^n + kb_j^n, \quad j \ge 1, \\ \text{with} \quad Q = \sum_{\sigma=-1}^{1} A_\sigma E^\sigma, \quad \text{and the shift operator } EU_j = U_{j+1}. \end{gathered} \tag{3.5a}$$

For the LW–scheme (3.2) we obtain the coefficients

$$\begin{aligned} A_0 &= I - kC - (\lambda A)^2 + \frac{1}{2}(kC)^2, \\ A_{\pm 1} &= \mp \frac{1}{2}\lambda A + \frac{1}{2}(\lambda A)^2 \pm \frac{1}{4}\lambda k(AC + CA), \\ b_j^n &= \frac{1}{2}(I - kC)F_j^n + \frac{1}{2}F_j^{n+1} - \frac{1}{4}\lambda A(F_{j+1}^n - F_{j-1}^n). \end{aligned}$$

Furthermore, we need a *discrete initial condition*

$$U_j^0 = f(jh), \qquad j = 0, 1, 2, \ldots \tag{3.5b}$$

and *boundary conditions* at $x = 0$ for the inflow components U^+

$$(U^+)_0^{n+1} = (U^+)_0^n - \frac{k}{2} K_0 \left[(U^-)_0^{n+1} + (U^-)_0^n \right] + g^{n+1} - g^n, \tag{3.5c}$$

obtained by a discretization (3.4c) using the trapezoidal rule. For the outflow data U^- we use the *horizontal extrapolation* (3.3a) to close the numerical scheme. These boundary condition can be written in the following form:

$$U_0^{n+1} = S_{-1} U_0^{n+1} + S_0 U_0^n + \tilde{g}^n, \tag{3.6}$$
$$\text{where} \quad S_{-1} = \sum_{\sigma=0}^{2} B_{\sigma,-1} E^\sigma, \qquad S_0 = B_{0,0},$$

with

$$B_{0,-1} = -\frac{k}{2} \begin{pmatrix} 0 & K_0 \\ 0 & 0 \end{pmatrix}, \qquad B_{1,-1} = \begin{pmatrix} 0 & 0 \\ 0 & 2I_{N-m} \end{pmatrix}, \qquad B_{2,-1} = \begin{pmatrix} 0 & 0 \\ 0 & -I_{N-m} \end{pmatrix},$$
$$B_{0,0} = \begin{pmatrix} I_m & -\frac{k}{2} K_0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{g}^n = \begin{pmatrix} g^{n+1} - g^n \\ 0 \end{pmatrix}.$$

Here we used the *discrete compatibility condition* : $(U^+)_0^0 = g^0$.

Remark 3.2. *The discretized boundary condition* (3.5c) *is of second order in* t *at* $t_{n+\frac{1}{2}} = (n + \frac{1}{2})\, k$*, since the centered difference quotient was used.*

We need for the stability analysis of the interior discretization the following definition.

Definition 3.1 (amplification matrix [23])**.** *We denote with*

$$\hat{Q}(\theta) := \sum_{\sigma=-1}^{1} A_\sigma e^{i\sigma\theta}, \qquad \theta \in \mathbb{R} \tag{3.7}$$

the amplification matrix *of* Q*.*

Practically one can determine the amplification matrix $\hat{Q}(\theta)$, by substituting U_j^n by $\hat{Q}^n e^{ij\theta}$ in the difference scheme and solve for $\hat{Q}$, cf. [37]. In the stability theory $\hat{Q}(\theta)$ should satisfy the following two assumptions.

Assumption 3.1 (stability for discrete IVP [23])**.** $\left|\hat{Q}(\theta)\right| \leq 1$.

Assumption 3.2 (dissipative scheme [37])**.** *The scheme (without the lower order terms, i.e. $C = 0$) is* dissipative of order $2r$*, i.e. there exist a constant $c > 0$, such that for the eigenvalues $\mu(\theta)$ of $\hat{Q}(\theta)$ holds*

$$\left|\mu(\theta)\right| \leq 1 - c \sin^{2r} \frac{\theta}{2}. \tag{3.8}$$

Note that this condition (3.8) is equivalent to (cf. [37])

$$\left|\mu(\theta)\right|^2 \leq 1 - c' \sin^{2r} \frac{\theta}{2}. \tag{3.9}$$

Assumption 3.1 guarantees, that the scheme (3.5a), (3.5b) is stable for the pure IVP. The *stability of the pure IVP* is, besides the well–posedness of the underlying analytic problems (Section 2.3), a *necessary condition* for the stability of the difference method for the IBVP. Assumption 3.2 assures, that high frequencies have no impact, i.e. the magnitude of the high frequency oscillations decreases in every step [23].

The Lax-Wendroff method to solve the pure IVP is *stable* for

$$\nu := \max_{\ell=1,\dots,N} \left| \frac{\lambda_\ell k}{h} \right| \leq 1, \tag{3.10}$$

which is the *CFL (Courant-Friedrichs-Levy) condition*; ν is called *Courant number*.

Now we want to determine the dissipation of the Lax-Wendroff scheme and calculate the amplification matrix $\hat{Q}(\theta)$ (with $C = 0$, $F = 0$):

$$\begin{aligned}
\hat{Q}^{n+1} e^{ij\theta} &= \hat{Q}^n e^{ij\theta} \left[I - \frac{1}{2} \Lambda\lambda \left(e^{i\theta} - e^{-i\theta} \right) + \frac{1}{2} (\Lambda\lambda)^2 \left(e^{i\theta} - 2 + e^{-i\theta} \right) \right] \\
&= \hat{Q}^n e^{ij\theta} \left[I - i\Lambda\lambda \sin\theta - (\Lambda\lambda)^2 (1 - \cos\theta) \right].
\end{aligned}$$

Doing so we obtain the amplification matrix $\hat{Q}(\theta) = I - i\Lambda\lambda \sin\theta - (\Lambda\lambda)^2 (1 - \cos\theta)$, that is a polynomial in Λ and thus has the eigenvalues

$$\mu_\ell(\theta) = 1 - i\lambda_\ell \lambda \sin\theta - (\lambda_\ell \lambda)^2 (1 - \cos\theta), \quad \ell = 1, \dots, N. \tag{3.11}$$

Here we denoted the eigenvalues of Λ with λ_ℓ. Finally, we have

$$\left|\mu_\ell(\theta)\right|^2 = 1 - 4(\lambda_\ell \lambda)^2 \left[1 - (\lambda_\ell \lambda)^2 \right] \sin^4 \frac{\theta}{2}, \quad \ell = 1, \dots, N,$$

where we used $(1 - \cos\theta)^2 = 4 \sin^4(\theta/2)$. Thus the difference scheme is following (3.9) *dissipative of order* 4, if $0 < |\lambda_\ell \lambda| < 1$ for all $\ell = 1, \dots, N$. Since Λ is regular, we have that the Assumptions 3.1 and 3.2 are satisfied, if the CFL condition (3.10) holds.

3.1.1. Estimate of the Solution of the Differential Equation

We prove an estimate for the solution that will serve as a motivation for a suitable stability definition to be used. Furthermore, we will use later a similar technique for the derivation of the estimates for the solutions to the difference approximations.

For $U, V : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}^n$ we define the following L^2*-inner products*

$$\langle U,V\rangle_x := \int_0^\infty U^*(x,t)V(x,t)\,dx, \qquad \langle U,V\rangle_t := \int_0^\infty U^*(x,t)V(x,t)\,dt,$$
$$\langle U,V\rangle_{x,t} := \int_0^\infty \int_0^\infty U^*(x,t)V(x,t)\,dx\,dt$$

and the *induced norms* as usual by $\|u\|_x^2 := \langle u,u\rangle_x$, etc. We denote the corresponding L^2 *spaces* wit $L^2(x)$, $L^2(t)$ and $L^2(x,t)$. In the sequel we will use the Laplace transformation; let us recall that the $\mathscr{L}$–transformation of a vector valued function is done componentwise.

Lemma 3.1 (Estimate of the Solution)**.** *Assume $f = 0$. There is a constant $\alpha_0 \geq 0$, such that $e^{-\alpha t}U(x,t) \in L^2(x,t)$, $e^{-\alpha t}U(0,t) \in L^2(t)$ for $\alpha > \alpha_0$, and there exists a constant c_0 with*

$$(\alpha-\alpha_0)\left\|e^{-\alpha t}U(0,t)\right\|_t^2 + (\alpha-\alpha_0)^2\left\|e^{-\alpha t}U(x,t)\right\|_{x,t}^2 \leq c_0^2\left[(\alpha-\alpha_0)\left\|e^{-\alpha t}g(t)\right\|_t^2 + \left\|e^{-\alpha t}F(x,t)\right\|_{x,t}^2\right]. \quad (3.12)$$

Proof. (of the estimate). The proof follows [18, Proof of Theorem 2.3].
A Laplace transformation of (3.4a) considering the initial condition (3.4b) yields the ODE

$$s\widehat{U}(x,s) - f(x) + \Lambda\widehat{U}_x(x,s) + B\widehat{U}(x,s) = \widehat{F}(x,s).$$

Thus we have with the $\mathscr{L}$-transformed boundary condition (3.4c) since $f = 0$

$$s\widehat{U}(x,s) = -\Lambda\widehat{U}_x(x,s) - B\widehat{U}(x,s) + \widehat{F}(x,s), \quad x > 0, \quad (3.13a)$$
$$\widehat{U^+}(0,s) = \widehat{g}(s) - \frac{1}{s}K_0\widehat{U^-}(0,s), \quad (3.13b)$$

with $s = \alpha + i\xi$, $\alpha > \alpha_0$ fixed. We define the matrix $H \in \mathbb{R}^{N\times N}$ by

$$H := \begin{pmatrix} \rho I_m & 0 \\ 0 & I_{N-m} \end{pmatrix}, \qquad 0 < \rho \leq 1, \quad \rho = \text{const.},$$

where ρ will be chosen suitable later. We obtain with $\alpha = \operatorname{Re} s > 0$ and the Cauchy-Schwarz inequality

$$\begin{aligned}\alpha\left\langle\widehat{U},H\widehat{U}\right\rangle_x &= -\operatorname{Re}\left\langle\widehat{U},H\Lambda\widehat{U}_x\right\rangle_x - \operatorname{Re}\left\langle\widehat{U},HB\widehat{U}\right\rangle_x + \operatorname{Re}\left\langle\widehat{U},H\widehat{F}\right\rangle_x \\ &\leq R + |B|\left\|\widehat{U}\right\|_x^2 + \frac{1}{2}\alpha\left\langle\widehat{U},H\widehat{U}\right\rangle_x + \frac{1}{2\alpha}\left\|\widehat{F}\right\|_x^2.\end{aligned} \quad (3.14)$$

Notice that for the estimate of the last term we used an inequality of the form

$$\begin{aligned} &\operatorname{Re}\langle a,b\rangle \le \frac{1}{2}\|a\|^2 + \frac{1}{2}\|b\|^2 \\ \Longrightarrow\quad &\operatorname{Re}\left\langle \sqrt{\alpha}\,a, b/\sqrt{\alpha}\right\rangle \le \frac{1}{2}\alpha\|a\|^2 + \frac{1}{2\alpha}\|b\|^2, \quad \alpha>0, \end{aligned}$$

and $\left\|H\widehat{U}\right\|_x^2 \le \left\langle \widehat{U}, H\widehat{U}\right\rangle_x$ for $0<\rho\le 1$. The term R is determined by integration by parts

$$\begin{aligned} R &= \frac{1}{2}\left(\widehat{U^+}(0,s)\right)^\top \rho\Lambda^+\widehat{U^+}(0,s) + \frac{1}{2}\left(\widehat{U^-}(0,s)\right)^\top \Lambda^-\widehat{U^-}(0,s) \\ &\le \frac{\rho}{2}\lambda_1\left|\widehat{U^+}(0,s)\right|^2 - \frac{|\lambda_{m+1}|}{2}\left|\widehat{U^-}(0,s)\right|^2 \\ &= -\frac{\delta}{2}\left|\widehat{U^+}(0,s)\right|^2 + \frac{\rho\lambda_1+\delta}{2}\left|\widehat{U^+}(0,s)\right|^2 - \frac{|\lambda_{m+1}|}{2}\left|\widehat{U^-}(0,s)\right|^2. \end{aligned} \tag{3.15}$$

Here, $\delta>0$ is a constant, that will be chosen suitable later. From the $\mathscr{L}$-transformed boundary condition (3.13b) we obtain the estimate

$$\left|\widehat{U^+}(0,s)\right|^2 \le 2\left|\widehat{g}(s)\right|^2 + 2\frac{|K_0|^2}{\alpha^2}\left|\widehat{U^-}(0,s)\right|^2, \qquad \xi\in\mathbb{R},\quad s=\alpha+i\xi,$$

that we use in (3.15). Now we can choose δ, ρ such small that

$$\begin{aligned} R &\le -\frac{\delta}{2}\left|\widehat{U^+}(0,s)\right|^2 - \underbrace{\left[\frac{|\lambda_{m+1}|}{2} - (\rho\lambda_1+\delta)\frac{|K_0|^2}{\alpha^2}\right]}_{\ge\frac{\delta}{2}}\left|\widehat{U^-}(0,s)\right|^2 + (\rho\lambda_1+\delta)\left|\widehat{g}(s)\right|^2 \\ &\le -\frac{\delta}{2}\left|\widehat{U}(0,s)\right|^2 + (\rho\lambda_1+\delta)\left|\widehat{g}(s)\right|^2. \end{aligned}$$

This is fulfilled if

$$\left(1+2\frac{|K_0|^2}{\alpha_0^2}\right)\delta \le |\lambda_{m+1}| - 2\rho\lambda_1\frac{|K_0|^2}{\alpha_0^2} \tag{3.16}$$

holds. Hence from (3.14) we obtain

$$\frac{1}{2}\alpha\left\langle \widehat{U}, H\widehat{U}\right\rangle_x \le -\frac{\delta}{2}\left|\widehat{U}(0,s)\right|^2 + (\rho\lambda_1+\delta)\left|\widehat{g}(s)\right|^2 + |B|\left\|\widehat{U}\right\|_x^2 + \frac{1}{2\alpha}\left\|\widehat{F}\right\|_x^2,$$

and after multiplication with 2α we have

$$\alpha\left[\alpha\left\langle \widehat{U}, H\widehat{U}\right\rangle_x - 2|B|\left\|\widehat{U}\right\|_x^2 + \delta\left|\widehat{U}(0,s)\right|^2\right] \le \left\|\widehat{F}\right\|_x^2 + 2\alpha(\rho\lambda_1+\delta)\left|\widehat{g}(s)\right|^2.$$

With $\rho\left\|\widehat{U}\right\|_x^2 \le \left\langle \widehat{U}, H\widehat{U}\right\rangle_x$ (since $\rho\le 1$), one obtains

$$\alpha\left[(\rho\alpha - 2|B|)\left\|\widehat{U}\right\|_x^2 + \delta\left|\widehat{U}(0,s)\right|^2\right] \le \left\|\widehat{F}\right\|_x^2 + 2\alpha(\rho\lambda_1+\delta)\left|\widehat{g}(s)\right|^2.$$

With $\alpha_0 := (2|B| + \varepsilon)/\rho$ for small $\varepsilon > 0$ and multiplication with $(\alpha - \alpha_0)/\alpha$ we get

$$(\alpha-\alpha_0)\delta\left|\widehat{U}(0,s)\right|^2 + (\alpha-\alpha_0)^2\rho\left\|\widehat{U}\right\|_x^2 \le \frac{\alpha-\alpha_0}{\alpha}\left\|\widehat{F}\right\|_x^2 + (\alpha-\alpha_0)2\left(\rho\lambda_1+\delta\right)\left|\widehat{g}(s)\right|^2.$$

Finally, the *Parseval equation* of the Laplace–transformation yields

$$\begin{aligned}(\alpha-\alpha_0)\delta\left\|e^{-\alpha t}U(0,t)\right\|_t^2 + (\alpha-\alpha_0)^2\rho\left\|e^{-\alpha t}U(x,t)\right\|_{x,t}^2 \\ \le (\alpha-\alpha_0)2\left(\rho\lambda_1+\delta\right)\left\|e^{-\alpha t}g(t)\right\|_t^2 + \underbrace{\frac{\alpha-\alpha_0}{\alpha}}_{\le 1}\left\|e^{-\alpha t}F(x,t)\right\|_{x,t}^2,\end{aligned}$$

and this gives with a suitable constant c_0 the statement (3.12). □

We note that when using the ABCs (3.4c) one can choose $\alpha_0 := 2|B|/\rho$, since from $B = 0$ follows $K_0 = 0$ and the critical term in (3.16) vanishes.

Corollary 3.1. *The problem* (3.4) *is strongly well–posed in the generalized sense [24].*

3.1.2. The Definition of the Stability

We denote with $\ell^2(x)$ the *space of all grid functions* $U = \{U_j\}_{j=0}^{\infty}$, $U_j = U(x_j)$, $x_j = jh$, $j = 0, 1, 2, \ldots$ with $h\sum_{j=0}^{\infty}\left|U_j\right|^2 < \infty$ and define the inner product and norm by

$$\langle U, V\rangle_h := h\sum_{j=0}^{\infty} U_j^* V_j\,, \qquad \|U\|_h^2 := \langle U, U\rangle_h\,.$$

Analogously, we define the spaces $\ell^2(t)$ and $\ell^2(x,t)$ and we define with

$$\langle U, V\rangle_k := k\sum_{n=0}^{\infty} U^*(t_n)V(t_n), \qquad \|U\|_k^2 := \langle U, U\rangle_k\,, \qquad t_n = nk,$$

$$\langle U, V\rangle_{h,k} := hk\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} U_j^*(t_n)V_j\,(t_n), \qquad \|U\|_{h,k}^2 := \langle U, U\rangle_{h,k}$$

the corresponding inner products and norms. We use the following definition of stability:

Definition 3.2 (GKS stability [18])**.** *For vanishing initial data* (3.5b)*, the approximation* (3.5) *is called* GKS–stable*, if there are constants $c_0 > 0$, $\alpha_0 \ge 0$, such that for all $t = t_n = nk$ and all $\alpha > \alpha_0$ the following holds*

$$\begin{aligned}\frac{\alpha-\alpha_0}{\alpha k+1}\left\|e^{-\alpha t}U_0\right\|_k^2 + \left(\frac{\alpha-\alpha_0}{\alpha k+1}\right)^2\left\|e^{-\alpha t}U\right\|_{h,k}^2 \\ \le c_0^2\left[\frac{\alpha-\alpha_0}{\alpha k+1}\left\|e^{-\alpha(t+k)}g_0\right\|_k^2 + \left\|e^{-\alpha(t+k)}b\right\|_{h,k}^2\right].\end{aligned} \tag{3.17}$$

3.2. The Perturbed Scheme

First we analyze the stability of the following *modified* LW–scheme, that is obtained, by neglecting terms of order $O(k)$, $O(k^2)$ in Q and b (cf. (3.5a)). This modified scheme reads

$$U_j^{n+1} = \tilde{Q}U_j^n + kd_j^n, \qquad j \geq 1\,; \quad \tilde{Q} = \sum_{\sigma=-1}^{1} \tilde{A}_\sigma E^\sigma, \tag{3.18}$$

$$\text{where} \quad \tilde{A}_0 = I - (\lambda A)^2, \qquad \tilde{A}_{\pm 1} = \mp\frac{1}{2}\lambda A + \frac{1}{2}(\lambda A)^2,$$

$$d_j^n = \frac{1}{2}(F_j^n + F_j^{n+1}) - \frac{1}{4}\lambda A(F_{j+1}^n - F_{j-1}^n)$$

supplied with the initial condition (3.5b) and the boundary conditions (3.3a), (3.5c).

The scheme (3.5) is a perturbation of order $O(k)$ of this scheme (3.18) but only on the interior grid. Hence the *Perturbation Theorem of the GKS theory* [18, Theorem 4.3] can be applied, which states that the GKS stability is invariant w.r.t. perturbations of order $O(k)$, i.e. for the stability analysis we can consider the *perturbed scheme* instead of the original scheme. For this purpose we decompose the matrix $\tilde{Q}$ according to the partition of Λ (cf. notation on page 4) as follows:

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}^{++} & \tilde{Q}^{+-} \\ \tilde{Q}^{-+} & \tilde{Q}^{--} \end{pmatrix}.$$

3.2.1. The Stability of the Perturbed Scheme

The inflow and outflow problems are *decoupled* on the interior grid for the perturbed scheme ($\tilde{Q}$ does not contain the 'coupling matrix' C), i.e. one can *solve successively* both problems. The perturbed scheme is obviously stable if and only if both inflow and outflow problems are stable. The outflow problem is *closed* and the solution of this problem yields the outflow data $(U^-)_0^{n+1}$, that is needed in the inflow condition (3.5c). In the sequel we consider the stability of both separate problems, the outflow and the inflow problem, and use the Z-Transformation of the solution, which is the discrete analogue of the $\mathscr{L}$-transformation:

Definition 3.3 (Z-Transformation [8])**.** *The formal correspondence of a series and a complex function given by*

$$\{u^n\} \longleftrightarrow \mathscr{Z}\{u^n\} := \hat{u}(z) := \sum_{n=0}^{\infty} z^{-n}u^n, \qquad |z| \geq R, \tag{3.19}$$

*is called Z-*transformation*. The function $\hat{u}(z)$ is called Z-*transform *of the series $\{u^n\}$, $n \in \mathbb{N}_0$.*

The stability of the outflow problem. This is a Lax–Wendroff discretization of $U_t^- + \Lambda^- U_x^- = F^-$ with *horizontal outflow extrapolation*. This scheme is *GKS–stable* according

to [13, Example 3.1], i.e. there exist constants $c_0 > 0$, $\alpha_0 \geq 0$, such that for all $t = t_n = nk$ and all $\alpha > \alpha_0$ holds

$$\frac{\alpha-\alpha_0}{\alpha k+1}\left\|e^{-\alpha t}U_0^-\right\|_k^2+\left(\frac{\alpha-\alpha_0}{\alpha k+1}\right)^2\left\|e^{-\alpha t}U^-\right\|_{h,k}^2 \leq c_0^2\left\|e^{-\alpha(t+k)}d^-\right\|_{h,k}^2. \tag{3.20}$$

A Z-transformation of

$$(U^-)_j^{n+1} = \tilde{Q}^{--}(U^-)_j^n + k(d^-)_j^n, \qquad j = 1,2,3,\ldots$$

and division by z yields the *resolvent equation*

$$\left(I - \frac{1}{z}\tilde{Q}^{--}\right)\widehat{U_j^-} = \widehat{d_j^-} \tag{3.21}$$

with

$$\widehat{U_j^-}(z) = k\sum_{n=0}^{\infty} z^{-n}(U^-)_j^n, \qquad \widehat{d_j^-}(z) = k^2\sum_{n=0}^{\infty} z^{-n-1}(d^-)_j^n, \qquad z = e^{(\alpha+i\omega)k}.$$

According to the *Characterization Theorem of the GKS Stability* [18, Theorem 4.2] there exist constants $\alpha_0^- \geq 0$, $c_1^- > 0$, such that (3.21) has a unique solution for all z with $|z| > e^{\alpha_0^- k}$ and all d^- with

$$k\left(\frac{|z|-e^{\alpha_0^- k}}{|z|}\right)\left|\widehat{U_0^-}(z)\right|^2+\left(\frac{|z|-e^{\alpha_0^- k}}{|z|}\right)^2\left\|\widehat{U^-}\right\|_h^2 \leq (c_1^-)^2\left\|\widehat{d^-}\right\|_h^2 \tag{3.22}$$

and this yields the estimate

$$\left|\widehat{U_0^-}(z)\right|^2 \leq \frac{|z|}{k\left(|z|-e^{\alpha_0^- k}\right)}(c_1^-)^2\left\|\widehat{d^-}\right\|_h^2. \tag{3.23}$$

The stability of the inflow problem. A discretization of the inhomogeneous absorbing boundary condition (3.4c) ist given by

$$(U^+)_0^n = g^n - \frac{k}{2}K_0\sum_{\ell=0}^{n-1}\left[(U^-)_0^\ell + (U^-)_0^{\ell+1}\right] \tag{3.24}$$

and a Z-transformation yields

$$\widehat{U_0^+}(z) = \widehat{g}(z) - \frac{k}{2}K_0\frac{z+1}{z-1}\widehat{U_0^-}(z). \tag{3.25}$$

The inflow problem is a Lax-Wendroff discretization of $U_t^+ + \Lambda^+ U_x^+ = F^+$ with inflow data $U_0^+ = \tilde{h}^+$. It is *stable* according to [13, Lemma 2.3], and due to the *Characterization*

Theorem [18, Theorem 4.2] there exists some constants $\alpha_0^+ \geq 0$, $c_1^+ > 0$, such that

$$k\left(\frac{|z|-e^{\alpha_0^+ k}}{|z|}\right)\left|\widehat{U_0^+}(z)\right|^2 + \left(\frac{|z|-e^{\alpha_0^+ k}}{|z|}\right)^2 \left\|\widehat{U^+}\right\|_h^2 \leq (c_1^+)^2\left[k\,\frac{|z|-e^{\alpha_0^+ k}}{|z|}\left|\widehat{\tilde{h}}\right|^2 + \left\|\widehat{d^+}\right\|_h^2\right] \quad (3.26)$$

for all z with $|z| > e^{\alpha_0^+ k}$. However, for proving the stability we need an estimate in terms of $\widehat{g}$, $\widehat{d}$ (and not $\widehat{\tilde{h}}$).

First we derive an *estimate of the inflow boundary condition*. We choose $\alpha_0 > \max(\alpha_0^-, \alpha_0^+) \geq 0$ and with $e^{\alpha_0 k} < |z| \leq |z-1|+1$ we have

$$k\left|\frac{z+1}{z-1}\right| \leq k\,\frac{|z-1|+2}{|z-1|} \leq k\left[1+\frac{2}{|z|-1}\right] \leq k + \frac{2k}{e^{\alpha_0 k}-1} \leq k + \frac{2}{\alpha_0},$$

since $e^{\alpha_0 k} - 1 \geq \alpha_0 k$. Hence, with some constant c_2 we obtain

$$\frac{k^2}{2}\left|\frac{z+1}{z-1}\right|^2 \leq c_2, \quad (3.27)$$

since we can restrict our considerations to the interesting range $0 < k \leq 1$.

The Z-transformed boundary condition (3.25) yields with (3.27) and (3.23)

$$\begin{aligned}\left|\widehat{U_0^+}(z)\right|^2 &\leq 2\left|\widehat{g}(z)\right|^2 + \frac{k^2}{2}\left|\frac{z+1}{z-1}\right|^2 \left|K_0\right|^2 \left|\widehat{U_0^-}(z)\right|^2 \\ &\leq 2\left|\widehat{g}(z)\right|^2 + c_2\left|K_0\right|^2 \left|\widehat{U_0^-}(z)\right|^2 \\ &\leq 2\left|\widehat{g}(z)\right|^2 + \frac{|z|}{k\left(|z|-e^{\alpha_0 k}\right)} c_2 (c_1^-)^2 \left|K_0\right|^2 \left\|\widehat{d^-}\right\|_h^2, \quad \forall\ |z| > e^{\alpha_0 k}.\end{aligned} \quad (3.28)$$

Remark 3.3. *If the estimate* (3.26) *holds for an* $\alpha_0^+ \geq 0$*, then it also holds for all* $\tilde{\alpha}_0^+ > \alpha_0^+$*. This is a direct consequence if this inequality is divided by* $(|z| - e^{\alpha_0^+ k})/|z|$*. An analogous statement holds for* (3.22).

Corollary 3.2. *The estimates* (3.22), (3.26) *also hold for the above chosen* $\alpha_0 > 0$ *instead of* α^- *or* α^+.

With $\widehat{\tilde{h}} = \widehat{U_0^+}$ we substitute (3.28) into (3.26) (with α_0) and obtain

$$k\left(\frac{|z|-e^{\alpha_0 k}}{|z|}\right)\left|\widehat{U_0^+}(z)\right|^2 + \left(\frac{|z|-e^{\alpha_0 k}}{|z|}\right)^2 \left\|\widehat{U^+}\right\|_h^2 \leq (c_1^+)^2\left[k\,\frac{|z|-e^{\alpha_0 k}}{|z|}\,2\left|\widehat{g}(z)\right|^2 + \left\|\widehat{d^+}\right\|_h^2 + c_2(c_1^-)^2\left|K_0\right|^2\left\|\widehat{d^-}\right\|_h^2\right], \quad (3.29)$$

which shows the GKS stability of the perturbed scheme and thus also proves the GKS stability of the original LW-scheme (3.5), since the following holds:

Remark 3.4. *It is easy to see, that the discrete inflow boundary conditions* (3.5c) *and* (3.24) *are equivalent und therefore the scheme* (3.5) *is GKS stable.*

Remark 3.5. *The above stability proof can be transfered to all explicit, dissipative, two–level, linear one step methods with horizontal outflow extrapolation* (3.3).

3.3. Numerical Results

In this section we want to verify numerically with an example the stability of the Lax-Wendroff scheme that was proven in Section 3.1. Afterwards, we will investigate the accuracy of the absorbing boundary conditions.

Example 3.1. *We consider as an example the hyperbolic system*

$$U_t + \underbrace{\begin{pmatrix} v+c & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v-c \end{pmatrix}}_{=\Lambda} U_x + \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{=C} U = 0, \qquad 0 \le x \le 1, \quad t \ge 0, \tag{3.30}$$

with $v = 0.2$, $c = 1$ *and absorbing boundary conditions of first order. Hence it is* $m = 2$, $N = 3$ *and as initial data we choose*

$$U_\ell(x,0) = f_\ell(x) := \begin{cases} \cos^2(\frac{\pi}{2}\frac{x-0.5}{0.45}) & |x-0.5| < 0.45, \\ 0 & \textit{else} \end{cases} \qquad \ell = 1,2,3. \tag{3.31}$$

The matrices $K_0 \in \mathbb{R}^{2\times 1}$, $K_1 \in \mathbb{R}^{1\times 2}$ *in the absorbing boundary conditions are*

$$K_0 = \begin{pmatrix} \dfrac{c-v}{2c}\, c_{13} \\ \dfrac{c-v}{c}\, c_{23} \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} \dfrac{v+c}{2c}\, c_{31} & \dfrac{v}{c}\, c_{32} \end{pmatrix} = \begin{pmatrix} 0.6 & 0 \end{pmatrix}. \tag{3.32}$$

Example 3.2. *As a second example we consider* (3.30) *with the skew-symmetric coefficients*

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -2 & 0 \end{pmatrix}. \tag{3.33}$$

For this case the matrices in K_0, K_1 *are given by*

$$K_0 = \begin{pmatrix} 0.4 \\ 1.6 \end{pmatrix}, \quad K_1 = \begin{pmatrix} -0.6 & -0.4 \end{pmatrix}. \tag{3.34}$$

The CFL–condition (3.10) *for these two test problems reads*

$$\lambda \le \frac{5}{6} = 0.8\bar{3}. \tag{3.35}$$

3.3.1. Numerical Verification of the Stability

Here we *modulate* the initial function (3.31) using a sine function:

$$U(x,0) = \sin(2\pi p x) f(x), \quad x \in [0, L]\,. \tag{3.36}$$

The parameter p is the number of periods of the sine term on $0 \le x \le 1$. We choose a mesh ratio λ, that fulfills the CFL condition (3.35), i.e. the inner scheme is stable: we want to investigate only the effects of the ABCs on the stability and we expect that this will strongly depend on the chosen modulation frequency.

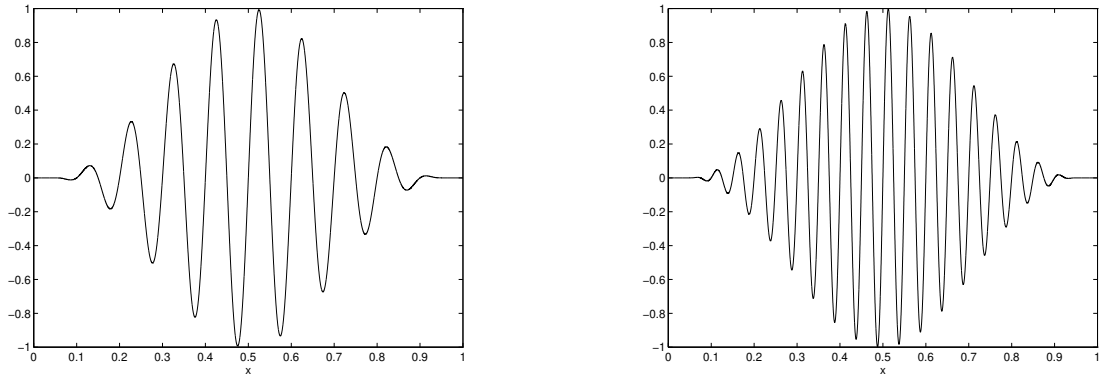

Figure 1: Modulated initial function (3.36) for $p = 10$ and $p = 20$.

Besides satisfying the CFL condition the spatial step size h must be chosen sufficiently small such that a high modulation frequency can be resolved. We use the following rule of thumb (with $L = 1$)

$$h = \frac{k}{\lambda} \le \frac{1}{8}\,\frac{L}{p} \quad \text{and hence} \quad k \le \frac{\lambda}{8p}\,.$$

The temporal evolution of the third component of Example 1 with modulated initial data is shown in Figure 2 for $p = 8$, $\lambda = 0.8$ and $\Delta t = 0.01$. For the third component $x = 1$ is inflow boundary and one can observe there the effects of the modulation in form of waves and these are not significantly reflected by the first order ABC.

Next we present the temporal evolution of the *discrete* L^2*-norm* $\|U\|_h^2 := h\sum_{j=0}^{J} \left|U_j\right|^2$ for different time steps $k = \Delta t$. Figure 3 shows the results for $\lambda = 0.8$ and $p = 10$, $p = 20$. The corresponding results for the second example are shown in Figure 4. In both examples the high modulation frequencies are 'smoothed out' by the ABCs and the L^2-norms decay quickly, since the wave packet leaves the computational interval $[0, 1]$ in both directions. The ABCs prevent significant reflections and the results illustrate the stability of the LW-scheme shown in Section 3.1.

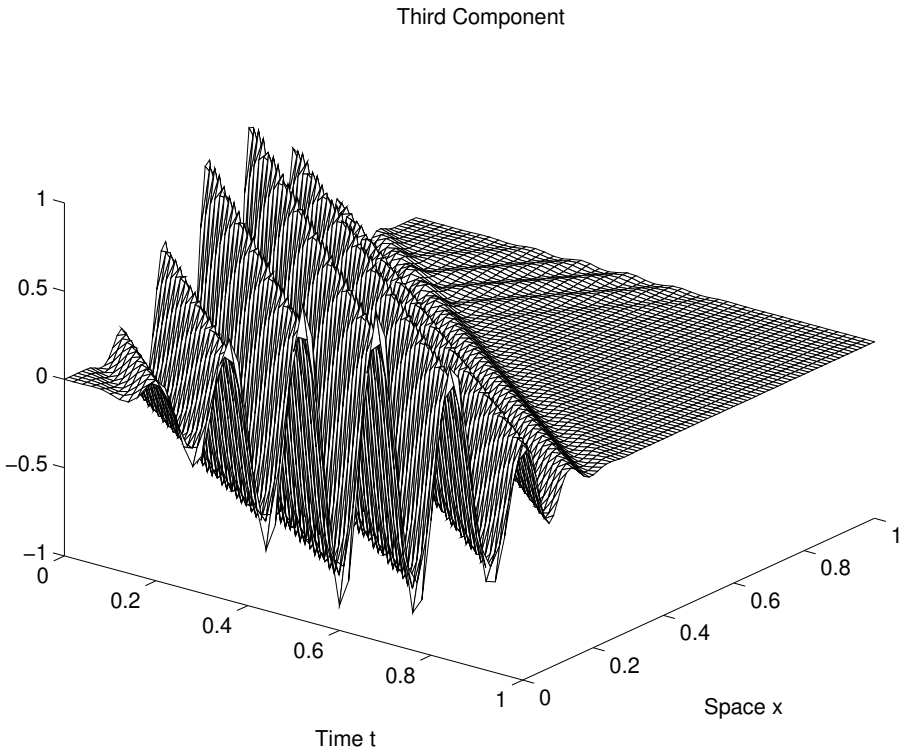

Figure 2: Third component of the solution of Example 3.1 for $p = 8$.

3.3.2. The Accuracy of the Absorbing Boundary Conditions

In this section we wan to compare the absorbing boundary conditions of order 0 and 1. A discretization of the zero order ABC (2.19a) at $x = 0$ yields

$$(U^+)_0^{n+1} = (U^+)_0^n. \tag{3.37}$$

Since the initial function (3.31) is compactly supported in the computational interval, (3.37) corresponds to a *homogeneous Dirichlet boundary condition*. The discrete first order ABC reads

$$\frac{(U^+)_0^{n+1} - (U^+)_0^n}{k} + K_0 (U^-)_0^{n+\frac{1}{2}} = 0$$

and thus we have

$$(U^+)_0^{n+1} = (U^+)_0^n - \frac{k}{2} K_0 \left((U^-)_0^n + (U^-)_0^{n+1} \right). \tag{3.38}$$

We determine the numerical reflections caused by the ABCs, the so–called *reflected part*, which is the difference of the solution to the IBVP and the solution to the pure IVP. Hereby, the whole space solution is simulated by using an significantly larger x–interval, such that the solution does not reach the boundary within the given time interval. We denote the reflected part with the index 'refl' and measure it in the following Figures 5 and 6 using the *relative* L^2*-norm* $\|U\|_{h,rel} := \|U\|_h \,/\, \left\|U^0\right\|_h$.

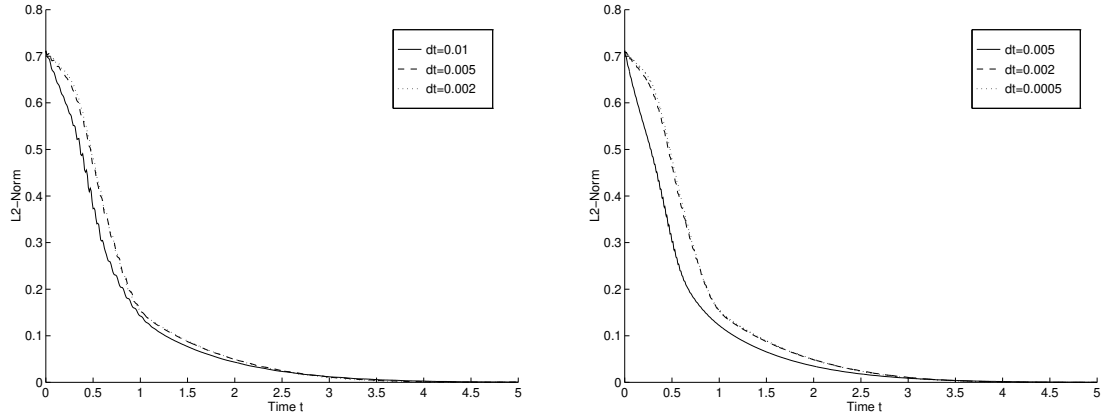

Figure 3: L^2-norms Example 3.1 for $p = 10$ and $p = 20$.

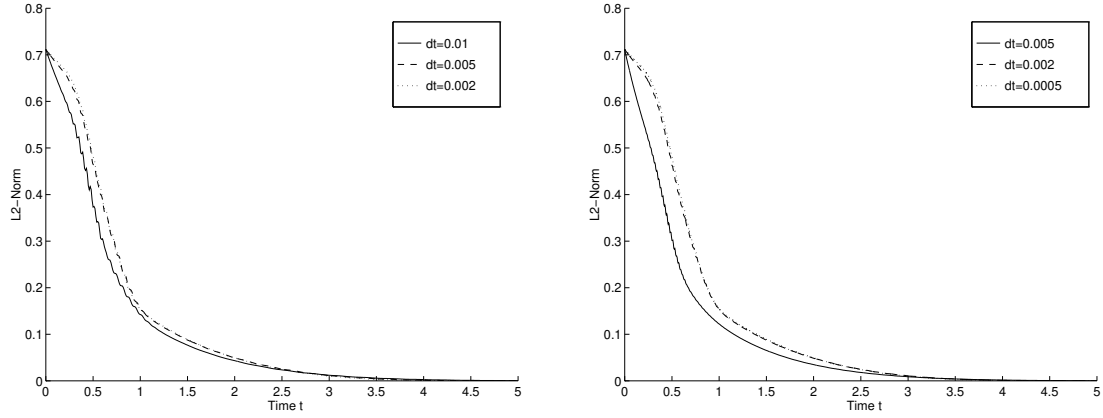

Figure 4: L^2-norms Example 3.2 for $p = 10$ and $p = 20$.

For both examples it turns out that the numerical reflections of the first order ABC are much smaller than the ones of the homogeneous Dirichlet BC. The improvement is around factor 50 or 75 for the first example and between factor 3 and 8 for the second example.

4. Numerical Results in 2D

In this section we want to show the numerical results for linear hyperbolic systems in 2D. For simplicity we restrict ourselves here to the case $h = \Delta x = \Delta y$ with a constant *mesh ratio* $\lambda := k/h$. We introduce the grid points $(x_j, y_\ell, t_n) = (jh, \ell h, nk)$, $j = 0, 1, 2, \ldots, J$, $\ell = \ldots, -1, 0, 1, 2, \ldots$, $n = 0, 1, 2, \ldots$ and denote with $U^n_{j,\ell} \in \mathbb{R}^N$ the approximations to the solution $U(x_j, y_\ell, t_n)$ at the grid points (x_j, y_ℓ).

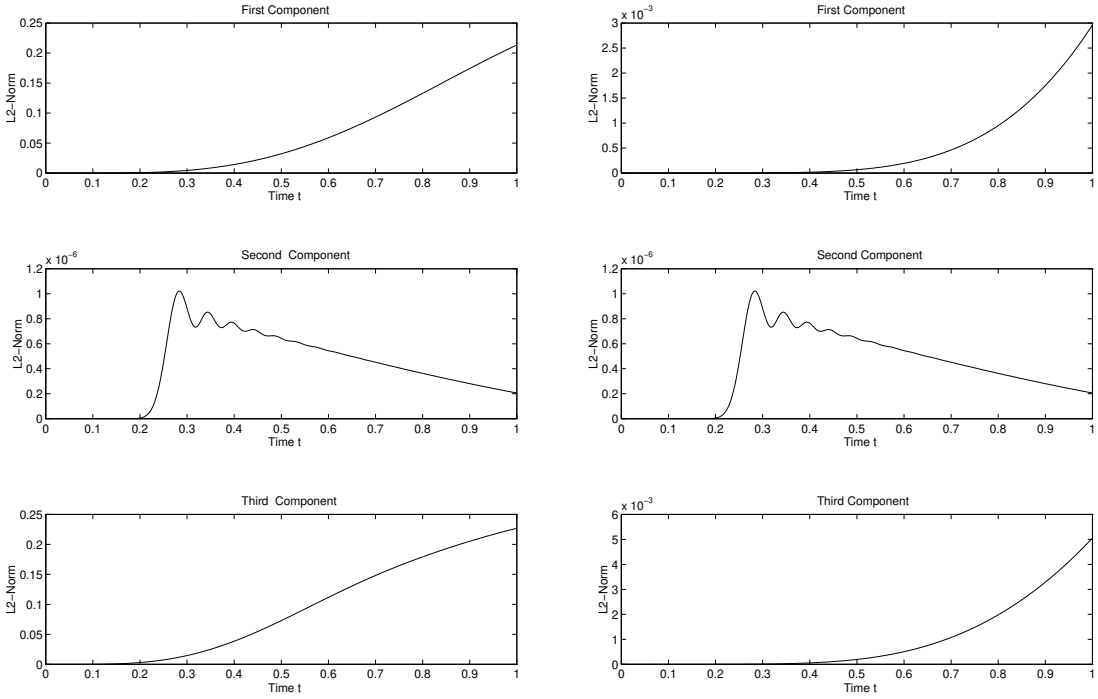

Figure 5: Example 3.1: Reflected part for homogeneous Dirichlet BC (left) and first order ABC (right) with $\lambda = 0.8,\ \Delta t = 0.001$

4.1. The Shallow–Water Equations

The *shallow–water equations* [44] describe the motion of an incompressible fluid if the depth is small in comparison to a typical horizontal length. They read:

$$\begin{aligned} u_t + uu_x + vu_y + \varphi_x - fv &= 0, \\ v_t + uv_x + vv_y + \varphi_y + fu &= 0, \\ \varphi_t + (\varphi u)_x + (\varphi v)_y &= 0. \end{aligned} \tag{4.1}$$

Here, $u = u(x,y,t)$ and $v = v(x,y,t)$ denote the *horizontal components of the velocity*, $f = f(x,y)$ is the *Coriolis parameter* and $\varphi = gh(x,y,t)$ is the *earth potential*, where g is the gravitational acceleration and h the surface height. The system (4.1) can be rewritten [44] as a conservation law:

$$\begin{pmatrix} \varphi u \\ \varphi v \\ \varphi \end{pmatrix}_t + \begin{pmatrix} \varphi u^2 + \frac{1}{2}\varphi^2 \\ \varphi uv \\ \varphi u \end{pmatrix}_x + \begin{pmatrix} \varphi uv \\ \varphi v^2 + \frac{1}{2}\varphi^2 \\ \varphi v \end{pmatrix}_y = \begin{pmatrix} f\varphi v \\ -f\varphi u \\ 0 \end{pmatrix}. \tag{4.2}$$

4.1.1. The linearized Shallow–Water Equations

The linearized form can be obtained from (4.1) by linearizing around a constant solution $\bar{U} = (\bar{u}, \bar{v}, \bar{\varphi})^\top$. For a solution U of (4.1) one uses the *perturbation ansatz* $U = \bar{U} + U'$ with a

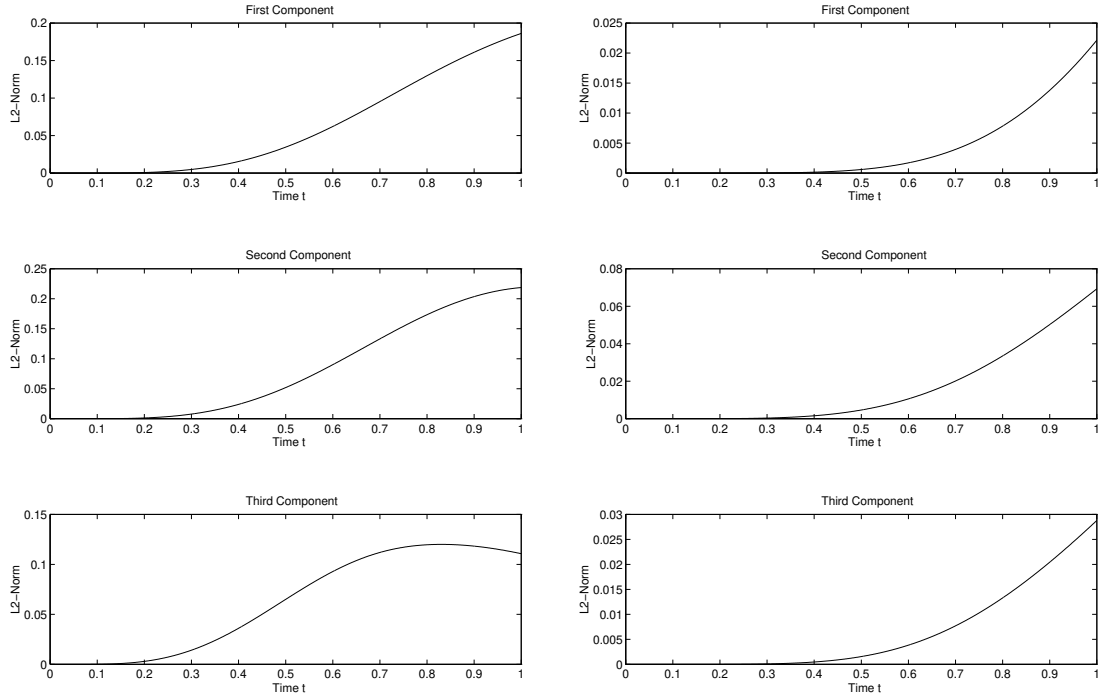

Figure 6: Example 3.2: Reflected part for homogeneous Dirichlet BC (left) and first order ABC (right) with $\lambda = 0.8$, $\Delta t = 0.001$

small perturbation $U' = (u', v', \varphi')^\top$ and neglect terms of second order in the perturbation, cf. [44] . Then the *linearized equation* reads:

$$\begin{pmatrix} u' \\ v' \\ \varphi' \end{pmatrix}_t + \begin{pmatrix} \bar{u} & 0 & 1 \\ 0 & \bar{u} & 0 \\ \bar{\varphi} & 0 & \bar{u} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ \varphi' \end{pmatrix}_x + \begin{pmatrix} \bar{v} & 0 & 0 \\ 0 & \bar{v} & 1 \\ 0 & \bar{\varphi} & \bar{v} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ \varphi' \end{pmatrix}_y + \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u' \\ v' \\ \varphi' \end{pmatrix} = 0. \quad (4.3)$$

The system matrices of (4.3) can be symmetrized simultaneously by a left multiplication with the matrix $\mathrm{diag}(\bar{\varphi}, \bar{\varphi}, 1)$ (note that $\bar{\varphi} > 0$), i.e. (4.3) is a *symmetrizable hyperbolic system* (cf. Definition 2.2). However, then the coefficient of the time derivative is unequal the identity and hence we propose the change of variables using $S := \mathrm{diag}(\sqrt{\bar{\varphi}}, \sqrt{\bar{\varphi}}, 1)$:

$$S \begin{pmatrix} \bar{u} & 0 & 1 \\ 0 & \bar{u} & 0 \\ \bar{\varphi} & 0 & \bar{u} \end{pmatrix} S^{-1} = \begin{pmatrix} \bar{u} & 0 & \sqrt{\bar{\varphi}} \\ 0 & \bar{u} & 0 \\ \sqrt{\bar{\varphi}} & 0 & \bar{u} \end{pmatrix} \quad \text{and} \quad S \begin{pmatrix} \bar{v} & 0 & 0 \\ 0 & \bar{v} & 1 \\ 0 & \bar{\varphi} & \bar{v} \end{pmatrix} S^{-1} = \begin{pmatrix} \bar{v} & 0 & 0 \\ 0 & \bar{v} & \sqrt{\bar{\varphi}} \\ 0 & \sqrt{\bar{\varphi}} & \bar{v} \end{pmatrix},$$

such that the system (4.3) transforms to

$$V_t + \begin{pmatrix} \bar{u} & 0 & c \\ 0 & \bar{u} & 0 \\ c & 0 & \bar{u} \end{pmatrix} V_x + \begin{pmatrix} \bar{v} & 0 & 0 \\ 0 & \bar{v} & c \\ 0 & c & \bar{v} \end{pmatrix} V_y + \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V = 0,$$

where $V := SU' = (u'\sqrt{\bar\varphi}, v'\sqrt{\bar\varphi}, \varphi')^\top = (cu', cv', \varphi')^\top$. The quantity $c := \sqrt{\bar\varphi}$ can be interpreted as '*speed of sound*'. To simplify the notation, we omit in the sequel the bars to mark the perturbed quantities. With $a := \bar u$, $b := \bar v$ we obtain

$$V_t + \begin{pmatrix} a & 0 & c \\ 0 & a & 0 \\ c & 0 & a \end{pmatrix} V_x + \begin{pmatrix} b & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix} V_y + \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V = 0, \tag{4.4}$$

with $V = (cu, cv, \varphi)^\top$ and the *physical restrictions* $0 < a^2 + b^2 < c^2$, $c > 0$. Furthermore, we assume for simplicity that f in (4.4) is constant.

4.1.2. The Absorbing Boundary Conditions

In order to apply the theory of Section 2.2, we have to diagonalize the coefficient of the x-derivative. This can be achieved with the orthogonal matrix T

$$T = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{-1}{\sqrt2} \end{pmatrix}. \tag{4.5}$$

Thus we obtain from (4.4) a new equation for $W := T^{-1}V$ of the form (2.9) :

$$W_t + \underbrace{\begin{pmatrix} a+c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a-c \end{pmatrix}}_{=\Lambda} W_x + \underbrace{\begin{pmatrix} b & \frac{c}{\sqrt2} & 0 \\ \frac{c}{\sqrt2} & b & \frac{-c}{\sqrt2} \\ 0 & \frac{-c}{\sqrt2} & b \end{pmatrix}}_{=B} W_y + \underbrace{\begin{pmatrix} 0 & \frac{-f}{\sqrt2} & 0 \\ \frac{f}{\sqrt2} & 0 & \frac{f}{\sqrt2} \\ 0 & \frac{-f}{\sqrt2} & 0 \end{pmatrix}}_{=C} W = 0.$$

For the case $a > 0$ (i.e. inflow situation at $x = 0$) the matrices $K_0 \in \mathbb{R}^{2\times1}$, $K_L \in \mathbb{R}^{1\times2}$ in the absorbing boundary conditions (2.19c) are given by

$$K_0 = \begin{pmatrix} \frac{c-a}{2c}\, c_{13} \\ \frac{c-a}{c}\, c_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{(c-a)f}{c\sqrt2} \end{pmatrix}, \quad K_L = \begin{pmatrix} \frac{a+c}{2c}\, c_{31} & \frac{a}{c}\, c_{32} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-af}{c\sqrt2} \end{pmatrix}, \tag{4.6}$$

and the matrices $X_0 \in \mathbb{R}^{2\times1}$, $X_L \in \mathbb{R}^{1\times2}$ are

$$X_0 = \begin{pmatrix} \frac{c-a}{2c}\, b_{13} \\ \frac{c-a}{c}\, b_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{(a-c)}{\sqrt2} \end{pmatrix}, \quad X_L = \begin{pmatrix} \frac{a+c}{2c}\, b_{31} & \frac{a}{c}\, b_{32} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-a}{\sqrt2} \end{pmatrix}. \tag{4.7}$$

Analogously, for the case $a < 0$ (i.e. outflow situation at $x = 0$) we obtain $K_0, X_0 \in \mathbb{R}^{1\times 2}$ and $K_L, X_L \in \mathbb{R}^{2\times 1}$:

$$K_0 = \begin{pmatrix} \frac{-a}{c}\, c_{12} & \frac{c-a}{2c}\, c_{13} \end{pmatrix} = \begin{pmatrix} \frac{af}{c\sqrt{2}} & 0 \end{pmatrix}, \quad K_L = \begin{pmatrix} \frac{a+c}{c}\, c_{21} \\ \frac{a+c}{2c}\, c_{31} \end{pmatrix} = \begin{pmatrix} \frac{(a+c)f}{c\sqrt{2}} \\ 0 \end{pmatrix}, \tag{4.8}$$

$$X_0 = \begin{pmatrix} \frac{-a}{c}\, b_{12} & \frac{c-a}{2c}\, b_{13} \end{pmatrix} = \begin{pmatrix} \frac{-a}{\sqrt{2}} & 0 \end{pmatrix}, \quad X_L = \begin{pmatrix} \frac{a+c}{c}\, b_{21} \\ \frac{a+c}{2c}\, b_{31} \end{pmatrix} = \begin{pmatrix} \frac{(a+c)}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{4.9}$$

A discretization of the first order ABC (2.19c) at $x = 0$ is then given by

$$\frac{(U^+)^{n+1}_{0,\ell} - (U^+)^n_{0,\ell}}{k} + X_0 \frac{(U^-)^{n+\frac{1}{2}}_{0,\ell+1} - (U^-)^{n+\frac{1}{2}}_{0,\ell-1}}{2h} + K_0 (U^-)^{n+\frac{1}{2}}_{0,\ell} = 0.$$

Such a discretization (especially of the time-derivative) seems reasonable for a scheme with two time levels t_n, t_{n+1}. Doing so, we arrive at the *discrete inflow boundary condition*

$$(U^+)^{n+1}_{0,\ell} = (U^+)^n_{0,\ell} - \frac{\lambda}{2} X_0 \left((U^-)^{n+\frac{1}{2}}_{0,\ell+1} - (U^-)^{n+\frac{1}{2}}_{0,\ell-1} \right) - kK_0 (U^-)^{n+\frac{1}{2}}_{0,\ell}, \tag{4.10}$$

where the quantities with half-integral time indices are defined by arithmetic averages. For the outflow data $(U^-)^{n+1}_{0,\ell}$ we use as in 1D a *horizontal extrapolation*

$$(U^-)^{n+1}_{0,\ell} = 2(U^-)^{n+1}_{1,\ell} - (U^-)^{n+1}_{2,\ell}. \tag{4.11}$$

The right boundary at $x = L$ is treated analogously.

4.1.3. The Boundary Conditions for the nonlinear Case

To formulate the boundary conditions we consider the shallow–water equations in the form

$$\begin{pmatrix} u \\ v \\ \varphi \end{pmatrix}_t + \begin{pmatrix} u & 0 & 1 \\ 0 & u & 0 \\ \varphi & 0 & u \end{pmatrix} \begin{pmatrix} u \\ v \\ \varphi \end{pmatrix}_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & 1 \\ 0 & \varphi & v \end{pmatrix} \begin{pmatrix} u \\ v \\ \varphi \end{pmatrix}_y + \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \varphi \end{pmatrix} = 0. \tag{4.12}$$

For a fixed t, y we freeze the coefficient matrices local at $x = 0$ and mark this with an index zero. Doing so, we can proceed as for linear equation Section 4.1.2 i.e. we symmetrize simultaneously using $S_0 := \mathrm{diag}(c_0, c_0, 1)$, $c_0 = \sqrt{\varphi_0}$

$$\begin{pmatrix} c_0 u \\ c_0 v \\ \varphi \end{pmatrix}_t + \underbrace{\begin{pmatrix} u_0 & 0 & c_0 \\ 0 & u_0 & 0 \\ c_0 & 0 & u_0 \end{pmatrix}}_{=A_0(u_0,c_0)} \begin{pmatrix} c_0 u \\ c_0 v \\ \varphi \end{pmatrix}_x + \underbrace{\begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & c_0 \\ 0 & c_0 & v_0 \end{pmatrix}}_{=B_0(v_0,c_0)} \begin{pmatrix} c_0 u \\ c_0 v \\ \varphi \end{pmatrix}_y + \underbrace{\begin{pmatrix} 0 & -f_0 & 0 \\ f_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=C_0(f_0)} \begin{pmatrix} c_0 u \\ c_0 v \\ \varphi \end{pmatrix} = 0$$

and diagonalize the system matrix A_0. Now the ABC depends on the sign of u_0 ab. The ABCs for $u_0 > 0$ and $u_0 < 0$ are presented in Section 4.1.2 In the case $u_0 = 0$ we treat the second component (belonging to the eigenvalue 0) like a outflow component and apply horizontal extrapolation (4.11). Again, the right boundary at $x = L$ is treated analogously.

4.2. Numerical Example in 2D

We consider the following linear system:

$$U_t + AU_x + BU_y + CU = F(x,y,t), \qquad 0 \le x \le L,\ -\infty < y < \infty,\ t \ge 0, \tag{4.13}$$

where A, B, C are constant $N \times N$–matrices. The Lax-Wendroff methods reads

$$\begin{aligned}
U_{j,\ell}^{n+1} = U_{j,\ell}^n &- \frac{1}{2}\lambda\left(A(U_{j+1,\ell}^n - U_{j-1,\ell}^n) + B(U_{j,\ell+1}^n - U_{j,\ell-1}^n)\right) - kCU_{j,\ell}^n \\
&+ \frac{1}{2}\lambda^2\left(A^2(U_{j+1,\ell}^n - 2U_{j,\ell}^n + U_{j-1,\ell}^n) + B^2(U_{j,\ell+1}^n - 2U_{j,\ell}^n + U_{j,\ell-1}^n)\right) \\
&+ \frac{1}{8}\lambda^2(AB+BA)(U_{j+1,\ell+1}^n - U_{j+1,\ell-1}^n - U_{j-1,\ell+1}^n + U_{j-1,\ell-1}^n) + \frac{1}{2}(kC)^2U_{j,\ell}^n \\
&+ \frac{1}{4}\lambda k\left((AC+CA)(U_{j+1,\ell}^n - U_{j-1,\ell}^n) + (BC+CB)(U_{j,\ell+1}^n - U_{j,\ell-1}^n)\right) \\
&+ \frac{1}{2}k(F_{j,\ell}^{n+1} + F_{j,\ell}^n) - \frac{1}{4}\lambda k\left(A(F_{j+1,\ell}^n - F_{j-1,\ell}^n) + B(F_{j,\ell+1}^n - F_{j,\ell-1}^n)\right) - \frac{1}{2}k^2CF_{j,\ell}^n.
\end{aligned}$$

Engquist and Majda conjectured [10, page 641], that the problem (4.13) with the ABCs (2.19c) at $x = 0$, $x = L$ is *well–posed*. We will give numerical evidence to this conjecture.

Remark 4.1 (Stability of the pure IVP)**.** *The LW–scheme to solve the pure IVP for $C = 0$ is* stable *(in the sense of [26, page 383]), if*

$$\left|\frac{\lambda_{\max} k}{h}\right| \le \frac{1}{2\sqrt{2}}. \tag{4.14}$$

Here, $\lambda_{\max} = \max(\rho(A), \rho(B))$, denotes the absolute greatest eigenvalue of A and B.

Example 4.1. *We use as an illustrative example the* linearized shallow–water equations *from Section 4.1.1 with $a = 0.1$, $b = -0.2$, $c = 1.2$, $f = 0$ and $L = 1$, i.e. it is $m = 2$, $N = 3$. As initial condition we use $u = v = 0$ and for φ the initial function*

$$\varphi(x,y,0) := \begin{cases} \cos^2\left(\frac{\pi}{2}\frac{|(x-0.5,y)^\top|}{0.45}\right) & |(x-0.5,y)^\top| < 0.45, \\ 0 & \text{else}. \end{cases} \tag{4.15}$$

For our example we obtain $\lambda_{\max} = \max(|a|, |a \pm c|, |b|, |b \pm c|) = |b - c| = 1.4$ and hence the necessary stability condition (4.14) *reads*

$$\lambda \le \frac{5}{14\sqrt{2}} \approx 0.2525. \tag{4.16}$$

In the sequel we will fix the mesh ratio to $\lambda = 0.25$ *and choose* $0 \leq x \leq 1$, $-3 \leq y \leq 3$ *as computational domain.*

First we will show how the solution evolve in time (using a time step $\Delta t = 0.01$). Figure 7 and Figure 8 present the time evolution of the potential φ using first order ABCs or *reflecting boundary conditions* $u = v = 0$ (and horizontal extrapolation for φ).

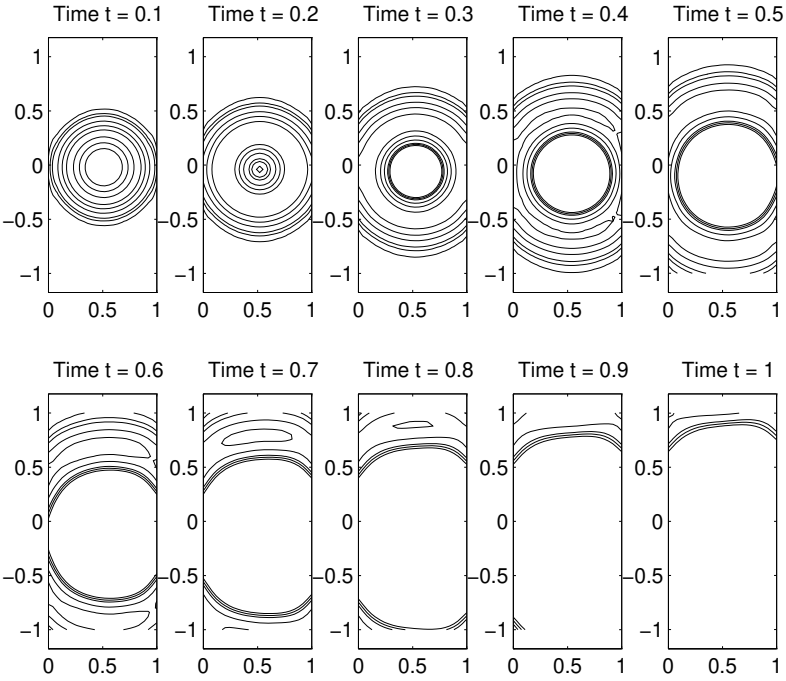

Figure 7: φ for $t = 0.1, 0.2, \ldots, 1$ using first order ABCs

Note that there is a drift in positive x-direction and negative y-direction due to the chosen values $a = 0.1$, $b = -0.2$ that corresponds to averaged horizontal velocities $\bar{u}$, $\bar{v}$

4.2.1. Numerical Investigation of the Stability

Analogue to the procedure in Section 3.3.1 we consider the *modulated initial function*

$$\varphi(x, y, 0) := \begin{cases} \sin\left(2\pi p |(x - 0.5, y)^\top|\right) \cos^2\left(\frac{\pi}{2} \frac{|(x-0.5,y)^\top|}{0.45}\right) & |(x - 0.5, y)^\top| < 0.45, \\ 0 & \text{else,} \end{cases}$$

where the parameter p is roughly speaking some measure for the frequency.

From Remark 4.1 we know that the *interior scheme* is stable, if condition (4.16) is satisfied and we choose λ accordingly, since we only want to study the effect of the ABCs. We

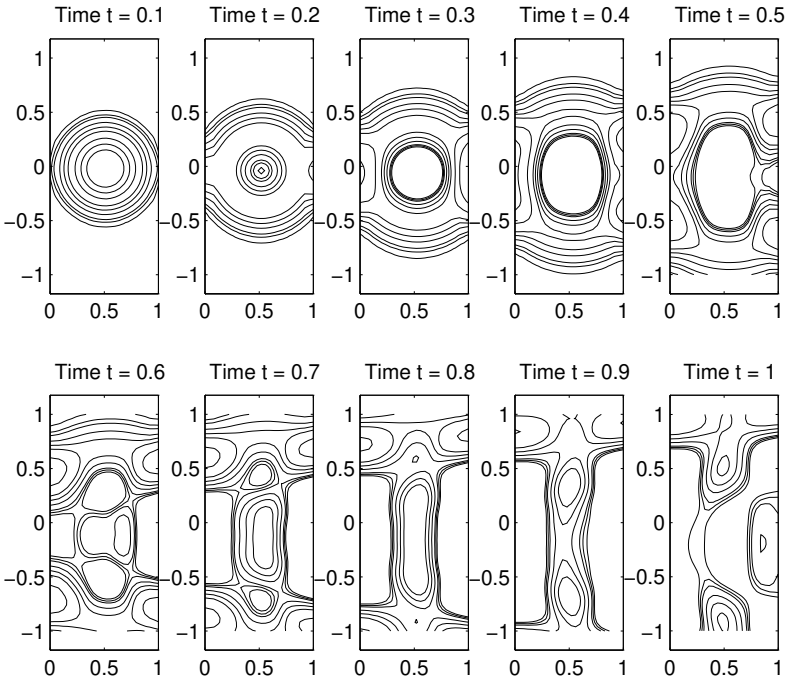

Figure 8: φ for $t = 0.1, 0.2, \ldots, 1$ using reflecting boundary conditions

expect that stability problems can only arise for essentially nonorthogonal impact angles of the waves and these problems will strongly depend on the frequency.

We plot or each component (u, v, φ) the temporal evolution of the *discrete* L^2*-norm*

$$\|U\|_{h,h}^2 := h^2 \sum_{\ell=-\infty}^{\infty} \sum_{j=0}^{J} \left|U_{j,\ell}\right|^2.$$

The summation range w.r.t. the y-coordinate is chosen sufficiently large in the implementation. The results for $\lambda = 0.25$, $\Delta t = 0.001$ and $s = 0.5$ are presented in Figure 9. One clearly observe the *oscillatory behavior* arising if waves with high modulation frequencies passes the boundary at $x = 0$, $x = 1$. For this example the scheme is stable if (4.16) is satisfied. Enlarging the mesh ratio λ leads to *instabilities* for $\lambda \geq 0.37$, cf. Figure 10.

4.2.2. The Accuracy of the Absorbing Boundary Conditions

The wave front reaches with increasing time the boundary $x = 0$ und $x = 1$ with increasing $|y|$ more and more transversally and it is expected that the numerical reflections induced by the ABCs will increase. This is due to the fact that in the derivation of the ABCs in Section 2.2 the Taylor expansion was made around an orthogonal impact angle.

We consider the numerical reflections of the ABCs of zero and first order and relativize them to the reflections of the reflecting boundary conditions. The *reflected part* is determined as in Section 3.3.2., In the following table it can be seen that the difference in the

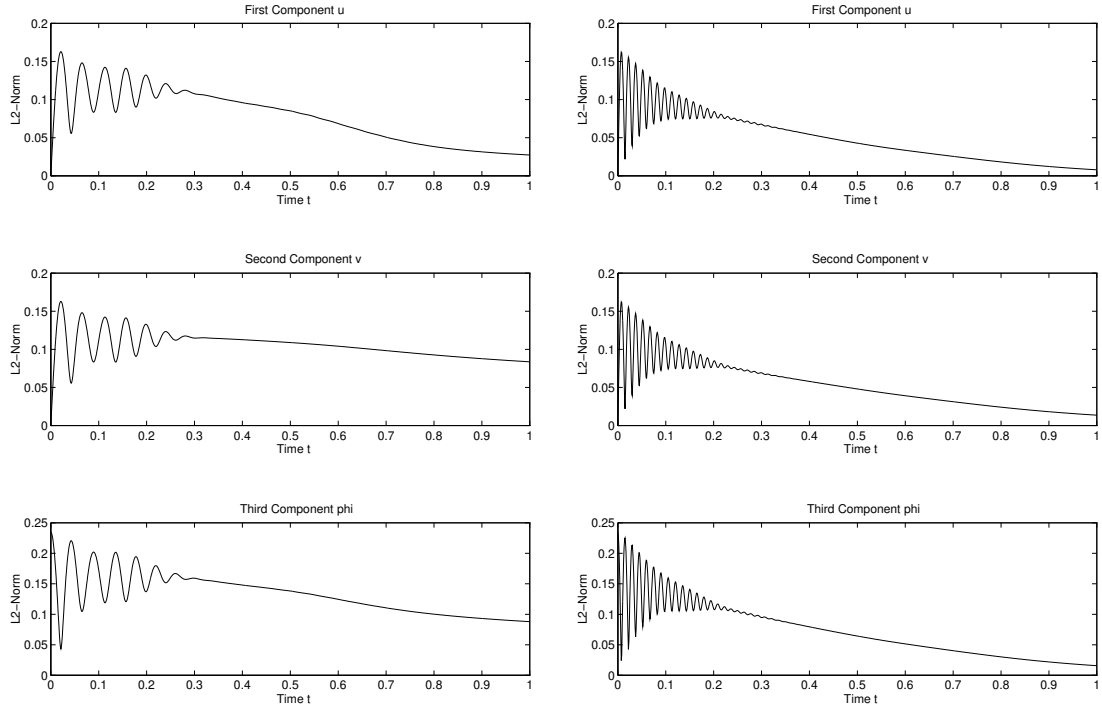

Figure 9: L^2-norms of each component for $\lambda = 0.25$ and $p = 10$ (left), $p = 30$ (right)

reflected part φ_{refl} between the ABCs of zero and first order is not very large (except for the second component). The percentages are relative to the L^2-norm of the reflected part when using reflecting boundary conditions.

BC at $x = 0$, $x = 1$	$\left\|u_{\text{refl}}\right\|_{h,h}$	$\left\|v_{\text{refl}}\right\|_{h,h}$	$\left\|\varphi_{\text{refl}}\right\|_{h,h}$
total reflection $u = v = 0$	0.07866	0.02605	0.07864
zero order ABC (2.19a)	0.00369 [4.7%]	0.01063 [40.8%]	0.00402 [5.1%]
first order ABC (2.19c)	0.00318 [4.0%]	0.00202 [7.8%]	0.00355 [4.5%]

Table 1: Reflected Part at $T = 0.3$ for $\lambda = 0.25$ and $\Delta t = 0.01$.

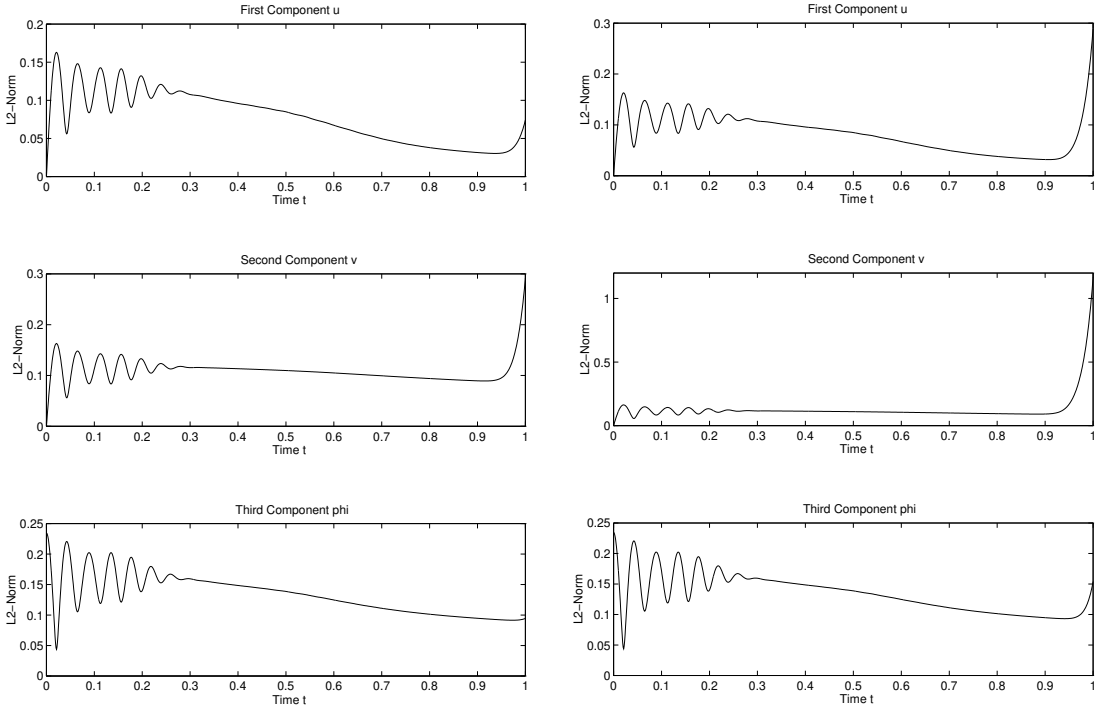

Figure 10: L^2-norms of each component for $p = 10$ and $\lambda = 0.37$ (left), $\lambda = 0.38$ (right)

Conclusion and Outlook

In this work we proved the strict well–posedness of hyperbolic systems with the classical absorbing boundary conditions of Engquist and Majda. We showed that these boundary conditions lead to a GKS–stable Lax–Wendroff-type finite difference scheme. The technique of the later proof can be easily transfered to other finite difference schemes.

Future research directions will include the modification of the much more simple stability conditions of Goldberg und Tadmor [12–15] to cover the situation at hand. The second direction of research would be to derive the absorbing boundary conditions on a purely discrete level, i.e. directly for the considered numerical method. We expect that the resulting *discrete absorbing boundary conditions* are better adapted to the interior scheme and have higher accuracy. First works in this direction can be found in [11, Section 5], [22], [45]. Finally, the study of absorbing boundary conditions for nonlinear hyperbolic equations is a challenging task, cf. [20] and [40, 41].

Acknowledgments

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