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Asymptotic behavior of a hydrodynamic system in the nematic liquid crystal flows

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Abstract

In this paper we study the long time behavior of the classical solutions to a hydrodynamical system modeling the flow of nematic liquid crystals. This system consists of a coupled system of Navier–Stokes equations and kinematic transport equations for the molecular orientations. By using a suitable Łojasiewicz–Simon type inequality, we prove the convergence of global solutions to single steady states as time tends to infinity. Moreover, we provide estimates for the convergence rate.

1 Introduction

We consider the following hydrodynamical system that models the flow of liquid crystal materials

$$v_t + v \cdot \nabla v - \nu \Delta v + \nabla P = -\lambda \nabla \cdot [\nabla d \odot \nabla d + (\Delta d - f(d)) \otimes d], \tag{1.1}$$

$$\nabla \cdot v = 0, \tag{1.2}$$

$$d_t + v \cdot \nabla d - d \cdot \nabla v = \gamma(\Delta d - f(d)), \tag{1.3}$$

in $Q \times (0, \infty)$. Here, Q is a unit square in $\mathbb{R}^n$, $(n = 2, 3)$ (the more general case $Q = \Pi_{i=1}^n (0, L_i)$ with different periods L_i in different directions can be treated in a similar way). v is the velocity field of the flow and d represents the averaged macroscopic/continuum molecular orientations in $\mathbb{R}^n$, $n = 2, 3$. P is a scalar function representing the pressure, which includes both the hydrostatic and the induced elastic part from the orientation field. The constants ν, λ and γ stand for viscosity, the competition between kinetic energy and potential energy, and macroscopic elastic relaxation time (Debroah number) for the molecular orientation field, respectively. $f(d) = \frac{1}{\eta^2}(|d|^2 - 1)d$ with $0 < \eta \leq 1$ may be seen as a penalty function to approximate the constraint $|d| = 1$, which is due to liquid crystal molecules being of similar size [18]. $f(d)$ is the gradient of the scalar valued function $F(d) = \frac{1}{4\eta^2}(|d|^2 - 1)^2$. $\nabla d \odot \nabla d$ denotes the $n \times n$ matrix whose (i, j)-th entry is given by $\nabla_i d \cdot \nabla_j d$, for $1 \leq i, j \leq n$. $\otimes$ is the usual Kronecker multiplication, e.g., $(a \otimes b)_{ij} = a_i b_j$ for $a, b \in \mathbb{R}^n$.

The above system is a simplified version of the Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals (cf. [2, 3, 8, 15, 16]). Generally speaking, the system is a macroscopic continuum description of the time evolutions of these materials influenced by both the flow field $v(x, t)$, and the microscopic orientational configuration $d(x, t)$, which can be derived from the coarse graining of the directions of rod–like liquid crystal molecules. Equation (1.1) is the conservation of linear momentum (the force balance equation). It combines a usual equation describing the flow of an isotropic fluid with an extra nonlinear coupling term, which is anisotropic. This extra term is the induced elastic stress from the elastic energy through the transport, represented by the third equation. Equation (1.2) represents incompressibility of the fluid. Equation (1.3) is associated with conservation of the angular momentum. The left hand side of (1.3) stands for the kinematic transport by the flow field, while the right hand side represents the internal relaxation due to the elastic energy. The continuum theory of liquid crystals due to Ericksen and Leslie was developed during the period of 1958 through 1968. Since then there has been a remarkable research in liquid crystals, both theoretically and experimentally.

In the context of hydrodynamics, the basic variable is the flow map (particle trajectory) $x(X,t)$. X is the original labeling (the Lagrangian coordinate) of the particle, which is also referred as the material coordinate. x is the current (Eulerian) coordinate, and is also called the reference coordinate. For a given velocity field $v(x,t)$, the flow map is defined by the ODE :

$$x_t = v(x(X,t),t), \quad x(X,0) = X.$$

To incorporate the elastic properties of the material, we need to introduce the deformation tensor

$$\mathcal{F}(X,t) = \frac{\partial x}{\partial X}(X,t).$$

This quantity is defined in the Lagrangian material coordinate and it satisfies

$$\frac{\partial F(X,t)}{\partial t} = \frac{\partial v(x(X,t),t)}{\partial X}.$$

In Eulerian coordinates, we define $\tilde{\mathcal{F}}(x,t) = \mathcal{F}(X,t)$. We shall use the notation $\tilde{\mathcal{F}}_{ij} = \frac{\partial x_i}{\partial X_j}$. By using the chain rule, the above equation can be transformed into the following transport equation for $\tilde{\mathcal{F}}$ (cf. [6]):

$$\tilde{\mathcal{F}}_t + (v \cdot \nabla)\tilde{\mathcal{F}} \quad = \quad \nabla v \tilde{\mathcal{F}}.$$

Without ambiguity, in the following text, we will not distinguish these two notations $\mathcal{F}$ and $\tilde{\mathcal{F}}$. If the liquid crystal has a rod-like shape, then transport of the direction of d can be expressed as

$$d(x(X,t),t) = \mathcal{F} d_0(X),$$

where $d_0(X)$ is the initial condition. This equation demonstrates the stretching of the director besides the transport along the trajectory. By taking full time derivative on both sides, we have (cf. [26])

$$\frac{D}{Dt} d(x(X,t),t) = \dot{\mathcal{F}} d_0(X) = \nabla v \mathcal{F} d_0 = \nabla v d = (d \cdot \nabla) v.$$

Hence, the total transport of the orientation vector d becomes

$$d_t + v \cdot \nabla d - d \cdot \nabla v,$$

which represents the covariant parallel transport with no-slip boundary condition between the rod-like particle and the fluid (cf. [10, 14]). In general, for a molecule of ellipsoidal shape with a finite aspect ratio, the transport of the main axis direction is represented by

$$d(x(X,t),t) = \mathbb{E} d_0(X),$$

where $\mathbb{E}$ is a linear combination of $\mathcal{F}$ and $\mathcal{F}^{-T}$ and satisfies the transport equation:

$$\mathbb{E}_t + (v \cdot \nabla)\mathbb{E} = (\alpha \nabla v + (1-\alpha)(-\nabla^T v))\mathbb{E}.$$

The spherical, rod-like and disc-like liquid crystal molecules correspond to $\alpha = \frac{1}{2}$, 1 and 0, respectively (cf. [1, 2, 19, 26]).

In this paper, we shall focus on the rod-like nematic liquid crystal molecules. Analysis for the case of general ellipsoidal molecules may be done in the future work. The system (1.1)–(1.3) can be derived from an energetic variational point of view. Consider the following action functional in terms of flow map $x(X,t)$:

$$\mathcal{A} = \int_0^T \int_{\Omega_0} \left[\frac{1}{2}|x_t|^2 - \lambda \frac{1}{2} |\mathcal{F}^{-T} \nabla_X \mathcal{F} d_0(X)|^2 - \lambda F(\mathcal{F} d_0(X)) \right] J dX dt,$$

where Ω_0 is the region at the initial time and $J = \det \frac{\partial x}{\partial X}$ is the Jacobian. It has been shown that by using the least action principle (Hamilton's principle), $\frac{\delta \mathcal{A}}{\delta x} = 0$, we can recover the system (1.1)–(1.3). We refer to [26] for details. It has been pointed that such a derivation using the variation with respect to domain, i.e., least action principle, is equivalent to the principle of virtual work.

In this paper, we will consider system (1.1)–(1.3) subject to the periodic boundary conditions

$$v(x+e_i) = v(x), \quad d(x+e_i) = d(x), \qquad \text{for } x \in \partial Q, \tag{1.4}$$

and initial conditions

$$v|_{t=0} = v_0(x) \ \text{ with } \nabla \cdot v_0 = 0, \quad d|_{t=0} = d_0(x), \quad \text{for } x \in Q. \tag{1.5}$$

Here, e_i $(i = 1, ..., n)$ are the canonical basis of $\mathbb{R}^n$. The global existence of weak/classical solutions to the system (1.1)–(1.5) for $n = 2$ or $n = 3$ with large viscosity assumption has been proven in [26]. It was pointed out that by supposing periodic boundary conditions, we can get rid of the boundary terms when integrating by parts, which is necessary in the derivation of higher-order energy inequalities. Without this assumption, some boundary terms would remain persistent and undermine the whole estimates. The corresponding initial boundary value problems are still open.

The main results of this paper are as follows

Theorem 1.1. *When $n = 2$, for any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, the unique global classical solution to problem (1.1)–(1.5) has the following property:*

$$\lim_{t \to +\infty} (\|v(t)\|_{H^1} + \|d(t) - d_\infty\|_{H^2}) = 0, \tag{1.6}$$

where d_∞ is a solution to the following nonlinear elliptic boundary value problem:

$$-\Delta d_\infty + f(d_\infty) = 0, \quad x \in Q, \quad d_\infty(x+e_i) = d_\infty(x), \quad x \in \partial Q. \tag{1.7}$$

Moreover, there exists a positive constant C depending on v_0, d_0, Q, d_∞ such that

$$\|v(t)\|_{H^1} + \|d(t) - d_\infty\|_{H^2} \le C(1+t)^{-\frac{\theta}{(1-2\theta)}}, \quad \forall\, t \ge 0. \tag{1.8}$$

Here, $\theta \in (0, 1/2)$ is the same constant as in the Łojasiewicz–Simon inequality (see Lemma 2.3 below).

In three dimensional case, we have the following results:

Theorem 1.2. *When $n = 3$, for any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, under the large viscosity assumption $\nu \ge \nu_0(\lambda, \gamma, v_0, d_0)$ (cf. (4.24)), the problem (1.1)–(1.5) admits a unique global classical solution enjoying the same properties as in Theorem 1.1.*

Theorem 1.3. *When $n = 3$, let $d^* \in H^2_p(Q)$ be an absolute minimizer of the functional*

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_\Omega F(d)dx \tag{1.9}$$

in the sense that $E(d^*) \le E(d)$ for all $d \in H^1_p(Q)$. There exists a constant $\sigma \in (0, 1]$, which may depend on λ, γ, ν, f, Q and d_*, such that for any initial data $(v_0, d_0) \in V \times H^2_p(Q)$ that satisfy $\|v_0\|_{H^1} + \|d_0 - d_*\|_{H^2} < \sigma$, problem (1.1)–(1.5) admits a unique global classical solution enjoying the same properties as in Theorem 1.1.

Remark 1.1. *Theorem 1.3 implies that if the initial velocity v_0 is small and the initial molecule orientation d_0 is sufficiently close to an absolute minimizer of the functional E, then problem (1.1)–(1.5) admits a unique global solution and the solution will converge to a certain equilibrium, which is not necessarily the absolute minimizer itself.*

Theorems 1.1–1.3 imply the uniqueness of asymptotic limit of system (1.1)–(1.5) whenever it admits a global bounded solution. The problem on uniqueness of asymptotic limit for nonlinear evolution equations, namely whether the global solution will converge to an equilibrium as time tends to infinity, has attracted a lot of interests among mathematicians for a long time. If the spacial dimension $n \ge 2$, it is known that the structure of the set of equilibria can be nontrivial and may form a continuum for certain physically reasonable nonlinearities. In particular, in our present case, under the periodic boundary conditions, one may expect that the dimension of the set of stationary solutions is at least n. This is because a shift in each variable should give another steady state (cf. [23]), e.g., if d_* is a steady state, so is $d_*(\cdot + \tau e_i)$, $1 \le i \le n$, $\tau \in \mathbb{R}^+$. In this case it is highly nontrivial to decide whether a given trajectory will converge to a single steady state. In 1983, L. Simon [25] made a breakthrough by proving that for a semilinear parabolic equation with a nonlinear term $f(x, u)$ being analytic in the unknown function u, its bounded global solution would converge to an equilibrium solution as $t \to \infty$. Simon's idea relies on a generalization of the Łojasiewicz inequality (cf. [21,22]) for analytic functions defined in the finite dimensional space $\mathbb{R}^m$. Since then, his original approach has been simplified and applied to prove convergence results for various evolution equations (see e.g., [4, 5, 7, 9, 11, 30] and the references therein). In order to apply the Łojasiewicz–Simon approach to our problem (1.1)–(1.5), we need to introduce a suitable Łojasiewicz–Simon type inequality for vector functions with periodic boundary condition (cf. Lemma 2.3). The corresponding result for small molecule system (cf. [18]) was shown in [20] under various boundary conditions (Dirichlet b.c./free–slip b.c.) and also for flows with changing fluid density.

Once we prove the convergence to an equilibrium, a natural subject for further study is the convergence rate. It is well known that an estimate in certain (lower–order) norm can usually be obtained directly from the Łojasiewicz–Simon approach (see, e.g., [7,31]). One can then in a straightforward way, obtains estimates in higher–order norms by using interpolation inequalities (cf. [7]), and consequently, the decay exponent deteriorates. We shall show that by using suitable energy estimates and constructing proper differential inequalities, it is possible to obtain the same estimates on the convergence rate in both higher and lower order norms. Our approach improves the previous results in the literature and can be applied to many other problems (cf. [5, 29, 30]).

The remaining part of this paper is organized as follows. In Section 2, we introduce the functional setting, some preliminary results and some technical lemmas. Section 3 is devoted to the two dimensional case. We prove the convergence of global solutions to single steady states as time tends to infinity and obtain an estimate on convergence rate. In Section 4, we consider the three dimensional case. The same convergence result is proved for two subcases, in which the global existence of classical solution can be obtained. In the finally Section 5, we briefly discuss the results for liquid crystal flows with non-vanishing average velocity.

2 Preliminaries

Recall the well–established functional setting for periodic problems (cf. [18, 27]):

$$\begin{aligned}
H_p^m(Q) &= \{v \in H^m(\mathbb{R}^n) \mid v(x+e_i) = v(x)\}, \\
\dot{H}_p^m(Q) &= H_p^m(Q) \cap \left\{ v : \int_Q v(x)dx = 0 \right\}, \\
H &= \{v \in L_p^2(Q),\ \nabla \cdot v = 0\}, \quad \text{where } L_p^2(Q) = H_p^0(Q), \\
V &= \{v \in \dot{H}_p^1(Q),\ \nabla \cdot v = 0\}, \\
V' &= \text{the dual of } V.
\end{aligned}$$

For simplicity, we denote the inner product on $L_p^2(Q)$ as well as H by $(\cdot,\cdot)$ and the associated norm by $\|\cdot\|$. We shall denote by C the genetic constants depending on

$\lambda, \gamma, \nu, Q, f$ and the initial data. Special dependence will be pointed out explicitly in the text if necessary. Since the parameters λ and γ do not play important roles in the proof, we set $\lambda = \gamma = 1$ for the sake of simplicity. The following embedding inequalities will be frequently used in the subsequent proofs:

Lemma 2.1. *(cf. [27]) If $n = 2$, we have*

$$\|u\|_{L^\infty(Q)} \le c\|u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{H^2(Q)}, \quad \forall\ u \in H^2_p(Q),$$

If $n = 3$, then

$$\|u\|_{L^\infty(Q)} \le c\|u\|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}_{H^2(Q)}, \quad \forall\ u \in H^2_p(Q).$$

Here, we note that $\|u\|_{H^2(Q)}$ can be estimated by $\|\Delta u\|$ and $\|u\|$ in sprit of the elliptic estimate (2.24).

The global existence of weak/classical solutions to system (1.1)–(1.5) has been proven in [26, Theorem 1.1]. More precisely, we have

Proposition 2.1. *Assume that $(v_0, d_0) \in V \times H^2_p(Q)$. Then, if either $n = 2$ or $n = 3$ with the large viscosity assumption $\nu \ge C(\lambda, \gamma, v_0, d_0)$ (see* (4.24)*), problem* (1.1)–(1.5) *admits a global solution such that*

$$v \in L^\infty(0, \infty; V), \quad d \in L^\infty(0, \infty; H^2), \tag{2.1}$$

and for any $T > 0$,

$$v \in L^2(0, T; H^2), \quad d \in L^2(0, T; H^3). \tag{2.2}$$

Moreover, $v, d \in C^\infty(Q)$ for all $t \in (0, T)$, (v, d) and all of their spatial derivatives are absolutely continuous in time.

The proof for Proposition 2.1 relies on a modified Galerkin method introduced in [18]. After generating a sequence of approximate solutions, one can use the Ladyzhenskaya method (cf. [13, 28]) to get a high–order energy estimate, which enables us to pass to the limit. Furthermore, a weak solution together with high–order derivative estimates implies a strong solution, i.e. $v \in L^2(0, T; H^2(Q))$ and $d \in L^2(0, T; H^3(Q))$. Finally, a bootstrap argument based on Serrin's result [24] (cf. also [12]) and Sobolev embedding theorems leads to the existence of classical solutions. Comparing with the small molecule system (cf. [18]), we now have different kinematic transport and accordingly one more stress term $(\Delta d - f(d)) \otimes d$ in the elastic stress in (1.1) and one more transport term $d \cdot \nabla v$ in (1.3). These bring extra technical difficulties to prove the existence result. For instance, $d \cdot \nabla v$ stands for the parallel transport, which includes both rotation and stretching effect of the director d. The stretching effect leads to the loss of maximum principle for the equation for d. On the other hand, the extra stress term $(\Delta d - f(d)) \otimes d$ cannot be suitably defined in the weak formulation as in [18], and thus the requirement that $d \in L^\infty(0,T;L^\infty(Q))$ must be imposed so that the problem is well-posed. For detailed discussions, we refer to [26].

The Lyapunov functional for problem (1.1)–(1.5) is

$$\mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + \frac{\lambda}{2}\|\nabla d(t)\|^2 + \lambda \int_Q F(d(t))dx. \tag{2.3}$$

Our system has the following *basic energy law*

$$\frac{d}{dt}\mathcal{E}(t) = -\nu\|\nabla v(t)\|^2 - \lambda\gamma\|\Delta d(t) - f(d(t))\|^2, \quad t \geq 0. \tag{2.4}$$

It is worth pointing out that, for different molecule shapes (and as a result, different kinematic transports), the system possesses the same energy dissipative law (2.4) (cf. [19]).

First, we shall show a continuous dependence result on initial data, from which the uniqueness of classical solutions to problem (1.1)–(1.5) follows.

Lemma 2.2. *Suppose that (v_i, d_i) $(i=1,2)$ are global solutions to problem* (1.1)–(1.5) *corresponding to initial data $(v_{0i}, d_{0i}) \in V \times H^2_p(Q)$ $(i=1,2)$. Moreover, we assume that for any $T>0$, the following estimate holds*

$$\|v_i(t)\|_{H^1} + \|d_i(t)\|_{H^2} \leq M, \quad \forall\, t \in [0,T]. \tag{2.5}$$

Then for any $t \in [0,T]$, we have

$$\begin{aligned}
& \|(v_1-v_2)(t)\|^2 + \|(d_1-d_2)(t)\|^2_{H^1} + \int_0^t (\nu\|\nabla(v_1-v_2)(\tau)\|^2 + \|\Delta(d_1-d_2)(\tau)\|^2)d\tau \\
\leq\ & 2e^{Ct}(\|v_{01}-v_{02}\|^2 + \|d_{01}-d_{02}\|^2_{H^1}),
\end{aligned} \tag{2.6}$$

where C is a constant depending on M but not on t.

Proof. Denote

$$\bar{v} = v_1 - v_2, \quad \bar{d} = d_1 - d_2. \tag{2.7}$$

Since (v_i, d_i) are solutions to problem (1.1)–(1.5), we have

$$v_{1t} + v_1\cdot\nabla v_1 - \nu\Delta v_1 + \nabla P_1 = -\nabla\cdot[\nabla d_1 \odot \nabla d_1 + (\Delta d_1 - f(d_1))\otimes d_1], \tag{2.8}$$

$$\nabla\cdot v_1 = 0, \tag{2.9}$$

$$d_{1t} + v_1\cdot\nabla d_1 - d_1\cdot\nabla v_1 = \Delta d_1 - f(d_1), \tag{2.10}$$

and

$$v_{2t} + v_2\cdot\nabla v_2 - \nu\Delta v_2 + \nabla P_2 = -\nabla\cdot[\nabla d_2 \odot \nabla d_2 + (\Delta d_2 - f(d_2))\otimes d_2], \tag{2.11}$$

$$\nabla\cdot v_2 = 0, \tag{2.12}$$

$$d_{2t} + v_2 \cdot \nabla d_2 - d_2 \cdot \nabla v_2 \quad = \quad \Delta d_2 - f(d_2). \tag{2.13}$$

Multiplying $v_1 - v_2$ with the subtraction of (2.11) from (2.8), and multiplying $(d_1 - d_2) - (\Delta d_1 - \Delta d_2)$ with the subtraction of (2.13) from (2.10), we add these two resultants together. By integration by parts, we infer from the periodic boundary conditions that

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\|\bar{v}\|^2 + \|\bar{d}\|^2 + \|\nabla \bar{d}\|^2\right) + \nu\|\nabla \bar{v}\|^2 + \|\nabla \bar{d}\|^2 + \|\Delta \bar{d}\|^2 \\
= \; & -(v_2 \cdot \nabla \bar{v}, \bar{v}) - (\bar{v} \cdot \nabla v_1, \bar{v}) - (\Delta d_2 \cdot \nabla \bar{d}, \bar{v}) + (\Delta d_2 \otimes \bar{d}, \nabla \bar{v}) \\
& -((f(d_1) - f(d_2)) \otimes d_1, \nabla \bar{v}) - (f(d_2) \otimes \bar{d}, \nabla \bar{v}) + (f(d_1) - f(d_2), \Delta \bar{d}) \\
& +(v_2 \cdot \nabla \bar{d}, \Delta \bar{d}) - (\bar{d} \cdot \nabla v_2, \Delta \bar{d}) - (f(d_1) - f(d_2), \bar{d}) - (\bar{v} \cdot \nabla d_1, \bar{d}) \\
& -(v_2 \cdot \nabla \bar{d}, \bar{d}) + (d_1 \cdot \nabla \bar{v}, \bar{d}) + (\bar{d} \cdot \nabla v_2, \bar{d}).
\end{aligned} \tag{2.14}$$

From assumption (2.5), $\|v\|_{H^1}$ and $\|d\|_{H^2}$ are uniformly bounded in $[0, T]$. Hence, by using the Sobolev embedding theorems, we can estimate the righthand side term by term (the calculation presented here is for the three dimensional case and it is also valid for two dimensional case).

$$\begin{aligned}
& |(v_2 \cdot \nabla \bar{v}, \bar{v})| + |(\bar{v} \cdot \nabla v_1, \bar{v})| \\
\le \; & \|v_2\|_{L^6}\|\nabla \bar{v}\|\|\bar{v}\|_{L^3} + \|\bar{v}\|_{L^4}^2\|\nabla v_1\| \\
\le \; & C\|\nabla \bar{v}\|(\|\nabla \bar{v}\|^{\frac{1}{2}}\|\bar{v}\|^{\frac{1}{2}} + \|\bar{v}\|) + C(\|\nabla \bar{v}\|^{\frac{3}{4}}\|\bar{v}\|^{\frac{1}{4}} + \|\bar{v}\|)^2 \\
\le \; & \varepsilon\|\nabla \bar{v}\|^2 + C\|\bar{v}\|^2.
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
& |(\Delta d_2 \cdot \nabla \bar{d}, \bar{v})| + |(\Delta d_2 \otimes \bar{d}, \nabla \bar{v})| \\
\le \; & \|\Delta d_2\|\|\nabla \bar{d}\|_{L^3}\|\bar{v}\|_{L^6} + \|\Delta d_2\|\|\bar{d}\|_{L^\infty}\|\nabla \bar{v}\| \\
\le \; & C(\|\Delta \bar{d}\|^{\frac{1}{2}}\|\nabla \bar{d}\|^{\frac{1}{2}} + \|\nabla \bar{d}\|)(\|\nabla \bar{v}\| + \|\bar{v}\|) + C(\|\Delta \bar{d}\|^{\frac{3}{4}}\|\bar{d}\|^{\frac{1}{4}} + \|\bar{d}\|)\|\nabla \bar{v}\| \\
\le \; & \varepsilon(\|\Delta \bar{d}\|^2 + \|\nabla \bar{v}\|^2) + C(\|\bar{d}\|_{H^1}^2 + \|\bar{v}\|^2).
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
& |((f(d_1) - f(d_2)) \otimes d_1, \nabla \bar{v})| + |(f(d_2) \otimes \bar{d}, \nabla \bar{v})| + |(f(d_1) - f(d_2), \Delta \bar{d})| \\
& +|(f(d_1) - f(d_2), \bar{d})| \\
\le \; & (\|f(d_1) - f(d_2)\|\|d_1\|_{L^\infty} + \|f(d_2)\|_{L^\infty}\|\bar{d}\|)\|\nabla \bar{v}\| + \|f(d_1) - f(d_2)\|(\|\Delta \bar{d}\| + \|\bar{d}\|) \\
\le \; & C(\|f'(\xi)\|_{L^\infty} + 1)\|\bar{d}\|\|\nabla \bar{v}\| + C\|f'(\xi)\|_{L^\infty}\|\bar{d}\|(\|\Delta \bar{d}\| + \|\bar{d}\|) \\
\le \; & \varepsilon(\|\nabla \bar{v}\|^2 + \|\Delta \bar{d}\|^2) + C\|\bar{d}\|^2,
\end{aligned} \tag{2.17}$$

where $\xi = ad_1 + (1 - a)d_2$ with $a \in [0, 1]$.

$$\begin{aligned}
& |(v_2 \cdot \nabla \bar{d}, \Delta \bar{d})| + |(\bar{d} \cdot \nabla v_2, \Delta \bar{d})| \\
\le \; & \|v_2\|_{L^6}\|\nabla \bar{d}\|_{L^3}\|\Delta \bar{d}\| + \|\nabla v_2\|\|\bar{d}\|_{L^\infty}\|\Delta \bar{d}\| \\
\le \; & C(\|\Delta \bar{d}\|^{\frac{1}{2}}\|\nabla \bar{d}\|^{\frac{1}{2}} + \|\nabla \bar{d}\|)\|\Delta \bar{d}\| + C(\|\Delta \bar{d}\|^{\frac{3}{4}}\|\bar{d}\|^{\frac{1}{4}} + \|\bar{d}\|)\|\Delta \bar{d}\| \\
\le \; & \varepsilon\|\Delta \bar{d}\|^2 + C\|\bar{d}\|_{H^1}^2.
\end{aligned} \tag{2.18}$$

$$
\begin{aligned}
& |(\bar{v}\cdot\nabla d_1,\bar{d})|+|(v_2\cdot\nabla\bar{d},\bar{d})|+|(d_1\cdot\nabla\bar{v},\bar{d})|+|(\bar{d}\cdot\nabla v_2,\bar{d})| \\
\le\ & \|\nabla d_1\|_{L^3}\|\bar{v}\|\|\bar{d}\|_{L^6}+\|v_2\|_{L^6}\|\nabla\bar{d}\|\|\bar{d}\|_{L^3}+\|d_1\|_{L^\infty}\|\nabla\bar{v}\|\|\bar{d}\|+\|\nabla v_2\|\|\bar{d}\|_{L^4}^2 \\
\le\ & \varepsilon\|\nabla\bar{v}\|^2+C(\|v\|^2+\|\bar{d}\|_{H^1}^2). \qquad (2.19)
\end{aligned}
$$

Choosing ε small enough in the above estimates, we infer from (2.14) that

$$
\frac{d}{dt}(\|\bar{v}\|^2+\|d\|_{H^1}^2)+\nu\|\nabla\bar{v}\|^2+\|\Delta\bar{d}\|^2\le C(\|v\|^2+\|\bar{d}\|_{H^1}^2), \tag{2.20}
$$

where C is a constant depending on $\|v_i\|_{H^1}$ and $\|d_i\|_{H^2}$ but not on t. By Gronwall's inequality, we can see that for any $t\in[0,T]$,

$$
\|\bar{v}(t)\|^2+\|d(t)\|_{H^1}^2+\int_0^t(\nu\|\nabla\bar{v}(\tau)\|^2+\|\Delta\bar{d}(\tau)\|^2)d\tau\le 2e^{Ct}(\|\bar{v}(0)\|^2+\|\bar{d}(0)\|_{H^1}^2) \tag{2.21}
$$

The proof is complete. □

Remark 2.1. *Since the global classical solution* (v,d) *to problem* (1.1)–(1.5) *obtained in Proposition 2.1 is uniformly bounded in* $H^1\times H^2$ *(cf. [26], see also Lemma 3.2 and Lemma 4.1 below), it immediately follows from Lemma 2.2 that this global solution is unique.*

Next, we look at the following elliptic problem with periodic boundary condition

$$
\begin{cases} -\Delta d+f(d)=0, & x\in Q,\\ d(x+e_i)=d(x), & x\in\partial Q. \end{cases} \tag{2.22}
$$

Define

$$
E(d):=\frac{1}{2}\|\nabla d\|^2+\int_Q F(d)dx. \tag{2.23}
$$

It is not difficult to see that the solution to (2.22) is a critical point of $E(d)$ and conversely the critical point of $E(d)$ is a solution to (2.22). Moreover, the solution to (2.22) is smooth. We recall the interior elliptic estimate, which states that for any $U_1\subset\subset U_2$ there is a constant $C>0$ depending only on U_1 and U_2 such that $\|d\|_{H^2(U_1)}\le C(\|\Delta d\|_{L^2(U_2)}+\|d\|_{L^2(U_2)})$. In our case, we can choose Q' to be the union of Q and its neighborhood copies. Then we have

$$
\|d\|_{H^2(Q)}\le C(\|\Delta d\|_{L^2(Q')}+\|d\|_{L^2(Q')})=9C(\|\Delta d\|_{L^2(Q)}+\|d\|_{L^2(Q)}). \tag{2.24}
$$

In order to apply the Łojasiewicz–Simon approach to prove the convergence to equilibrium, we shall introduce a suitable Łojasiewicz–Simon type inequality that is related to our problem. In particular, we have

Lemma 2.3. [Łojasiewicz–Simon inequality] *Let* ψ *be a critical point of* $E(d)$. *Then there exist constants* $\theta\in(0,\frac{1}{2})$ *and* $\beta>0$ *depending on* ψ *such that for any* $d\in H^1_p(Q)$ *satisfying* $\|d-\psi\|_{H^1(Q)}<\beta$, *it holds*

$$
\|-\Delta d+f(d)\|_{(H^1_p(Q))'}\ge|E(d)-E(\psi)|^{1-\theta}, \tag{2.25}
$$

where $(H^1_p(Q))'$ *is the dual space of* $H^1_p(Q)$.

Remark 2.2. *Lemma 2.3 can be viewed as an extended version of Simon's result [25] for scalar functions using the* L^2*-norm. For the proof of this result, we refer to [9, Chapter 2, Theorem 5.2].*

The following lemma turns out to useful in the study of long-time behavior of solutions to evolution equations.

Lemma 2.4. [31, Lemma 6.2.1] *Let* T *be given with* $0 < T \le +\infty$*. Suppose that* $y(t)$ *and* $h(t)$ *are nonnegative continuous functions defined on* $[0, T]$*, which satisfy the following conditions:*

$$\frac{dy}{dt} \le c_1 y^2 + c_2 + h(t), \quad \int_0^T y(t)dt \le c_3, \quad \int_0^T h(t)dt \le c_4,$$

where c_i $(i = 1, 2, 3, 4)$ *are given nonnegative constants. Then for any* $r \in (0, T)$*, the following estimate holds:*

$$y(t + r) \le \Big(\frac{c_3}{r} + c_2 r + c_4\Big) e^{c_1 c_3}, \quad \forall\, t \in [0, T - r].$$

Furthermore, if $T = +\infty$*, then*

$$\lim_{y \to +\infty} y(t) = 0.$$

3 Long-time Behavior in Two Dimensional Case

In this section, we prove the convergence of global solutions to single steady states as time tends to infinity in the two dimensional case. In 2-D case, an important property for the global solution to problem (1.1)–(1.5) is the following high–order energy inequality, which played a crucial role in the proof of global existence result in [26].

Denote

$$A(t) = \|\nabla v(t)\|^2 + \lambda \|\Delta d(t) - f(d(t))\|^2. \tag{3.1}$$

We recall that it has been assumed that $\lambda = \gamma = 1$. Besides, since viscosity ν does not play a crucial role in the 2-D case, we also set $\nu = 1$ in this section for the sake of simplicity. Then we have (cf. [26, (45)])

Lemma 3.1. *In two dimensional case, the following inequality holds for the classical solution* (v, d) *to problem* (1.1)–(1.5)*:*

$$\frac{d}{dt}A(t) + \frac{1}{2}(\|\Delta v(t)\|^2 + \|\nabla(\Delta d(t) - f(d(t)))\|^2) \le C(A^2(t) + 1), \quad \forall\, t \ge 0, \tag{3.2}$$

where C *is a constant depending on* $f, Q, \|v_0\|, \|d_0\|_{H^1(Q)}$.

3.1 Convergence to Equilibrium

Based on the high–order energy inequality (3.2), we are able to show the decay property of the velocity field v.

Lemma 3.2. *For any $t \geq 0$, the following uniform estimate holds*

$$\|v(t)\|_{H^1} + \|d(t)\|_{H^2} \leq C, \tag{3.3}$$

where C is a constant depending on $f, Q, \|v_0\|_{H^1}, \|d_0\|_{H^2}$. Furthermore,

$$\lim_{t \to +\infty} (\|v(t)\|_{H^1} + \| - \Delta d(t) + f(d(t))\|) = 0. \tag{3.4}$$

Proof. It follows from the basic energy law (2.3) that

$$\mathcal{E}(t) + \int_0^t A(\tau) d\tau \leq \mathcal{E}(0) < \infty, \quad \forall\, t \geq 0. \tag{3.5}$$

(3.5) implies the uniform estimate

$$\|v(t)\| + \|d(t)\|_{H^1} \leq C, \quad \forall\, t \geq 0. \tag{3.6}$$

Concerning the uniform bound (3.3), we take $r = 1$ in Lemma 2.4 to get

$$\|\nabla v(t)\| + \| - \Delta d(t) + f(d(t))\| \leq C, \quad \forall\, t \geq 1, \tag{3.7}$$

where C does not depend on t. On the other hand, for any $t \in [0, 1]$, it follows from (3.2) and the fact $\int_0^1 A(t)dt \leq C$ that

$$\sup_{0 \leq t \leq 1} A(t) \leq e^{\int_0^1 A(t)dt} A(0) + C \leq C. \tag{3.8}$$

Besides, from the continuous embedding $H^1 \hookrightarrow L^p$ $(1 \leq p < \infty)$ and (3.6) we have

$$\|\Delta d\| \leq \| - \Delta d + f(d)\| + \|f(d)\| \leq \| - \Delta d + f(d)\| + C\left(1 + \|d\|_{L^6}^3\right) \leq C. \tag{3.9}$$

Therefore, (3.3) follows from (3.7)–(3.9). Furthermore, (3.5) together with Lemma 3.1 and Lemma 2.4 yields that

$$\lim_{t \to +\infty} (\|\nabla v(t)\| + \|\Delta d(t) - f(d(t))\|) = 0. \tag{3.10}$$

By the Poincaré inequality for $v \in V$, we conclude (3.4). The proof is complete. $\square$

Let $\mathcal{S}$ be the set

$$\mathcal{S} = \{(0, u) \mid -\Delta u + f(u) = 0, \text{ in } Q,\ u(x + e_i) = u(x) \text{ on } \partial Q\}.$$

The ω-limit set of $(v_0, d_0) \in V \times H_p^2(Q) \subset L_p^2(Q) \times H_p^1(Q)$ is defined as follows:

$$\begin{aligned} \omega(v_0, d_0) = \{(v_\infty(x), d_\infty(x)) \mid & \text{ there exists } \{t_n\} \nearrow \infty \text{ such that} \\ & (v(t_n), d(t_n)) \to (v_\infty, d_\infty) \text{ in } L^2(Q) \times H^1(Q), \text{ as } t_n \to +\infty\}. \end{aligned}$$

We infer from Lemma 3.2 that

Proposition 3.1. *$\omega(v_0, d_0)$ is a nonempty bounded subset in $H^1_p(Q) \times H^2_p(Q)$, which is compact in $L^2_p(Q) \times H^1_p(Q)$. Besides, all asymptotic limiting points (v_∞, d_∞) of problem (1.1)–(1.5) belong to $\mathcal{S}$. In other words, $\omega(v_0, d_0) \subset \mathcal{S}$.*

In what follows, we prove the convergence for director field d. For any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, it follows from Lemma 3.2 that $\|d\|_{H^2}$ is uniformly bounded. Proposition 3.1 implies that there is an increasing unbounded sequence $\{t_n\}_{n\in\mathbb{N}}$ and a function d_∞ such that

$$\lim_{t_n \to +\infty} \|d(t_n) - d_\infty\|_{H^1} = 0. \tag{3.11}$$

Moreover, d_∞ satisfies the equation

$$-\Delta d_\infty + f(d_\infty) = 0, \quad x \in \Omega, \quad d_\infty(x + e_i) = d_\infty(x) \text{ on } \partial Q. \tag{3.12}$$

We prove the convergence result following a simple argument first introduced in [11] in which the key observation is that after a certain time t_0, $d(t)$ will fall into a certain small neighborhood of d_∞ and stay there forever.

From the basic energy law (2.3), we can see that $\mathcal{E}(t)$ is decreasing on $[0, \infty)$, and it has a finite limit as time goes to infinity because it is nonegative. Therefore, it follows from (3.4) and (3.11) that

$$\lim_{t_n \to +\infty} \mathcal{E}(t_n) = E(d_\infty). \tag{3.13}$$

On the other hand, we can infer from (2.3) that $\mathcal{E}(t) \geq E(d_\infty)$, for all $t > 0$, and the equal sign holds if and only if, for all $t > 0$, $v = 0$ and d solves problem (3.12).

We now consider all possibilities.

Case 1. If there is a $t_0 > 0$ such that $\mathcal{E}(t_0) = E(d_\infty)$, then for all $t > t_0$, we deduce from (2.3) that

$$\|\nabla v\| \equiv 0, \quad \|-\Delta d + f(d)\| \equiv 0. \tag{3.14}$$

It follows from (1.3), (3.14) and the Sobolev embedding theorem that for $t \geq t_0$, $\|d_t\| = 0$. Namely, d is independent of time for all $t \geq t_0$. Due to (3.11), we conclude that $d(t) \equiv d_\infty$ for $t \geq t_0$.

Case 2. For all $t > 0$, we suppose that $\mathcal{E}(t) > E(d_\infty)$. First we assume that the following claim holds true.

Proposition 3.2. *There is a $t_0 > 0$ such that for all $t \geq t_0$, $\|d(t) - d_\infty\|_{H^1} < \beta$. Namely, for all $t \geq t_0$, $d(t)$ satisfies the condition in Lemma 2.3.*

In this case, it follows from Lemma 2.3 that

$$|E(d) - E(d_\infty)|^{1-\theta} \leq \|-\Delta d + f(d)\|_{(H^1_p)'} \leq \|-\Delta d + f(d)\|, \quad \forall\, t \geq t_0. \tag{3.15}$$

The fact $\theta \in (0, \frac{1}{2})$ implies that $0 < 1-\theta < 1$, $2(1-\theta) > 1$. Consequently,

$$\|v\|^{2(1-\theta)} = \|v\|^{2(1-\theta)-1}\|v\| \le C\|v\|.$$

Then we infer from the basic inequality

$$(a+b)^{1-\theta} \le a^{1-\theta} + b^{1-\theta}, \qquad \forall\, a, b \ge 0$$

that

$$\begin{aligned}
(\mathcal{E}(t) - E(d_\infty))^{1-\theta} &\le \left(\frac{1}{2}\|v\|^2 + |E(d) - E(d_\infty)|\right)^{1-\theta} \\
&\le \left(\frac{1}{2}\|v\|^2 + \|-\Delta d + f(d)\|^{\frac{1}{1-\theta}}\right)^{1-\theta} \\
&\le \left(\frac{1}{2}\right)^{1-\theta}\|v\|^{2(1-\theta)} + \|-\Delta d + f(d)\| \\
&\le C\|v\| + \|-\Delta d + f(d)\|. \qquad (3.16)
\end{aligned}$$

Therefore, a direct calculation yields that

$$\begin{aligned}
-\frac{d}{dt}(\mathcal{E}(t) - E(d_\infty))^{\theta} &= -\theta(\mathcal{E}(t) - E(d_\infty))^{\theta-1}\frac{d}{dt}\mathcal{E}(t) \\
&\ge \frac{C\theta(\|\nabla v\| + \|-\Delta d + f(d)\|)^2}{C\|v\| + \|-\Delta d + f(d)\|} \\
&\ge C_1(\|\nabla v\| + \|-\Delta d + f(d)\|), \quad \forall\, t \ge t_0, \qquad (3.17)
\end{aligned}$$

where C_1 is a constant depending on v_0, d_0, Q and θ.
Integrating from t_0 to t, we get

$$\begin{aligned}
&(\mathcal{E}(t) - E(d_\infty))^{\theta} + C_1\int_{t_0}^{t}(\|\nabla v(\tau)\| + \|-\Delta d(\tau) + f(d(\tau))\|)d\tau \\
\le\ &(\mathcal{E}(t_0) - E(d_\infty))^{\theta} < \infty, \quad \forall\, t \ge t_0. \qquad (3.18)
\end{aligned}$$

Since $\mathcal{E}(t) - E(d_\infty) \ge 0$, we conclude that

$$\int_{t_0}^{\infty}(\|\nabla v(\tau)\| + \|-\Delta d(\tau) + f(d(\tau))\|)d\tau < \infty. \qquad (3.19)$$

On the other hand, it follows from equation (1.3), (3.3) and Sobolev embedding theorems that

$$\begin{aligned}
\|d_t\| &\le \|v\cdot\nabla d\| + \|d\cdot\nabla v\| + \|-\Delta d + f(d)\| \\
&\le \|v\|_{L^4}\|\nabla d\|_{L^4} + \|d\|_{L^\infty}\|\nabla v\| + \|-\Delta d + f(d)\| \\
&\le C\|\nabla v\| + \|-\Delta d + f(d)\|. \qquad (3.20)
\end{aligned}$$

Hence,

$$\int_{t_0}^{\infty} \|d_t(\tau)\| d\tau < +\infty, \tag{3.21}$$

which easily implies that as $t \to +\infty$, $d(t)$ converges in $L^2(Q)$. This and (3.11) indicate that

$$\lim_{t\to+\infty} \|d(t) - d_\infty\| = 0. \tag{3.22}$$

Since $d(t)$ is uniformly bounded in $H^2(Q)$ (cf. (3.3)), by standard interpolation inequality we have

$$\lim_{t\to+\infty} \|d(t) - d_\infty\|_{H^1} = 0. \tag{3.23}$$

On the other hand, the uniform bound of d in $H^2(Q)$ implies the weak convergence

$$d(t) \rightharpoonup d_\infty, \quad \text{in } H^2(Q). \tag{3.24}$$

However, the decay property of the quantity $A(t)$ (cf. Lemma 3.2) could tell us more. Namely, we could get strong convergence of d in H^2. To see this, we keep in mind that

$$\begin{aligned}
\|\Delta d - \Delta d_\infty\| &\leq \|\Delta d - \Delta d_\infty - f(d) + f(d_\infty)\| + \|f(d) - f(d_\infty)\| \\
&\leq \|\Delta d - f(d)\| + \|f'(\xi)\|_{L^4}\|d - d_\infty\|_{L^4} \\
&\leq \|\Delta d - f(d)\| + C\|d - d_\infty\|_{H^1}.
\end{aligned} \tag{3.25}$$

The above estimate together with (3.4) and (3.23) yields

$$\lim_{t\to+\infty} \|d(t) - d_\infty\|_{H^2} = 0. \tag{3.26}$$

To finish the proof, it remains to show that Proposition 3.2 always holds true for the global solution $d(t)$ to system (1.1)–(1.5). Define

$$\bar{t}_n = \sup\{\ t > t_n |\ \|d(s) - d_\infty\|_{H^1} < \beta,\ \forall\ s \in [t_n, t]\}. \tag{3.27}$$

It follows from (3.11) that for any $\varepsilon \in (0, \beta)$, there exists an integer N such that when $n \geq N$,

$$\|d(t_n) - d_\infty\|_{H^1} < \varepsilon, \tag{3.28}$$

$$\frac{1}{C_1}(\mathcal{E}(t_n) - E(d_\infty))^\theta < \varepsilon. \tag{3.29}$$

On the other hand, we know that the orbit of the classical solution d is continuous in H^1. It follows from (3.3) that $d \in L^\infty(0, +\infty; H^2)$. As a consequence, $d \in L^2(t, t+1; H^2)$ for any $t \geq 0$. The basic energy law and (3.20) imply $d_t \in L^2(t, t+1; L^2)$. Thus, for any $t \geq 0$, it holds $d \in C([t, t+1]; H^1)$. The continuity of the orbit of d in H^1 and (3.28) yield that

$$\bar{t}_n > t_n, \quad \text{for all } n \geq N. \tag{3.30}$$

Then there are two possibilities:

(i). If there exists $n_0 \geq N$ such that $\bar{t}_{n_0} = +\infty$, then from the previous discussions in Case 1 and Case 2, the theorem is proved.

(ii) Otherwise, for all $n \geq N$, we have $t_n < \bar{t}_n < +\infty$, and for all $t \in [t_n, \bar{t}_n]$, $E(d_\infty) < \mathcal{E}(t)$. Then from (3.18) with t_0 being replaced by t_n, and t being replaced by $\bar{t}_n$, we obtain from (3.29) that

$$\int_{t_n}^{\bar{t}_n} (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau < \varepsilon. \tag{3.31}$$

Thus, it follows that (cf. (3.20))

$$\begin{aligned}
\|d(\bar{t}_n) - d_\infty\| &\leq \|d(t_n) - d_\infty\| + \int_{t_n}^{\bar{t}_n} \|d_t(\tau)\| d\tau \\
&\leq \|d(t_n) - d_\infty\| + C \int_{t_n}^{\bar{t}_n} (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau \\
&< C\varepsilon,
\end{aligned} \tag{3.32}$$

which implies that $\lim_{n\to+\infty} \|d(\bar{t}_n) - d_\infty\| = 0$. Since $d(t)$ is relatively compact in H^1, there exists a subsequence of $\{d(\bar{t}_n)\}$, still denoted by $\{d(\bar{t}_n)\}$ converging to d_∞ in H^1, i.e., when n is sufficiently large,

$$\|d(\bar{t}_n) - d_\infty\|_{H^1} < \beta,$$

which contradicts the definition of $\bar{t}_n$ that $\|d(\bar{t}_n) - d_\infty\|_{H^1} = \beta$.

Summing up, we have considered all the possible cases and the conclusion (1.6) is proved. □

3.2 Convergence Rate

In this part, we shall prove the estimate for convergence rate (1.8). This can be achieved by several steps.

Step 1. As has been shown in the literature (cf. [7,31]), an estimate on the convergence rate in certain lower–order norm could be obtained directly from the Łojasiewicz–Simon approach. From Lemma 2.3 and (3.17), we have

$$\frac{d}{dt}(\mathcal{E}(t) - E(d_\infty)) + C_1(\mathcal{E}(t) - E(d_\infty))^{2(1-\theta)} \leq 0, \quad \forall\, t \geq t_0, \tag{3.33}$$

which implies

$$\mathcal{E}(t) - E(d_\infty) \leq C(1+t)^{-\frac{1}{1-2\theta}}, \quad \forall\, t \geq t_0. \tag{3.34}$$

Integrating (3.17) on (t,∞), where $t \geq t_0$, it follows from (3.20) that

$$\int_t^\infty \|d_t(\tau)\| d\tau \leq \int_t^\infty (C\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau \leq C(1+t)^{-\frac{\theta}{1-2\theta}}. \quad (3.35)$$

By adjusting the constant C properly, we obtain

$$\|d(t) - d_\infty\| \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad t \geq 0. \quad (3.36)$$

Step 2. In Step 1, we only obtain the convergence rate of d (in L^2). Unlike for the temperature variable in some phase–field systems (cf. [30] and references cited therein), although we have got some decay information for the velocity field v such that

$$\int_t^\infty \|\nabla v(\tau)\| d\tau \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad (3.37)$$

we are not able to prove convergence rate of v directly. This is because now v satisfies a Navier–Stokes equation which is much more complicated than the heat equation for the temperature variable in phase–field systems. As a result, one cannot easily obtain relation between $\|\nabla v\|$ and v_t (in certain possible norm) from the equation (1.1) itself. However, it is possible to achieve our goal by using the idea in [30] where we use higher–order energy estimates and construct proper differential inequalities (cf. also [5, 29]). Besides, the convergence rate of d in higher order norm can be proved simultaneously.

The steady state corresponding to problem (1.1)–(1.5) satisfies the following system:

$$\begin{aligned} v_\infty \cdot \nabla v_\infty - \nu \Delta v_\infty + \nabla P_\infty &= -\nabla \cdot [\nabla d_\infty \odot \nabla d_\infty + (\Delta d_\infty - f(d_\infty)) \otimes d_\infty], && (3.38)\\ \nabla \cdot v_\infty &= 0, && (3.39)\\ v_\infty \cdot \nabla d_\infty - d_\infty \cdot \nabla v_\infty &= \Delta d_\infty - f(d_\infty), && (3.40)\\ v_\infty(x+e_i) = v_\infty(x), &\qquad d_\infty(x+e_i) = d_\infty(x). && (3.41) \end{aligned}$$

Lemma 3.2 implies that all limiting points of system (1.1)–(1.5) have the form $(0, d_\infty) \in \mathcal{S}$. As a result, system (3.38)–(3.41) can be reduced to

$$\begin{aligned} \nabla P_\infty + \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) &= -\nabla d_\infty \cdot \Delta d_\infty, && (3.42)\\ -\Delta d_\infty + f(d_\infty) &= 0, && (3.43) \end{aligned}$$

with periodic boundary condition for d_∞. In (3.42), we use the fact that

$$\nabla \cdot (\nabla d_\infty \odot \nabla d_\infty) = \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) + \nabla d_\infty \cdot \Delta d_\infty.$$

Subtracting the stationary problem (3.42)–(3.43) from the evolution problem (1.1)–(1.5), we obtain that

$$v_t + v \cdot \nabla v - \nu \Delta v + \nabla (P - P_\infty) + \nabla \left(\left(\frac{|\nabla d|^2}{2} \right) - \left(\frac{|\nabla d_\infty|^2}{2} \right) \right)$$

$$
\begin{aligned}
&= -\nabla\cdot[(\Delta d - f(d))\otimes d] - \nabla d\cdot\Delta d + \nabla d_\infty\cdot\Delta d_\infty, && (3.44)\\
\nabla\cdot v &= 0, && (3.45)\\
d_t + v\cdot\nabla d - d\cdot\nabla v &= \Delta(d-d_\infty) - f(d) + f(d_\infty). && (3.46)
\end{aligned}
$$

Multiplying (3.44) by v and (3.46) by $-\Delta d + f(d) = -\Delta(d-d_\infty) + f(d) - f(d_\infty)$, respectively, integrating over Q, and adding the results together, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac12\|v\|^2 + \frac12\|\nabla d - \nabla d_\infty\|^2 + \int_Q [F(d) - F(d_\infty) - f(d_\infty)(d-d_\infty)]dx\right)\\
&\quad +\nu\|\nabla v\|^2 + \|\Delta d - f(d)\|^2\\
&= (v, \nabla d_\infty\cdot\Delta d_\infty)\\
&= (v, \nabla d_\infty\cdot(\Delta d_\infty - f(d_\infty))) + (v\cdot\nabla d_\infty, f(d_\infty))\\
&= 0. \qquad (3.47)
\end{aligned}
$$

Multiplying (3.46) by $d-d_\infty$ and integrating in Ω, we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|d-d_\infty\|^2 + \|\nabla(d-d_\infty)\|^2\\
&= -(v\cdot\nabla d, d-d_\infty) + (d\cdot\nabla v, d-d_\infty) - (f(d)-f(d_\infty), d-d_\infty) := I_1. \qquad (3.48)
\end{aligned}
$$

The right hand side can be estimated as follows

$$
\begin{aligned}
|I_1| &\le \|v\|_{L^4}\|\nabla d\|_{L^4}\|d-d_\infty\| + \|\nabla v\|\|d\|_{L^\infty}\|d-d_\infty\| + \|f'(\xi)\|_{L^\infty}\|d-d_\infty\|^2\\
&\le C\|\nabla v\|\|d-d_\infty\| + C(\|\nabla(d-d_\infty)\|^{\frac12}\|d-d_\infty\|^{\frac12} + \|d-d_\infty\|)^2\\
&\le \varepsilon_1\|\nabla v\|^2 + \frac12\|\nabla(d-d_\infty)\|^2 + C\|d-d_\infty\|^2. \qquad (3.49)
\end{aligned}
$$

Multiplying (3.48) by $\alpha>0$ and adding the resultant to (3.47), using (3.49), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac12\|v\|^2 + \frac12\|\nabla d - \nabla d_\infty\|^2 + \frac{\alpha}{2}\|d-d_\infty\|^2 + \int_Q (F(d) - F(d_\infty))dx\right.\\
&\quad \left. - \int_Q f(d_\infty)(d-d_\infty)dx\right) + (\nu - \alpha\varepsilon_1)\|\nabla v\|^2 + \|\Delta d - f(d)\|^2 + \frac{\alpha}{2}\|\nabla(d-d_\infty)\|^2\\
&\le C\alpha\|d-d_\infty\|^2. \qquad (3.50)
\end{aligned}
$$

On the other hand, by the Taylor's expansion, we have

$$
F(d) = F(d_\infty) + f(d_\infty)(d-d_\infty) + f'(\xi)(d-d_\infty)^2, \qquad (3.51)
$$

where $\xi = ad + (1-a)d_\infty$ with $a\in[0,1]$.
Then we deduce that

$$
\begin{aligned}
&\left|\int_Q [F(d) - F(d_\infty)dx - f(d_\infty)(d-d_\infty)]dx\right| = \left|\int_Q f'(\xi)(d-d_\infty)^2 dx\right|\\
&\le \|f'(\xi)\|_{L^\infty}\|d-d_\infty\|^2 \le C_2\|d-d_\infty\|^2. \qquad (3.52)
\end{aligned}
$$

Let us define now, for $t \geq 0$,

$$\begin{aligned} y(t) &= \frac{1}{2}\|v(t)\|^2 + \frac{1}{2}\|\nabla d(t) - \nabla d_\infty\|^2 + \frac{\alpha}{2}\|d(t) - d_\infty\|^2 + \int_Q (F(d(t))dx - F(d_\infty))dx \\ &\quad - \int_Q f(d_\infty)(d(t) - d_\infty)dx. \end{aligned} \tag{3.53}$$

In (3.50) and (3.53), we choose

$$\alpha \geq 1 + 2C_2 > 0, \quad \varepsilon_1 = \frac{\nu}{2\alpha}. \tag{3.54}$$

As a result,

$$y(t) + C_2\|d - d_\infty\|^2 \geq \frac{1}{2}(\|v\|^2 + \|d - d_\infty\|_{H^1}^2). \tag{3.55}$$

Furthermore, we infer from (3.55) that for certain constants $C_3, C_4 > 0$,

$$\frac{d}{dt}y(t) + C_3 y(t) \leq C_4\|d - d_\infty\|^2 \leq C(1+t)^{-\frac{2\theta}{1-2\theta}}. \tag{3.56}$$

By Gronwall's inequality, we have (cf. [29, 30])

$$y(t) \leq C(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \geq 0, \tag{3.57}$$

which together with (3.55) implies that

$$\|v(t)\| + \|d(t) - d_\infty\|_{H^1} \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \geq 0. \tag{3.58}$$

Step 3. In the last step, we shall prove the convergence rate in the same space where the initial data stay. In Section 3.1, it has been proved that, once we could obtain the uniform bound of d in H^2, we are able to obtain strong convergence of d in H^2 instead of weak convergence. By reinvestigating the higher–order energy estimate for the subtracted system (3.44)–(3.46) (cf. also Lemma 3.1), we can obtain a further result, which provides the same rate estimate of (v, d) in $H^1 \times H^2$ as (3.58).

Taking the time derivative of $A(t)$, we obtain (cf. also [26, (43)])

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}A(t) + (\nu\|\Delta v\|^2 + \|\nabla(\Delta d - f(d)\|^2) \\ = \;& (\Delta v, v \cdot \nabla v) + (\Delta v, \nabla \cdot (\nabla d \odot \nabla d)) + (\Delta v, \nabla \cdot ((\Delta d - f(d)) \otimes d)) \\ &+ (\nabla(\Delta d - f(d)), \nabla(v \cdot \nabla d)) - (\nabla(\Delta d - f(d)), \nabla(d \cdot \nabla v)) \\ &+ (\Delta d - f(d), f'(d)d_t) \\ = \;& I_2 + \ldots + I_7. \end{aligned} \tag{3.59}$$

Noticing that we have got uniform bounds for $\|v\|_{H^1}$ and $\|d\|_{H^2}$ before (see Lemma 3.2), in what follows we estimate I_i $(i = 2, ..., 7)$ term by term.

$$|I_2| = |(\Delta v, v \cdot \nabla v)| \leq \|\Delta v\|\|v\|_{L^\infty}\|\nabla v\|$$

$$
\begin{aligned}
&\leq C\|\Delta v\|^{\frac{3}{2}}\|\nabla v\|^{\frac{3}{2}} \leq C\|\Delta v\|^{\frac{3}{2}}\|\nabla v\|^{\frac{1}{2}} \\
&\leq \varepsilon\|\Delta v\|^2 + C_\varepsilon\|\nabla v\|^2. \qquad (3.60)
\end{aligned}
$$

Before estimating I_3, we first look at the following estimate

$$
\begin{aligned}
\|\nabla\Delta d\| &\leq \|\nabla(\Delta d - f(d))\| + \|f'(d)\nabla d\| \leq \|\nabla(\Delta d - f(d))\| + \|f'(d)\|_{L^4}\|\nabla d\|_{L^4} \\
&\leq \|\nabla(\Delta d - f(d))\| + C\|\Delta d\|^{\frac{1}{2}}\|\nabla d\|^{\frac{1}{2}} \leq \|\nabla(\Delta d - f(d))\| + C. \qquad (3.61)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|I_3| &= |(\Delta v, \nabla\cdot(\nabla d \odot \nabla d))| = |(\Delta v, \nabla\cdot(\nabla d \odot \nabla d)) - (\Delta v, \nabla P_\infty)| \\
&= |(\Delta v, \nabla d \cdot \Delta d) - (\Delta v, \nabla d_\infty \cdot \Delta d_\infty)| \\
&\leq |(\Delta v, \nabla d \cdot \Delta d) - (\Delta v, \nabla d \cdot \Delta d_\infty)| + |(\Delta v, \nabla d \cdot \Delta d_\infty) - (\Delta v, \nabla d_\infty \cdot \Delta d_\infty)| \\
&=: I_{3a} + I_{3b}. \qquad (3.62)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
I_{3a} &= |(\Delta v, \nabla d \cdot \Delta d) - (\Delta v, \nabla d \cdot \Delta d_\infty)| \\
&\leq \|\Delta v\|\|\nabla d\|_{L^\infty}\|\Delta d - \Delta d_\infty\| \\
&\leq C\|\Delta v\|\|\Delta d - \Delta d_\infty\|\|\nabla d\|^{\frac{1}{2}}\|\nabla\Delta d\|^{\frac{1}{2}} \\
&\leq C\|\Delta v\|(\|\Delta d - f(d)\| + \|f(d) - f(d_\infty)\|)(\|\nabla(\Delta d - f(d))\| + C)^{\frac{1}{2}} \\
&\leq \varepsilon\|\Delta v\|^2 + C_\varepsilon(\|\Delta d - f(d)\| + \|f(d) - f(d_\infty)\|)^2(\|\nabla(\Delta d - f(d))\| + C) \\
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 \\
&\quad + C_\varepsilon[1 + (\|\Delta d - f(d)\| + \|f(d) - f(d_\infty)\|)^2](\|\Delta d - f(d)\| + \|f(d) - f(d_\infty)\|)^2 \\
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon(\|\Delta d - f(d)\| + \|f(d) - f(d_\infty)\|)^2 \\
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon\|\Delta d - f(d)\|^2 + C_\varepsilon\|d - d_\infty\|^2, \qquad (3.63)
\end{aligned}
$$

$$
\begin{aligned}
I_{3b} &= |(\Delta v, \nabla d \cdot \Delta d_\infty) - (\Delta v, \nabla d_\infty \cdot \Delta d_\infty)| \\
&\leq \|\Delta v\|\|\nabla(d - d_\infty)\|_{L^\infty}\|\Delta d_\infty\| \\
&\leq \|\Delta v\|\|\nabla(d - d_\infty)\|^{\frac{1}{2}}\|\nabla\Delta(d - d_\infty)\|^{\frac{1}{2}} \\
&\leq \varepsilon\|\Delta v\|^2 + C_\varepsilon\|\nabla(d - d_\infty)\|\|\nabla\Delta(d - d_\infty)\| \\
&\leq \varepsilon\|\Delta v\|^2 + C_\varepsilon\|\nabla(d - d_\infty)\|\|\nabla(\Delta d - f(d))\| \\
&\quad + C_\varepsilon\|\nabla(d - d_\infty)\|\|\nabla(f(d) - f(d_\infty))\| \\
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon\|\nabla(d - d_\infty)\|^2 \\
&\quad + C_\varepsilon\|\nabla(d - d_\infty)\|(\|f'(d)\nabla(d - d_\infty)\| + \|(f'(d) - f'(d_\infty))\nabla d_\infty\|) \\
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon\|\nabla(d - d_\infty)\|^2 \\
&\quad + C_\varepsilon\|\nabla(d - d_\infty)\|(\|f'(d)\|_{L^\infty}\|\nabla(d - d_\infty)\| + \|f'(d) - f'(d_\infty)\|_{L^4}\|\nabla d_\infty\|_{L^4}) \\
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon\|\nabla(d - d_\infty)\|^2 \\
&\quad + C_\varepsilon\|\nabla(d - d_\infty)\|\|f''(\xi)\|_{L^\infty}\|d - d_\infty\|_{L^4}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 \\
&\quad + C_\varepsilon\|\nabla(d-d_\infty)\|^2 + C_\varepsilon\|\nabla(d-d_\infty)\|^{\frac{3}{2}}\|d-d_\infty\|^{\frac{1}{2}} \\
&\leq \varepsilon\|\Delta v\|^2 + \varepsilon\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon\|\nabla(d-d_\infty)\|^2 + C_\varepsilon\|d-d_\infty\|^2. \qquad (3.64)
\end{aligned}
$$

Next,

$$
\begin{aligned}
I_4 + I_6 &= (\Delta v, \nabla\cdot((\Delta d - f(d))\otimes d)) - (\nabla(\Delta d - f(d)), \nabla(d\cdot\nabla v)) \\
&= -(d\cdot\nabla\Delta v, \Delta d - f(d)) + (\Delta d - f(d), \Delta(d\cdot\nabla v)) \\
&= -(d\cdot\nabla\Delta v, \Delta d - f(d)) + \int_Q (\Delta d_i - f_i)\Delta(d_j\nabla_j v_i)\,dx \\
&= -(d\cdot\nabla\Delta v, \Delta d - f(d)) \\
&\quad + \int_Q (\Delta d_i - f_i)(\Delta d_j \nabla_j v_i + 2\nabla_k d_j\nabla_k\nabla_j v_i + d_j\Delta\nabla_j v_i)\,dx \\
&= -(d\cdot\nabla\Delta v, \Delta d - f(d)) + \int_Q (\Delta d_i - f_i)(d_j\nabla_j\Delta v_i)\,dx + (\Delta d - f, \Delta d\cdot\nabla v) \\
&\quad + 2\int_Q (\Delta d_i - f_i)(\nabla_k d_j\nabla_k\nabla_j v_i)\,dx \\
&= (\Delta d - f(d), \Delta d\cdot\nabla v) + 2\int_Q (\Delta d_i - f_i)(\nabla_k d_j\nabla_k\nabla_j v_i)\,dx \\
&= (\Delta d - f(d), \Delta d\cdot\nabla v) + 2\int_Q (\Delta d_i - f_i)\left[(\nabla d_j)_k\nabla_k(\nabla_j v)_i\right]\,dx \\
&= (\Delta d - f(d), \Delta d\cdot\nabla v) + 2(\Delta d - f(d), (\nabla d\cdot\nabla)\cdot\nabla v) \\
&=: \tilde{I}_4 + \tilde{I}_6. \qquad (3.65)
\end{aligned}
$$

$$
\begin{aligned}
|\tilde{I}_4| &\leq \|\Delta d - f(d)\|_{L^4}\|\Delta d\|\|\nabla v\|_{L^4} \\
&\leq C(\|\nabla(\Delta d - f(d))\|^{\frac{1}{2}}\|\Delta d - f(d)\|^{\frac{1}{2}} + \|\Delta d - f(d)\|)(\|\Delta v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}} + \|\nabla v\|) \\
&\leq \varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon(\|\Delta d - f(d)\|^2 + \|\nabla v\|^2). \qquad (3.66)
\end{aligned}
$$

$$
\begin{aligned}
|\tilde{I}_6| &\leq C\|\Delta d - f(d)\|_{L^4}\|\nabla d\|_{L^4}\|D^2 v\| \\
&\leq C(\|\nabla(\Delta d - f(d))\|^{\frac{1}{2}}\|\Delta d - f(d)\|^{\frac{1}{2}} + \|\Delta d - f(d)\|)\|\Delta v\| \\
&\leq \varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon\|\Delta d - f(d)\|^2. \qquad (3.67)
\end{aligned}
$$

As a result,

$$
\begin{aligned}
|I_4 + I_6| &= |\tilde{I}_4 + \tilde{I}_6| \leq |\tilde{I}_4| + |\tilde{I}_6| \\
&\leq 2\varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon(\|\Delta d - f(d)\|^2 + \|\nabla v\|^2). \qquad (3.68)
\end{aligned}
$$

$$
\begin{aligned}
|I_6| &= |(\nabla(\Delta d - f(d)), \nabla(v\cdot\nabla d))| \\
&\leq \|\nabla(\Delta d - f(d))\|(\|\nabla v\|\|\nabla d\|_{L^\infty} + \|v\|_{L^\infty}\|D^2 d\|_{L^2})
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{2}\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon(\|\nabla\Delta d\|\|\nabla d\|\|\nabla v\|^2 + C_\varepsilon\|\Delta v\|\|v\| \\
&\leq \frac{\varepsilon}{2}\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon\|\nabla(\Delta d - f(d))\|\|\nabla v\|^2 + C_\varepsilon\|\nabla v\|^2 \\
&\quad +\varepsilon\|\Delta v\|^2 + C_\varepsilon\|v\|^2 \\
&\leq \varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon\|\nabla v\|^2 + C_\varepsilon\|v\|^2.
\end{aligned}
\tag{3.69}
$$

$$
\begin{aligned}
I_7 &= (\Delta d - f(d), f'(d)d_t) \\
&= -(\Delta d - f(d), f'(d)v\cdot\nabla d) + (\Delta d - f(d), f'(d)d\cdot\nabla v) \\
&\quad +(\Delta d - f(d), f'(d)(\Delta d - f(d))) \\
&=: I_{7a} + I_{7b} + I_{7c}.
\end{aligned}
\tag{3.70}
$$

$$
\begin{aligned}
|I_{7a}| &\leq \|\Delta d - f(d)\|\|f'(d)\|_{L^\infty}\|v\|_{L^4}\|\nabla d\|_{L^4} \\
&\leq C\|\Delta d - f(d)\|^2 + C\|\nabla v\|^2, \\
|I_{7b}| &\leq \|\Delta d - f(d)\|\|f'(d)\|_{L^\infty}\|d\|_{L^\infty}\|\nabla v\| \\
&\leq C\|\Delta d - f(d)\|^2 + C\|\nabla v\|^2, \\
|I_{7c}| &\leq \|\Delta d - f(d)\|^2\|f'(d)\|_{L^\infty} \leq C\|\Delta d - f(d)\|^2.
\end{aligned}
$$

Therefore,

$$
|I_7| \leq |I_{7a}| + |I_{7b}| + |I_{7c}| \leq C\|\Delta d - f(d)\|^2 + C\|\nabla v\|^2. \tag{3.71}
$$

Summing up, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}A(t) + (\nu - 6\varepsilon)\|\Delta v\|^2 + (1 - 5\varepsilon)\|\nabla(\Delta d - f(d)\|^2 \\
\leq\;& C_5(\|\nabla v\|^2 + \|\Delta d - f(d)\|^2) + C_6(\|v\|^2 + \|d - d_\infty\|_{H^1}^2),
\end{aligned}
\tag{3.72}
$$

where C_5, C_6 are positive constants depending on $\varepsilon, \|\nabla v\|, \|d\|_{H^2}, \|d_\infty\|_{H^2}$. Taking

$$
\varepsilon \in \left(0, \frac{1}{12}\min\{1, \nu\}\right),
$$

we have

$$
\frac{1}{2}\frac{d}{dt}A(t) + \frac{\nu}{2}\|\Delta v\|^2 + \frac{1}{2}\|\nabla(\Delta d - f(d)\|^2 \leq C_5 A(t) + C_6(\|v\|^2 + \|d - d_\infty\|_{H^1}^2). \tag{3.73}
$$

Recalling (3.55), multiplying (3.73) by a small positive constant α_1 and adding the resultant to (3.50), we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\Big(y(t) + \frac{\alpha_1}{2}A(t)\Big) + \frac{\alpha_1\nu}{2}\|\Delta v\|^2 + \frac{\alpha_1}{2}\|\nabla(\Delta d - f(d)\|^2 \\
&+ (\nu - \alpha\varepsilon_1 - C_5\alpha_1)\|\nabla v\|^2 + (1 - C_5\alpha_1)\|\Delta d - f(d)\|^2 \\
&+\frac{\alpha}{2}\|\nabla(d - d_\infty)\|^2
\end{aligned}
$$

$$\leq \ C\alpha\|d-d_\infty\|^2 + C_6\alpha_1(\|v\|^2 + \|d-d_\infty\|^2_{H^1}). \tag{3.74}$$

Constants α, ε_1 have been chosen as in (3.54). Then we can take

$$\alpha_1 \in \left(0, \min\left\{\frac{1}{2C_5}, \frac{\nu}{4C_5}\right\}\right). \tag{3.75}$$

Using the estimates (3.57), (3.58), we infer from (3.74) that

$$\begin{aligned} \frac{d}{dt}\left(y(t) + \frac{\alpha_1}{2}A(t)\right) + C_7\left(y(t) + \frac{\alpha_1}{2}A(t)\right) \ &\leq \ C_7 y(t) + C_8(\|v\|^2 + \|d-d_\infty\|^2_{H^1}) \\ &\leq \ C_9(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \geq 0. \end{aligned} \tag{3.76}$$

Again using Gronwall's inequality, we have

$$y(t) + \frac{\alpha_1}{2}A(t) \leq C_{10}(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \geq 0, \tag{3.77}$$

which together with (3.57) implies that (cf. (3.55))

$$\begin{aligned} 0 \leq A(t) \ &\leq \ -\frac{2}{\alpha_1}y(t) + \frac{2C_{10}}{\alpha_1}(1+t)^{-\frac{2\theta}{1-2\theta}} \leq \frac{2C_2}{\alpha_1}\|d-d_\infty\|^2 + \frac{2C_{10}}{\alpha_1}(1+t)^{-\frac{2\theta}{1-2\theta}} \\ &\leq \ C_{11}(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \geq 0. \end{aligned} \tag{3.78}$$

The above estimate yields

$$\|\nabla v(t)\| + \|\Delta d(t) - f(d(t))\| \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \geq 0. \tag{3.79}$$

Recalling (3.25), it follows from (3.79) that

$$\|\Delta d(t) - \Delta d_\infty\| \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \geq 0. \tag{3.80}$$

Summing up, we can deduce the required estimate (1.8) from (3.58), (3.79) and (3.80).

The proof of Theorem 1.1 is complete. □

4 Results in Three Dimensional Case

In this section we prove the corresponding results in 3-D case, namely, Theorem 1.2 and Theorem 1.3. When the space dimension is three, we can still set $\lambda = \gamma = 1$ for the sake of simplicity (cf. [18,26]). However, to prove Theorem 1.2, viscosity ν plays an essential role, which can not be neglected. The largeness of ν is needed to guarantee the existence of the global solution.

The following property is useful to understand the asymptotic behavior of the solutions to problem (1.1)–(1.5).

Theorem 4.1. *For any $R > 0$, whenever*

$$\|\nabla v\|^2(0) + \|\Delta d - f(d)\|^2(0) \le R,$$

there exists a constant $\varepsilon_0 \in (0,1)$, depending on ν, f, Q and R, such that either

(1) problem (1.1)–(1.5) *has a unique global classical solution (v, d) with uniform estimate*

$$\|v(t)\|_{H^1(Q)} + \|d(t)\|_{H^2(Q)} \le C, \quad \forall\, t \ge 0, \tag{4.1}$$

or

(2) there is a $T_* \in (0, \infty)$ such that

$$\mathcal{E}(T_*) \le \mathcal{E}(0) - \varepsilon_0,$$

where

$$\mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + \frac{1}{2}\|\nabla d(t)\|^2 + \int_Q F(d(t))\, dx.$$

Proof. Suppose that (v, d) is a weak solution of problem (1.1)–(1.5). First, we can see from the basic energy law that

$$\mathcal{E}(t) \le \mathcal{E}(0), \quad \forall\, t \ge 0, \tag{4.2}$$

which implies

$$\|v(t)\| + \|d(t)\|_{H^1} \le C, \quad \forall\, t \ge 0. \tag{4.3}$$

By a direct calculation, we have (cf. (3.59), (3.65))

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}A(t) + (\nu\|\Delta v\|^2 + \|\nabla(\Delta d - f(d)\|^2) \\
= \; & (\Delta v, v \cdot \nabla v) + (\Delta v, \nabla \cdot (\nabla d \odot \nabla d)) + (\nabla(\Delta d - f(d)), \nabla(v \cdot \nabla d)) \\
& +(\Delta d - f(d), \Delta d \cdot \nabla v) + 2(\Delta d - f(d), (\nabla d \cdot \nabla) \cdot \nabla v) \\
& +(\Delta d - f(d), f'(d) d_t).
\end{aligned} \tag{4.4}$$

We remark that the above (formal) calculation is valid for the classical solutions, but it can be justified by a proper approximating procedure (cf. [18, 26]). We now estimate the right-hand side term by term. Since we only know the uniform estimate of $\|v\|$ and $\|d\|_{H^1}$, the estimates we get are different from those in the previous section.

$$\begin{aligned}
|(\Delta v, v \cdot \nabla v)| & \le \|\Delta v\| \|v\|_{L^\infty} \|\nabla v\| \le C\|\Delta v\|(\|\Delta v\|^{\frac{3}{4}} + 1)\|\nabla v\| \\
& \le \frac{\nu}{8}\|\Delta v\|^2 + \frac{C}{\nu^7}\|\nabla v\|^8.
\end{aligned} \tag{4.5}$$

Similar to (3.61), in the 3-D case,

$$\|\nabla \Delta d\| \le \|\nabla(\Delta d - f(d))\| + \|f'(d)\|_{L^3}\|\nabla d\|_{L^6} \le \|\nabla(\Delta d - f(d))\| + C(\|\Delta d\| + 1)$$

$$\leq \|\nabla(\Delta d - f(d))\| + C\|\Delta d - f(d)\| + C. \tag{4.6}$$

As a result,

$$\begin{aligned}
&|(\Delta v, \nabla\cdot(\nabla d \odot \nabla d))| \\
= \;& (\Delta v, \Delta d \nabla d) \leq \|\Delta d\|\|\Delta v\|\|\nabla d\|_{L^\infty} \\
\leq \;& C\|\Delta d\|\|\Delta v\|(\|\nabla \Delta d\|^{\frac{3}{4}} + 1) \\
\leq \;& C|\Delta d\|\|\Delta v\|(\|\nabla(\Delta d - f(d))\|^{\frac{3}{4}} + C\|\Delta d - f(d)\|^{\frac{3}{4}} + C) \\
\leq \;& \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^8 + 1).
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&|(\nabla(\Delta d - f(d)), \nabla(v\cdot\nabla d))| \\
\leq \;& \|\nabla(\Delta d - f(d))\|(\|\nabla v\|\|\nabla d\|_{L^\infty} + \|v\|_{L^\infty}\|D^2 d\|) \\
\leq \;& C\|\nabla(\Delta d - f(d))\|[\|\nabla v\|(\|\nabla\Delta d\|^{\frac{3}{4}} + 1) + C(\|\Delta v\|^{\frac{3}{4}} + 1)(\|\Delta d\| + 1)] \\
\leq \;& \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C(\|\nabla v\|^8 + \|\Delta d - f(d)\|^8 + 1).
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&|(\Delta d - f(d), \Delta d\cdot\nabla v)| \\
\leq \;& \|\Delta d - f(d)\|_{L^3}\|\Delta d\|\|\nabla v\|_{L^6} \\
\leq \;& C(\|\nabla(\Delta d - f(d))\|^{\frac{1}{2}}\|\Delta d - f(d)\|^{\frac{1}{2}} + \|\Delta d - f(d)\|) \\
& \times(\|\Delta d - f(d)\| + 1)(\|\Delta v\| + \|\nabla v\|) \\
\leq \;& \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^6 + \|\nabla v\|^2 + 1).
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&2|(\Delta d - f(d), (\nabla d\cdot\nabla)\cdot\nabla v)| \\
\leq \;& C\|\Delta d - f\|_{L^3}\|\nabla d\|_{L^6}\|v\|_{H^2} \\
\leq \;& C(\|\nabla(\Delta d - f(d))\|^{\frac{1}{2}}\|\Delta d - f(d)\|^{\frac{1}{2}} + \|\Delta d - f(d)\|)(\|\Delta d\| + 1)(\|\Delta v\| + 1) \\
\leq \;& \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C\|\Delta d - f(d)\|^6 + C.
\end{aligned} \tag{4.10}$$

As before, the last term on the right-hand side can be expressed into three terms

$$\begin{aligned}
(\Delta d - f(d), f'(d)d_t) \;=\; & -(\Delta d - f(d), f'(d)v\cdot\nabla d) + (\Delta d - f(d), f'(d)d\cdot\nabla v) \\
& +(\Delta d - f(d), f'(d)(\Delta d - f(d))).
\end{aligned} \tag{4.11}$$

Then we have

$$\begin{aligned}
|(\Delta d - f(d), f'(d)v\cdot\nabla d)| \;\leq\; & \|\Delta d - f(d)\|\|f'(d)\|_{L^3}\|\nabla d\|_{L^6}\|v\|_{L^\infty} \\
\leq\; & C\|\Delta d - f(d)\|(\|\Delta d\| + 1)(\|\Delta v\|^{\frac{3}{4}} + 1) \\
\leq\; & \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^{\frac{16}{5}} + 1),
\end{aligned} \tag{4.12}$$

$$|(\Delta d - f(d), f'(d)d\cdot\nabla v)| \;\leq\; \|\Delta d - f(d)\|\|f'(d)\|_{L^3}\|d\|_{L^\infty}\|\nabla v\|_{L^6}$$

$$
\begin{aligned}
&\le C\|\Delta d - f(d)\|(\|\Delta d\|+1)(\|\Delta v\|+1)\\
&\le \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^4+1),
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
& |(\Delta d - f(d), f'(d)(\Delta d - f(d)))|\\
=\ & \|f'(d)\|_{L^3}\|\Delta d - f(d)\|_{L^3}^2\\
\le\ & C(\|\nabla(\Delta d - f(d))\|^{\frac12}\|\Delta d - f(d)\|^{\frac12} + \|\Delta d - f(d)\|)^2\\
\le\ & \frac18\|\nabla(\Delta d - f(d))\|^2 + C\|\Delta d - f(d)\|^2.
\end{aligned} \tag{4.14}
$$

Summing up, we can conclude that

$$
\frac{d}{dt}A(t) \le C_*(A(t)^4+1), \tag{4.15}
$$

where C_* is a constant that only depends on ν, f, Q, $\|v_0\|$ and $\|d_0\|_{H^1}$.

If the initial data satisfy

$$
A(0) = \|\nabla v_0\|^2 + \|\Delta d_0 - f(d_0)\|^2 \le R,
$$

we consider the following initial value problem for a nonlinear ordinary differential equation:

$$
\frac{d}{dt}Y(t) = C_*(Y(t)^4+1), \quad Y(0) = A(0) \le R.
$$

We denote by $I = [0, T_{max})$ the maximal existence interval of $Y(t)$ such that

$$
\lim_{t\to T_{max}^-} Y(t) = \infty.
$$

On the other hand, it is easy to see that for any $t \in I$, $0 \le A(t) \le Y(t)$. Consequently, $A(t)$ exists on I. Moreover, T_{max} is determined by $Y(0)$ and C_* such that $T_{max} = T_{max}(Y(0), C_*)$ is increasing when $Y(0) \ge 0$ is decreasing. We can take $t_0 = \frac12 T_{max}(R, C_*) > 0$. Then it follows that $Y(t)$ as well as $A(t)$ is uniformly bounded on $[0, t_0]$. This fact together with the argument in [26] and Lemma 2.2 implies the local existence of a unique (classical) solution of problem (1.1)–(1.5) at least on $[0, t_0]$.

If (2) is not true, we have

$$
\mathcal{E}(t) \ge \mathcal{E}(0) - \varepsilon_0, \quad \forall\, t \ge 0.
$$

From the basic energy law, we infer that

$$
\int_0^\infty \int_Q \left(\nu|\nabla v(t)|^2 + |\Delta d(t) - f(d(t))|^2\right)\, dxdt \le \varepsilon_0, \quad \forall\, t \ge 0.
$$

Hence, there exists a $t_* \in [\frac{t_0}{2}, t_0]$ such that

$$
\nu\|\nabla v(t_*)\|^2 + \|\Delta d(t_*) - f(d(t_*))\|^2 \le \frac{2\varepsilon_0}{t_0}.
$$

Choosing ε_0 such that

$$\frac{2}{\min\{1,\nu\}}\frac{\varepsilon_0}{t_0} \le R,$$

we have $A(t_*) \le R$. Taking t_* as the initial time, we infer from the above argument that $A(t)$ is uniformly bounded at least on $[0, \frac{3t_0}{2}] \subset [0, t_* + t_0]$. Moreover, the bound only depends on R, C_* but not on the length of existence interval. We can extend the local unique classical solution step by step to infinity such that

$$A(t) \le C, \quad \forall\, t \ge 0, \tag{4.16}$$

where C is uniform in time. The proof is complete. $\square$

Remark 4.1. *Theorem 4.1 implies that if the energy $\mathcal{E}$ does not 'drop' too fast, problem* (1.1)-(1.5) *admits a global unique classical solution. This assumption can be verified for certain special cases, which are stated in the following corollaries.*

Corollary 4.1. *Let $d^* \in H^2_p(Q)$ be an absolute minimizer of the functional*

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_Q F(d)dx$$

in the sense that $E(d^*) \le E(d)$ for all $d \in H^1_p(Q)$. There exists a constant $\sigma \in (0,1]$ that may depend on ν, f, Q and d_* such that for initial data $(v_0, d_0) \in V \times H^2_p(Q)$ satisfying

$$\|v_0\|_{H^1} + \|d_0 - d_*\|_{H^2} \le \sigma,$$

problem (1.1)–(1.5) *admits a unique global classical solution.*

Proof. Without loss of generality, we shall assume that $\sigma \le 1$. From assumption

$$||v_0||_{H^1} + ||d_0 - d^*||_{H^2} \le \sigma \le 1,$$

we infer that

$$\begin{aligned}
& \nu\|v_0\|^2_{H^1} + \|\Delta d_0 - f(d_0)\|^2 \\
\le\ & \nu\|v_0\|^2_{H^1} + 2\|\Delta d_0 - \Delta d^*\|^2 + 2\|f(d_0) - f(d^*)\|^2 \\
\le\ & K_1(\|v_0\|_{H^1} + \|d_0 - d^*\|_{H^2})^2 \\
\le\ & K_1.
\end{aligned} \tag{4.17}$$

In addition, since d^* is the absolute minimizer of $E(d)$, we have

$$\begin{aligned}
\mathcal{E}(0) - \mathcal{E}(t) &\le \mathcal{E}(0) - E(d(t)) \le \mathcal{E}(0) - E(d^*) \\
&\le \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\left(\|\nabla d_0\|^2 - \|\nabla d_*\|^2\right) + \int_Q F(d_0) - F(d_*)dx \\
&\le \frac{1}{2}\sigma^2 + C\sigma
\end{aligned}$$

$$\leq \ K_2\sigma.$$

Here K_1 and K_2 are positive constants that only depend on d_*, ν, f (not on σ).

Take $R = K_1$, $\varepsilon_0 = K_2\sigma$ respectively. We can apply Theorem 4.1 by choosing

$$\sigma = \min\left\{1,\ \frac{K_1}{4K_2}T_{max}(K_1, C_*)\min\{1,\nu\}\right\}. \tag{4.18}$$

The proof is complete. □

Corollary 4.1 implies that if the initial velocity v_0 is small in H^1 and initial director d_0 is properly close to the absolute minimizer d_* of functional $E(d)$ in H^2, problem (1.1)–(1.5) admits a unique global classical solution. However, from the proof of Theorem 4.1 we can somewhat relax the ßmallnessrequirement from $H^1 \times H^2$ to $L^2 \times H^1$.

Corollary 4.2. *Let $d^* \in H^2_p(Q)$ be an absolute minimizer of the functional*

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_Q F(d)dx$$

in the sense that $E(d^*) \leq E(d)$ for all $d \in H^1_p(Q)$. For any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, there exists a constant $\sigma \in (0,1]$, which depends on ν, f, Q, $\|v_0\|_{H^1}$ and $\|d_0\|_{H^2}$ such that if

$$\|v_0\| + \|d_0 - d_*\|_{H^1} \leq \sigma,$$

problem (1.1)–(1.5) *admits a unique global classical solution.*

Proof. Without loss of generality, we assume that $\sigma \leq 1$. Set

$$K_1 := \nu\|\nabla v_0\|^2 + \|\Delta d_0 - f(d_0)\|^2 < \infty. \tag{4.19}$$

Moreover, we have

$$\mathcal{E}(0) - \mathcal{E}(t) \leq K_2\sigma,$$

where K_2 is a positive constant that only depend on d_*, ν, f (not on σ). As in the proof of Corollary 4.1, we take $R = K_1$, $\varepsilon_0 = K_2\sigma$ and choose

$$\sigma = \min\left\{1,\ \frac{K_1}{4K_2}T_{max}(K_1, C_*)\min\{1,\nu\}\right\}. \tag{4.20}$$

The conclusion follows from Theorem 4.1. Here, we note that now σ depends on the norm of $\|v_0\|_{H^1}$ and $\|d_0\|_{H^2}$ while in Corollary 4.1, σ only depends on d_*. □

In what follows, we proceed to prove the conclusions in Theorem 1.2 and Theorem 1.3. First, we have the following result for both cases.

Proposition 4.1. *In three space dimension case, for the unique classical solution* (v,d) *obtained in Proposition 2.1, Corollary 4.1 and Corollary 4.2, it holds*

$$\lim_{t\to+\infty} A(t) = 0. \tag{4.21}$$

Proof. (1) For the large viscosity case (cf. Proposition 2.1), after refining the argument in [26], we indeed have the following differential inequality (the detailed calculation is left to interested readers):

Lemma 4.1. *We consider 3-D case. Set* $\tilde{A}(t) = A(t) + 1$. *For arbitrary* $\nu_0 > 0$, *if* $\nu \geq \nu_0 > 0$, *then the following inequality holds for the classical solution* (v,d) *to problem* (1.1)–(1.5)*:*

$$\frac{d}{dt}\tilde{A}(t) \leq -\left(\nu - M_1\nu^{\frac{1}{2}}\tilde{A}(t)\right)\|\Delta v\|^2 - \left(1 - \frac{M_1}{\nu^{\frac{1}{4}}}\tilde{A}(t)\right)\|\nabla(\Delta d - f(d))\|^2 + M_2\tilde{A}(t), \tag{4.22}$$

where M_1, M_2 *are constants depending on* $f, |Q|, \|v_0\|, \|d_0\|_{H^1}$, M_2 *may also depend on* ν_0.

Based on Lemma 4.1, we can show the uniform estimate for $A(t)$ by using the idea in [18]. It follows from the basic energy law that

$$\int_t^{t+1} \tilde{A}(\tau)d\tau \leq \int_t^{t+1} A(\tau)d\tau + 1 \leq \tilde{M}, \quad \forall\ t \geq 0, \tag{4.23}$$

where $\tilde{M} > 0$ is a constant depending only on $\|v_0\|, \|d_0\|_{H^1}$. Take ν large enough satisfying

$$\nu^{\frac{1}{4}} \geq M_1(\tilde{A}(0) + M_2\tilde{M} + 4\tilde{M}) + 1. \tag{4.24}$$

Then by (4.22), there must be some $T_0 > 0$ such that

$$\nu - M_1\nu^{\frac{1}{2}}\tilde{A}(t) \geq 0, \quad 1 - \frac{M_1\tilde{A}(t)}{\nu^{\frac{1}{4}}} \geq 0,$$

for all $t \in [0, T_0]$. Moreover, on $[0, T_0]$,

$$\frac{d}{dt}\tilde{A}(t) \leq M_2\tilde{A}(t). \tag{4.25}$$

Denote $T_* = \sup T_0$. First we show that $T_* \geq 1$ by a contradiction argument.
If $T_* < 1$, then

$$\tilde{A}(T_*) \leq \tilde{A}(0) + M_2\int_0^1 \tilde{A}(t)dt \leq \tilde{A}(0) + M_2\tilde{M}.$$

On the other hand, from the definition of T_*, we have

$$\nu < \max\{M_1^2\tilde{A}^2(T_*), M_1^4\tilde{A}^4(T_*)\},$$

which contradicts (4.24).

Next, if $T_* < +\infty$, (4.23) implies that there is a $t_1 \in [T_* - \frac{1}{2}, T_*]$ such that

$$\tilde{A}(t_1) \le 4\tilde{M}. \tag{4.26}$$

As a result,

$$\tilde{A}(T_*) \le 4\tilde{M} + M_2 \int_{t_1}^{T_*} \tilde{A}(t)dt \le 4\tilde{M} + M_2\tilde{M}. \tag{4.27}$$

From the definition of T_*, we have

$$\nu < \max\{M_1^2 \tilde{A}^2(T_*), M_1^4 \tilde{A}^4(T_*)\},$$

which together with (4.27) also yields a contradiction with (4.24). Hence, we have the uniform estimate

$$\tilde{A}(t) \le \frac{\nu^{\frac{1}{2}}}{M_1}, \quad \forall\, t \ge 0, \tag{4.28}$$

which implies that

$$A(t) \le C, \quad \forall\, t \ge 0, \tag{4.29}$$

where C is a constant depending on $f, |Q|, \|v_0\|_{H^1}, \|d_0\|_{H^2}$. Thus, we infer from (4.22) that

$$\frac{d}{dt}A(t) = \frac{d}{dt}\tilde{A}(t) \le M_2\tilde{A}(t) \le C.$$

Due to the fact that $A(t) \in L^1(0,\infty)$ (cf. (3.5)), we can conclude that (4.21) holds.

(2) Now we consider the near equilibrium case. When the assumptions in Theorem 4.1 (or Corollary 4.1 / Corollary 4.2) are satisfied, we have (4.16) holds for all $t \ge 0$. Thus (4.15) implies

$$\frac{d}{dt}A(t) \le C_*(A^4(t) + 1) \le C, \quad \forall\, t \ge 0.$$

By the same argument as in (1), we obtain (4.21).

The proof is complete. $\square$

After previous preparations, we can proceed to prove the results in Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2 and Theorem 1.3. Based on Proposition 4.1, for both large viscosity case and near equilibrium case, one can argue exactly as in Section 3.1 to conclude that

$$\lim_{t\to+\infty} (\|v\|_{H^1} + \|d - d_\infty\|_{H^2}) = 0. \tag{4.30}$$

Notice that we now have uniform bounds for $\|v\|_{H^1}$ and $\|d\|_{H^2}$. Then we are able to show the estimate on convergence rate (1.8) for both cases. To this end, we can check the argument for 2-D case step by step. By applying corresponding Sobolev embedding theorems in 3-D, we can see that all calculations in Section 3.2 are valid with minor modifications. Hence, the details are omitted here. We complete the proofs for Theorem 1.2 and Theorem 1.3. $\square$

5 Remark on the Flow with Non-vanishing Average Velocity

We briefly discuss the flows with non-vanishing average velocity. Due to the periodic boundary condition (1.4), by integration of (1.1) over Q, we get

$$\frac{d}{dt}\left(\frac{1}{|Q|}\int_Q v(t)dx\right) = 0, \tag{5.1}$$

which implies

$$m_v := \frac{1}{|Q|}\int_Q v(t)dx \equiv \frac{1}{|Q|}\int_Q v_0 dx, \quad \forall\, t \geq 0, \tag{5.2}$$

where $|Q|$ is the measure of Q.

Our main results (Theorems 1.1–1.3) in this paper are valid for the flow with vanishing average velocity (see the definition of function space V), namely, $m_v = 0$. In that case, we can apply the Poincaré inequality to $v \in V$ such that $\|v\| \leq C\|\nabla v\|$. This enables us to show that under the dissipations of system (1.1)–(1.5), the velocity of the flows will tend to zero and the director d will converge to a steady state.

When the non-vanishing average flow v is considered, as for the single Navier–Stokes equation (cf. [27]), we set

$$v = \tilde{v} + m_v. \tag{5.3}$$

Then we transform problem (1.1)–(1.5) into the following system for variables $\tilde{v}$ and d:

$$\begin{aligned} \tilde{v}_t + \tilde{v}\cdot\nabla\tilde{v} - \nu\Delta\tilde{v} + m_v\cdot\nabla\tilde{v} + \nabla P &= -\lambda\nabla\cdot[\nabla d \odot \nabla d + (\Delta d - f(d))\otimes d], & (5.4)\\ \nabla\cdot\tilde{v} &= 0, & (5.5)\\ d_t + \tilde{v}\cdot\nabla d + m_v\cdot\nabla d - d\cdot\nabla\tilde{v} &= \gamma(\Delta d - f(d)), & (5.6) \end{aligned}$$

subject to the corresponding periodic boundary conditions and initial conditions

$$\begin{aligned} &\tilde{v}(x+e_i) = \tilde{v}(x), \quad d(x+e_i) = d(x), \qquad \text{for } x \in \partial Q, & (5.7)\\ &\tilde{v}|_{t=0} = \tilde{v}_0(x) = v_0(x) - m_v, \quad \text{with } \nabla\cdot v_0 = 0, \quad d|_{t=0} = d_0(x), \quad \text{for } x \in Q. & (5.8) \end{aligned}$$

Introduce the new energy functional

$$\tilde{\mathcal{E}}(t) = \frac{1}{2}\|\tilde{v}\|^2 + \frac{\lambda}{2}\|\nabla d\|^2 + \lambda\int_Q F(d)dx. \tag{5.9}$$

It is not difficult to check that system (5.4)–(5.8) still enjoys the *basic energy law*

$$\frac{d}{dt}\tilde{\mathcal{E}}(t) = -\nu\|\nabla\tilde{v}\|^2 - \lambda\gamma\|\Delta d - f(d)\|^2, \quad t \geq 0. \tag{5.10}$$

By a similar argument, we can still prove the global existence of classical solution $(\tilde{v}, d)$ to problem (5.4)–(5.8) under the same assumptions as in Proposition 2.1, Theorem 4.1, Corollary 4.1 and Corollary 4.2. Moreover, we can prove the same higher-order energy inequalities like Lemma 3.1, Lemma 4.1 and (4.15) for $(\tilde{v}, d)$.

As far as the long-time behavior of the global solution is concerned, following a similar arguments in previous sections, we can conclude that

$$\lim_{t\to+\infty}(\|\tilde{v}(t)\|_{H^1} + \|\Delta d(t) - f(d(t))\|) = 0. \tag{5.11}$$

Recalling (5.3), we infer from (5.11) that

$$\lim_{t\to+\infty}\|v(t) - m_v\|_{H^1} = 0. \tag{5.12}$$

However, in general we cannot conclude anything about the convergence of d like in Theorems 1.1–1.3. (5.11) implies that the 'limit' of d, which is denoted by $\hat{d}$, will satisfy $\Delta\hat{d} - f(\hat{d}) = 0$ with corresponding periodic boundary condition. Let us look at the 'limiting' case such that $v = \hat{v} = m_v$ and $d = \hat{d}$. It follows from (1.5) that

$$\frac{D}{Dt}\hat{d} = \hat{d}_t + \hat{v}\cdot\nabla\hat{d} = 0. \tag{5.13}$$

Consequently, $\hat{d}$ is purely transported and it $(i.e., \hat{d}(x(X,t),t))$ remains unchanged when the molecule moves through a flow field with velocity m_v. However, the local rate of change $\hat{d}_t$ may not be zero, since the convective rate of change may not vanish. Hence, in the Eulerian coordinates, or in Q, $\hat{d}(x,t)$ may change in time. As a result, there might be no steady state for the director. Obviously, this is different from the situation in the previous sections, where all the three rates of change are vanishing in the limiting case. We can look at a simple example. In the case of periodic boundary condition ($Q = \Pi_{i=1}^2(0,1)$), let $\hat{v} = (1,0)^T$ and $\hat{d}(x,0) = \hat{d}_0(x)$ for $x \in Q$. We can see that in the Eulerian coordinates, $\hat{d}(x,t)$ $(x \in Q)$ is a periodic function in time such that for $t \geq 0$, $\hat{d}(x,t) = \hat{d}(x,t+1)$.

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