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Damage of nonlinearly elastic materials at small strain – Existence and regularity results –

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Abstract

In this paper an existence result for energetic solutions of rate-independent damage processes is established and the temporal regularity of the solution is discussed. We consider a body consisting of a physically nonlinearly elastic material undergoing small deformations and partial damage. The present work is a generalization of [MiR06] concerning the properties of the stored elastic energy density as well as the suitable Sobolev space for the damage variable: While previous work assumes that the damage variable z satisfies $z \in W^{1,r}(\Omega)$ with $r > d$ for $\Omega \subset \mathbb{R}^d$, we can handle the case $r > 1$ by a new technique for the construction of joint recovery sequences. Moreover, this work generalizes the temporal regularity results to physically nonlinearly elastic materials by analyzing Lipschitz- and Hölder-continuity of solutions with respect to time.

1 Introduction

Damage describes the creation and growth of cracks and voids on the micro-level of a solid material. This process can be investigated by means of continuum damage mechanics, which goes back on Kachanov in 1958. Within this approach, an inner variable, the damage variable, is incorporated to the constitutive law, where it describes the influence of damage on the elastic behavior of the material. In this paper we treat the case of isotropic damage, which presumes a uniform orientation distribution of the cracks and voids in the material. Hence the damage variable is a scalar-valued function of time and space $z : [0,T] \times \Omega \to [0,1]$, where $z(t,x)$ is defined as the volume fraction at time t of the undamaged material in a neighborhood of a material point x in the reference configuration $\Omega \subset \mathbb{R}^d$. Thus, the values of the function z range between 0 and 1, where $z(t,x) = 1$ means no damage and $z(t,x) = 0$ stands for maximal damage in the neighborhood of the point $x \in \Omega$ at time t. We consider damage as a unidirectional process, so that $\partial_t z(t,x) \leq 0$ for a.e. $(t,x) \in [0,T] \times \Omega$. In the sense of [MiR06, MRZ07] we treat partial damage only, which means that $z = 0$ does not mean that all material is disintegrated. Instead we have in mind that the material consists of two constituents, like a matrix and fibers, where only one of them may experience damage. Thus, for $z = 0$ the material is still able to support arbitrary stresses without further damage.

The model that is analyzed in this paper is based on one proposed by Frémond and Nedjar to describe the damage of concrete, see [Fré02] chap. 12. It consists of a functional representing the free energy of the body and a dissipation potential accounting for the energy dissipated by the damage process. However, we restrict our analysis to the rate-independent case and neglect viscous effects.

The free energy depends on time $t \in [0,T]$, the damage variable $z \in [0,1]$ and – in the small strain case – on the linearized Green-St. Venant strain tensor $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$, where $u : \Omega \to \mathbb{R}^d$ is the displacement field. The free energy is defined via three different energy terms:

$$\mathcal{E}(t,u,z) := \int_\Omega W(x,e(u),z)\,\mathrm{d}x + \frac{\kappa}{r}\int_\Omega |\nabla z|^r\,\mathrm{d}x - \langle l(t),u\rangle\,, \tag{1.1}$$

where the first term in (1.1) denotes the stored elastic energy, which is determined by the stored elastic energy density $W : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [0,1] \to \mathbb{R}_\infty$. The properties of W will be specified in Subsection 3.1 more precisely. The second term in (1.1) involves the gradient of damage and takes into account microscopic interactions, i.e. it considers the influence of damage in a point x on its neighborhood. Furthermore, $\kappa > 0$ denotes the so-called factor of influence of damage. The third term in formula (1.1) represents the work of external loadings, which may comprise both volume and surface forces.

The dissipation potential is considered to be of the following form:

$$\mathcal{R}(\dot z) := \int_\Omega R(x,\dot z)\,\mathrm{d}x\,,\quad \text{where } R(x,v) := \begin{cases} \varrho(x)|v| & \text{if } v \in (-\infty,0] \\ \infty & \text{if } v > 0 \end{cases} \tag{1.2}$$

with $\varrho \in L^\infty(\Omega)$ satisfying $0 < \varrho_0 \le \varrho(x)$ for a.e. $x \in \Omega$.

This definition of the dissipation potential accounts for the unidirectionality of the damage process: Only those damage variables, that describe an increase of damage, lead to finite dissipation. Moreover, the dissipation potential defined via (1.2) is rate-independent, since it is homogeneous of degree one, i.e.: $R(x,\alpha w) = \alpha R(x,w)$ for every $\alpha > 0$ and every $w \in \mathbb{R}$. Hence, the dissipation potential generates a so-called dissipation distance:

$$\mathcal{D}(z_0,z_1) = \mathcal{R}(z_1 - z_0)\,. \tag{1.3}$$

Specifying a suitable state space $\mathcal{Q}$, the triple $(\mathcal{Q},\mathcal{E},\mathcal{D})$ is called a rate-independent system and our aim is to construct energetic solutions $(u,z) : [0,T] \to \mathcal{Q}$. They are defined by satisfying the global energy balance (E) and the global stability (S) of Definition 2.1. These main tools for the energetic approach are explained in Section 2. Section 3 provides the assumptions that are made on the setting of the damage process throughout this paper and it contains the existence result and its proof. In Section 4 the temporal regularity of energetic solutions is analyzed under the assumption of additional convexity properties on the free energy. Finally, Section 5 discusses classes of free energies known in engineering, which fit into the framework of our setting.

One main difference to previous works on the existence analysis of rate-independent processes [FrM06, MiP07, MPP08] is, that we do not claim a growth property on the stored elastic energy density of the form $c_1|e|^p - C \le W(x,e,z) \le c_2|e|^p + \tilde C$ for constants $c_1, c_2, C, \tilde C > 0$ and $1 < p < \infty$, which would lead under the assumption of convexity to a growth condition on the stresses of the form

(H4*) $|\partial_e W(x,e,z)| \le c(|e|^{p-1} + \tilde c)$ for constants $c, \tilde c > 0$.

This condition is not applicable for our purposes, since we want to allow for stored elastic energy densities used in literature [Ser93] to describe strain hardening:

$$W(e,z) := \frac{1+z}{4}(\operatorname{tr} e)^2 + |e^D|^{\tilde{p}}, \tag{1.4}$$

for a constant $1 < \tilde{p} < \infty$ and the deviator $e^D := e - \frac{\operatorname{tr} e}{d}\mathrm{Id}$. In Section 5 it is demonstrated that W satisfies $c_1|e|^2 - C \leq W(e,z)$, but (H4*) is not fulfilled for the exponent 2. Therefore we use the alternative stress control:

(H4) $|\partial_e W(x,e,z)| \leq c(W(x,e,z)+\tilde{c})$ for constants $c, \tilde{c} > 0$.

The main challenge of analyzing the damage problem lies in the discontinuity of the dissipation distance $\mathcal{D}$ arising from the unidirectionality of the damage process. Compared to [FrM06, MiP07, MPP08], where the dissipation distance was assumed to be (weakly) continuous, another method is required for proving the stability of limit states, see (C2) in the abstract existence theorem 2.4 and Section 3.2.5. The possibly infinite valued dissipation distance does not allow to pass to the limit along a stable sequence in stability condition (S), see Definitions 2.1 and 2.2 as well as Section 3.2.5. To overcome this problem the so-called joint recovery condition was introduced in [MiR06, MRS08], see here Section 3.2.5. It is based on the construction of a recovery sequence, recovering the stability inequality for the limit jointly in all variables. Applying this method, the existence of an energetic solution of $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ defined by (1.1), (1.3) was proven for $r > d$ using the continuous embedding of $W^{1,r}(\Omega) \subset C(\bar{\Omega})$. In this work we provide a more delicate construction for the joint recovery sequences that allows to handle weak convergence in $W^{1,r}(\Omega)$ with $r \in (1,\infty)$.

In Section 4, following the ideas in [MiT04] we prove the temporal continuity of energetic solutions under the assumption of uniform convexity of the free energy on sublevels. In [MiT04] it was proven, that an energetic solution is Lipschitz-continuous with respect to time, if the free energy satisfies a uniform convexity inequality of the form

$$\mathcal{E}(t, \theta q_1 + (1-\theta) q_2) \leq \theta \mathcal{E}(t, q_1) + (1-\theta q_2)\mathcal{E}(t, q_2) - c\theta(1-\theta)\|q_1 - q_2\|^\alpha . \tag{1.5}$$

with $\alpha = 2$, which must hold for all $q_1, q_2 \in \mathcal{Q}$. We only claim that (1.5) holds on sublevels of $\mathcal{E}(t,\cdot)$, i.e. c depends on the sublevel. We allow for $\alpha \geq 2$ and prove Hölder-continuity of the energetic solution with respect to time. In Section 5 we demonstrate for an example that $\alpha = 2$ is restricted to free energies $\mathcal{E}$ of (sub-)quadratic growth with respect to the state q. Free energies being of super-quadratic growth with respect to one of the state components may satisfy more general uniform convexity inequalities with $\alpha > 2$.

2 Energetic formulation

We analyze the damage problem within its energetic formulation. For this, we fix a state space $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$, which is assumed to be a weakly closed subset of a reflexive Banach space. Our approach is solely based on the free energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$

and the dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_\infty$. We search for an energetic solution $q : [0,T] \to \mathcal{Q}$, which is supposed to satisfy the global stability condition (S) and the global energy balance (E).

Definition 2.1 (Energetic solution) *A function $q = (u, z) : [0,T] \to \mathcal{Q}$ is called an energetic solution for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, if $t \mapsto \partial_t \mathcal{E}(t,q) \in L^1((0,T))$ and if for all $t \in [0,T]$ we have $\mathcal{E}(t, q(t)) < \infty$, stability* (S) *and energy balance* (E)*:*

$$\text{for all } \tilde{q} = (\tilde{u}, \tilde{z}) \in \mathcal{Q} \text{ holds}: \quad \mathcal{E}(t, q(t)) \le \mathcal{E}(t, \tilde{q}) + \mathcal{D}(z(t), \tilde{z})\,; \tag{S}$$

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [s,t]) = \mathcal{E}(s, q(s)) + \textstyle\int_s^t \partial_t \mathcal{E}(\xi, q(\xi)) \mathrm{d}\xi\,, \tag{E}$$

where $\mathrm{Diss}_{\mathcal{D}}(z, [s,t]) := \sup_{\substack{\text{all part. of } [s,t] \\ M \in \mathbb{N}}} \sum_{j=1}^{M} \mathcal{D}(z(\xi_{j-1}), z(\xi_j))$.

Stability inequality (S) suggests to introduce sets of stable states.

Definition 2.2 (Set of stable states, stable sequence) *The set of stable states at time $t \in [0,T]$ is defined by:*

$$\mathcal{S}(t) := \{ q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) < \infty, \forall \tilde{q} \in \mathcal{Q} : \mathcal{E}(t,q) \le \mathcal{E}(t,\tilde{q}) + \mathcal{D}(z,\tilde{z}) \}\,.$$

A sequence $(t_k, q_k)_{k \in \mathbb{N}} \subset [0,T] \times \mathcal{Q}$ is called a stable sequence if (i) and (ii) hold:

(i) $\sup_{k \in \mathbb{N}} \{ \mathcal{E}(t_k, q_k) \} < \infty$, *i.e. there is a constant $E \in \mathbb{R}$ such that*

$$q_k \in L_E(t_k) := \{ q \in \mathcal{Q} \,|\, \mathcal{E}(t_k, q) \le E \}\,, \tag{2.1}$$

(ii) $q_k \in \mathcal{S}(t_k)$ *for every $k \in \mathbb{N}$.*

In order to guarantee the existence of an energetic solution, certain general assumptions have to be made on $\mathcal{E}$ and $\mathcal{D}$, see also [MaM05, MRS08].

The energy $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ has to fulfill the following conditions:

$$\begin{aligned} &\textit{Compactness of energy sublevels:} \quad \forall\, t \in [0,T]\ \forall\, E \in \mathbb{R} : \\ &\qquad L_E(t) := \{ q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) \le E \} \text{ is weakly seq. compact.} \end{aligned} \tag{E1}$$

$$\begin{aligned} &\textit{Uniform control of the power: } \exists\, c_0 \in \mathbb{R}\ \exists\, c_1 > 0\ \forall\, (t_q, q) \in [0,T] \times \mathcal{Q} \text{ with } \mathcal{E}(t_q, q) < \infty : \\ &\qquad \mathcal{E}(\cdot, q) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t \mathcal{E}(t,q)| \le c_1 (c_0 + \mathcal{E}(t,q)) \text{ for all } t \in [0,T]. \end{aligned} \tag{E2}$$

Condition (E2) enables to apply Gronwall's lemma in order to derive a Lipschitz-estimate for $\mathcal{E}$ with respect to time:

$$|\mathcal{E}(t,q) - \mathcal{E}(s,q)| \le \left(\mathrm{e}^{c_1 |t-s|} - 1 \right) (\mathcal{E}(t,q) + c_0) \le \mathrm{e}^{c_1 T} (\mathcal{E}(t,q) + c_0) |t-s|\,. \tag{2.2}$$

Hence, if $\mathcal{E}(t,q) < E$ for $E \in \mathbb{R}$, then, for $c_E := \mathrm{e}^{c_1 T}(E + c_0)$, estimate (2.2) implies

$$|\mathcal{E}(t,q) - \mathcal{E}(s,q)| \le c_E |t-s|\,. \tag{2.3}$$

The abstract existence theory requires the following general assumptions on the dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$:

$$\textit{Quasi-distance:} \quad \forall\, z_1, z_2, z_3 \in \mathcal{Z} : \quad \begin{aligned} &\mathcal{D}(z_1, z_2) = 0 \;\Leftrightarrow z_1 = z_2 \;\text{ and} \\ &\mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3); \end{aligned} \tag{D1}$$

$$\textit{Semi-continuity:} \quad \mathcal{D} :\ \mathcal{Z}\times\mathcal{Z} \to [0,\infty] \text{ is weakly seq. lower semi-continuous.} \tag{D2}$$

Remark 2.3 *$\mathcal{D}$ is an extended quasi-distance on $\mathcal{Z}$, since all metric axioms except of symmetry are satisfied and since the value ∞ is allowed. $\mathcal{D}$ on $\mathcal{Q}$ is a pseudo-distance or semi-distance, because for $q_1 = (u_1, z_1)$, $q_2 = (u_2, z_2)$ the property $\mathcal{D}(z_1, z_2) = 0$ not necessarily implies $q_1 = q_2$.*

Conditions (E1), (E2) and (D1), (D2) are useful to state an abstract existence result for the energetic formulation of rate-independent problems. This abstract version of the main existence theorem was developed within the works [MaM05, FrM06, MRS08].

Theorem 2.4 (Abstract main existence theorem) *Let $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfy conditions* (E1)*,* (E2) *and* (D1)*,* (D2)*. Moreover, let the following compatibility conditions hold: For every stable sequence $(t_k, q_k)_{k\in\mathbb{N}}$ with $t_k \to t$, $q_k \rightharpoonup q$ in $[0,T] \times \mathcal{Q}$ we have*

$$\partial_t\mathcal{E}(t, q_k) \to \partial_t\mathcal{E}(t, q)\,, \tag{C1}$$

$$q \in \mathcal{S}(t)\,. \tag{C2}$$

Then, for each $q_0 \in \mathcal{S}(0)$ there exists an energetic solution $q : [0,T] \to \mathcal{Q}$ for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfying $q(0) = q_0$.

The proof of Theorem 2.4 is based on a time-discretization, where conditions (E1), (D2) ensure the existence of a minimizer for the time-incremental minimization problem at each time-step. For a given partition $\Pi := \{0 = t_0 < t_1 < \ldots < t_M = T\}$, for every $k = 1, \ldots, M$ we have to

$$\text{find } q_k \in \operatorname{Argmin}\{\mathcal{E}(t_k, \tilde{q}) + \mathcal{D}(z_{k-1}, \tilde{z}) \,|\, \tilde{q} = (\tilde{u}, \tilde{z}) \in \mathcal{Q}\}\,. \tag{IP}$$

One then defines a piecewise constant interpolant q^Π with $q^\Pi(t) := q_{k-1}$ for $t \in [t_{k-1}, t_k)$ and $q^\Pi(T) = q_M$. Choosing a sequence $(\Pi_m)_{m\in\mathbb{N}}$ of partitions, where the fineness of Π_m tends to 0 as $m \to \infty$, it is possible to apply Helly's selection principle to the sequence $(q^{\Pi_m})_{m\in\mathbb{N}}$. Then, it is shown that the limit function fulfills the properties (S) and (E) of an energetic solution. See e.g. [MRS08] for a detailed proof.

3 Existence analysis for the damage model

The aim in this section is to prove the existence of an energetic solution for the damage problem by applying the abstract existence theorem 2.4 on this setup. Thereto, we introduce general assumptions on the given data like the domain $\Omega \subset \mathbb{R}^d$, the external loadings and the stored elastic energy density.

3.1 Assumptions and the existence result

We consider a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz-boundary $\partial\Omega$ modeling a non-linearly elastic material. This body undergoes a damage process driven by exterior forces $l(t)$, which may change with time. Furthermore, the body is assumed to be fixed at one part Γ_D of its boundary $\partial\Omega$ with positive $(d-1)$-dimensional measure $\mathcal{L}^{d-1}(\Gamma_D) > 0$, such that the displacement field $\tilde{u} : \Omega \to \mathbb{R}^d$ is prescribed there: $\tilde{u} = u_D(t)$ on Γ_D for $t \in [0,T]$. This means that we allow for time-dependent Dirichlet conditions, where the Dirichlet boundary Γ_D itself is fixed in time. From now on we write $u_D(t)$ also for the given extention into the domain Ω of the function u_D specifying the Dirichlet condition on the boundary. Hence, using the splitting $\tilde{u} = u + u_D(t)$, we define the state $q = (u,z)$ and the free energy

$$\mathcal{E}(t,u,z) = \int_\Omega W(x, e(u){+}e_D(t), z)\,\mathrm{d}x + \frac{\kappa}{r}\int_\Omega |\nabla z|^r\,\mathrm{d}x - \langle l(t), u{+}u_D(t)\rangle\,, \tag{3.1}$$

where $u = 0$ on Γ_D, such that $u{+}u_D(t) = u_D(t)$ on Γ_D. Moreover, $e(u) := \frac{1}{2}(\nabla u{+}\nabla u^\top)$ and $e_D(t) := \frac{1}{2}(\nabla u_D(t){+}\nabla u_D(t)^\top)$ denote the linearized strain tensor of u and $u_D(t)$ respectively.

We make the following general assumptions on the domain Ω and the given data u_D, l :

$$\left.\begin{aligned}
&\textbf{(A1)}\quad \Omega \text{ is a bounded Lipschitz-domain, } \Gamma_D \subset \partial\Omega \text{ with } \mathcal{L}^{d-1}(\Gamma_D) > 0,\\
&\textbf{(A2)}\quad u_D \in \mathrm{C}^1([0,T], W^{1,\infty}(\Omega,\mathbb{R}^d)) \text{ with } c_D{:=}\|u_D\|_{\mathrm{C}^1([0,T],W^{1,\infty}(\Omega,\mathbb{R}^d)\cap W^{1,p}(\Omega,\mathbb{R}^d))},\\
&\textbf{(A3)}\quad l \in \mathrm{C}^1([0,T], W^{-1,p'}(\Omega,\mathbb{R}^d)) \text{ with } c_l := \|l\|_{\mathrm{C}^1([0,T],W^{-1,p'}(\Omega,\mathbb{R}^d))}.
\end{aligned}\right\} \tag{3.2}$$

Here $p' = p/(p-1)$, where $p \in (1,\infty)$ will be fixed in (H3) below.

Furthermore, we claim the following hypotheses on the stored elastic energy density:

(H1) Carathéodory-function: $W(x,\cdot,\cdot) \in \mathrm{C}^0(\mathbb{R}^{d\times d}_{\mathrm{sym}} {\times} [0,1])$ for a.e. $x \in \Omega$ and $W(\cdot,e,z)$ is measurable in Ω.

(H2) Convexity: For every $(x,z) \in \Omega \times [0,1]$ the function $W(x,\cdot,z)$ is convex.

(H3) Coercivity: There are constants c_1, $C > 0$, and $1 < p < \infty$ such that for all $(x,e,z) {\in} \Omega {\times} \mathbb{R}^{d\times d}_{\mathrm{sym}} {\times} [0,1]$ we have $c_1|e|^p - C \le W(x,e,z)$.

(H4) Stress control: For all $(x,z) \in \Omega \times [0,1]$ we have $W(x,\cdot,z) \in \mathrm{C}^1(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and there exist constants $c{>}0, \tilde{c}{\ge}0$ such that for all $(x,e,z) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [0,1]$ we have

$$|\partial_e W(x,e,z)| \le c(W(x,e,z) + \tilde{c})\,.$$

(H5) Monotonicity: There are constants $k > 0$, $\tilde{k} \le 0$ so that for all (x,e,z), $(x,e,\tilde{z}) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [0,1]$ with $z \le \tilde{z}$ we have

$$W(x,e,z) \le W(x,e,\tilde{z}) \le k(W(x,e,z) + \tilde{k})\,.$$

Hypotheses (H1)-(H3) will ensure condition (E1). Hypothesis (H4) is the basis to prove Lipschitz-estimate (2.3). The first estimate in assumption (H5) reflects the physical property of damage, that an increase of damage decreases the stored elastic energy. The second estimate in (H5) states that the remaining elastic properties after all damage has occurred are still comparable to the undamaged material. This assumption is reasonable, because we only treat partial damage in our analysis. Total damage would neither allow for the second inequality in (H5) nor for coercivity (H4), since for a completely disintegrated body the displacement field has no meaning any longer.

In view of hypothesis (H4) we choose the space of admissible displacements as

$$\mathcal{U} := \{u \in W^{1,p}(\Omega,\mathbb{R}^d) \,|\, u = 0 \text{ on } \Gamma_D\}. \tag{3.3}$$

Under consideration of formula (3.1) we put the set of admissible damage variables

$$\mathcal{Z} := \{z \in W^{1,r}(\Omega) \,|\, 0 \le z \le 1 \text{ a.e. in } \Omega\} \tag{3.4}$$

and $\mathcal{Q} := \mathcal{U} \times \mathcal{Z}$ indicates the set of admissible states. By $\mathcal{X} := W^{1,p}(\Omega,\mathbb{R}^d) \times W^{1,r}(\Omega)$ with its strong topology we denote the Banach space that specifies the topology for weak convergence. Within the analysis we will consider the convergence of sequences $(q_k)_{k\in\mathbb{N}} \subset \mathcal{Q}$ to a limit q with respect to the weak topology of $\mathcal{X}$ and we will indicate the weak convergence in $\mathcal{X}$ by $q_k \rightharpoonup q$ in $\mathcal{X}$.

With these tools at hand we state the existence theorem for the damage problem.

Theorem 3.1 (Existence theorem for the damage problem) *Let* $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ *be given as above. Let* $\mathcal{E}$ *be defined via* (3.1) *such that* (3.2) *and (H1)-(H5) hold. Let* $\mathcal{D}$ *be given by* (1.2) *and* (1.3). *Then, for the rate-independent damage process defined by* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *there exists an energetic solution for any initial state* $q_0 \in \mathcal{S}(0)$.

The proof of Theorem 3.1 is carried out in Section 3.2. The main difficulty lies in the missing weak continuity of the dissipation distance, which especially complicates the proof of the compatibility conditions (C1) and (C2), see Sections 3.2.4, 3.2.5.

3.2 Proof of the existence theorem for the damage problem

In this subsection the assumptions (E1), (E2), (D1), (D2) and (C1), (C2) of the abstract main existence theorem 2.4 are checked. An analysis similar to ours is given in [MiP07, MaM08, MPP08]. As our damage model allows for more general assumptions in (H1)-(H5) we repeat all steps for the readers convenience. In particular, previous work (e.g. [MPP08]) assumes (H2) and (H4*), where (H4*) ensures that $\partial_A W(x,A,z) \in L^{p'}(\Omega,\mathbb{R}^{d\times d})$, which is not guaranteed by (H4).

For a shorter notation in the proofs we introduce the following abbreviations:

$$\left.\begin{aligned} \mathcal{I}(t,u,z) &:= \int_\Omega W(x,e(u)+e_D(t),z)\,\mathrm{d}x, \\ \mathcal{C}(z) &:= \tfrac{\kappa}{r}\int_\Omega |\nabla z|^r\,\mathrm{d}x, \\ \mathcal{J}(t,u,z) &:= \mathcal{I}(t,u,z) - \langle l(t), u+u_D(t)\rangle, \end{aligned}\right\} \tag{3.5}$$

such that

$$\mathcal{E}(t,u,z) = \mathcal{I}(t,u,z) + \mathcal{C}(z) - \langle l(t), u + u_D(t)\rangle = \mathcal{J}(t,u,z) + \mathcal{C}(z). \tag{3.6}$$

A basic tool in the proofs is Korn's inequality, which holds for functions $u \in \mathcal{U} \subset W^{1,p}(\Omega,\mathbb{R}^d)$ for $\mathcal{U}$ defined by (3.3).

Theorem 3.2 (Korn's inequality [GeS86]) *Let $\Omega \subset \mathbb{R}^d$ and $\Gamma_D \subset \partial\Omega$, satisfy (A1) and let $1 < p < \infty$. There is a constant $C_K = C_K(\Omega,p)$ such that for every $v \in \mathcal{U}$ the following estimate holds:*

$$\|v\|_{W^{1,p}(\Omega,\mathbb{R}^d)} \le C_K \|e(v)\|_{L^p(\Omega,\mathbb{R}^{d\times d})} . \tag{3.7}$$

3.2.1 Compactness of the energy sublevels (E1)

In the following, the weak sequential compactness of the energy sublevels is established using the standard approach in the direct method of the calculus of variations.

Lemma 3.3 *Let the assumptions* (3.2) *and* (H1)-(H5) *hold. Then there exist constants* c_3, $C_3 > 0$ *such that* $\mathcal{E}(t,\cdot,\cdot) : \mathcal{U}\times\mathcal{Z} \to \mathbb{R}$ *satisfies a growth estimate of the form*

$$\mathcal{E}(t,u,z) \ge c_3 \left(\|u\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} + \|z\|^r_{W^{1,r}(\Omega)} \right) - C_3 \ \text{ for all } (u,z) \in \mathcal{U}\times\mathcal{Z} . \tag{3.8}$$

Proof: For $(x,e,z,A) \in \Omega \times \mathbb{R}^{d\times d}_{\text{sym}} \times [0,1] \times \mathbb{R}^d$ we set

$$\overline{W}(x,e,z,A) := W(x,e,z) + \frac{\kappa}{r}|A|^r .$$

Let $u \in \mathcal{U}$. Using hypotheses (A2), (A3), (H3), Young's and Korn's inequality we get

$$\begin{aligned}
\mathcal{E}(t,u,z) &= \int_\Omega \overline{W}(x, e(u){+}e_D(t), z, \nabla z)\,\mathrm{d}x - \langle l(t), u{+}u_D(t)\rangle \\
&\ge c_1(\|e(u)\|_{L^p}{-}c_D)^p - (C{+}\frac{\kappa}{r})\mathcal{L}^d(\Omega) - c_l(\|u\|_{W^{1,p}}{+}c_D) + \frac{\kappa}{r}\|z\|^r_{W^{1,r}} \\
&\ge c_1(2^{1-p}\|e(u)\|^p_{L^p}{-}c_D^p) - (C{+}\frac{\kappa}{r})\mathcal{L}^d(\Omega) - c_l(\|u\|_{W^{1,p}}{+}c_D) + \frac{\kappa}{r}\|z\|^r_{W^{1,r}} \\
&\ge \frac{2^{1-p}c_1}{C_K^p}\|u\|^p_{W^{1,p}} - (C{+}\frac{\kappa}{r})\mathcal{L}^d(\Omega) - c_1 c_D^p - \frac{1}{p'}\left(\frac{c_l}{\varepsilon}\right)^{p'} - \frac{(\varepsilon\|u\|_{W^{1,p}})^p}{p} - c_l c_D + \frac{\kappa}{r}\|z\|^r_{W^{1,r}} \\
&\ge \frac{2^{-p}c_1}{C_K^p}\|u\|^p_{W^{1,p}} + \frac{\kappa}{r}\|z\|^r_{W^{1,r}} - (C{+}\frac{\kappa}{r})\mathcal{L}^d(\Omega) - c_1 c_D^p - c_l c_D - \frac{1}{p'}\left(\frac{c_l}{\varepsilon}\right)^{p'} ,
\end{aligned} \tag{3.9}$$

where Young's inequality with $\varepsilon := \left(\frac{2^{-p}c_1 p}{C_K^p}\right)^{\frac{1}{p}}$ lead to the third inequality of (3.9).

This proves (3.8) with suitable c_3 and C_3. ∎

Proposition 3.4 *Let assumptions* (3.2) *as well as* (H1)-(H5) *hold. Then* $\mathcal{E}(t,\cdot,\cdot)$ *is weakly sequentially lower semicontinuous with respect to the weak topology of* $\mathcal{X}$ *and its sublevels* $L_E(t)$ *are weakly sequentially compact in* $\mathcal{X}$*.*

Proof: First, we obtain that $\mathcal{C}(\cdot) : W^{1,r}(\Omega) \to \mathbb{R}$ is bounded from below by 0 and lower semicontinuous, since every sequence L^r-converging sequence contains a subsequence that converges pointwise a.e. by Riesz' convergence theorem. Moreover, $\mathcal{C}(\cdot)$ is convex and hence weakly sequentially lower semicontinuous by [Dac89] p. 49, Th. 1.2.. Furthermore, [Dac89] p. 74 states the weak sequential lower semicontinuity of $J(\xi, \eta) = \int_\Omega W(x, \xi(x), \eta(x))\,\mathrm{d}x$ for $\eta = z$, $\xi = e(u)$ on $W^{1,p}(\Omega, \mathbb{R}^d) \times L^r(\Omega)$ if hypotheses (H1)-(H3) are satisfied, because the compact embedding of $W^{1,r}(\Omega) \Subset L^r(\Omega)$ by Rellich's embedding theorem implies the strong L^r-convergence of a sequence converging weakly in $W^{1,r}(\Omega)$. Hence, $\mathcal{E}$ is weakly sequentially lower semicontinuous on $\mathcal{X}$.

Let now $(u_k, z_k)_{k\in\mathbb{N}} \subset L_E(t) \subset \mathcal{Q}$. Then estimate (3.8) yields

$$\|e(u_k)\|_{W^{1,p}(\Omega,\mathbb{R}^d)} + \|z_k\|_{W^{1,r}(\Omega)} \le \left(\frac{E{+}C_3}{c_3}\right)^{\frac{1}{p}} + \left(\frac{E{+}C_3}{c_3}\right)^{\frac{1}{r}} . \tag{3.10}$$

Since the spaces $W^{1,p}(\Omega, \mathbb{R}^d)$, $W^{1,r}(\Omega)$ are real, reflexive Banach spaces for $1 < p, r < \infty$, the sequence $(u_k, z_k)_{k\in\mathbb{N}}$ contains subsequence converging weakly in $\mathcal{X}$. In particular, due to the compact embedding of $\mathcal{X}$ into $L^p(\Omega, \mathbb{R}^d) \times L^r(\Omega)$ and Riesz' convergence theorem we find a further subsequence converging pointwise a.e. in Ω with their limits $z \in \mathcal{Z}$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ with $u{=}0$ on Γ_D. Since the weak sequential lower semicontinuity of $\mathcal{E}(t, \cdot)$ on $\mathcal{X}$ is equivalent to the weak sequential closedness of its sublevels, see [Dac89, Thm. 2.1, p. 28], the limit (u, z) of the subsequence is an element of $L_E(t)$. This proves that the sublevels are weakly sequentially compact, i.e. (E1) and it implies that $u \in \mathcal{U}$. ■

Remark 3.5 (Existence, uniqueness of minimizers) *As a direct consequence of Proposition 3.4 one obtains the existence of a minimizer for the minimization problems*

$$\min_{(\tilde u,\tilde z)\in\mathcal{Q}} (\mathcal{E}(t, \tilde u, \tilde z){+}\mathcal{D}(z, \tilde z)), \ \min_{\tilde u\in\mathcal{U}} \mathcal{J}(t, \tilde u, z) \ \textit{and} \ \min_{\tilde q\in\mathcal{Q}} \mathcal{J}(t, \tilde q)$$

for all $t{\in}[0, T]$ *and all* $z{\in}\mathcal{Z}$*, as well as for the time-incremental problems* (IP) *in every time step. This implies that the stable sets* $\mathcal{S}(t)$ *are non-empty for every* $t{\in}[0, T]$*. If strict convexity is claimed in* (H2)*, then the minimizers* $u{\in}\mathcal{U}$ *of* $\mathcal{J}(t, \cdot, z)$ *are even unique.*

3.2.2 Control of the power of the energy (E2)

In this subsection condition (E2) is proven under the assumptions (3.2) and (H1)-(H4).

As a first step we derive a Lipschitz-estimate for the stored elastic energy density.

Lemma 3.6 (Lipschitz-estimate for W) *Let* W *satisfy* (H2) *and* (H4)*. Then for every* $(x, z) \in \Omega \times [0, 1]$ *and any* $e, \tilde e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *it holds:*

$$|W(x, \tilde e, z) - W(x, e, z)| \le \frac{c}{2}(W(x, e, z) + W(x, \tilde e, z) + 2\tilde c)|\tilde e - e| . \tag{3.11}$$

Proof: Under consideration of (H2) and (H4) we obtain for $\alpha \in [0,1]$:

$$
\begin{aligned}
|W(x,\tilde{e},z)-W(x,e,z)| &= \left|\int_0^1 \partial_e W(x,(e+\alpha(\tilde{e}-e)),z):(\tilde{e}-e)\mathrm{d}\alpha\right| \\
&\le \int_0^1 c(\alpha(W(x,\tilde{e},z)+\tilde{c})+(1-\alpha)(W(x,e,z)+\tilde{c}))|\tilde{e}-e|\,\mathrm{d}\alpha \\
&= \frac{c}{2}(W(x,\tilde{e},z)+\tilde{c})|\tilde{e}-e|+\frac{c}{2}(W(x,e,z)+\tilde{c})|\tilde{e}-e|\,,
\end{aligned}
$$

which gives the result. ■

Now, we are in a position to prove condition (E2).

Theorem 3.7 *Let* (H2)-(H4) *and* (3.2) *be satisfied. Then there exist constants* $c_0 \ge 0$, $c_1 > 0$ *such that for every* $(t_q, q) \in [0,T] \times \mathcal{Q}$ *with* $\mathcal{E}(t_q, q) < \infty$ *holds:*

$$
\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T]), \text{ where}
$$

$$
\partial_t \mathcal{E}(t,q) = \int_\Omega \partial_e W(x,e(u)+e_D(t),z):\dot{e}_D(t)\,\mathrm{d}x - \langle \dot{l}(t), u+u_D(t)\rangle - \langle l(t), \dot{u}_D(t)\rangle \tag{3.12}
$$

$$
\text{and} \quad |\partial_t \mathcal{E}(t,q)| \le c_1(\mathcal{E}(t,q)+c_0) \quad \text{for every } t \in [0,T]\,. \tag{3.13}
$$

Proof: Note that the assumption $\mathcal{E}(t_q, q) =: E_q < \infty$ for some $t_q \in [0,T]$ together with (A2), (A3) and (H4) yields $\mathcal{E}(t,q) < \tilde{E}_q < \infty$ for every t in a sufficiently small neighborhood $\mathcal{U}(t_q) \subset [0,T]$ of t_q, since $\mathcal{E}(\cdot,q)$ as the sum and composition of the continuous functions $l(\cdot)$, $u_D(\cdot)$, $W(x,\cdot,z)$, $\langle\cdot,\cdot\rangle$ and $\int_\Omega(\cdot)\mathrm{d}x$ is a continuous function itself. In a first step, we prove that the time-derivative $\partial_t\mathcal{E}(\cdot,q)$ exists in $\mathcal{U}(t_q)$. In this neighborhood the estimate (3.13) can be derived as a second step. We will obtain that the constants are independent of t_q and $\mathcal{U}(t_q)$. This allows us to apply Gronwall's lemma and Lipschitz-estimate (2.3) uniformly in each neighborhood of any time t_q with finite energy. Thus, $\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T])$ follows.

Now, we prove the existence of $\partial_t\mathcal{E}(t,q)$ for $t \in \mathcal{U}(t_q)$. Thereto we define for $t \in \mathcal{U}(t_q)$

$$
h(x,t,\alpha) := \begin{cases} \frac{1}{\alpha}\left(W(x,e(u)+e_D(t+\alpha),z) - W(x,e(u)+e_D(t),z)\right) & \text{if } \alpha \neq 0 \\ \partial_e W(x,e(u)+e_D(t),z) : \dot{e}_D(t) & \text{if } \alpha = 0 \end{cases}
$$

and we must show that $h(x,t,\cdot) \in \mathrm{C}^0([-\alpha_t,\alpha_t])$ for α_t suitably. By the mean value theorem of differentiability, we know the existence of $\tilde{\alpha}=\tilde{\alpha}(\alpha)$ for every $\alpha \in [-\alpha_t,\alpha_t]$, such that

$$
\begin{aligned}
&\frac{1}{\alpha}\left(W(x,e(u)+e_D(t+\alpha),z)-W(x,e(u)+e_D(t),z)\right) \\
&= \partial_e W(x,e(u)+e_D(t+\tilde{\alpha}),z) : \dot{e}_D(t+\tilde{\alpha}) \\
&\quad \to \partial_e W(x,e(u)+e_D(t),z) : \dot{e}_D(t) \quad \text{as } \alpha,\ \tilde{\alpha} \to 0 \text{ by (H4) and (A2)}\,.
\end{aligned} \tag{3.14}
$$

In order to show that the integrals converge as well, we are going to apply the dominated convergence theorem. Thereto we estimate by (A2) and (H4)

$$
|(3.14)| \le c_D c\,(W(x,e(u)+e_D(t+\tilde{\alpha}),z)+\tilde{c}) \to c_D c\,(W(x,e(u)+e_D(t),z)+\tilde{c})
$$

as α, $\tilde{\alpha} \to 0$ due to (A2) and (H4). By Lipschitz-estimate (3.11), (A2) and (A3) we have

$$\begin{aligned} &\left| \int_\Omega W(x, e(u) + e_D(t+\tilde{\alpha}), z) - W(x, e(u) + e_D(t), z) \,\mathrm{d}x \right| \\ &\le \|e_D(t+\tilde{\alpha}) - e_D(t)\|_{L^\infty(\Omega,\mathbb{R}^{d\times d})} \left(2c\tilde{c}\mathcal{L}^d(\Omega) + \mathcal{E}(t,u,z) + \mathcal{E}(t+\tilde{\alpha},u,z) + 2c_l c_D\right) \overset{\tilde{\alpha}\to 0}{\longrightarrow} 0, \end{aligned} \tag{3.15}$$

since $\mathcal{E}(t+\tilde{\alpha}, u, z) < \tilde{E}_q$ for every $t+\tilde{\alpha} \in \mathcal{U}(t_q)$. The differentiability of $\langle l(t), u+u_D(t)\rangle$ is ensured by (A2), (A3). Thus we have proven the existence of $\partial_t\mathcal{E}(\cdot, q)$ in $\mathcal{U}(t_q)$.

By (3.8) we find an upper estimate for $\|e(u)+e_D(t)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})}$ in terms of $\mathcal{E}(t,q)$:

$$\begin{aligned} \|e(u)+e_D(t)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})} &\le 2^{p-1}\left(\|e(u)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})} + c_D^p\right) \\ &\le 2^{p-1}\left(\frac{\mathcal{E}(t,q)+C_3}{c_3} + c_D^p\right) =: A_1\mathcal{E}(t,q) + B_1 \end{aligned} \tag{3.16}$$

This estimate will be used in the following to get (3.13). We have

$$|\partial_t\mathcal{E}(t,q)| \le \left| \int_\Omega \partial_e W(x, e(u)+e_D(t), z) : \dot{e}_D(t)\,\mathrm{d}x \right| + |\langle \dot{l}(t), u+u_D(t)\rangle| + |\langle l(t), \dot{u}_D(t)\rangle|\,,$$

where the loading terms are treated with Korn's and Young's inequality as in the proof of (3.8), such that one obtains an estimate of the form

$$|\langle \dot{l}(t), u+u_D(t)\rangle| + |\langle l(t), \dot{u}_D(t)\rangle| \le A_2\mathcal{E}(t,q) + B_2\,. \tag{3.17}$$

Application of (H4) to the stored elastic energy term yields

$$\begin{aligned} &\left| \int_\Omega \partial_e W(x, e(u)+e_D(t), z) : \dot{e}_D(t)\,\mathrm{d}x \right| \le c_D c\left(\mathcal{I}(t,q) + \tilde{c}\mathcal{L}^d(\Omega)\right) \\ &\le c_D c\left(\mathcal{E}(t,q) + c_l\|u\|_{W^{1,p}(\Omega,\mathbb{R}^{d\times d})} + c_l c_D + \tilde{c}\mathcal{L}^d(\Omega)\right) \\ &= A_3\left(\mathcal{E}(t,q) + \|u\|_{W^{1,p}(\Omega,\mathbb{R}^{d\times d})}\right) + B_3\,. \end{aligned} \tag{3.18}$$

Applying Korn's inequality (3.7) to $\|u\|_{W^{1,p}(\Omega,\mathbb{R}^{d\times d})}$ leads to the estimate

$$\begin{aligned} &\left| \int_\Omega \partial_e W(x, e(u)+e_D(t), z) : \dot{e}_D(t)\,\mathrm{d}x \right| \le (3.18) \\ &\le A_3\left(\mathcal{E}(t,q) + C_K\|e(u)+e_D(t)\|_{L^p(\Omega,\mathbb{R}^{d\times d})}\right) + A_3C_Kc_D + B_3 \\ &\le A_4(1 + \|e(u)+e_D(t)\|_{L^p(\Omega,\mathbb{R}^{d\times d})})^p + A_3\,\mathcal{E}(t,q) + B_3 \\ &\le A_4 2^{p-1}(1 + \|e(u)+e_D(t)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})}) + A_3\,\mathcal{E}(t,q) + B_3 \\ &\le A_4 2^{p-1}(1 + A_1\mathcal{E}(t,q) + B_1) + A_3\,\mathcal{E}(t,q) + B_3 \\ &= A_5\,\mathcal{E}(t,q) + B_5\,, \end{aligned} \tag{3.19}$$

where (3.16) has been applied to obtain the last inequality. Combining (3.17), (3.19) yields the desired estimate (3.13). ■

3.2.3 Proof of the abstract assumptions on the dissipation distance

Now, we show that a dissipation distance that refers to a rate-independent damage process satisfies the assumptions (D1) and (D2).

Theorem 3.8 *The dissipation distance* $\mathcal{D}$ *on* $\mathcal{Z}$ *given by* (1.2), (1.3) *satisfies* (D1), (D2).

Proof: **Ad (D1):** By (1.2) we have $\mathcal{D}(z_1, z_2) \geq \varrho_0 \|z_2 - z_1\|_{L^1(\Omega)}$. Hence, $\mathcal{D}(z_1, z_2) = 0$ implies $z_1 = z_2$. Let now $z_1, z_2, z_3 \in \mathcal{Z}$ to show that the triangle-inequality holds. If its right-hand side is infinite, then the inequality is satisfied trivially. For a finite right-hand side $z_1 \geq z_2 \geq z_3$ is necessary and hence we even obtain equality.
Ad (D2): To show sequential lower semicontinuity, let $z_{0_k} \rightharpoonup z_0$, $z_{1_k} \rightharpoonup z_1$ in $W^{1,r}(\Omega)$ and put $w_k := z_{1_k} - z_{0_k}$, $w := z_1 - z_0$. Assume that $\liminf_{k\to\infty} \mathcal{D}(z_0, z_1) < \infty$, otherwise the inequality trivially holds. For a subsequence that attains the limit inferior, i.e. $w_k \leq 0$ for all $k \in \mathbb{N}$, we obtain that

$$|\mathcal{D}(z_{0_k}, z_{1_k}) - \mathcal{D}(z_0, z_1)| \leq \|\varrho\|_{L^\infty(\Omega)} \|w_k - w\|_{L^1(\Omega)} \to 0 \text{ as } k \to \infty$$

due to the compact embedding $W^{1,r}(\Omega) \Subset L^1(\Omega)$. Thus $\mathcal{D}(z_0, z_1) \leq \liminf_{k\to\infty} \mathcal{D}(z_{0_k}, z_{1_k})$. ∎

3.2.4 Convergence of the time-derivative of the energies (C1)

The aim in this subsection is to prove the first compatibility condition.

Theorem 3.9 *Let hypotheses* (H1)-(H5), (3.2) *and* (D1), (D2) *hold true. Then, for every stable sequence* $(t_k, q_k)_{k\in\mathbb{N}} \subset [0, T] \times \mathcal{Q}$ *with* $t_k \to t$ *and* $q_k \rightharpoonup q$ *in* $\mathcal{X}$ *we have*

$$\partial_t \mathcal{E}(t, q_k) \to \partial_t \mathcal{E}(t, q) \ . \tag{C1}$$

Proof: Since $\mathcal{C}(z) := \int_\Omega \frac{\kappa}{r} |\nabla z|^r \, \mathrm{d}x$ does not depend on time t we have $\partial_t \mathcal{E}(t, q) = \partial_t \mathcal{J}(t, q)$ for $\mathcal{J}(t, q) = \mathcal{I}(t, q) - \langle l(t), u(t) + u_D(t) \rangle$. As the last term is linear, it is sufficient to prove Theorem 3.9 for $\mathcal{I}$.

The following two properties, shown in separate lemmas later on, are utilized to obtain the convergence result:

(**P1**) It holds $\mathcal{I}(t, u_k, z_k) \to \mathcal{I}(t, u, z)$ for every stable sequence $(t_k, u_k, z_k)_{k\in\mathbb{N}}$, where $t_k \to t$, $(u_k, z_k) \rightharpoonup (u, z)$ in $\mathcal{X}$, see Lemma 3.10.

(**P2**) For $q \in L_E(0)$ the derivatives $\partial_t \mathcal{I}(\cdot, q)$ are uniformly continuous, see Lemma 3.11.

Using properties (P1) and (P2) we are able to apply Proposition 3.3 of [FrM06] to $\mathcal{I}$ and conclude $\partial_t \mathcal{I}(t, q_k) \to \partial_t \mathcal{I}(t, q)$. Thus, (C1) is established. ∎

In the following, the two properties (P1) and (P2) from the proof of Theorem 3.9 are verified. Property (P1) is a consequence of

Lemma 3.10 *Let $(t_k, u_k, z_k)_{k\in\mathbb{N}}$ be a stable sequence with $t_k \to t$, $(u_k, z_k) \rightharpoonup (u, z)$ in $\mathcal{X}$ as $k \to \infty$ and let* (H1)-(H5) *and* (3.2) *hold. Then*

$$\mathcal{J}(t, u, z_k) \to \mathcal{J}(t, u, z) \quad \textit{and} \quad \mathcal{J}(t, u_k, z_k) \to \mathcal{J}(t, u, z) \quad \textit{as } k \to \infty\,.$$

Proof: As a first step, we show that $\mathcal{J}(t, u, z_k) \to \mathcal{J}(t, u, z)$.

Note that $W(\cdot, e(u)+e_D(t), z_k) \xrightarrow{\mathcal{L}^d} W(\cdot, e(u)+e_D(t), z)$, since every subsequence $(W(\cdot, e(u)+e_D(t), z_{k_l}))_{l\in\mathbb{N}}$ contains a further subsequence that converges pointwise a.e.. This is due to the continuity of W with respect to z and Riesz' convergence theorem. By (H5) we obtain for every $k \in \mathbb{N}$ that

$$W(x, e(u)+e_D(t), z_k) \le k(W(x, e(u)+e_D(t), 0) + \tilde{k}) \le k(W(x, e(u)+e_D(t), z) + \tilde{k})\,.$$

Moreover, we have

$$\begin{aligned}\int_\Omega (W(x, e(u)+e_D(t), z) + \tilde{k})\,\mathrm{d}x &\le \mathcal{E}(t, u, z) + \tilde{k}\mathcal{L}^d(\Omega) + c_l(\|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)} + c_D)\\ &\le \liminf_{k\to\infty}(\mathcal{E}(t_k, u_k, z_k) + c_E|t - t_k|) + \tilde{k}\mathcal{L}^d(\Omega) + c_l(\|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)} + c_D) < \infty\end{aligned}$$

by lower semicontinuity, (2.1) and (2.3). The dominated convergence theorem now yields $\mathcal{J}(t, u, z_k) \to \mathcal{J}(t, u, z)$. Since u_k minimizes $\mathcal{J}(t_k, \cdot, z_k)$ and since (2.1), (2.3) hold, we infer

$$\mathcal{J}(t, u_k, z_k) - c_E|t_k - t| \le \mathcal{J}(t_k, u_k, z_k) \le \mathcal{J}(t, u, z_k) + c_E|t_k - t| \to \mathcal{J}(t, u, z)$$

and by weak sequential lower semicontinuity we conclude $\mathcal{J}(t_k, u_k, z_k) \to \mathcal{J}(t, u, z)$. ■

The next lemma refers to property (P2) from the proof of Theorem 3.9. It is based on the fact that the given data are continuously differentiable on the compact time interval $[0, T]$ by (A2), (A3) in (3.2), and hence they and their time-derivatives are uniformly continuous.

Lemma 3.11 (Uniform continuity of the powers of $\mathcal{I}$) *Let* (H1)-(H5) *and* (3.2) *be satisfied. Then, for each E, $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $q \in \mathcal{Q}$ with $\mathcal{E}(0, q) < E$ it holds:*

$$\textit{If } |t - s| < \delta \textit{ then } |\partial_t\mathcal{I}(t, q) - \partial_t\mathcal{I}(s, q)| < \varepsilon\,.$$

Proof: Due to (A2) and (A3) we find for every $\tilde{\varepsilon} > 0$ a $\tilde{\delta} > 0$ such that for all s, $t \in [0, T]$ with $|s-t| < \tilde{\delta}$ we have $\|u_D(s)-u_D(t)\|_{W^{1,\infty}(\Omega,\mathbb{R}^d)} + \|\dot{u}_D(s)-\dot{u}_D(t)\|_{W^{1,\infty}(\Omega,\mathbb{R}^d)} < \tilde{\varepsilon}$. Choose now $\varepsilon, E > 0$ and let $(u, z) \in L_E(0)$. By Lemma 3.8 we obtain for $t = 0$:

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)} \le \left(\frac{\mathcal{E}(0, u, z)+C_3}{c_3}\right)^{\frac{1}{p}} \le \left(\frac{E+C_3}{c_3}\right)^{\frac{1}{p}} =: \tilde{B}\,.$$

This shows that functions $u+u_D(t)$ with $(u, z) \in L_E(0)$ are uniformly bounded for every $t \in [0, T]$, since $\|u+u_D(t)\|_{W^{1,p}(\Omega,\mathbb{R}^d)} \le \|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)} + \|u_D(t)\|_{W^{1,p}(\Omega,\mathbb{R}^d)} \le \tilde{B}+c_D =: B$.

Furthermore we estimate

$$
\begin{aligned}
&|\partial_t\mathcal{I}(t,q)-\partial_t\mathcal{I}(s,q)| \\
&\leq \left|\int_\Omega \partial_e W(x,e(u){+}e_D(t),z) : (\dot e_D(t)-\dot e_D(s))\,\mathrm{d}x\right| && (3.20)\\
&+\left|\int_\Omega (\partial_e W(x,e(u){+}e_D(t),z)-\partial_e W(x,e(u){+}e_D(s),z)) : \dot e_D(s)\,\mathrm{d}x\right|\,. && (3.21)
\end{aligned}
$$

In view of (H3), (H4) and Lipschitz-estimate (2.3) we see that

$$
\begin{aligned}
(3.20) &\leq \|\partial_e W(\cdot,e(u){+}e_D(t),z)\|_{L^1(\Omega)}\|\dot e_D(t)-\dot e_D(s)\|_{L^\infty(\Omega,\mathbb{R}^{d\times d})}\\
&\leq (\mathcal{E}(0,q)+C\mathcal{L}^d(\Omega)+c_ET+c_lB)\|\nabla\dot u_D(t)-\nabla\dot u_D(s)\|_{L^\infty(\Omega,\mathbb{R}^{d\times d})}<\frac{\varepsilon}{2}\,,
\end{aligned}
$$

if only $|t{-}s|{<}\tilde\delta_1$ is sufficiently small. Moreover we have $e(u){+}e_D(s)\to e(u){+}e_D(t)$ $\mathcal{L}^d$-a.e.. Keeping in mind the continuity of $\partial_e W(x,\cdot,z)$ by (H2) we choose $\tilde\varepsilon_2:=\frac{\varepsilon}{2c_D}$ so that

$$
(3.21)=\|\partial_e W(\cdot,e(u){+}e_D(t),z)-\partial_e W(\cdot,e(u){+}e_D(s),z)\|_{L^1(\Omega,\mathbb{R}^{d\times d})}<\tilde\varepsilon_2
$$

for $|s-t|<\tilde\delta_2$ sufficiently small. Hence we obtain $(3.21)<\frac{\varepsilon}{2}$ if $|s-t|<\tilde\delta_2$. Altogether we conclude that $|\partial_t\mathcal{I}(s,q)-\partial_t\mathcal{I}(t,q)|<\varepsilon$ if $|s-t|<\delta:=\min\{\tilde\delta_1,\tilde\delta_2\}$. ■

3.2.5 Closedness of the stable sets (C2) and joint recovery condition

In the framework of damage we have to cope with a dissipation distance that is not weakly continuous on $W^{1,r}(\Omega)$. Hence it is not possible to show (C2) directly as in [FrM06, MiP07], where weak continuity is essential. Like in [MiR06, MRS08] we get (C2) via the so-called joint recovery condition.

Definition 3.12 (Joint recovery condition)
The rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfies the joint recovery condition if for all stable sequences $(t_k,q_k)_{k\in\mathbb{N}}=(t_k,u_k,z_k)_{k\in\mathbb{N}}\subset[0,T]\times\mathcal{Q}$ with $(t_k,q_k)\rightharpoonup(t,q)$ in $[0,T]\times\mathcal{X}$ and for every $\hat q=(\hat u,\hat z)\in\mathcal{Q}$ there is a sequence $(\hat q_k)_{k\in\mathbb{N}}=(\hat u_k,\hat z_k)_{k\in\mathbb{N}}$ with $\hat q_k\rightharpoonup\hat q$ in $\mathcal{X}$ and

$$
\limsup_{k\to\infty}\left(\mathcal{E}(t_k,\hat q_k)+\mathcal{D}(z_k,\hat z_k)-\mathcal{E}(t_k,q_k)\right)\leq\mathcal{E}(t,\hat q)+\mathcal{D}(z,\hat z)-\mathcal{E}(t,q)\,. \qquad \text{(JRC)}
$$

This condition implies (C2), which is also called the closedness of the sets of stable states, since (JRC) is equivalent to

$$
\mathcal{E}(t,q){-}\mathcal{E}(t,\hat q){-}\mathcal{D}(z,\hat z)\leq\liminf_{k\to\infty}\underbrace{(\mathcal{E}(t_k,q_k){-}\mathcal{E}(t_k,\hat q_k){-}\mathcal{D}(z_k,\hat z_k))}_{K_k}
$$

and $K_k\leq 0$ by $q_k\in\mathcal{S}(t_k)$ for every $k\in\mathbb{N}$.

In the case $\mathcal{D}(z,\hat{z})$, the joint recovery sequence has to be constructed in such a manner that $\mathcal{D}(z_k,\hat{z}_k)<\infty$ is satisfied for every $k\in\mathbb{N}$. Otherwise the left-hand side in (JRC) is too big. In fact, we will enforce $\mathcal{D}(z_k,\hat{z}_k)\to\mathcal{D}(z,\hat{z})$, which follows from $z_k\rightharpoonup z$ and $\hat{z}_k\rightharpoonup\hat{z}$ only if the additional constraint $\hat{z}_k\le z_k$ holds.

For this end, case $1<r\le d$ requires substantially new ideas compared to [MiR06], where the embedding $W^{1,r}(\Omega)\Subset \mathrm{C}^0(\bar{\Omega})$ was used. In that case, the finiteness of the dissipation distance can be easily achieved by choosing $\hat{z}_k:=(z_k-\|z_k-z\|_\infty)^+$, with

$$(f)^+:=\max\{0,f\}\,.$$

The compact embedding $W^{1,r}(\Omega)\Subset \mathrm{C}(\overline{\Omega})$ ensures that $\|z_k-z\|_\infty\to 0$ as $k\to\infty$. In the following, the result of [MiR06] is extended to the case of $1<r<\infty$ by constructing the joint recovery sequence in such a manner that the compact embedding $W^{1,r}(\Omega)\Subset \mathrm{C}(\overline{\Omega})$ is not needed for the proof of estimate (JRC).

For the construction of a joint recovery sequence we will entirely use that the superposition of a $W^{1,r}$-function with the Lipschitz-continuous function $\max\{0,f\}:\mathbb{R}\to\mathbb{R}$ again gives a $W^{1,r}$-function:

Lemma 3.13 (Superposition lemma, [MaM72]) *Let $g:\mathbb{R}\to\mathbb{R}$ be Lipschitz-continuous and $v\in W^{1,r}(\Omega)$. Then $g\circ v\in W^{1,r}(\Omega)$ and*

$$\nabla(g\circ v)(x)=g'(v(x))\nabla v(x)\quad \textit{for a.a. } x\in\Omega\,.$$

The following result establishes the compatibility condition (C2).

Theorem 3.14 (Joint recovery condition for $1<r<\infty$) *Let* (H1)-(H5) *hold. Then, the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfies the joint recovery condition. Hence, if $(t_k,q_k)_{k\in\mathbb{N}}$ is a stable sequence with $t_k\to t$, $q_k\rightharpoonup q$ in $\mathcal{X}$, then $q\in\mathcal{S}(t)$, i.e.* (C2) *holds.*

Proof: Let $(u_k,z_k)_{k\in\mathbb{N}}\subset\mathcal{U}\times\mathcal{Z}$ with $u_k\rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^d)$ and $z_k\rightharpoonup z$ in $W^{1,r}(\Omega)$. Choose $\hat{q}\in\mathcal{Q}$ such that $\hat{q}\in L_E(t)$ for some $E\in\mathbb{R}$, otherwise (JRC) trivially holds. Now we distinguish between the following two cases:
Case A: Let $\hat{q}=(\hat{u},\hat{z})\in\mathcal{Q}$ be such that there exists a $\mathcal{L}^d$-measurable set $B\subset\Omega$ with $\mathcal{L}^d(B)>0$ and $\hat{z}>z$ on B. Then $\mathcal{D}(z,\hat{z})=\infty$ and (JRC) holds.

Case B: Let $\hat{q}=(\hat{u},\hat{z})\in\mathcal{Q}$ be such that $\hat{z}\le z$ a.e. in Ω. Then, $\mathcal{D}(z,\hat{z})=\int_\Omega\varrho(z-\hat{z})\,\mathrm{d}x<\infty$.
To construct a joint recovery sequence we put $\hat{u}_k:=\hat{u}$ for every $k\in\mathbb{N}$ and

$$\hat{z}_k:=\min\left\{(\hat{z}-\delta_k)^+,\,z_k\right\}:=\begin{cases}(\hat{z}-\delta_k)^+ & \text{if } (\hat{z}-\delta_k)^+\le z_k\\ z_k & \text{if } (\hat{z}-\delta_k)^+> z_k\end{cases},\tag{3.22}$$

where $0<\delta_k\overset{\mathbb{R}}{\to}0$ will be chosen suitably in step 2. Thus, $\hat{z}_k\le z_k$ a.e. and therefore $\mathcal{D}(z_k,\hat{z}_k)<\infty$ for every $k\in\mathbb{N}$. Besides, it holds $\hat{z}_k(x)<\hat{z}(x)\le z(x)$ for a.e. $x\in\Omega$ with $\hat{z}(x)\neq 0$. Again we have $\hat{z}_k=z_k+\max\{0,(\hat{z}-\delta_k)^+-z_k\}\in W^{1,r}(\Omega)$ by Lemma 3.13.

For a joint recovery sequence constructed by (3.22) we can in general only prove weak convergence in $W^{1,r}(\Omega)$. This can be seen from Example 3.16 below the proof.

It holds $\mathcal{E}(t_k,\hat{q}_k) \le \mathcal{E}(t_k,\hat{q}) + \mathcal{C}(\hat{z}_k) \le \hat{c}$ due to $\hat{q} \in L_E(t)$ and estimate (2.3) for $\hat{q}$. Furthermore, (2.3) provides a uniform Lipschitz-constant for $(\hat{q}_k)_{k\in\mathbb{N}}$ such that

$$\mathcal{E}(t_k,\hat{q}_k) + \mathcal{D}(z_k,\hat{z}_k) - \mathcal{E}(t_k,q_k) \le \mathcal{E}(t,\hat{q}_k) + \mathcal{D}(z_k,\hat{z}_k) - \mathcal{E}(t,q_k) + 2L|t_k - t| \,, \tag{3.23}$$

where L is the maximum of the uniform Lipschitz-constants for $(q_k)_{k\in\mathbb{N}}$ and $(\hat{q}_k)_{k\in\mathbb{N}}$. Since $|t_k - t| \to 0$, inequality (JRC) holds if we can prove

$$\limsup_{k\to\infty} \left(\mathcal{E}(t,\hat{q}_k) + \mathcal{D}(z_k,\hat{z}_k) - \mathcal{E}(t,q_k)\right) \le \mathcal{E}(t,\hat{q}) + \mathcal{D}(z,\hat{z}) - \mathcal{E}(t,q)\,. \tag{JRC*}$$

In order to show (JRC*) we take into account that

$$\begin{aligned}&\limsup_{k\to\infty}(\mathcal{E}(t,\hat{q}_k) + \mathcal{D}(z_k,\hat{z}_k) - \mathcal{E}(t,q_k))\\ &\le \limsup_{k\to\infty}\mathcal{I}(t,\hat{q}_k) - \liminf_{k\to\infty}\mathcal{I}(t,q_k) + \limsup_{k\to\infty}\mathcal{D}(z_k,\hat{z}_k) + \limsup_{k\to\infty}(\mathcal{C}(\hat{z}_k) - \mathcal{C}(z_k))\\ &\quad -\langle l(t),\hat{u}-u\rangle\end{aligned} \tag{3.24}$$

and estimate these limits in separate steps.

For a shorter notation in the subsequent steps, we now introduce the abbreviation $[f < g] := \{x \in \Omega \,|\, f(x) < g(x)\}$ with an analogous meaning for $\le, >$ and $\ge$.

Step 1: We prove that $\hat{z}_k \rightharpoonup \hat{z}$ in $W^{1,r}(\Omega)$ as $k \to \infty$.
By construction the sequence $(\hat{z}_k)_{k\in\mathbb{N}}$ is uniformly bounded in $W^{1,r}(\Omega)$. Thus, there is a weakly convergent subsequence $\hat{z}_{k_l} \rightharpoonup \tilde{z} \in W^{1,r}(\Omega)$. Due to the compact embedding this subsequence converges strongly in $L^r(\Omega)$ and by Riesz' convergence theorem it has a further subsequence converging pointwise a.e. in Ω. This last subsequence has to converge $\hat{z_{k_{l\,m}}} \to \hat{z}$ a.e. in Ω by definition of $\hat{z}_k$. Hence, we obtain $\tilde{z} = \hat{z}$ and therefore $\hat{z}_k \to \hat{z}$ in $L^r(\Omega)$. Since $(\hat{z}_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,r}(\Omega)$, the same arguments also yield $\hat{z}_k \rightharpoonup \hat{z}$ in $W^{1,r}(\Omega)$.

Step 2: We show that $\limsup_{k\to\infty}(\mathcal{C}(\hat{z}_k) - \mathcal{C}(z_k)) \le \mathcal{C}(\hat{z}) - \mathcal{C}(z)$:
For the calculation of the limit, the domain Ω is decomposed as follows:

$$\Omega = A_k \cup B_k \text{ with } B_k = [(\hat{z} - \delta_k)^+ > z_k] \text{ and } A_k = \Omega\backslash B_k.$$

Thereby it holds $B_k = [(\hat{z} - \delta_k)^+ > z_k] \subset [(z-\delta_k)^+ > z_k] \subset [|z - z_k| \ge \delta_k]$. By application of Markov's inequality in estimate (**M**) we can now determine $(\delta_k)_{k\in\mathbb{N}}$ in such a way that $\mathcal{L}^d([(\hat{z} - \delta_k)^+ > z_k]) \to 0$ as $k \to \infty$:

$$\mathcal{L}^d([(\hat{z} - \delta_k)^+ > z_k]) \;\le\; \mathcal{L}^d([|z - z_k| \ge \delta_k]) \overset{(\mathbf{M})}{\le} \frac{1}{\delta_k^r}\int_\Omega |z - z_k|^r \,\mathrm{d}x \quad \overset{!}{\to} 0 \,,$$

if, for instance, $\delta_k := \|z_k - z\|_{L^r(\Omega)}^{\frac{1}{r}}$. Note that Markov's inequality is only applicable if $\delta_k > 0$. But $\|z_k - z\|_{L^r(\Omega)} = 0$ implies $\mathcal{L}^d([|z_k - z| > 0]) = 0$ and hence $\mathcal{L}^d(B_k) \to 0$ as

$k \to \infty$ is guaranteed. For $A_k = \Omega \backslash B_k$ we have $\mathcal{L}^d(A_k) \to \mathcal{L}^d(\Omega)$ as $k \to \infty$. Using the characteristic functions of these sets

$$I_{A_k}(x) := \begin{cases} 1 & \text{if } x \in A_k \\ 0 & \text{if } x \in B_k \end{cases}$$

and Lemma 3.15 from below we find $I_{A_k} \nabla z_k \rightharpoonup \nabla z$ in $L^r(\Omega, \mathbb{R}^d)$. By weak sequential lower semicontinuity we conclude

$$\begin{aligned} \limsup_{k\to\infty}(\mathcal{C}(\hat{z}_k) - \mathcal{C}(z_k)) &= \limsup_{k\to\infty} \int_{A_k} (|\nabla(\hat{z} - \delta_k)|^r - |\nabla z_k|^r)\,\mathrm{d}x \\ &\le \int_\Omega |\nabla \hat{z}|^r\,\mathrm{d}x - \liminf_{k\to\infty} \int_\Omega |I_{A_k} \nabla z_k|^r\,\mathrm{d}x \le \mathcal{C}(\hat{z}) - \mathcal{C}(z) \,. \end{aligned}$$

Step 3: Estimation of the remaining terms in line (3.24):
To calculate $\limsup_{k\to\infty} \mathcal{I}(t, \hat{u}, \hat{z}_k)$ we choose a subsequence $(\hat{z}_{k_l})_{l\in\mathbb{N}} \subset (\hat{z}_k)_{k\in\mathbb{N}}$ such that $\hat{z}_{k_l} \to \hat{z}\, \mathcal{L}^d$-a.e.. Since $W(x,e,\cdot) \in \mathrm{C}^0([0,1])$ cf. (H1) we have that $W(\cdot, e(\hat{u}) + e_D(t), \hat{z}_{k_l}) \to W(\cdot, e(\hat{u}) + e_D(t), \hat{z})\ \mathcal{L}^d$-a.e.. Furthermore, by (H5) we infer that $W(x, e(\hat{u}) + e_D(t), \hat{z}_{k_l}) \le k(W(x, e(\hat{u}) + e_D(t), \hat{z}) + \tilde{k}) \in L^1(\Omega)$. Then, the dominated convergence theorem gives $\mathcal{I}(t, \hat{u}, \hat{z}_{k_l}) \to \mathcal{I}(t, \hat{u}, \hat{z})$.

The estimate $-\liminf_{k\to\infty} \mathcal{I}(t, q_k) \le -\mathcal{I}(t,q)$ is obvious by the weak sequential lower semicontinuity of $\mathcal{I}(t,\cdot)$.

In view of the definition of the joint recovery sequence it holds $\hat{z}_k \le z_k$ for every $k \in \mathbb{N}$ and therefore

$$\lim_{k\to\infty} \mathcal{D}(z_k, \hat{z}_k) = \lim_{k\to\infty} \int_\Omega R(\hat{z}_k - z_k)\,\mathrm{d}x = \int_\Omega R(\hat{z} - z)\,\mathrm{d}x = \mathcal{D}(z, \hat{z}) \,,$$

by continuity of R, since both $z_k \to z$ and $\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$ as $k \to \infty$.

Hence inequality (JRC*) is proven. ■

It remains to show the lemma applied in step 2 of the above proof.

Lemma 3.15 *Let $\mathcal{L}^d(A_k) \to \mathcal{L}^d(\Omega)$ and $f_k \rightharpoonup f$ in $L^r(\Omega, \mathbb{R}^d)$ as $k \to \infty$. Then*

$$I_{A_k} f_k \rightharpoonup f \quad \text{as } k \to \infty \,.$$

Proof: Let $\varphi \in L^{r'}(\Omega, \mathbb{R}^d)$. First, we prove that $\varphi_k := I_{A_k}\varphi \to \varphi$ in $L^{r'}(\Omega, \mathbb{R}^d)$:

$$\|\varphi_k - \varphi\|^{r'}_{L^{r'}(\Omega,\mathbb{R}^d)} = \int_{\Omega\backslash A_k} |\varphi|^{r'}\,\mathrm{d}x \to 0 \quad \text{as } k \to \infty, \text{ since } \mathcal{L}^d(\Omega\backslash A_k) \to 0.$$

Hence, for every $\varphi \in L^{r'}(\Omega, \mathbb{R}^d)$ we have $\int_\Omega I_{A_k} f_k \cdot \varphi\,\mathrm{d}x = \int_\Omega f_k \cdot \varphi_k\,\mathrm{d}x \to \int_\Omega f \cdot \varphi\,\mathrm{d}x$, since $\varphi_k \to \varphi$ in $L^{r'}(\Omega, \mathbb{R}^d)$ and $f_k \rightharpoonup f$ in $L^r(\Omega, \mathbb{R}^d)$. ■

Now, we give an example on a weakly converging sequence, where the method (3.22) generates a weakly converging recovery sequence, that does not converge strongly.

Example 3.16 *Consider* $\Omega = \{(r,\phi)\,|\,0 \le r < 1,\, 0 \le \phi \le 2\pi\}$ *and*

$$z_k(r) := \begin{cases} kr & \text{for } 0 \le r \le \frac{1}{2k}, \\ \frac{1}{2} & \text{for } \frac{1}{2k} < r < 1, \end{cases} \qquad k \in \mathbb{N}\,. \tag{3.25}$$

Then $z_k \rightharpoonup z = \frac{1}{2}$ *in* $H^1(\Omega)$. *For* $\hat{z} := \frac{1}{4}$ *the joint recovery sequence constructed by* (3.22) *satisfies* $\hat{z}_k \rightharpoonup \hat{z}$ *in* $H^1(\Omega)$, *but* $\|\hat{z}_k - \hat{z}\|^2_{H^1(\Omega)} \to \frac{\pi}{16}$.

However, the sequence in (3.25) *may not be stable. Thus, it still might be possible to prove strong convergence of a recovery sequence where a stable sequence has been used in* (3.22).

4 On the temporal regularity of energetic solutions

The proof of the abstract existence theorem 2.4 for energetic solutions is based on a generalized version of Helly's selection principle, see [MaM05]. This formulation provides a universal temporal regularity result for the inner variable, namely to be of bounded variation in time. For the displacement field one obtains in general boundedness and measurability with respect to time. This is due to the fact that the interpolants of the solutions for the time-incremental problems (IP), which approximate the energetic solution, are both bounded and measurable in time, see e.g. [FrM06, MiR06]. In fact, we have

$$\begin{aligned} &z \in \mathrm{BV}([0,T], L^1(\Omega)) \cap L^\infty([0,T], W^{1,r}(\Omega))\,, \\ &u \in L^\infty([0,T], W^{1,p}(\Omega))\,. \end{aligned}$$

The BV estimate comes from the estimate $\mathrm{Var}_{L^1(\Omega)}(z,[r,s]) \le \frac{1}{\varrho_0}\mathrm{Diss}_{\mathcal{D}}(z,[r,s]) < \infty$, which is a consequence of the energy balance.

In fact, the monotonicity $z(t_1,x) \ge z(t_2,x)$ for $t_1 < t_2$ implies $\mathrm{Var}_{L^1(\Omega)}(z,[r,s]) = \int_\Omega z(r,x) - z(s,x)\,\mathrm{d}x \le |\Omega|$. The L^∞ bound for $q = (u,z)$ in $W^{1,p}(\Omega,\mathbb{R}^d) \times W^{1,r}(\Omega)$ is a consequence of the energy bound $\mathcal{E}(t,q(t)) \le E_*$.

It was first obtained in [MiT04] that the temporal regularity of the energetic solution can be improved, if $\mathcal{E}$ has additional convexity properties. For the case that $\mathcal{E}(\cdot,\cdot)$ is strictly convex in both, the strain tensor and the inner variable, one obtains that an energetic solution is continuous in time. Furthermore, it is proven in [MiT04] that even Lipschitz-continuity can be achieved for energies that are uniformly convex of the form

$$\mathcal{E}(t,\theta q_1 + (1-\theta)q_2) \le \theta\mathcal{E}(t,q_1) + (1-\theta)\mathcal{E}(t,q_2) - c\theta(1-\theta)\|q_1 - q_2\|^\alpha_{\mathcal{Q}} \text{ for } \theta \in [0,1],\ q_1,q_2 \in \mathcal{Q}, \tag{4.1}$$

with some constant $c > 0$ and $\alpha = 2$. In Section 4.2 we will see that (4.1) depends on the choice of $\|\cdot\|_{\mathcal{Q}}$ and that uniform convexity is not restricted to the exponent $\alpha = 2$. In a lemma we provide properties of stored elastic energy densities that lead to uniform convexity on sublevels with an exponent $\alpha \ge 2$. In such a situation we prove Hölder-continuity with respect to time.

Before we go into the analysis we provide an example of an energy density W that satisfies all the assumptions from above and additionally the uniform convexity conditions that will be used later. The fact that joint convexity is compatible with damage models was first exploited in [Rou08].

Example 4.1 *The simplest example for a suitable W generating a uniformly convex energy functional is given in the form*

$$W(x, e, z) = \frac{1}{2\big(1+\eta(1-z)\big)^\gamma}\, e{:}\mathbb{B}{:}e + \frac{a}{2}\, z^2,$$

where $\eta, a > 0$, $\gamma \in (0,1)$, and $\mathbb{B}$ is a symmetric and positive definite linear operator on $\mathbb{R}^{d\times d}_{\text{sym}}$. Such densities are discussed in detail in Section 5.1.

4.1 Temporal continuity

The first result provides continuity in time, which means that energetic solutions cannot have jumps. The idea is to use that under the assumption of strict convexity energetic solutions $q : [0,T] \to \mathcal{Q}$ have weak left and right limits $q_+(t)$ and $q_-(t)$ for all t. Moreover, it can be shown that $q_-(t)$, $q(t)$, and $q_+(t)$ must be minimizers of the functional $q \mapsto \mathcal{E}(t,q)+\mathcal{D}(q_-(t),q)$. By strict continuity one then concludes that all three values must be the same and weak continuity follows. Strong continuity is concluded by an argument of Visintin (cf. [Vis84]), which allows us to convert weak convergence and energy convergence into strong convergence by exploiting the strict convexity once again.

We now develop the details. We first provide a result that does not explicitly use the strict convexity of $\mathcal{E}(t,\cdot)$; for stable states $q = (u,z) \in \mathcal{S}(t)$ it only requires the uniqueness of the minimizer of $\mathcal{E}(t,\cdot,z)$, which then is u.

Lemma 4.2 (Jump relations) *Assume that $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfies (E1)—(C2). Moreover,*

$$\forall\, t \in [0,T]\ \forall\, q = (u,z) \in \mathcal{S}(t) : \quad \{u\} = \operatorname*{Argmin}_{\widetilde{u}\in\mathcal{U}} \mathcal{E}(t,\widetilde{u},z). \tag{4.2}$$

Then, for all $t \in [0,T]$ the weak limits $q_-(t) = \text{w-lim}_{\tau\to t^-} q(\tau)$ and $q_+(t) = \text{w-lim}_{\tau\to t^+} q(\tau)$ (where $q_-(0) := q(0)$ and $q_+(T) = q(T)$) exists and satisfy

$$\begin{aligned} &\mathcal{E}(t,q_-(t)) = \mathcal{E}(t,q(t)) + \mathcal{D}(q_-(t),q(t)),\ \ \mathcal{E}(t,q(t)) = \mathcal{E}(t,q_+(t)) + \mathcal{D}(q(t),q_+(t)),\\ &\textit{and}\ \ \mathcal{D}(q_-(t),q_+(t)) = \mathcal{D}(q_-(t),q(t)) + \mathcal{D}(q(t),q_+(t)). \end{aligned} \tag{4.3}$$

Proof: From $\text{Diss}_{\mathcal{D}}(z,[0,T]) < \infty$ we conclude that the limits $z_-(t) = \text{w-lim}_{\tau\to t^-} z(\tau)$ and $z_+(t) = \text{w-lim}_{\tau\to t^+} z(\tau)$ exist, cf. [MaM05]. Now, fix t, choose $v_\pm \in \mathcal{U}$ and subsequences $(t_k^\pm)_{k\in\mathbb{N}}$ such that $u(t_k^\pm) \rightharpoonup v_\pm$, where $t_k^\pm \to t$ with $\pm(t_k^\pm - t) > 0$. Then, (C2) guarantees $(v_\pm, z_\pm(t)) \in \mathcal{S}(t)$. Exploiting the assumption (4.2) we find that $v^\pm$ are uniquely determined and cannot depend on the subsequence. Hence, the function $u : [0,T] \to \mathcal{U}$ has the desired left-hand and right-hand limits $u_\pm(t)$ in the weak sense.

To obtain the desired energy identities (4.3) we exploit the energy balance

$$\mathcal{E}(s, q(s)) + \mathrm{Diss}_{\mathcal{D}}(z, [r, s]) = \mathcal{E}(r, q(r)) + \int_r^s \partial_\tau \mathcal{E}(\tau, q(\tau))\,\mathrm{d}\tau, \quad 0 \le r < s \le T.$$

For the first identity in (4.3) we let $s = t$ and consider $r \to t^-$. Using the obvious relation $\mathrm{Diss}_{\mathcal{D}}(z, [r, t]) \to \mathcal{D}(z_-(t), z(t))$ we find

$$\mathcal{E}(t, q(t)) + \mathcal{D}(z_-(t), z(t)) \le \limsup_{r \to t^-} \mathcal{E}(r, q(r)) \le \mathcal{E}(t, q_-(t)) \le \mathcal{E}(t, q(t)) + \mathcal{D}(z_-(t), z(t)),$$

where the second estimate follows from the stability $\mathcal{E}(r, q(r)) \le \mathcal{E}(r, q_-(t)) + \mathcal{D}(z(r), z_-(t))$ by taking the limit $r \to t^-$, while the third estimate is just the stability of $q_-(t)$. This establishes the first estimate in (4.3).

The second identity in (4.3) follows by setting $r = t$ and taking the limit $s \to t^+$:

$$\begin{aligned}\mathcal{E}(t, q_+(t)) + \mathcal{D}(z(t), z_+(t)) &\le \liminf_{s \to t^+} \mathcal{E}(s, q(s)) + \mathcal{D}(z(t), z(s)) \\ &= \mathcal{E}(t, q(t)) + 0 \le \mathcal{E}(t, q_+(t)) + \mathcal{D}(z(t), z_+(t)),\end{aligned}$$

where we first used lower semicontinuity (E1), then the energy balance, and finally the stability of $q(t)$. Thus, the second identity in (4.3) holds.

The third identity in (4.3) follows from (D1) and the first two identities:

$$\begin{aligned}\mathcal{D}(z_-(t), z_+(t)) &\le \mathcal{D}(z_-(t), z(t)) + \mathcal{D}(z(t), z_+(t)) \\ &= \mathcal{E}(t, q_-(t)) - \mathcal{E}(s, q_+(s)) \;\le\; \mathcal{D}(z_-(t), z_+(t)),\end{aligned}$$

where the last estimate uses the stability of $q_-(t)$. ■

The next result provides the continuity of the energetic solutions under the assumption that the functionals $\mathcal{E}(t, \cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ and $\mathcal{D}(z, \cdot) : \mathcal{Z} \to [0, \infty]$ are convex. In fact, the proof only uses the weaker property that for stable states $q \in \mathcal{S}$ the functional $\widetilde{q} \mapsto \mathcal{E}(t, \widetilde{q}) + \mathcal{D}(z, \widetilde{z})$ has a unique minimizer, see [MiR08].

Theorem 4.3 (Continuity by strict convexity) *Let the assumptions of the existence theorem 3.1 hold. Moreover, assume that $W : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [0, 1] \to \mathbb{R}$ is strictly convex. Then, any energetic solution $q : [0, T] \to \mathcal{Q}$ is (norm) continuous with respect to time, i.e. $q \in \mathrm{C}^0([0, T], \mathcal{Q})$.*

Proof: We first observe that for each $t \in [0, T]$ the functional $\mathcal{E}(t, \cdot)$ is strictly convex, since it is obtained by integration over the strictly convex density $(e, z, A) \mapsto W(x, e{+}e_D(t, x), z) - \frac{\kappa}{r}|A|^r$ and the linear term $l(t)$ with arguments $(e, z, A) = (e(u), z, \nabla z)$ depending linearly on $(u, z) \in \mathcal{Q}$. Moreover, for each $z \in \mathcal{Z}$ the mapping $\widetilde{z} \mapsto \mathcal{D}(z, \widetilde{z})$ is convex. Thus, for each $t \in [0, T]$ the functional

$$\mathcal{Q} \ni \widetilde{q} = (\widetilde{u}, \widetilde{z}) \;\mapsto\; \mathcal{E}(t, \widetilde{q}) + \mathcal{D}(z_-(t), \widetilde{z}),$$

has a unique minimizer.

Exploiting the jump relations (4.3) we easily find that $q_-(t)$, $q(t)$, and $q_+(t)$ all provide the same value $\mathcal{E}(t, q_-(t))$, which must be the global minimum by the stability of $q_-(t)$. Hence, the three values must coincide, and Lemma 4.2 allows us to conclude weak continuity of $q : [0, T] \to \mathcal{Q}$, namely $q(\tau) \rightharpoonup q(t)$ for $\tau \to t$.

Applying the jump relations (4.3) once again we also have $\mathcal{E}(\tau, q(\tau)) \to \mathcal{E}(t, q(t))$ for $\tau \to t$. Fixing t and employing (2.2) we also obtain $\mathcal{E}(t, q(\tau)) \to \mathcal{E}(t, q(t))$. Thus, we are able to apply the following Proposition 4.4 to the family $V(\tau) = (e(u(\tau)) + e_D(t), z(\tau), A(\tau))$, which provides the following strong convergence in $L^p(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}) \times L^r(\Omega) \times L^r(\Omega; \mathbb{R}^d)$:

$$(e(u(\tau)) + e_D(t), z(\tau), \nabla z(\tau)) \to (e(u(\tau)) + e_D(t), z(\tau), \nabla z(t)).$$

Using Korn's inequality (3.7) the desired strong convergence $q(\tau) \to q(t)$ in $\mathcal{Q}$ follows. ■

The following result was used in the proof above. Since it is only a slight variant of [Vis84, §2 & Th. 8], we leave the details to the reader.

Proposition 4.4 *Let Ω satisfy* (A1) *and* $\mathbf{C}$ *be a nonempty, closed, convex subset of* $\mathcal{V} := L^p(\Omega, \mathbb{R}^K)$, $1 \le p < \infty$, $d \ge 1$. *Let $\phi : \Omega \times \mathbb{R}^K \to [0, \infty]$ be a Carathéodory function such that $\phi(x, \cdot)$ is strictly convex on $\mathbb{R}^K$ a.e. on Ω. For $V \in \mathbf{C}$ set $\Phi(V) := \int_\Omega \phi(x, V(x))\,\mathrm{d}x$. Then, the following holds:*

$$\left.\begin{array}{r} V_k \rightharpoonup V \ \text{in } \mathcal{V}, \\ \Phi(V_k) \to \Phi(V), \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} V_k \to V \ \text{in } \mathcal{V}, \\ \phi(\cdot, V_k(\cdot)) \to \phi(\cdot, V(\cdot)) \ \text{in } L^1(\Omega). \end{array}\right.$$

4.2 Temporal Hölder- and Lipschitz-continuity

In this section we generalize the ideas developed in [MiT04, MiR07], where Lipschitz continuity with respect to time was derived. Our generalization has two aspects. First we emphasize that the convexity properties can be formulated with respect to a norm $\|\cdot\|_\mathcal{V}$ that may differ significantly from that in the underlying space $\mathcal{Q}$, which was chosen to be as small as possible as long as we keep the coercivity of $\mathcal{E}$, see (E1). Second we generalize the notion of uniform convexity by allowing for a weaker lower bound in (4.4). Previous work asked $\alpha = 2$ and $\beta = 1$ and enforced the condition on whole $\mathcal{Q}$, while we only pose it on sublevels.

After we have established the main abstract result in Theorem 4.5, we will show how the main assumptions can be satisfied for integral functionals in Lemma 4.6. Examples and applications to damage will be given in Section 5.

Theorem 4.5 (Temporal Hölder continuity) *Assume for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ that $\mathcal{Q}$ is a closed convex subset of a Banach space. Let $L_E(t) = \{q \in \mathcal{Q} \,|\, \mathcal{E}(t, q) \le E\}$. Moreover, there are $\alpha \ge 2$, $\beta \le 1$ such that for all E_* there exist C_*, $c_* > 0$ so that for all $t \in [0, T]$, $q_0, q_1 \in L_{E_*}(t)$ and all $\theta \in [0, 1]$ the following holds:*

$$\begin{aligned} &\mathcal{E}(t, q_\theta) + \mathcal{D}(z_0, z_\theta) + c_*\theta(1-\theta)\|q_1 - q_0\|^\alpha_\mathcal{V} \\ &\quad \le (1-\theta)\big(\mathcal{E}(t, q_0) + \mathcal{D}(z_0, z_0)\big) + \theta\big(\mathcal{E}(t, q_1) + \mathcal{D}(z_0, z_1)\big) \end{aligned} \tag{4.4a}$$

$$|\partial_t\mathcal{E}(t, q_1) - \partial_t\mathcal{E}(t, q_0)| \le C_*\|q_1 - q_0\|^\beta_\mathcal{V}, \tag{4.4b}$$

where $(u_\theta, z_\theta) = q_\theta = (1{-}\theta)q_0 + \theta q_1$.

Then, any energetic solution $q : [0,T] \to \mathcal{Q}$ *of* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *is Hölder continuous from* $[0,T]$ *to* $\mathcal{V}$ *with the exponent* $1/(\alpha{-}\beta)$*, i.e. there is a constant* $C_\mathrm{H} > 0$ *such that*

$$\|q(s){-}q(t)\|_\mathcal{V} \leq C_\mathrm{H}|t{-}s|^{1/(\alpha-\beta)} \quad \textit{for all } s,t \in [0,T]\,. \tag{4.5}$$

Proof: We proceed in three steps. First we derive an improved stability condition (S), where an additional term of the form $c_*\theta(1{-}\theta)\|q_1{-}q_0\|_\mathcal{V}^\alpha$ appears on the left-hand side. Second, following [MiT04, MiR07], we derive an estimate for $\|q(s){-}q(t)\|_\mathcal{V}$ and finally we use a differential inequality to obtain (4.5).

Step 1. Improved stability estimate:
Choose E_* such that $\mathcal{E}(t, q(t)) \leq E_*$ for all t. For fixed $s, t \in [0,T]$ we apply (4.4a) with $q_0 = q(t)$ and $q_1 = q(s)$. By the stability of $q(t)$ we find

$$\begin{aligned}\mathcal{E}(t, q_0) &\leq \mathcal{E}(t, q_\theta) + \mathcal{D}(z, z_\theta)\\ &\leq (1{-}\theta)\mathcal{E}(t, q_0) + \theta\big(\mathcal{E}(t, q_1){+}\mathcal{D}(z_0, z_1)\big) - c_*\theta(1{-}\theta)\|q_1{-}q_0\|_\mathcal{V}^\alpha.\end{aligned}$$

After subtracting $\mathcal{E}(t, q_0)$ from both sides we may divide by θ and pass to the limit $\theta \to 0^+$. Recalling $q_0 = q(t)$ and $q_1 = q(s)$ this leads to

$$\mathcal{E}(t, q(t)) + c_*\|q(t){-}q(s)\|_\mathcal{V}^\alpha \leq \mathcal{E}(t, q(s)) + \mathcal{D}(z(t), z(s)), \tag{4.6}$$

which is the desired improved stability estimate. (In fact, in place of $q(s)$ we could have taken any $\widetilde{q}$ with $\mathcal{E}(t, \widetilde{q}) \leq E_*$; or vice versa, we could have weakened condition (4.4) by assuming it only for stable states.)

Step 2. Estimate for $\|q(t){-}q(s)\|_\mathcal{V}$*:*
Now we assume $0 \leq s \leq t \leq T$ and interchange the role of s and t in (4.6). Employing $\mathcal{D}(z(s), z(t)) \leq \mathrm{Diss}_\mathcal{D}(z; [s,t])$ and the energy balance we find

$$\begin{aligned}c_*\|q(t){-}q(s)\|_\mathcal{V}^\alpha &\leq \mathcal{E}(s, q(t)) + \mathcal{D}(z(s), z(t)) - \mathcal{E}(s, q(s))\\ &\leq \mathcal{E}(s, q(t)) - \mathcal{E}(t, q(t)) + \mathcal{E}(t, q(t)) + \mathrm{Diss}_\mathcal{D}(z; [s,t]) - \mathcal{E}(s, q(s))\\ &= \int_s^t \partial_\xi\mathcal{E}(\xi, q(t)){-}\partial_\xi\mathcal{E}(\xi, q(\xi))\,\mathrm{d}\xi \leq \int_s^t C_*\|q(t){-}q(\xi)\|_\mathcal{V}^\beta\,\mathrm{d}\xi,\end{aligned}$$

where we used (4.4b) in the last estimate.

Step 3. Hölder estimate:
Putting $h(\tau) := \int_{t-\tau}^t \|q(\xi) - q(t)\|_\mathcal{V}^\beta \mathrm{d}\xi$ for $\tau \in [0, t-s]$ yields $h'(\tau) \leq \big(\frac{C_*}{c_*}h(\tau)\big)^{\beta/\alpha}$. Using $h(0) = 0$ leads to $h(\tau) \leq C_1\tau^{\alpha/(\alpha-\beta)}$ with a constant C_1 depending only on C_*, c_*, α, and β. Hence we conclude

$$\|q(s) - q(t)\|_\mathcal{V} = h'(t{-}s)^{1/\beta} \leq \Big(\tfrac{C_*}{c_*}h(t{-}s)\Big)^{1/\alpha} \leq \Big(\tfrac{C_*C_1}{c_*}h(t{-}s)\Big)^{1/\alpha}(t{-}s)^{1/(\alpha-\beta)},$$

which is the desired result. ■

We now discuss a few results which are useful to establish the assumptions in (4.4) for integral functionals.

Lemma 4.6 (On the convexity assumptions)
(A) Assume that $\mathcal{D}(z_0,\cdot):\mathcal{Z}\to[0,\infty]$ *and* $\mathcal{C}:\mathcal{Q}\to\mathbb{R}_\infty$ *are convex and that* $\mathcal{W}:\mathcal{Q}\to\mathbb{R}_\infty$ *satisfies the following:*

$$\begin{aligned}\forall\, E_*\ \exists\, C_W, c_w>0\ \forall\, q_0,q_1 \text{ with } \mathcal{W}(q_0),\mathcal{W}(q_1)\le w_*\ \forall\,\theta\in[0,1]:\\ \mathcal{W}((1{-}\theta)q_0{+}\theta q_1)+c_w\theta(1{-}\theta)\|q_1{-}q_0\|_{\mathcal{V}}^\alpha\le(1{-}\theta)\mathcal{W}(q_0)+\theta\mathcal{W}(q_1).\end{aligned}\tag{4.7}$$

Then, with $\mathcal{E}(t,\cdot)=\mathcal{W}+\mathcal{C}$ *condition* (4.4a) *holds.*
(B) For $j\in\{1,\dots,m\}$ *let* $V_j\in\{\mathbb{R},\mathbb{R}^d,\mathbb{R}^{d\times d}\}$ *and let* $\mathbb{V}:=\times_{j=1}^m V_j$. *Assume that* $\mathbb{W}:\Omega{\times}\mathbb{V}\to[0,\infty]$ *is a Carathéodory function and that there exist* $k\in\{0,1,...,m\}$, $C_1,c_1,c_0>0$ *and* $p_j>1$ *with* $p_j\ge2$ *for* $j\le k$ *and* $p_j<2$ *for* $j>k$ *such that for a.a.* $x\in\Omega$ *and all* b, b^0, $b^1\in\mathbb{V}$ *the following estimates hold:*

$$\mathbb{W}(x,b)\ge c_0\sum_{j=1}^m|b_j|^{p_j}-C_1,\tag{4.8a}$$

$$\begin{aligned}&c_1\theta(1{-}\theta)\Big(\textstyle\sum_{j=1}^k|b_j^1{-}b_j^0|^{p_j}+\sum_{j=k+1}^m\frac{|b_j^1-b_j^0|^2}{(1+\mathbb{W}(x,b^0)+\mathbb{W}(x,b^1))^{\gamma_j}}\Big)\\&\qquad\le(1{-}\theta)\mathbb{W}(x,b^0)+\theta\mathbb{W}(x,b^1)-\mathbb{W}(x,(1{-}\theta)b^0{+}\theta b^1),\end{aligned}\tag{4.8b}$$

where $\gamma_j=(2{-}p_j)/p_j\in(0,1)$. *Then, with* $\mathcal{V}=\times_{j=1}^m L^{p_j}(\Omega)$ *and* $\mathcal{W}(v)=\int_\Omega\mathbb{W}(x,v(x))\,\mathrm{d}x$ *the condition* (4.7) *holds with* $\alpha=\max\{p_1,...,p_k,2\}$.
(C) Assume that for a.a. $x\in\Omega$ *we have* $\mathbb{W}(x,\cdot)\in\mathrm{C}^1(\mathbb{V})$ *and that there is a constant* $c_*>0$ *such that the following holds for all* $b^0,b^1\in\mathbb{V}$:

$$\begin{aligned}&\mathbb{W}(x,b^1)-\mathbb{W}(x,b^0)-\partial_b\mathbb{W}(b^0)\cdot(b^1{-}b^0)\\&\ge c_*\sum_{j=1}^k|b_j^1{-}b_j^0|^{p_j}{+}c_*\sum_{j=k+1}^m\frac{|b_j^1{-}b_j^0|^2}{(1+\mathbb{W}(x,b^0)+\mathbb{W}(x,b^1))^{\gamma_j}}\end{aligned}\tag{4.9}$$

for p_j, γ_j *as in part (B). Then* $\mathbb{W}$ *satisfies* (4.8b).
(D) Let $\mathbb{P}:\Omega{\times}\mathbb{R}^m\to\mathbb{R}$ *be a Carathéodory function satisfying*

$$|\mathbb{P}(x,b)|\le C_2\mathbb{W}(x,b)+C_3,\tag{4.10a}$$

$$|\mathbb{P}(b^1){-}\mathbb{P}(b^0)|\le C_4\sum_{j=1}^m(1{+}\mathbb{W}(x,b^0){+}\mathbb{W}(x,b^1))^{\delta_j}|b_j^1{-}b_j^0|,\tag{4.10b}$$

where $\delta_j=(p_j{-}1)/p_j\in(0,1)$ *and* $\mathbb{W}$ *fulfills* (4.8). *For* $\mathcal{W}(v)<\infty$ *define* $\mathcal{P}(v)=\int_\Omega\mathbb{P}(x,v(x))\,\mathrm{d}x$. *Then, for each* E_* *there exists* $C_*^{\mathcal{P}}$ *such that for all* $v_0,v_1\in\mathcal{V}$ *with* $\mathcal{W}(v_0)$, $\mathcal{W}(v_1)\le E_*$ *we have* $|\mathcal{P}(v_1)-\mathcal{P}(v_0)|\le C_*^{\mathbb{P}}\|v_1{-}v_0\|_{\mathcal{V}}$.

Proof: Part (A) follows simply by using the convexity of $\mathcal{D}(z,\cdot)$ and $\mathcal{C}$ and adding it to the estimate provided by (4.7).

For Part (B) be first note that $\mathcal{W}(v^0)$, $\mathcal{W}(v^1)\le E_*$ together with (4.8a) implies that there is a constant Λ_* such that

$$\|v_j^n\|_{L^{p_j}(\Omega)}\le\Lambda_*\quad\text{for } n\in\{0,1\}\text{ and } j=1,...,m.$$

Setting $b^n = v^n(x)$ and integrating both sides of (4.8b) over the domain Ω it remains to estimate the left-hand side from below. For $j > k$ we derive a so-called reverse Hölder's inequality for the the quotient $u^2/N^{-\gamma}$ via

$$\int_\Omega u^{2/(1+\gamma)}\,\mathrm{d}x \leq \Big(\int_\Omega u^2/N^\gamma\,\mathrm{d}x\Big)^{1/(1+\gamma)}\Big(\int_\Omega N\,\mathrm{d}x\Big)^{\gamma/(1+\gamma)}$$

where $u = |v_j^1(x) - v_j^0|$ and $N = 1 + \mathbb{W}(v^0) + \mathbb{W}(v^1)$. This provides the lower bound

$$\begin{aligned}&(1-\theta)\mathcal{W}(v^0) + \theta\mathcal{W}(v^1) - \mathcal{W}((1-\theta)v^0 + \theta v^1)\\ &\geq c_1\theta(1-\theta)\Big(\textstyle\sum_{j=1}^k \|v_j^1 - v_j^0\|_{L^{p_j}}^{p_j} + \sum_{j=k+1}^m \frac{\|v_j^1 - v_j^0\|_{L^{p_j}}^2}{(|\Omega| + 2E_*)^{\gamma_j}}\Big).\end{aligned}$$

Since $\alpha = \max\{p_1, ..., p_k, 2\}$ the desired lower bound (4.7) follows from $\rho \leq \alpha$ and from $\|v_j^1 - v_j^0\|_{L^{p_j}}^\rho \geq \|v_j^1 - v_j^0\|_{L^{p_j}}^\alpha/(2\Lambda_*)^{\alpha-\rho}$.

To establish Part (C) we let $b^\theta = (1-\theta)b^0 + \theta b^1$ and apply (4.9) with b^0 replaced by b^θ. Dropping x for notational simplicity and using $b^1 - b^\theta = (1-\theta)(b^1 - b^0)$ we find

$$\begin{aligned}&\mathbb{W}(b^1) - \mathbb{W}(b^\theta) - (1-\theta)\partial_b\mathbb{W}(b^\theta)\cdot(b^1 - b^0)\\ &\geq c_*\sum_{j=1}^k (1-\theta)^{p_j}|b_j^1 - b_j^0|^{p_j} + c_*(1-\theta)^2\sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(b^1) + \mathbb{W}(b^\theta))^{\gamma_j}}.\end{aligned} \tag{4.11}$$

Similarly, we may replace b^0 by b^1 in (4.9) by b^θ and b^0, respectively, and find, using $b^0 - b^\theta = -\theta(b^1 - b^0)$,

$$\begin{aligned}&\mathbb{W}(b^0) - \mathbb{W}(b^\theta) + \theta\partial_b\mathbb{W}(b^\theta)\cdot(b^1 - b^0)\\ &\geq c_*\sum_{j=1}^k \theta^{p_j}|b_j^1 - b_j^0|^{p_j} + c_*\theta^2\sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(b^0) + \mathbb{W}(b^\theta))^{\gamma_j}}.\end{aligned} \tag{4.12}$$

Multiplying (4.11) by θ and (4.12) by $1-\theta$ and adding the results, the term with the partial derivative cancels and we obtain

$$\begin{aligned}&(1-\theta)\mathbb{W}(b^0) + \theta\mathbb{W}(b^1) - \mathbb{W}(b^\theta)\\ &\geq c_*\sum_{j=1}^k \big(\theta(1-\theta)^{p_j} + (1-\theta)\theta^{p_j}\big)|b_j^1 - b_j^0|^{p_j} + c_*\theta(1-\theta)\sum_{j=k+1}^m A_j(\theta, b^1, b_0)|b_j^1 - b_j^0|^2\end{aligned}$$

where $A_j(\theta, b^1, b_0) = \dfrac{1-\theta}{(1 + \mathbb{W}(b^1) + \mathbb{W}(b^\theta))^{\gamma_j}} + \dfrac{\theta}{(1 + \mathbb{W}(b^0) + \mathbb{W}(b^\theta))^{\gamma_j}}$.

Since $\theta(1-\theta)^{p_j} + (1-\theta)\theta^{p_j} \geq \theta(1-\theta)/2^{p_j}$ it suffices to estimate the terms A_j from below. Letting $w_n = \mathbb{W}(b^n)$ convexity gives $\mathbb{W}(b^\theta) \leq (1-\theta)w_0 + \theta w_1$. Using $\theta \in [0, 1]$ we find

$$\begin{aligned}A_j(\theta, b^1, b_0) \;&\geq\; \frac{1-\theta}{(1 + (1+\theta)w_1 + (1-\theta)w_0)^{\gamma_j}} + \frac{\theta}{(1 + (2-\theta)w_1 + \theta w_0)^{\gamma_j}}\\ &\geq\; \Big(\frac{1-\theta}{(1+\theta)^{\gamma_j}} + \frac{\theta}{(2-\theta)^{\gamma_j}}\Big)\frac{1}{(1 + w_1 + w_0)^{\gamma_j}} \;\geq\; \frac{(2/3)^{\gamma_j}}{(1 + w_1 + w_0)^{\gamma_j}}.\end{aligned}$$

Thus, (4.8) is established and Part (C) is proved.

Part (D) follows by a direct application of Hölder's inequality providing

$$|\mathcal{P}(v^1)-\mathcal{P}(v^0)| \leq C_4 \sum_{j=1}^{m} \big(|\Omega|+2E_*\big)^{\delta_j} \|v_j^1-v_j^0\|_{L^{p_j}} \leq C_*^{\mathcal{P}} \|v^1-v^0\|_{\mathcal{V}}$$

with $C_*^{\mathcal{P}} = \max\{(|\Omega|+2E_*)^{\delta_j} \,|\, j = 1, ..., m\}$. ■

Note that Part (D) will be applied to $\mathcal{P}(q) = \partial_t \mathcal{E}(t, q)$ which is given in (3.12). Clearly, the linear term involving $\dot{l}(t)$ can be estimated directly. Thus, for fixed $t \in [0, T]$ the density $\mathbb{P}$ will have the form

$$\mathbb{P}(x, e, z) = \partial_e W(x, e{+}e_D(t, x)){:}\dot{e}_D(t, x),$$

where e_D is given in (A2) of (3.2), see before Corollary 5.4 for more details.

5 Examples

In this section we give examples on stored elastic energy densities that are well known from engineering literature and that satisfy the hypotheses stated in Section 3.1. Moreover, we provide examples fitting to the setup of Lemma 4.6. To simplify notations we drop the explicit dependence on the material coordinates $x \in \Omega$. Of course, the result generalize to heterogeneous materials, if all the estimates are uniform as assumed in the previous sections.

5.1 Elastic energy densities with additional convexity properties

Sections 5.1.1 and 5.1.2 deal with examples on the different types of convexity. They all use Part (C) of Lemma 4.6.

5.1.1 Examples on joint convexity, strict convexity and uniform convexity

In the modeling of damage the inner variable often influences the stored elastic energy density in form of a product. The function $\widehat{W}$ analyzed in the following was first introduced in [Rou08]. There, it was shown that such product can be jointly convex in the two variables e and z. With regard to Lemma 4.6 we summarize several properties of $\widehat{W}$ in the next lemma.

Lemma 5.1 *For $g \in C^2([0, 1], (0, 1])$, $a \geq 0$ and $\mathbb{B} \in \mathbb{R}^{(d\times d)\times(d\times d)})$ symmetric and positive definite let*

$$W(e, z) := \frac{1}{2g(z)}\, e{:}\mathbb{B}{:}e + \frac{a}{2}\, z^2,$$

where we further assume $1 = g(0) > g(1) > 0$, $g'(z) \leq 0$ and $g''(z) \leq -\gamma \leq 0$ for $z \in [0, 1]$. Then, $W : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [0, 1] \to \mathbb{R}$ is convex and there exists a constant $C > 0$ such *that for all* e, $\widehat{e}, z$, *and* $\widehat{z}$ *we have*

$$|\partial_e W(e,z)| \leq C\,(\widehat{W}(e,z){+}1)\,, \tag{5.1}$$
$$|\partial_e W(e,z){-}\partial_e W(\widehat{e},\widehat{z})| \leq C|e{-}\widehat{e}| + C\left(1{+}W(e,z){+}W(\widehat{e},\widehat{z})\right)^{1/2}|z{-}\widehat{z}|\,. \tag{5.2}$$

If additionally $a > 0$ *and* $\gamma > 0$*, then there exists* $c_* > 0$ *such that*

$$W(\widehat{e},\widehat{z}) - W(e,z) - \partial_e W(e,z){:}(\widehat{e}{-}e) - \partial_z W(e,z)(\widehat{z}{-}z) \geq \frac{c_*}{2}\left(|\widehat{e}{-}e|^2 + |\widehat{z}{-}z|^2\right). \tag{5.3}$$

Proof: The estimates (5.1) and (5.2) follow easily from the linear structure $\partial_e W(e,z) = \frac{1}{g(z)}\mathbb{B}{:}e$ and the positive definiteness of $\mathbb{B}$, namely $W(e,z) \geq c_1|e|^2$ for all e and z.

To establish the convexity properties we calculate the Hessian D^2W explicitly. Omitting the argument z in g and its derivatives we obtain

$$D^2W(e,z)\left[\binom{E}{Z},\binom{E}{Z}\right] = \frac{1}{g^3}(gE{-}g'Ze){:}\mathbb{B}{:}(gE{-}g'Ze) + \frac{-g''}{2g^2}e{:}\mathbb{B}{:}eZ^2 + aZ^2, \tag{5.4}$$

which provides convexity since all terms on the right-hand side are nonnegative.

To derive strict convexity we let $\delta(z) = g'(z)/g(z) \in [-\delta_0, \delta_0]$ and use $g''(z) \leq -\gamma < 0$ to find c_2, $c_3 > 0$ such that

$$\begin{aligned} D^2W(e,z)\left[\binom{E}{Z},\binom{E}{Z}\right] &\geq c_2|E{-}\delta Ze|^2 + c_3|e|^2Z^2 + aZ^2 \\ &\geq \frac{c_2\varepsilon}{1+\varepsilon}|E|^2 + (c_3{-}\varepsilon\delta_0^2c_2)|e|^2Z^2 + aZ^2. \end{aligned}$$

Choosing $\varepsilon = c_3/(\delta_0^2c_2)$ we obtain (5.3) with $c_* = \min\{a, c_2c_3/(c_3{+}\delta_0^2)\}$ employing the classical convexity arguments. ■

The above lemma states that the stored energy density $W(e,z) = \frac{1}{\eta-z}\,e{:}\mathbb{B}{:}e + az^2/2$ with $\eta > 1$, $a \geq 0$, and $\mathbb{B}$ symmetric and positive definite is convex. For $a = 0$, it is not strictly convex, since $W(0,z) = 0$ for $z \in [0,1]$. For $a > 0$ we gain strict convexity but still do not have uniform convexity for W on $\mathbb{R}^{d\times d}_{\text{sym}} \times [0,1]$, since $g'' \equiv 0$, i.e., $\gamma = 0$. For C^2 functions uniform convexity is equivalent to $D^2W(e,z)\left[\binom{E}{Z},\binom{E}{Z}\right] \geq c_*(|E|^2{+}Z^2)$ for some fixed $c_* > 0$. However, inserting $(E,Z) = (\delta e, 1)$ into the formula (5.4) gives $D^2W(e,z)\left[\binom{\delta e}{1},\binom{\delta e}{1}\right] = a$, while $|\delta e|^2{+}1$ may be arbitrarily big, since $\delta(z) = g'(z)/g(z) = -1/(\eta{-}z) < 0$.

5.1.2 More examples on uniform convexity

In this section we construct an example for uniform convex stored elastic energy densities that have variables being parts of the strain tensor, like its deviator, mean strain or a single component. Thereto we will use the functions introduced in the lemma below.

Lemma 5.2 *Let* $V \in \{\mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d\times d}\}$ *have the scalar product* $A_1{\cdot}A_2 \in \mathbb{R}$ *for all* $A_1, A_2 \in V$. *For* κ, $\varepsilon > 0$, *and* $p \in (1,\infty)$ *let* $Z_{p\kappa\varepsilon}(A) := \frac{\kappa}{p}(\varepsilon{+}|A|^2)^{\frac{p}{2}}$ *for* $A \in V$. *Then there exist* *constants* $c_{p\kappa\varepsilon}$, C_p, $k_{p\kappa\varepsilon} > 0$ *and* $\lambda_p \in \{0, \varepsilon\}$ *such that for all* $A_1, A_2, A \in V$ *we have*

$$Z_{p\kappa\varepsilon}(A_1) - Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}(A_2) \cdot (A_1 - A_2) + c_{p\kappa\varepsilon}(\lambda_p + |A_1| + |A_2|)^{p-2}|A_1 - A_2|^2, \tag{5.5}$$

$$|\partial_A Z_{p\kappa\varepsilon}(A)| \leq C_p(Z_{p\kappa\varepsilon}(A) + 1) \tag{5.6}$$

$$|\partial_A Z_{p\kappa\varepsilon}(A_1) - \partial_A Z_{p\kappa\varepsilon}(A_2)| \leq \begin{cases} k_{p\kappa\varepsilon}|A_1 - A_2| & \text{if } 1 < p < 2 \\ k_{p\kappa\varepsilon}(\sqrt{\varepsilon} + |A_1| + |A_2|)^{p-2}|A_1 - A_2| & \text{if } p \geq 2 \end{cases} \tag{5.7}$$

Proof: In the proof we omit the subscripts p, κ, and ε. Direct computations give

$$\partial_A Z(A_2) \cdot A_1 = \kappa(\varepsilon + |A_2|^2)^{\frac{p-2}{2}} A_2 \cdot A_1,$$
$$\partial_A^2 Z(A_2)[A_1, A_3] = (p-2)\kappa(\varepsilon + |A_2|^2)^{\frac{p-4}{2}}(A_2 \cdot A_1)(A_2 \cdot A_3) + \kappa(\varepsilon + |A_2|^2)^{\frac{p-2}{2}} A_1 \cdot A_3 .$$

Estimate (5.5) can be verified by a Taylor expansion of $\xi \mapsto Z(A_2 + \xi(A_1 - A_2))$ in the point $\xi = 0$ with a remainder term of order 2 using the ideas of [Kne04].

Estimate (5.6) is obtained, with $C_p = p^{(p-1)/p}$, via

$$|\partial_A Z(A)| \leq \kappa(\varepsilon + |A|^2)^{\frac{p-2}{2}}(\varepsilon + |A|^2)^{\frac{1}{2}} = (pZ(A))^{(p-1)/p} \leq C_p(Z(A) + 1).$$

In the following we carry out the proof estimate (5.7) using a Taylor expansion of $f(\xi) := \partial_A Z(A_2 + \xi(A_1 - A_2))$ in the point $\xi = 0$ with a remainder term of order 1 :

$$|\partial_A Z(A_1) - \partial_A Z(A_2)| = |f(1) - f(0)| \leq \int_0^1 \left|\tfrac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right| \mathrm{d}\xi .$$

We let $A^\xi := A_2 + \xi(A_1 - A_2)$. For $1 < p \leq 2$ we have

$$\begin{aligned} \left|\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right| &= \left|\partial_A^2 Z(A^\xi)[A_1 - A_2, \cdot]\right| \\ &\leq \Big((2-p)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-4}{2}}|A^\xi|^2 + \kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}}\Big)|A_1 - A_2| \\ &\leq (3-p)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}}|A_1 - A_2| \leq (3-p)\kappa\varepsilon^{\frac{p-2}{2}}|A_1 - A_2|. \end{aligned}$$

This provides the upper estimate in (5.7). Similarly, for $p \geq 2$ we have

$$\begin{aligned} &\left|\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right| = \left|\partial_A^2 Z_{p\kappa\varepsilon}(A^\xi)[A_1 - A_2, \cdot]\right| \\ &\leq (p-1)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}}|A_1 - A_2| \leq (p-1)\kappa(\sqrt{\varepsilon} + |A_1| + |A_2|)^{p-2}|A_1 - A_2|, \end{aligned}$$

which is gives the lower estimate in (5.7). ■

We introduce linear, operators $g_i : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}$ of the form:

deviator:	$g_i(e) = e^D := e - \frac{\mathrm{tr}\, e}{d}\mathrm{Id}$	(5.8a)
volumetric strain:	$g_i(e) := \frac{\mathrm{tr}\, e}{d}\mathrm{Id}$	(5.8b)
kl-th component of e :	$g_i(e) := e_{kl}\mathrm{M}_{kl}$ for $k, l \in 1, \ldots, d$,	(5.8c)
	where M_{kl} has the entry 1 at position kl and 0 else.	

These operators are used in the lemma below.

Lemma 5.3 *For $1 < q, p_i, r, \tilde{r} < \infty$, $\varepsilon_q, \varepsilon_i, \kappa_q, \kappa_i, \kappa, \tilde{\kappa} > 0$, and $\tilde{\varepsilon} \geq 0$ let*

$$\begin{aligned}\overline{W}(t,x,e,z,A) := \widehat{W}(e{+}e_D(t,x),z){+}Z_{q\kappa_q\varepsilon_q}(e{+}e_D(t,x))\\ +\sum_{i=2}^{N} Z_{p_i\kappa_i\varepsilon_i}(g_i(e{+}e_D(t,x))){+}Z_{\tilde{r}\tilde{\kappa}\tilde{\varepsilon}}(z){+}Z_{r\kappa 0}(A),\end{aligned}\tag{5.9}$$

where $\widehat{W}$ is as in Lemma 5.1 with $\gamma, a > 0$ and the linear operators $g_i : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}$ are as in (5.8)*. Then, $\overline{W}$ satisfies* (4.8) *and $\partial_t\overline{W}$ satisfies* (4.10)*.*

Proof: We put $b = (e, e, g_2(e), \ldots, g_N(e), z, A)$ and $m = N + 3$. Then we have $\mathbb{W}(b) = \overline{W}(e, z, A)$ and $\mathbb{P}(b) = \partial_t\overline{W}(e{+}e_D, z, A) = \partial_e\overline{W}(e{+}e_D, z, A){:}\dot{e}_D$. The latter relation is due to the chain rule and the fact that g_i are linear, self-adjoint and idempotent:

$$\begin{aligned}&\partial_t Z_{p\kappa\varepsilon}(g_i(e{+}e_D)) = \partial_{g_i(e)}Z_{p\kappa\varepsilon}(g_i(e{+}e_D)){:}g_i(\dot{e}_D)\\ &= \kappa(\varepsilon{+}|g_i(e{+}e_D)|^2)^{\frac{p-2}{2}}g_i(g_i(e{+}e_D)){:}\dot{e}_D = \partial_{g_i(e)}Z_{p\kappa\varepsilon}(g_i(e{+}e_D)){:}\dot{e}_D\,.\end{aligned}\tag{5.10}$$

For $i = 1, 2$ let A_i be a component of b^i. Inequality (4.8a) is obvious, so that we only prove (4.8b) in detail by showing (4.9). From (5.5) in Lemma 5.2 we derive for $p \geq 2$ that $Z_{p\kappa\varepsilon}(A_1){-}Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}{\cdot}(A_1{-}A_2){+}c_{p\kappa\varepsilon}|A_1{-}A_2|^p$ and for $1 < p < 2$:

$$\begin{aligned}&Z_{p\kappa\varepsilon}(A_1){-}Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}(A_2){\cdot}(A_1{-}A_2){+}c_{p\kappa\varepsilon}(\lambda_p{+}|A_1|{+}|A_2|)^{p-2}|A_1{-}A_2|^2\\ &\geq \partial_A Z_{p\kappa\varepsilon}(A_2){\cdot}(A_1{-}A_2){+}\kappa^{\frac{2-p}{2}}(\lambda_p\kappa{+}(\varepsilon{+}|A_1|^2)^{\frac{p}{2}}{+}(\varepsilon{+}|A_2|^2)^{\frac{p}{2}})^{\frac{p-2}{2}}|A_1{-}A_2|^2\\ &\geq \partial_A Z_{p\kappa\varepsilon}(A_2){\cdot}(A_1{-}A_2){+}\kappa^{\frac{2-p}{2}}\min\{1,\lambda_p\kappa\}^{\frac{p-2}{p}}(1{+}\mathbb{W}(b^1){+}\mathbb{W}(b^2))^{\frac{p-2}{p}},\end{aligned}$$

which proves $\gamma_j = (2{-}p_j)/p_j$ for $j = 1, \ldots, m$. In view of (5.3) this proves (4.8b).

(4.10a) holds, since $|\partial_t\widehat{W}(e{+}e_D, z)|{=}|\partial_e\widehat{W}(e{+}e_D, z){:}\dot{e}_D|{\leq}c_D\tilde{c}(\overline{W}(t, e, z, A){+}1)$ by (5.1) and since $|\partial_t Z_{p\kappa\varepsilon}(g_i(e{+}e_D))|{=}|\partial_{g_i(e)}Z_{p\kappa\varepsilon}(g_i(e{+}e_D)){:}\dot{e}_D|{\leq}c_D C_{p\kappa\varepsilon}(Z_{p\kappa\varepsilon}(g_i(e{+}e_D)){+}1)$ due to (5.10), (5.6).

Inequality (4.10b) follows from (5.2) together with (5.7), since in the case $p \geq 2$ it holds

$$\begin{aligned}&|\partial_A Z_{p\kappa\varepsilon}(A_1){-}\partial_A Z_{p\kappa\varepsilon}(A_2)| \leq k_{p\kappa\varepsilon}(\varepsilon^{\frac{p}{2p}}{+}(\varepsilon{+}|A_1|^2)^{\frac{p}{2p}}{+}(\varepsilon{+}|A_2|^2)^{\frac{p}{2p}})^{p-2}|A_1{-}A_2|\\ &\leq k_{p\kappa\varepsilon}3^{\frac{p-1}{p}}(\varepsilon^{\frac{p}{2}}{+}(\varepsilon{+}|A_1|^2)^{\frac{p}{2}}{+}(\varepsilon{+}|A_2|^2)^{\frac{p}{2}})^{\frac{p-2}{p}}|A_1{-}A_2|\\ &\leq k_{p\kappa\varepsilon}3^{\frac{p-1}{p}}\max\{1,\varepsilon^{\frac{p-1}{2}}\}(1{+}\mathbb{W}(b^1){+}\mathbb{W}(b^2))^{\frac{p-1}{p}}|A_1{-}A_2|\,.\end{aligned}$$

■

For the free energy resulting from the density in (5.9) the space $\mathcal{V}$ of Lemma 4.6 is

$$\mathcal{V} := L^{\max\{2,q\}}(\Omega,\mathbb{R}^{d\times d}) \times \textstyle\bigtimes_{i=2}^{N} L^{p_i}(\Omega,\mathbb{R}^{d\times d}) \times L^{\max\{2,\tilde{r}\}}(\Omega,\mathbb{R}) \times L^r(\Omega,\mathbb{R}^d).$$

Hence, in estimate (4.4b) the term $|\partial_t\langle l(t), u_1{-}u_0\rangle|$ has to be estimated from above by $c_l C_K\|e(u_1){-}e(u_0)\|_{L^{\tilde{p}}(\Omega,\mathbb{R}^{d\times d})}$ with $\tilde{p} = \max\{2, q\}$ using Korn's inequality. Furthermore, we conclude by Lemma (4.6) that $\overline{W}$ is uniformly convex with the exponent $\alpha = \max\{2, q, p_i, r, \tilde{r}\,|\, i = 1, \ldots, N\}$ and by (4.10), that $\beta = 1$. Hence, by Theorem

4.5, the corresponding free energy functional is Lipschitz-continuous if $\alpha = 2$ and Hölder-continuous with the exponent $1/(\alpha-1)$ if $\alpha > 2$.

Furthermore, we mention that, if $\dot{u}_D(t) \equiv 0$ for all $t \in [0,T]$, then (4.4b) reduces to $|\partial_t\langle l(t), u_1-u_0\rangle| \leq c_l\|e(u_1)-e(u_0)\|_{W^{1,p}(\Omega,\mathbb{R}^d)}$ for p as in coercivity inequality (H3), which may satisfy $1 < p \leq 2$. Hence, if $\tilde{r} = r$ one can choose $\mathcal{V} = \mathcal{X}$, so that one can obtain the Hölder-estimate with respect to $\|\cdot\|_{\mathcal{X}}$.

Finally, we note that $\overline{W}$ fulfills all the hypotheses (H1)-(H5). For $\mathbb{B} \in L^\infty(\Omega, \mathbb{R}^{(d\times d)\times(d\times d)})$ we obtain that $\overline{W}$ is measurable in Ω and continuous with respect to (e,z,A), such that (H1) holds. Clearly, coercivity (H3) holds for the exponent $p \in (1, \max\{2,q\})$ and (H5), i.e. the monotonicity with respect to z is also given. Hypothesis (H4) holds due to (5.1), (5.6). If $q < p_i$ and $2 < p_i$ for some $i\in\{2,\ldots,N\}$, then (H4*) cannot be verified. But for $\overline{W}(t,x,e,z,A) = W(t,x,e,z)+Z_{2\kappa 0}(A)$ with W as in Lemma 5.1 (H4*) also holds true.

Corollary 5.4 *If the assumptions of Theorem 3.1 hold with $r \leq 2$ and if W is given as in Lemma 5.1, then all energetic solutions $q : [0,T] \to \mathcal{Q}$ satisfy $q \in C^{Lip}([0,T], \mathcal{V})$ with $\mathcal{V} = W^{1,2}(\Omega;\mathbb{R}^{d\times d}) \times W^{1,r}(\Omega)$.*

5.2 Damage of concrete

In the style of [Fré02, p. 319], where a model describing the damage of concrete is introduced, we consider here a stored elastic energy density of the form

$$W(e,z) := \mu|e|^2 + \varphi_-\big(\operatorname{tr}(-e)^+\big) + z\varphi_+\big(\operatorname{tr}(e)^+\big), \tag{5.11}$$

where $\mu > 0$ is the shear modulus. The functions $\varphi_\pm : [0,\infty) \to [0,\infty)$ only see the volume changes. They are convex and continuously differentiable with $\varphi_\pm(0) = 0$ and $|\varphi'_\pm(x)| \leq c(\varphi_\pm(x)+\hat{c})$ for constants $c, \hat{c} > 0$. Since damage mostly occurs under extension and compression corresponds to $\operatorname{tr}(e) < 0$, the function φ_- is not coupled to damage. However, φ_+ is premultiplied by z, since tension forces in concrete easily produces damage.

It is obvious that $W : \mathbb{R}^{d\times d}_{\mathrm{sym}}\times[0,1] \to \mathbb{R}$ satisfies (H1), (H3) and (H5). Convexity condition (H2) holds, since $\operatorname{tr}(\cdot)$ is linear, $\varphi_\pm$ are convex and $(\pm(\cdot))^+$ are convex as well. To demonstrate (H4) we use $\partial_e(\pm\operatorname{tr}(e)^+){:}\tilde{e} = \operatorname{sgn}(\pm\operatorname{tr}(e)^+)\mathrm{Id}{:}\tilde{e}$. Applying the chain rule on $\varphi_\pm\big(\operatorname{tr}(\pm e(u))^+\big)$ we conclude that

$$\begin{aligned}
|\partial_e W(e,z)| &= |2\mu e + \varphi'_-\big(\operatorname{tr}(-e)^+\big)\operatorname{sgn}(-\operatorname{tr}(e)^+)\mathrm{Id} + z\varphi'_+\big(\operatorname{tr}(e)^+\big)\operatorname{sgn}(\operatorname{tr}(e)^+)\mathrm{Id}| \\
&\leq \mu(|e|^2+1) + dc\big(\varphi_-\big(\operatorname{tr}(-e)^+\big)+\hat{c}\big) + zdc_+\big(\varphi_+\big(\operatorname{tr}(e)^+\big)+\hat{c}\big) \\
&\leq \max\{1,dc\}\big(W(e,z) + \max\{1,\hat{c}\}\big)\,.
\end{aligned}$$

5.3 Ramberg-Osgood materials

This section deals with Ramberg-Osgood materials, which are defined by energy densities composed similarly to (5.9), but formulated in terms of the complementary energy density depending on the stresses instead of the strains. Anyhow, in the following it is explained that the corresponding stored elastic energy density of Ramberg-Osgood materials can not be controlled by (H3) together with (H4*) but does satisfy (H3) together with estimate (H4). As introduced in [OsR43], Ramberg-Osgood materials can be described by a constitutive relation of power-law type formulated in terms of the complementary energy density

$$W_{\rm cp}:\mathbb{R}^{d\times d}\to\mathbb{R}:\sigma\mapsto\frac{1}{2}\sigma{:}\mathbb{A}:\sigma+\frac{a}{p'}|\sigma^D|^{p'},\tag{5.12}$$

which depends on the linearized 2nd Piola-Kirchhoff stress tensor σ and its deviatoric part $\sigma^D:=\sigma-\frac{1}{d}\operatorname{tr}\sigma\,\mathrm{Id}$. Thereby $a\in\mathbb{R}^+$, $2<p'<\infty$, and $\mathbb{A}\in\mathbb{R}^{(d\times d)\times(d\times d)}$ is symmetric, positive definite with constants $0<c_1^{\mathbb{A}}<c_2^{\mathbb{A}}$ such that $c_1^{\mathbb{A}}|e|^2\le e{:}\mathbb{A}{:}e\le c_2^{\mathbb{A}}|e|^2$. The complementary energy and the stored elastic energy, which depends on the strain tensor $e\in\mathbb{R}^{d\times d}_{\rm sym}$, are linked by a Legendre transform, i.e.:

$$W(e)=\sup_{\sigma\in\mathbb{R}^{d\times d}_{\rm sym}}\{\sigma:e-W_{\rm cp}(\sigma)\}\quad\text{so that}\quad\partial_eW(e)=\sigma\text{ and }\partial_\sigma W_{\rm cp}(\sigma)=e.\tag{5.13}$$

See [Zei85] Chap. 51 and [EkT76] Prop. IX 2.1. for more details. This relation together with (5.12) yields $e=\partial_\sigma W_{\rm cp}(\sigma)=\mathbb{A}(x):\sigma+a|\sigma^D|^{p'-2}\sigma^D$, which is used to check the hypotheses (H2)-(H4). In view of the first relation in (5.13), convexity is easily obtained for $W(\cdot)$. Furthermore, we derive the coercivity inequality:

$$\begin{aligned}W(e)&\ge\sup\nolimits_{\sigma\in\mathbb{R}^{d\times d}_{\rm sym}}\Big\{\sigma:e-\tfrac{c_2^{\mathbb{A}}}{2}|\sigma|^2-\tfrac{a}{p'}|\sigma^D|^{p'}\Big\}\\&=\sup\nolimits_{\sigma\in\mathbb{R}^{d\times d}_{\rm sym}}\Big\{\sigma^D:e^D-\tfrac{c_2^{\mathbb{A}}}{2}|\sigma^D|^2-\tfrac{a}{p'}|\sigma^D|^{p'}+\tfrac{1}{d^2}\operatorname{tr}\sigma\operatorname{tr}e-\tfrac{c_2^{\mathbb{A}}}{2}(\operatorname{tr}\sigma)^2\Big\}\\&=\sup\nolimits_{t\in\mathbb{R}}\Big\{\tfrac{t}{d^2}\operatorname{tr}e-\tfrac{c_2^{\mathbb{A}}}{2}t^2\Big\}+\sup\nolimits_{\tau\in\mathbb{R}^{d\times d}_{\rm dev}}\Big\{\tau:e^D-\tfrac{c_2^{\mathbb{A}}}{2}|\tau|^2-\tfrac{a}{p'}|\tau|^{p'}\Big\}\\&=\tfrac{1}{2d^4c_2^{\mathbb{A}}}(\operatorname{tr}e)^2+\sup\nolimits_{t\ge0}\Big\{t|e^D|-\tfrac{c_2^{\mathbb{A}}}{2}t^2-\tfrac{a}{p'}t^{p'}\Big\}\\&\ge\tfrac{1}{2d^4c_2^{\mathbb{A}}}(\operatorname{tr}e)^2+\sup\nolimits_{t\ge0}\Big\{t|e^D|-t^{p'}(\tfrac{2a}{p'})+C_1\Big\}\\&=\tfrac{1}{2d^4c_2^{\mathbb{A}}}(\operatorname{tr}e)^2+\tfrac{|e^D|^p}{p(2a)^{p-1}}-C_1\;\ge\;\min\Big\{\tfrac{1}{2d^4c_2^{\mathbb{A}}},\tfrac{1}{p(c_2^{\mathbb{A}}+a)^{p-1}}\Big\}|e|^p-C_2,\end{aligned}\tag{5.14}$$

where Young's inequality $t^2\le bt^{p'}+C_b$ has been used for the second estimate. The last inequality results from $1<p\le2$. Hence, (H3) holds for the exponent $p=\frac{p'}{p'-1}$. On the other hand we obtain with the same technique

$$\begin{aligned}W(e)\;&\le\;\sup_{\sigma\in\mathbb{R}^{d\times d}_{\rm sym}}\Big\{\sigma:e-\frac{c_1^{\mathbb{A}}}{2}|\sigma|^2-\frac{a}{p'}|\sigma^D|^{p'}\Big\}\le\frac{(\operatorname{tr}e)^2}{2c_1^{\mathbb{A}}d^4}+\frac{|e^D|^2}{8c_1^{\mathbb{A}}}+\frac{(p'-1)|e^D|^p}{2p'(2a)^{p-1}}\\&\le\;3\max\Big\{\frac{1}{2c_1^{\mathbb{A}}d^4},\frac{1}{8c_1^{\mathbb{A}}},\frac{(p'-1)}{2p'(2a)^{p-1}}\Big\}(|e|^2+2)\,,\end{aligned}$$

which yields $|\partial_eW(e)|\le c(|e|{+}\tilde c)$ due to convexity. Thus, (H3) and (H4*) are not satisfied for the same exponent. But (H3) in combination with (H4) holds, since (5.14) gives

$$|\partial_eW(e)|\le c(|e|{+}\tilde c)\le c\Big(\big(\frac{1}{c_0}(W(e){+}C_2)\big)^{1/p}+\tilde c\Big)\le c_1(W(e){*}\tilde c_1).$$

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