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Trace formulae for dissipative and coupled scattering systems

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Abstract

For scattering systems consisting of a (family of) maximal dissipative extension(s) and a selfadjoint extension of a symmetric operator with finite deficiency indices, the spectral shift function is expressed in terms of an abstract Titchmarsh-Weyl function and a variant of the Birman-Krein formula is proved.

Contents

1 Introduction

The main objective of this paper is to apply and to extend results from [9] and [10] on scattering matrices and spectral shift functions for pairs of selfadjoint or maximal dissipative extensions of a symmetric operator A with finite deficiency indices in a Hilbert space $\mathfrak{H}$.

Let us first briefly recall some basic concepts. For a pair of selfadjoint operators H and H_0 in $\mathfrak{H}$ the wave operators $W_\pm(H, H_0)$ of the scattering system $\{H, H_0\}$ are defined by

$$W_\pm(H, H_0) = s - \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}P_{ac}(H_0),$$

where $P^{ac}(H_0)$ is the projection onto the absolutely continuous subspace of the unperturbed operator H_0. If for instance the resolvent difference

$$(H - z)^{-1} - (H_0 - z)^{-1} \in \mathfrak{S}_1, \qquad z \in \rho(H) \cap \rho(H_0) \tag{1.1}$$

is a trace class operator, then it is well known that the wave operators $W_\pm(H, H_0)$ exist and are isometric in $\mathfrak{H}$, see, e.g. [53]. The scattering operator $S(H, H_0)$ of the scattering system $\{H, H_0\}$ is defined by

$$S(H, H_0) = W_+(H, H_0)^* W_-(H, H_0).$$

$S(H, H_0)$ commutes with H_0 and is unitary on the absolutely continuous subspace of H_0. Therefore $S(H, H_0)$ is unitarily equivalent to a multiplication operator induced by a family $S(H, H_0; \lambda)$ of unitary operators in the spectral representation of H_0. This family is usually called the scattering matrix of the scattering system $\{H, H_0\}$ and is one of the most important quantities in the analysis of scattering processes.

Another important object in scattering and perturbation theory is the so-called spectral shift function introduced by M.G. Krein in [33]. For the case $\mathrm{dom}\,(H) = \mathrm{dom}\,(H_0)$ and $V = H - H_0 \in \mathfrak{S}_1$ a spectral shift function ξ of the pair $\{H, H_0\}$ was defined with the help of the perturbation determinant

$$D_{H/H_0}(z) := \det\left((H - z)(H_0 - z)^{-1}\right). \tag{1.2}$$

Since $\lim_{|\Im\mathrm{m}\,(z)|\to\infty} D_{H/H_0}(z) = 1$ a branch of $z \mapsto \ln(D_{H/H_0}(z))$ in the upper half plane $\mathbb{C}_+$ is fixed by the condition $\ln(D_{H/H_0}(z)) \to 0$ as $\Im\mathrm{m}\,(z) \to \infty$ and the spectral shift function is then defined by

$$\xi(\lambda) = \frac{1}{\pi}\Im\mathrm{m}\left(\ln\left(D_{H/H_0}(\lambda + i0)\right)\right) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\Im\mathrm{m}\left(\ln\left(D_{H/H_0}(\lambda + i\varepsilon)\right)\right). \tag{1.3}$$

M.G. Krein proved that $\xi \in L_1(\mathbb{R}, d\lambda)$, $\|\xi\|_{L_1} \le \|V\|_1$, and that the trace formula

$$\mathrm{tr}\left((H - z)^{-1} - (H_0 - z)^{-1}\right) = -\int_{\mathbb{R}} \frac{\xi(\lambda)}{(\lambda - z)^2}\, d\lambda \tag{1.4}$$

holds for all $z \in \rho(H) \cap \rho(H_0)$. It turns out that the scattering matrix and the spectral shift function of the pair $\{H, H_0\}$ are related via the Birman-Krein formula:

$$\det\bigl(S(H, H_0; \lambda)\bigr) = \exp\bigl(-2\pi i\xi(\lambda)\bigr) \qquad \text{for a.e.} \quad \lambda \in \mathbb{R}. \tag{1.5}$$

The trace formula and the Birman-Krein formula can be extended to the case that only the resolvent difference (1.1) of H and H_0 is trace class. Namely, then there exists a real measurable function $\xi \in L^1(\mathbb{R},(1+\lambda^2)^{-1}d\lambda)$ such that (1.4) and (1.5) hold. However, in this situation it is not immediately clear how the perturbation determinant in (1.3) has to be replaced.

In Section 2 we propose a possible solution of this problem for pairs of selfadjoint extensions A_0 and A_Θ of a densely defined symmetric operator A with finite deficiency indices. Observe that here the resolvent difference is even a finite rank operator. In order to describe the pair $\{A_\Theta, A_0\}$ and a corresponding spectral shift function we use the notion of boundary triplets and associated Weyl functions. More precisely, we choose a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* and a selfadjoint parameter Θ in $\mathcal{H}$ such that $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and $A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$ holds. If $M(\cdot)$ is the Weyl function associated with this boundary triplet it is shown in Theorem 2.4 (see also [9] and [34] for special cases) that a spectral shift function $\xi(\cdot)$ of the pair $\{A_\Theta, A_0\}$ can be chosen as

$$\begin{aligned} \xi_\Theta(\lambda) &= \frac{1}{\pi}\Im\mathrm{m}\,\big(\mathrm{tr}\big(\log(M(\lambda+i0)-\Theta)\big)\big) \\ &= \frac{1}{\pi}\Im\mathrm{m}\,\big(\ln\big(\det(M(\lambda+i0)-\Theta)\big)\big) + 2k, \quad k\in\mathbb{Z}. \end{aligned} \tag{1.6}$$

By comparing (1.6) with (1.3) it is clear that $\det\big(M(z)-\Theta\big)$ plays a similar role as the perturbation determinant (1.2) for additive perturbations. Moreover, a simple proof of the Birman-Krein formula (1.5) in this situation is obtained in Section 2.5 by using the representation

$$S_\Theta(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(\Theta - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))} \tag{1.7}$$

of the scattering matrix $S_\Theta(\cdot) = S(A_\Theta, A_0;\cdot)$ of the scattering system $\{A_\Theta, A_0\}$ from [9], cf. also the work [7] by V.M. Adamyan and B.S. Pavlov.

These results are generalized to maximal dissipative extensions in Section 3. Let again A be a symmetric operator in $\mathfrak{H}$ with finite deficiency and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. If D is a dissipative matrix in $\mathcal{H}$, $\Im\mathrm{m}\,(D) \le 0$, then $A_D = A^* \upharpoonright \ker(\Gamma_1 - D\Gamma_0)$ is a maximal dissipative extension of A. For the scattering system $\{A_D, A_0\}$ the wave operators $W_\pm(A_D, A_0)$, the scattering operator $S(A_D, A_0)$ and the scattering matrix $S(A_D, A_0;\lambda)$ can be defined similarly as in the selfadjoint case. It turns out that the representation (1.7) extends to the dissipative case. More precisely, the Hilbert space $L^2(\mathbb{R}, \mathcal{H}_\lambda, d\lambda)$, $\mathcal{H}_\lambda := \mathrm{ran}\,(\Im\mathrm{m}\,(M(\lambda+i0))$, performs a spectral representation of the absolutely part A_0^{ac} of A_0 and the scattering matrix $S_D(\cdot) := S(A_D, A_0;\cdot)$ of the scattering system $\{A_D, A_0\}$ admits the representation

$$S_D(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))},$$

cf. [10, Theorem 3.8]. With the help of a minimal selfadjoint dilation $\widetilde{K}$ of A_D in the Hilbert space $\mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$, $\mathcal{H}_D := \mathrm{ran}\,(\Im\mathrm{m}\,(D))$, we verify that there is a spectral shift function η_D of the pair $\{A_D, A_0\}$ such that the trace formula

$$\mathrm{tr}\,\big((A_D - z)^{-1} - (A_0 - z)^{-1}\big) = -\int_\mathbb{R} \frac{\eta_D(\lambda)}{(\lambda - z)^2}\,d\lambda, \quad z\in\mathbb{C}_+,$$

holds and this spectral shift function $\eta_D(\cdot)$ admits the representation

$$\begin{aligned}\eta_D(\lambda) &= \frac{1}{\pi}\Im\mathrm{m}\left(\mathrm{tr}\big(\log(M(\lambda+i0)-D)\big)\right)\\ &= \frac{1}{\pi}\Im\mathrm{m}\left(\ln\big(\det(M(\lambda+i0)-D)\big)\right)+2k, \quad k\in\mathbb{Z},\end{aligned}$$

cf. Theorem 3.3. In Section 3.4 we show that the Birman-Krein formula holds in the modified form

$$\det(S_D(\lambda)) = \det(W_{A_D}(\lambda-i0)))\exp\big(-2\pi i\eta_D(\lambda)\big)$$

for a.e. $\lambda\in\mathbb{R}$, where $z\mapsto W_{A_D}(z)$, $z\in\mathbb{C}_-$, is the characteristic function of the maximal dissipative operator A_D. Since by [1, 2, 3, 4] the limit $W_{A_D}(\lambda-i0)^*$ can be regarded as the scattering matrix $S^{LP}(\cdot)$ of an appropriate Lax-Phillips scattering system one gets finally the representation

$$\det(S_D(\lambda)) = \overline{\det(S^{LP}(\lambda)))}\exp\big(-2\pi i\eta_D(\lambda)\big) \tag{1.8}$$

for a.e. $\lambda\in\mathbb{R}$. The results correspond to similar results for additive dissipative perturbations, [5, 42, 43, 44, 49, 50, 51].

In Section 4 so-called open quantum system with finite rank coupling are investigated. Here we follow the lines of [10]. From the mathematical point of view these open quantum systems are closely related to the Krein-Naimark formula for generalized resolvents and the Štrauss family of extensions of a symmetric operator. Recall that the Krein-Naimark formula establishes a one-to-one correspondence between the generalized resolvents $z\mapsto P_{\mathfrak{H}}(\widetilde{L}-z)^{-1}\restriction\mathfrak{H}$ of the symmetric operator A, that is, the compressed resolvents of selfadjoint extensions $\widetilde{L}$ of A in bigger Hilbert spaces, and the class of Nevanlinna families $\tau(\cdot)$ via

$$P_{\mathfrak{H}}(\widetilde{L}-z)^{-1}\restriction\mathfrak{H} = (A_0-z)^{-1}-\gamma(z)(\tau(z)+M(z))^{-1}\gamma(\bar{z})^*.$$

Here $\Pi_A=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is a boundary triplet for A^* and γ and M are the corresponding γ-field and Weyl function, respectively. It can be shown that the generalized resolvent coincides pointwise with the resolvent of the Štrauss extension

$$A_{-\tau(z)} := A^*\restriction\ker\big(\Gamma_1+\tau(z)\Gamma_0\big),$$

i.e., $P_{\mathfrak{H}}(\widetilde{L}-z)^{-1}\restriction\mathfrak{H}=(A_{-\tau(z)}-z)^{-1}$ holds, and that for $z\in\mathbb{C}_+$ each extension $A_{-\tau(z)}$ of A is maximal dissipative in $\mathfrak{H}$.

Under additional assumptions $\tau(\cdot)$ can be realized as the Weyl function corresponding to a densely defined closed simple symmetric operator T with finite deficiency indices in some Hilbert space $\mathfrak{G}$ and a boundary triplet $\Pi_T=\{\mathcal{H},\Upsilon_0,\Upsilon_1\}$ for T^*. Then the selfadjoint (exit space) extension $\widetilde{L}$ of A can be recovered as a coupling of the operators A and T corresponding to a coupling of the boundary triplets Π_A and Π_T (see [17] and formula (4.6) below). We prove in Theorem 4.2 that for such systems there exists a spectral shift function $\widetilde{\xi}(\cdot)$ given by

$$\widetilde{\xi}(\lambda) = \frac{1}{\pi}\Im\mathrm{m}\left(\mathrm{tr}\big(\log(M(\lambda+i0)+\tau(\lambda+i0))\big)\right)$$

and that the modified trace formula

$$\begin{aligned}&\mathrm{tr}\left((A_{-\tau(z)}-z)^{-1}-(A_0-z)^{-1}\right)+\\ &\qquad\mathrm{tr}\left((T_{-M(z)}-z)^{-1}-(T_0-z)^{-1}\right) = -\int_{\mathbb{R}}\frac{1}{(\lambda-z)^2}\,\widetilde{\xi}(\lambda)\,d\lambda\end{aligned}$$

holds for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Let $T_0 = T^* \upharpoonright \ker(\Upsilon_0)$ be the selfadjoint extension of T in $\mathfrak{G}$ corresponding to the boundary mapping Υ_0. With the help of the channel wave operators

$$W_\pm(\widetilde{L}, A_0) = s - \lim_{t \to \pm\infty} e^{it\widetilde{L}} e^{-itA_0} P^{ac}(A_0)$$
$$W_\pm(\widetilde{L}, T_0) = s - \lim_{t \to \pm\infty} e^{it\widetilde{L}} e^{-itT_0} P^{ac}(T_0)$$

one then defines the channel scattering operators

$$S_{\mathfrak{H}} := W_+(\widetilde{L}, A_0)^* W_-(\widetilde{L}, A_0) \quad \text{and} \quad S_{\mathfrak{G}} := W_+(\widetilde{L}, T_0)^* W_-(\widetilde{L}, T_0).$$

The corresponding channel scattering matrices $S_{\mathfrak{H}}(\lambda)$ and $S_{\mathfrak{G}}(\lambda)$ are studied in Section 4.3. Here we express these scattering matrices in terms of the functions M and τ in the spectral representations $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ and $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\lambda)})$ of A_0^{ac} and T_0^{ac}, respectively, and finally, with the help of these representations the modified Birman-Krein formula

$$\det(S_{\mathfrak{H}}(\lambda)) = \overline{\det(S_{\mathfrak{G}}(\lambda))} \exp\bigl(-2\pi i \widetilde{\xi}(\lambda)\bigr)$$

is proved in Theorem 4.6.

2 Self-adjoint extensions and scattering

In this section we consider scattering systems consisting of two selfadjoint extensions of a densely defined symmetric operator with equal finite deficiency indices in a separable Hilbert space. We generalize a result on the representation of the spectral shift function of such a scattering system from [9] and we give a short proof of the Birman-Krein formula in this setting.

2.1 Boundary triplets and closed extensions

Let A be a densely defined closed symmetric operator with equal deficiency indices $n_\pm(A) = \dim\ker(A^* \mp i) \leq \infty$ in the separable Hilbert space $\mathfrak{H}$. We use a concept of a boundary triplet for A^* in order to describe of the closed extensions $A_\Theta \subset A^*$ of A in $\mathfrak{H}$, see [30] and also [20, 22].

Definition 2.1 *A triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is called a* boundary triplet *for the adjoint operator* A^* *if* $\mathcal{H}$ *is a Hilbert space and* $\Gamma_0, \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$ *are linear mappings such that*

(i) *the abstract second Green's identity,*

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g),$$

holds for all $f, g \in \operatorname{dom}(A^*)$ *and*

(ii) *the mapping* $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(A^*) \longrightarrow \mathcal{H} \times \mathcal{H}$ *is surjective.*

We refer to [20] and [22] for a detailed study of boundary triplets and recall only some important facts. First of all a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists since the deficiency indices $n_\pm(A)$ of A are assumed to be equal. Then $n_\pm(A) = \dim \mathcal{H}$ holds. We note that a boundary triplet for A^* is not unique. Namely, if $\Pi' = \{\mathcal{G}', \Gamma_0', \Gamma_1'\}$ is another boundary triplet for A^*, then there exists a boundedly invertible operator $W = (W_{ij})_{i,j=1}^2 \in [\mathcal{G} \oplus \mathcal{G}, \mathcal{G}' \oplus \mathcal{G}']$ with the property

$$W^* \begin{pmatrix} 0 & -iI_{\mathcal{G}'} \\ iI_{\mathcal{G}'} & 0 \end{pmatrix} W = \begin{pmatrix} 0 & -iI_{\mathcal{G}} \\ iI_{\mathcal{G}} & 0 \end{pmatrix},$$

such that

$$\begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$$

holds. Here and in the following we write $[\mathfrak{K}, \mathcal{K}]$ for the set of bounded everywhere defined linear operators acting from a Hilbert space $\mathfrak{K}$ into a Hilbert space $\mathcal{K}$. For brevity we write $[\mathcal{K}]$ if $\mathcal{K} = \mathfrak{K}$.

An operator A' is called a *proper extension* of A if A' is closed and satisfies $A \subseteq A' \subseteq A^*$. In order to describe the set of proper extensions of A with the help of a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* we have to consider the set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$, that is, the set of closed linear subspaces of $\mathcal{H} \times \mathcal{H}$. Linear operators in $\mathcal{H}$ are identified with their graphs, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators is viewed as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$. For the usual definitions of the linear operations with linear relations, the inverse, the resolvent set and the spectrum we refer to [23]. Recall that the adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of a linear relation Θ in $\mathcal{H}$ is defined as

$$\Theta^* := \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\} \tag{2.1}$$

and Θ is said to be *symmetric* (*selfadjoint*) if $\Theta \subseteq \Theta^*$ (resp. $\Theta = \Theta^*$). Note that definition (2.1) extends the definition of the adjoint operator. A linear relation Θ is called *dissipative* if $\Im\mathrm{m}\,(g', g) \leq 0$ holds for all $\binom{g}{g'} \in \Theta$ and Θ is said to be *maximal dissipative* if Θ is dissipative and each dissipative extension of Θ coincides with Θ itself. In this case the upper half plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \Im\mathrm{m}\,\lambda > 0\}$ belongs to the resolvent set $\rho(\Theta)$. Furthermore, a linear relation Θ is called *accumulative* (*maximal accumulative*) if $-\Theta$ is dissipative (resp. maximal dissipative). For a maximal accumulative relation Θ we have $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \Im\mathrm{m}\,\lambda < 0\} \subset \rho(\Theta)$.

With a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* one associates two selfadjoint extensions of A defined by

$$A_0 := A^* \upharpoonright \ker(\Gamma_0) \quad \text{and} \quad A_1 := A^* \upharpoonright \ker(\Gamma_1).$$

A description of all proper extensions of A is given in the next proposition. Note also that the selfadjointness of A_0 and A_1 is a consequence of Proposition 2.2 (ii).

Proposition 2.2 *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$\Theta \mapsto A_\Theta := \Gamma^{-1}\Theta = \left\{ f \in \mathrm{dom}\,(A^*) : \ \Gamma f = (\Gamma_0 f, \Gamma_1 f)^\top \in \Theta \right\} \tag{2.2}$$

establishes a bijective correspondence between the set $\widetilde{\mathcal{C}}(\mathcal{H})$ and the set of proper extensions of A. Moreover, for $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ the following assertions hold.

(i) $(A_\Theta)^* = A_{\Theta^*}$.

(ii) A_Θ *is symmetric (selfadjoint) if and only if* Θ *is symmetric (resp. selfadjoint).*

(iii) A_Θ *is dissipative (maximal dissipative) if and only if* Θ *is dissipative (resp. maximal dissipative).*

(iv) A_Θ *is accumulative (maximal accumulative) if and only if* Θ *is accumulative (resp. maximal accumulative).*

(v) A_Θ *is disjoint with* A_0*, that is* $\mathrm{dom}\,(A_\Theta) \cap \mathrm{dom}\,(A_0) = \mathrm{dom}\,(A)$*, if and only if* $\Theta \in \mathcal{C}(\mathcal{H})$*. In this case the extension* A_Θ *in* (2.2) *is given by*

$$A_\Theta = A^* \upharpoonright \ker\big(\Gamma_1 - \Theta\Gamma_0\big). \tag{2.3}$$

We note that (2.3) holds also for linear relations Θ if the expression $\Gamma_1 - \Theta\Gamma_0$ is interpreted in the sense of linear relations.

In the following we shall often be concerned with simple symmetric operators. Recall that a symmetric operator is said to be *simple* if there is no nontrivial subspace which reduces it to a selfadjoint operator. By [32] each symmetric operator A in $\mathfrak{H}$ can be written as the direct orthogonal sum $\widehat{A} \oplus A_s$ of a simple symmetric operator $\widehat{A}$ in the Hilbert space

$$\widehat{\mathfrak{H}} = \mathrm{clospan}\big\{\ker(A^* - \lambda) : \lambda \in \mathbb{C}\backslash\mathbb{R}\big\}$$

and a selfadjoint operator A_s in $\mathfrak{H} \ominus \widehat{\mathfrak{H}}$. Here $\mathrm{clospan}\{\cdot\}$ denotes the closed linear span of a set. Obviously A is simple if and only if $\widehat{\mathfrak{H}}$ coincides with $\mathfrak{H}$.

2.2 Weyl functions and resolvents of extensions

Let, as in Section 2.1, A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices. If $\lambda \in \mathbb{C}$ is a point of regular type of A, i.e. $(A - \lambda)^{-1}$ is bounded, we denote the *defect subspace* of A by $\mathcal{N}_\lambda = \ker(A^* - \lambda)$. The following definition can be found in [19, 20, 22].

Definition 2.3 *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for the operator* A^* *and let* $A_0 = A^* \upharpoonright \ker(\Gamma_0)$*. The operator-valued functions* $\gamma(\cdot) : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ *and* $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ *defined by*

$$\gamma(\lambda) := \big(\Gamma_0 \upharpoonright \mathcal{N}_\lambda\big)^{-1} \qquad \textit{and} \qquad M(\lambda) := \Gamma_1\gamma(\lambda), \quad \lambda \in \rho(A_0), \tag{2.4}$$

are called the γ-field *and the* Weyl function*, respectively, corresponding to the boundary triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

It follows from the identity $\mathrm{dom}\,(A^*) = \ker(\Gamma_0) \dotplus \mathcal{N}_\lambda$, $\lambda \in \rho(A_0)$, where as above $A_0 = A^* \upharpoonright \ker(\Gamma_0)$, that the γ-field $\gamma(\cdot)$ in (2.4) is well defined. It is easily seen that both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the relations

$$\gamma(\mu) = \big(I + (\mu - \lambda)(A_0 - \mu)^{-1}\big)\gamma(\lambda), \qquad \lambda, \mu \in \rho(A_0), \tag{2.5}$$

and

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda), \qquad \lambda, \mu \in \rho(A_0), \tag{2.6}$$

are valid (see [20]). The identity (2.6) yields that $M(\cdot)$ is a *Nevanlinna function*, that is, $M(\cdot)$ is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and takes values in $[\mathcal{H}]$, $M(\lambda) = M(\overline{\lambda})^*$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $\Im m\,(M(\lambda))$ is a nonnegative operator for all λ in the upper half plane $\mathbb{C}_+$. Moreover, it follows that $0 \in \rho(\Im m\,(M(\lambda)))$ holds. It is important to note that if the operator A is simple, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ uniquely up to unitary equivalence, cf. [19, 20].

In the case that the deficiency indices $n_+(A) = n_-(A)$ are finite the Weyl function M corresponding to the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a matrix-valued Nevanlinna function in the finite dimensional space $\mathcal{H}$. From [24, 25] one gets the existence of the (strong) limit

$$M(\lambda + i0) = \lim_{\epsilon \to +0} M(\lambda + i\epsilon)$$

from the upper half plane for a.e. $\lambda \in \mathbb{R}$.

Let now $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with γ-field $\gamma(\cdot)$ and Weyl function $M(\cdot)$. The spectrum and the resolvent set of a proper (not necessarily selfadjoint) extension of A can be described with the help of the Weyl function. If $A_\Theta \subseteq A^*$ is the extension corresponding to $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ via (2.2), then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ ($\sigma_i(A_0)$, $i = p, c, r$) if and only if $0 \in \rho(\Theta - M(\lambda))$ (resp. $0 \in \sigma_i(\Theta - M(\lambda))$, $i = p, c, r$). Moreover, for $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* \tag{2.7}$$

holds. Formula (2.7) is a generalization of the known Krein formula for canonical resolvents. We emphasize that it is valid for any proper extension of A with a non-empty resolvent set. It is worth to note that the Weyl function can also be used to investigate the absolutely continuous and singular continuous spectrum of proper extensions of A, cf. [14].

2.3 Spectral shift function and trace formula

M.G. Krein's spectral shift function introduced in [33] is an important tool in the spectral and perturbation theory of selfadjoint operators, in particular scattering theory. A detailed review on the spectral shift function can be found in, e.g. [12, 13]. Furthermore we mention [26, 27, 28] as some recent papers on the spectral shift function and its various applications.

Recall that for any pair of selfadjoint operators H_1, H_0 in a separable Hilbert space $\mathfrak{H}$ such that the resolvents differ by a trace class operator,

$$(H_1 - \lambda)^{-1} - (H_0 - \lambda)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \tag{2.8}$$

for some (and hence for all) $\lambda \in \rho(H_1) \cap \rho(H_0)$, there exists a real valued function $\xi(\cdot) \in L^1_{loc}(\mathbb{R})$ satisfying the conditions

$$\operatorname{tr}\big((H_1 - \lambda)^{-1} - (H_0 - \lambda)^{-1}\big) = -\int_{\mathbb{R}} \frac{1}{(t - \lambda)^2}\,\xi(t)\,dt, \tag{2.9}$$

$\lambda \in \rho(H_1) \cap \rho(H_0)$, and

$$\int_{\mathbb{R}} \frac{1}{1 + t^2}\,\xi(t)\,dt < \infty, \tag{2.10}$$

cf. [12, 13, 33]. Such a function ξ is called a *spectral shift function* of the pair $\{H_1, H_0\}$. We emphasize that ξ is not unique, since simultaneously with ξ a function $\xi + c$, $c \in \mathbb{R}$, also satisfies both conditions (2.9) and (2.10). Note that the converse also holds, namely, any two spectral shift functions for a pair of selfadjoint operators $\{H_1, H_0\}$ satisfying (2.8) differ by a real constant. We remark that (2.9) is a special case of the general formula

$$\operatorname{tr}\left(\phi(H_1) - \phi(H_0)\right) = \int_{\mathbb{R}} \phi'(t)\,\xi(t)\,dt, \tag{2.11}$$

which is valid for a wide class of smooth functions, cf. [47] for a large class of such functions $\phi(\cdot)$.

In Theorem 2.4 below we find a representation for the spectral shift function ξ_Θ of a pair of selfadjoint operators A_Θ and A_0 which are both assumed to be extensions of a densely defined closed simple symmetric operator A with equal finite deficiency indices. For that purpose we use the definition

$$\log(T) := -i \int_0^\infty \left((T + it)^{-1} - (1 + it)^{-1} I_{\mathcal{H}}\right) dt \tag{2.12}$$

for an operator T in a finite dimensional Hilbert space $\mathcal{H}$ satisfying $\Im\mathrm{m}\,(T) \geq 0$ and $0 \notin \sigma(T)$, see, e.g. [26, 48]. A straightforward calculation shows that the relation

$$\det(T) = \exp\bigl(\operatorname{tr}\bigl(\log(T)\bigr)\bigr) \tag{2.13}$$

holds. Observe that

$$\operatorname{tr}\bigl(\log(T)\bigr) = \log\bigl(\det(T)\bigr) + 2k\pi i \tag{2.14}$$

holds for some $k \in \mathbb{Z}$. In [9, Theorem 4.1] it was shown that if $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* with $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and $A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$ is a selfadjoint extension of A which corresponds to a selfadjoint matrix Θ in $\mathcal{H}$, then the limit $\lim_{\epsilon\to+0} \log\left(M(\lambda + i\epsilon) - \Theta\right)$ exist for a.e. $\lambda \in \mathbb{R}$ and

$$\xi_\Theta(\lambda) := \frac{1}{\pi} \Im\mathrm{m}\,\bigl(\operatorname{tr}\bigl(\log(M(\lambda + i0) - \Theta)\bigr)\bigr) \tag{2.15}$$

defines a spectral shift function for the pair $\{A_\Theta, A_0\}$. We emphasize that Θ was assumed to be a matrix in [9], so that ξ_Θ in (2.15) is a spectral shift function only for special pairs $\{A_\Theta, A_0\}$. Theorem 2.4 below extends the result from [9] to the case of a selfadjoint relation Θ and hence to arbitrary pairs of selfadjoint extensions $\{A_\Theta, A_0\}$ of A.

To this end we first recall that any selfadjoint relation Θ in $\mathcal{H}$ can be written in the form

$$\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty \tag{2.16}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_{\mathrm{op}} \oplus \mathcal{H}_\infty$, where Θ_{op} is a selfadjoint operator in $\mathcal{H}_{\mathrm{op}} := \overline{\operatorname{dom}\Theta}$ and Θ_∞ is a pure relation in $\mathcal{H}_\infty := (\operatorname{dom}\Theta)^\perp$, that is,

$$\Theta_\infty = \left\{ \begin{pmatrix} 0 \\ h' \end{pmatrix} : h' \in \mathcal{H}_\infty \right\}. \tag{2.17}$$

Since in the following considerations the space $\mathcal{H}$ is finite dimensional we have $\mathcal{H}_{\mathrm{op}} = \operatorname{dom}\Theta = \operatorname{dom}\Theta_{\mathrm{op}}$ and Θ_{op} is a selfadjoint matrix. If $M(\cdot)$ is the Weyl function corresponding to a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, then

$$M_{\mathrm{op}}(\lambda) := P_{\mathrm{op}} M(\lambda) \iota_{\mathrm{op}}\,, \tag{2.18}$$

is a $[\mathcal{H}_{\rm op}]$-valued Nevanlinna function. Here $P_{\rm op}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\rm op}$ and $\iota_{\rm op}$ denotes the canonical embedding of $\mathcal{H}_{\rm op}$ in $\mathcal{H}$. One verifies that

$$\big(\Theta - M(\lambda)\big)^{-1} = \iota_{\rm op}\left(\Theta_{\rm op} - M_{\rm op}(\lambda)\right)^{-1} P_{\rm op} \tag{2.19}$$

holds for all $\lambda \in \mathbb{C}_+$. The following result generalizes [9, Theorem 4.1], see also [34] for a special case.

Theorem 2.4 *Let A be a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal finite deficiency indices, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and let $M(\cdot)$ be the corresponding Weyl function. Furthermore, let $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and let $A_\Theta = A^* \upharpoonright \Gamma^{-1}\Theta$, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, be a selfadjoint extension of A in $\mathfrak{H}$. Then the limit*

$$\lim_{\epsilon \to +0} \log\big(M_{\rm op}(\lambda + i\epsilon) - \Theta_{\rm op}\big)$$

exists for a.e. $\lambda \in \mathbb{R}$ and the function

$$\xi_\Theta(\lambda) := \frac{1}{\pi}\Im \mathrm{m}\left(\mathrm{tr}\big(\log(M_{\rm op}(\lambda + i0) - \Theta_{\rm op})\big)\right) \tag{2.20}$$

is a spectral shift function for the pair $\{A_\Theta, A_0\}$ with $0 \le \xi_\Theta(\lambda) \le \dim \mathcal{H}_{\rm op}$.

Proof. Since $\lambda \mapsto M_{\rm op}(\lambda) - \Theta_{\rm op}$ is a Nevanlinna function with values in $[\mathcal{H}_{\rm op}]$ and $0 \in \rho(\Im\mathrm{m}\,(M_{\rm op}(\lambda)))$ for all $\lambda \in \mathbb{C}_+$, it follows that $\log(M_{\rm op}(\lambda) - \Theta_{\rm op})$ is well-defined for all $\lambda \in \mathbb{C}_+$ by (2.12). According to [26, Lemma 2.8] the function $\lambda \mapsto \log(M_{\rm op}(\lambda) - \Theta_{\rm op})$, $\lambda \in \mathbb{C}_+$, is a $[\mathcal{H}_{\rm op}]$-valued Nevanlinna function such that

$$0 \le \Im\mathrm{m}\left(\log(M_{\rm op}(\lambda) - \Theta_{\rm op})\right) \le \pi I_{\mathcal{H}_{\rm op}}$$

holds for all $\lambda \in \mathbb{C}_+$. Hence the limit $\lim_{\epsilon \to +0} \log(M_{\rm op}(\lambda + i\epsilon) - \Theta_{\rm op})$ exists for a.e. $\lambda \in \mathbb{R}$ (see [24, 25] and Section 2.2) and $\lambda \mapsto \mathrm{tr}(\log(M_{\rm op}(\lambda) - \Theta_{\rm op}))$, $\lambda \in \mathbb{C}_+$, is a scalar Nevanlinna function with the property

$$0 \le \Im\mathrm{m}\left(\mathrm{tr}(\log(M_{\rm op}(\lambda) - \Theta_{\rm op}))\right) \le \pi \dim \mathcal{H}_{\rm op}, \quad \lambda \in \mathbb{C}_+,$$

that is, the function ξ_Θ in (2.20) satisfies $0 \le \xi_\Theta(\lambda) \le \dim \mathcal{H}_{\rm op}$ for a.e. $\lambda \in \mathbb{R}$.

In order to show that (2.9) holds with H_1, H_0 and ξ replaced by A_Θ, A_0 and ξ_Θ, respectively, we note that the relation

$$\frac{d}{d\lambda}\mathrm{tr}\big(\log(M_{\rm op}(\lambda) - \Theta_{\rm op})\big) = \mathrm{tr}\left((M_{\rm op}(\lambda) - \Theta_{\rm op})^{-1}\frac{d}{d\lambda}M_{\rm op}(\lambda)\right) \tag{2.21}$$

is true for all $\lambda \in \mathbb{C}_+$. This can be shown in the same way as in the proof of [9, Theorem 4.1]. From (2.6) we find

$$\gamma(\bar\mu)^*\gamma(\lambda) = \frac{M(\lambda) - M(\bar\mu)^*}{\lambda - \mu}, \qquad \lambda, \mu \in \mathbb{C}\backslash\mathbb{R},\ \lambda \ne \mu, \tag{2.22}$$

and passing in (2.22) to the limit $\mu \to \lambda$ one gets

$$\gamma(\bar\lambda)^*\gamma(\lambda) = \frac{d}{d\lambda}M(\lambda). \tag{2.23}$$

Making use of formula (2.7) for the canonical resolvents this implies

$$\begin{aligned}\operatorname{tr}\big((A_\Theta-\lambda)^{-1}-(A_0-\lambda)^{-1}\big)&=-\operatorname{tr}\big((M(\lambda)-\Theta)^{-1}\gamma(\bar\lambda)^*\gamma(\lambda)\big)\\&=-\operatorname{tr}\left((M(\lambda)-\Theta)^{-1}\frac{d}{d\lambda}M(\lambda)\right)\end{aligned}\tag{2.24}$$

for all $\lambda\in\mathbb{C}_+$. With respect to the decomposition $\mathcal{H}=\mathcal{H}_{\rm op}\oplus\mathcal{H}_\infty$ the operator

$$\big(M(\lambda)-\Theta\big)^{-1}\frac{d}{d\lambda}M(\lambda)=\iota_{\mathcal{H}_{\rm op}}\big(M_{\rm op}(\lambda)-\Theta_{\rm op}\big)^{-1}P_{\rm op}\frac{d}{d\lambda}M(\lambda)$$

is a 2×2 block matrix where the entries in the lower row are zero matrices and the upper left corner is given by

$$(M_{\rm op}(\lambda)-\Theta_{\rm op})^{-1}\frac{d}{d\lambda}M_{\rm op}(\lambda).$$

Therefore (2.24) becomes

$$\begin{aligned}\operatorname{tr}\big((A_\Theta-\lambda)^{-1}-(A_0-\lambda)^{-1}\big)&=-\operatorname{tr}\left((M_{\rm op}(\lambda)-\Theta_{\rm op})^{-1}\frac{d}{d\lambda}M_{\rm op}(\lambda)\right),\\&=-\frac{d}{d\lambda}\operatorname{tr}\big(\log(M_{\rm op}(\lambda)-\Theta_{\rm op})\big),\end{aligned}\tag{2.25}$$

where we have use (2.21).

Further, by [26, Theorem 2.10] there exists a $[\mathcal{H}_{\rm op}]$-valued measurable function $t\mapsto\Xi_{\Theta_{\rm op}}(t)$, $t\in\mathbb{R}$, such that $\Xi_{\Theta_{\rm op}}(t)=\Xi_{\Theta_{\rm op}}(t)^*$ and $0\le\Xi_{\Theta_{\rm op}}(t)\le I_{\mathcal{H}_{\rm op}}$ for a.e. $\lambda\in\mathbb{R}$ and the representation

$$\log(M_{\rm op}(\lambda)-\Theta_{\rm op})=C+\int_{\mathbb{R}}\Xi_{\Theta_{\rm op}}(t)\left((t-\lambda)^{-1}-t(1+t^2)^{-1}\right)dt,\quad\lambda\in\mathbb{C}_+,$$

holds with some bounded selfadjoint operator C. Hence

$$\operatorname{tr}\big(\log(M_{\rm op}(\lambda)-\Theta_{\rm op})\big)=\operatorname{tr}(C)+\int_{\mathbb{R}}\operatorname{tr}\left(\Xi_{\Theta_{\rm op}}(t)\right)\left((t-\lambda)^{-1}-t(1+t^2)^{-1}\right)dt$$

for $\lambda\in\mathbb{C}_+$ and we conclude from

$$\begin{aligned}\xi_\Theta(\lambda)&=\lim_{\epsilon\to+0}\frac{1}{\pi}\Im\mathrm{m}\left(\operatorname{tr}(\log(M_{\rm op}(\lambda+i\epsilon)-\Theta_{\rm op}))\right)\\&=\lim_{\epsilon\to+0}\frac{1}{\pi}\int_{\mathbb{R}}\operatorname{tr}\left(\Xi_{\Theta_{\rm op}}(t)\right)\epsilon\big((t-\lambda)^2+\epsilon^2\big)^{-1}dt\end{aligned}$$

that $\xi_\Theta(\lambda)=\operatorname{tr}(\Xi_{\Theta_{\rm op}}(\lambda))$ is true for a.e. $\lambda\in\mathbb{R}$. Therefore we have

$$\frac{d}{d\lambda}\operatorname{tr}\big(\log(M_{\rm op}(\lambda)-\Theta_{\rm op})\big)=\int_{\mathbb{R}}(t-\lambda)^{-2}\xi_\Theta(t)\,dt$$

and together with (2.25) we immediately get the trace formula

$$\operatorname{tr}\big((A_\Theta-\lambda)^{-1}-(A_0-\lambda)^{-1}\big)=-\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\xi_\Theta(t)\,dt.$$

The integrability condition (2.10) holds because of [26, Theorem 2.10]. This completes the proof of Theorem 2.4. □

2.4 A representation of the scattering matrix

Let again A be a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal finite deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. Let Θ be a selfadjoint relation in $\mathcal{H}$ and let $A_\Theta = A^* \upharpoonright \Gamma^{-1}\Theta$ be the corresponding selfadjoint extension of A in $\mathfrak{H}$. Since $\dim \mathcal{H}$ is finite by (2.7)

$$\dim\Big(\operatorname{ran}\left((A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1}\right)\Big) < \infty, \quad \lambda \in \rho(A_\Theta) \cap \rho(A_0),$$

and therefore the pair $\{A_\Theta, A_0\}$ forms a so-called *complete scattering system*, that is, the *wave operators*

$$W_\pm(A_\Theta, A_0) := \text{s-}\lim_{t \to \pm\infty} e^{itA_\Theta} e^{-itA_0} P^{ac}(A_0),$$

exist and their ranges coincide with the absolutely continuous subspace $\mathfrak{H}^{ac}(A_\Theta)$ of A_Θ, cf. [8, 31, 52, 53]. $P^{ac}(A_0)$ denotes the orthogonal projection onto the absolutely continuous subspace $\mathfrak{H}^{ac}(A_0)$ of A_0. The *scattering operator* S_Θ of the *scattering system* $\{A_\Theta, A_0\}$ is then defined by

$$S_\Theta := W_+(A_\Theta, A_0)^* W_-(A_\Theta, A_0).$$

If we regard the scattering operator as an operator in $\mathfrak{H}^{ac}(A_0)$, then S_Θ is unitary, commutes with the absolutely continuous part

$$A_0^{ac} := A_0 \upharpoonright \operatorname{dom}(A_0) \cap \mathfrak{H}^{ac}(A_0)$$

of A_0 and it follows that S_Θ is unitarily equivalent to a multiplication operator induced by a family $\{S_\Theta(\lambda)\}_{\lambda \in \mathbb{R}}$ of unitary operators in a spectral representation of A_0^{ac}, see e.g. [8, Proposition 9.57]. This family is called the *scattering matrix* of the scattering system $\{A_\Theta, A_0\}$.

We recall a representation theorem for the scattering matrix $\{S_\Theta(\lambda)\}_{\lambda \in \mathbb{R}}$ in terms of the Weyl function $M(\cdot)$ of the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ from [9]. For this we consider the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathcal{H})$, where $d\lambda$ is the Lebesgue measure on $\mathbb{R}$. Further, we set

$$\mathcal{H}_{M(\lambda)} := \operatorname{ran}\left(\Im\mathrm{m}\,(M(\lambda))\right), \quad M(\lambda) := M(\lambda + i0), \tag{2.26}$$

which defines subspaces of $\mathcal{H}$ for a.e. $\lambda \in \mathbb{R}$. By $P_{M(\lambda)}$ we denote the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{M(\lambda)}$. The family $\{P_{M(\lambda)}\}_{\lambda \in \mathbb{R}}$ is measurable. Hence $\{P_{M(\lambda)}\}_{\lambda \in \mathbb{R}}$ induces a multiplication operator P_M on $L^2(\mathbb{R}, d\lambda, \mathcal{H})$ defined by

$$(P_M f)(\lambda) = P_{M(\lambda)} f(\lambda), \qquad f \in L^2(\mathbb{R}, d\lambda, \mathcal{H}), \tag{2.27}$$

which is an orthogonal projection. The subspace $\operatorname{ran}(P_M)$ is denoted by $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ in the following. We remark that $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ can be regarded as the direct integral of the Hilbert spaces $\mathcal{H}_{M(\lambda)}$, that is,

$$L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)}) = \int^\oplus \mathcal{H}_{M(\lambda)}\, d\lambda.$$

The following theorem was proved in [9].

Theorem 2.5 *Let A be a densely defined closed simple symmetric operator with equal finite deficiency indices in the separable Hilbert space $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with corresponding Weyl function $M(\cdot)$. Furthermore, let $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and let $A_\Theta = A^* \upharpoonright \Gamma^{-1}\Theta$, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, be a selfadjoint extension of A in $\mathfrak{H}$. Then the following holds:*

(i) *A_0^{ac} is unitarily equivalent to the multiplication operator with the free variable in the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$.*

(ii) *In the spectral representation $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ of A_0^{ac} the scattering matrix $\{S_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{A_\Theta, A_0\}$ admits the representation*

$$S_\Theta(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(\Theta - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))} \tag{2.28}$$

for a.e. $\lambda \in \mathbb{R}$, where $M(\lambda) = M(\lambda + i0)$.

In the next corollary we find a slightly more convenient representation of the scattering matrix $\{S_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{A_\Theta, A_0\}$ for the case that Θ is a selfadjoint relation which is decomposed in the form $\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty$ with respect to $\mathcal{H} = \mathcal{H}_{\mathrm{op}} \oplus \mathcal{H}_\infty$, cf. (2.16) and (2.17). If $M(\cdot)$ is the Weyl function corresponding to the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, then the function

$$\lambda \mapsto M_{\mathrm{op}}(\lambda) = P_{\mathrm{op}} M(\lambda)\iota_{\mathrm{op}}$$

from (2.18) is a $[\mathcal{H}_{\mathrm{op}}]$-valued Nevanlinna function, and the subspaces

$$\mathcal{H}_{M_{\mathrm{op}}(\lambda)} := \mathrm{ran}\,\big(\Im\mathrm{m}\,(M_{\mathrm{op}}(\lambda + i0))\big)$$

of $\mathcal{H}_{M(\lambda)}$ are defined as in (2.26).

Corollary 2.6 *Let the assumptions be as in Theorem 2.5, let $\mathcal{H}_{M_{\mathrm{op}}(\lambda)}$ be as above and $\mathcal{H}^\infty_{M(\lambda)} := \mathcal{H}_{M(\lambda)} \ominus \mathcal{H}_{M_{\mathrm{op}}(\lambda)}$. Then there exists a family $V(\lambda) : \mathcal{H}_{M(\lambda)} \to \mathcal{H}_{M(\lambda)}$ of unitary operators such that the representation*

$$S_\Theta(\lambda) = V(\lambda)\left\{I_{\mathcal{H}^\infty_{M(\lambda)}} \oplus S_{\Theta_{\mathrm{op}}}(\lambda)\right\}V(\lambda)^* \tag{2.29}$$

holds with

$$S_{\Theta_{\mathrm{op}}}(\lambda) = I_{\mathcal{H}_{M_{\mathrm{op}}(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(M_{\mathrm{op}}(\lambda))}\big(\Theta_{\mathrm{op}} - M_{\mathrm{op}}(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M_{\mathrm{op}}(\lambda))} \tag{2.30}$$

for a.e. $\lambda \in \mathbb{R}$.

Proof. Using (2.28) and (2.19) we find the representation

$$S_\Theta(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(M(\lambda))}\,\iota_{\mathrm{op}}\,\big(\Theta_{\mathrm{op}} - M_{\mathrm{op}}(\lambda)\big)^{-1}P_{\mathrm{op}}\sqrt{\Im\mathrm{m}\,(M(\lambda))} \tag{2.31}$$

for a.e. $\lambda \in \mathbb{R}$. From the polar decomposition of $\sqrt{\Im\mathrm{m}\,(M(\lambda))}\,\iota_{\mathrm{op}}$ we obtain a family of isometric mappings $V_{\mathrm{op}}(\lambda)$ from $\mathcal{H}_{M_{\mathrm{op}}(\lambda)}$ onto

$$\mathrm{ran}\,\Big(\sqrt{\Im\mathrm{m}\,(M(\lambda))}\,\iota_{\mathrm{op}}\Big) \subset \mathcal{H}_{M(\lambda)}$$

defined by

$$V_{\mathrm{op}}(\lambda)\sqrt{\Im\mathrm{m}\,(M_{\mathrm{op}}(\lambda))} := \sqrt{\Im\mathrm{m}\,(M(\lambda))}\,\iota_{\mathrm{op}}\,.$$

Hence we find

$$\begin{aligned} S_\Theta(\lambda) = I_{\mathcal{H}_{M(\lambda)}} &+ 2iV_{\mathrm{op}}(\lambda)\sqrt{\Im\mathrm{m}\,(M_{\mathrm{op}}(\lambda))} \\ &\times \left(\Theta_{\mathrm{op}} - M_{\mathrm{op}}(\lambda)\right)^{-1}\sqrt{\Im\mathrm{m}\,(M_{\mathrm{op}}(\lambda))}V_{\mathrm{op}}(\lambda)^* \end{aligned}$$

for a.e. $\lambda \in \mathbb{R}$. Since the Hilbert space $\mathcal{H}_{M(\lambda)}$ is finite dimensional there is an isometry $V_\infty(\lambda)$ acting from $\mathcal{H}^\infty_{M(\lambda)} = \mathcal{H}_{M(\lambda)} \ominus \mathcal{H}_{M_{\mathrm{op}}(\lambda)}$ into $\mathcal{H}_{M(\lambda)}$ such that $V(\lambda) := V_\infty(\lambda) \oplus V_{\mathrm{op}}(\lambda)$ defines a unitary operator on $\mathcal{H}_{M(\lambda)}$. This immediately yields (2.29). □

2.5 Birman-Krein formula

An important relation between the spectral shift function and the scattering matrix for a pair of selfadjoint operators for the case of a trace class perturbation was found in [11] by Birman and Krein. Subsequently, this relation was called the Birman-Krein formula. Under the assumption that A_Θ and A_0 are selfadjoint extensions of a densely defined symmetric operator A with finite deficiency indices and A_Θ corresponds to a selfadjoint matrix Θ via a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* a simple proof for the Birman-Krein formula

$$\det(S_\Theta(\lambda)) = \exp\bigl(-2\pi i \xi_\Theta(\lambda)\bigr)$$

was given in [9]. Here $\xi_\Theta(\cdot)$ is the spectral shift function of the pair $\{A_\Theta, A_0\}$ defined by (2.15) and the scattering matrix $\{S_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ is given by (2.28).

The following theorem generalizes [9, Theorem 4.1] to the case of a selfadjoint relation Θ (instead of a matrix), so that the Birman-Krein formula is verified for all pairs of selfadjoint extensions of the underlying symmetric operator.

Theorem 2.7 *Let A be a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal finite deficiency indices, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and let $M(\cdot)$ be the corresponding Weyl function. Furthermore, let $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and let $A_\Theta = A^* \upharpoonright \Gamma^{-1}\Theta$, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, be a selfadjoint extension of A in $\mathfrak{H}$. Then the spectral shift function $\xi_\Theta(\cdot)$ in* (2.20) *and the scattering matrix $\{S_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ of the pair $\{A_\Theta, A_0\}$ are related via*

$$\det\bigl(S_\Theta(\lambda)\bigr) = \exp\bigl(-2\pi i \xi_\Theta(\lambda)\bigr) \tag{2.32}$$

for a.e. $\lambda \in \mathbb{R}$.

Proof. To verify the Birman-Krein formula we note that by (2.13)

$$\begin{aligned} &\exp\bigl(-2i\Im\mathrm{m}\bigl(\mathrm{tr}(\log(M_{\mathrm{op}}(\lambda) - \Theta_{\mathrm{op}}))\bigr)\bigr) \\ &\quad = \exp\bigl(-\mathrm{tr}(\log(M_{\mathrm{op}}(\lambda) - \Theta_{\mathrm{op}}))\bigr)\exp\Bigl(\overline{\mathrm{tr}(\log(M_{\mathrm{op}}(\lambda) - \Theta_{\mathrm{op}}))}\Bigr) \\ &\quad = \frac{\overline{\det(M_{\mathrm{op}}(\lambda) - \Theta_{\mathrm{op}})}}{\det(M_{\mathrm{op}}(\lambda) - \Theta_{\mathrm{op}})} = \frac{\det(M_{\mathrm{op}}(\lambda)^* - \Theta_{\mathrm{op}})}{\det(M_{\mathrm{op}}(\lambda) - \Theta_{\mathrm{op}})} \end{aligned}$$

holds for all $\lambda \in \mathbb{C}_+$. Hence we find

$$\exp\bigl(-2\pi i \xi_\Theta(\lambda)\bigr) = \frac{\det\bigl(M_{\rm op}(\lambda + i0)^* - \Theta_{\rm op}\bigr)}{\det\bigl(M_{\rm op}(\lambda + i0) - \Theta_{\rm op}\bigr)} \tag{2.33}$$

for a.e. $\lambda \in \mathbb{R}$, where $M_{\rm op}(\lambda + i0) := \lim_{\epsilon\to+0} M_{\rm op}(\lambda + i\epsilon)$ exists for a.e. $\lambda \in \mathbb{R}$. It follows from the representation of the scattering matrix in Corollary 2.6 and the identity $\det(I + AB) = \det(I + BA)$ that

$$\begin{aligned}
&\det S_\Theta(\lambda)\\
&\quad= \det\Bigl(I_{\mathcal{H}_{\rm op}} + 2i\bigl(\Im{\rm m}\,(M_{\rm op}(\lambda+i0))\bigr)\bigl(\Theta_{\rm op} - M_{\rm op}(\lambda+i0)\bigr)^{-1}\Bigr)\\
&\quad= \det\Bigl(I_{\mathcal{H}_{\rm op}} + \bigl(M_{\rm op}(\lambda+i0) - M_{\rm op}(\lambda+i0)^*\bigr)\bigl(\Theta_{\rm op} - M_{\rm op}(\lambda+i0)\bigr)^{-1}\Bigr)\\
&\quad= \det\Bigl(\bigl(M_{\rm op}(\lambda+i0)^* - \Theta_{\rm op}\bigr)\cdot\bigl(M_{\rm op}(\lambda+i0) - \Theta_{\rm op}\bigr)^{-1}\Bigr)\\
&\quad= \frac{\det\bigl(M_{\rm op}(\lambda+i0)^* - \Theta_{\rm op}\bigr)}{\det\bigl(M_{\rm op}(\lambda+i0) - \Theta_{\rm op}\bigr)}
\end{aligned}$$

holds for a.e. $\lambda \in \mathbb{R}$. Comparing this with (2.33) we obtain (2.32). □

3 Dissipative scattering systems

In this section we investigate scattering systems consisting of a maximal dissipative and a selfadjoint operator, which are both extensions of a common symmetric operator with equal finite deficiency indices. We shall explicitly construct a so-called dilation of the maximal dissipative operator and we calculate the spectral shift function of the dissipative scattering system with the help of this dilation. It will be shown that the scattering matrix of the dissipative scattering system and this spectral shift function are connected via a modified Birman-Krein formula.

3.1 Selfadjoint dilations of maximal dissipative operators

Let A be a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal finite deficiency indices $n_+(A) = n_-(A) = n < \infty$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $A_0 = A^* \upharpoonright \ker(\Gamma_0)$, be a boundary triplet for A^* and let $D \in [\mathcal{H}]$ be a dissipative $n \times n$-matrix, i.e. $\Im{\rm m}\,(D) \le 0$. Then by Proposition 2.2 (iii) the closed extension

$$A_D = A^* \upharpoonright \ker(\Gamma_1 - D\Gamma_0)$$

of A corresponding to $\Theta = D$ via (2.2) is maximal dissipative, that is, A_D is dissipative and maximal in the sense that each dissipative extension of A_D in $\mathfrak{H}$ coincides with A_D. Observe that $\mathbb{C}_+$ belongs to $\rho(A_D)$. For $\lambda \in \rho(A_D) \cap \rho(A_0)$ the resolvent of the extension A_D is given by

$$(A_D - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\bigl(D - M(\lambda)\bigr)^{-1}\gamma(\bar\lambda)^*, \tag{3.1}$$

cf. (2.7). With respect to the decomposition

$$D = \Re\mathrm{e}\,(D) + i\Im\mathrm{m}\,(D)$$

we decompose $\mathcal{H}$ into the orthogonal sum of the finite dimensional subspaces $\ker(\Im\mathrm{m}\,(D))$ and $\mathcal{H}_D := \mathrm{ran}\,(\Im\mathrm{m}\,(D))$,

$$\mathcal{H} = \ker(\Im\mathrm{m}\,(D)) \oplus \mathcal{H}_D, \tag{3.2}$$

and denote by P_D the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_D$ and by ι_D the canonical embedding of $\mathcal{H}_D$ into $\mathcal{H}$. Since $\Im\mathrm{m}\,(D) \leq 0$ the selfadjoint matrix

$$-P_D \Im\mathrm{m}\,(D)\,\iota_D \in [\mathcal{H}_D]$$

is strictly positive and therefore (see, e.g. [18, 22]) the function

$$\lambda \mapsto \begin{cases} -iP_D\Im\mathrm{m}\,(D)\iota_D, & \lambda \in \mathbb{C}_+, \\ iP_D\Im\mathrm{m}\,(D)\,\iota_D, & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.3}$$

can be realized as the Weyl function corresponding to a boundary triplet of a symmetric operator.

Here the symmetric operator and boundary triplet can be made more explicit, cf. [10, Lemma 3.1]. In fact, let G be the symmetric first order differential operator in the Hilbert space $L^2(\mathbb{R}, \mathcal{H}_D)$ defined by

$$(Gg)(x) = -ig'(x), \qquad \mathrm{dom}\,(G) = \big\{g \in W_2^1(\mathbb{R}, \mathcal{H}_D) \,:\, g(0) = 0\big\}. \tag{3.4}$$

Then G is simple, $n_\pm(G) = \dim \mathcal{H}_D$ and the adjoint operator $G^*g = -ig'$ is defined on

$$\mathrm{dom}\,(G^*) = W_2^1(\mathbb{R}_-, \mathcal{H}_D) \oplus W_2^1(\mathbb{R}_+, \mathcal{H}_D).$$

Moreover, the triplet $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$, where

$$\begin{aligned} \Upsilon_0 g &:= \frac{1}{\sqrt{2}}\big(-P_D\Im\mathrm{m}\,(D)\,\iota_D\big)^{-\frac{1}{2}}\big(g(0+) - g(0-)\big), \\ \Upsilon_1 g &:= \frac{i}{\sqrt{2}}\big(-P_D\Im\mathrm{m}\,(D)\,\iota_D\big)^{\frac{1}{2}}\big(g(0+) + g(0-)\big), \end{aligned} \tag{3.5}$$

$g \in \mathrm{dom}\,(G^*)$, is a boundary triplet for G^* and the extension $G_0 := G^* \restriction \ker(\Upsilon_0)$ of G is the usual selfadjoint first order differential operator in $L^2(\mathbb{R}, \mathcal{H}_D)$ with domain $\mathrm{dom}\,(G_0) = W_2^1(\mathbb{R}, \mathcal{H}_D)$ and $\sigma(G_0) = \mathbb{R}$. It is not difficult to see that the defect subspaces of G are given by

$$\ker(G^* - \lambda) = \begin{cases} \mathrm{span}\,\{x \mapsto e^{i\lambda x}\chi_{\mathbb{R}_+}(x)\xi \,:\, \xi \in \mathcal{H}_D\}, & \lambda \in \mathbb{C}_+, \\ \mathrm{span}\,\{x \mapsto e^{i\lambda x}\chi_{\mathbb{R}_-}(x)\xi \,:\, \xi \in \mathcal{H}_D\}, & \lambda \in \mathbb{C}_-, \end{cases}$$

and therefore it follows that the Weyl function $\tau(\cdot)$ corresponding to the boundary triplet $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$ is given by

$$\tau(\lambda) = \begin{cases} -iP_D\Im\mathrm{m}\,(D)\,\iota_D, & \lambda \in \mathbb{C}_+, \\ iP_D\Im\mathrm{m}\,(D)\,\iota_D, & \lambda \in \mathbb{C}_-. \end{cases} \tag{3.6}$$

Let A be the densely defined closed simple symmetric operator in $\mathfrak{H}$ from above and let G be the first order differential operator in (3.4). Clearly

$$K := \begin{pmatrix} A & 0 \\ 0 & G \end{pmatrix}$$

is a densely defined closed simple symmetric operator in the separable Hilbert space

$$\mathfrak{K} := \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$$

with equal finite deficiency indices $n_\pm(K) = n_\pm(A) + n_\pm(G) = n + \dim \mathcal{H}_D < \infty$ and the adjoint is

$$K^* = \begin{pmatrix} A^* & 0 \\ 0 & G^* \end{pmatrix}.$$

The elements in $\mathrm{dom}\,(K^*) = \mathrm{dom}\,(A^*) \oplus \mathrm{dom}\,(G^*)$ will be written in the form $f \oplus g$, $f \in \mathrm{dom}\,(A^*)$, $g \in \mathrm{dom}\,(G^*)$. It is straightforward to check that $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}_D$,

$$\widetilde{\Gamma}_0(f \oplus g) := \begin{pmatrix} \Gamma_0 f \\ \Upsilon_0 g \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1(f \oplus g) := \begin{pmatrix} \Gamma_1 f - \Re\mathrm{e}\,(D)\Gamma_0 f \\ \Upsilon_1 g \end{pmatrix}, \tag{3.7}$$

$f \oplus g \in \mathrm{dom}\,(K^*)$, is a boundary triplet for K^*. If $\gamma(\cdot), \nu(\cdot)$ and $M(\cdot), \tau(\cdot)$ are the γ-fields and Weyl functions of the boundary triplets $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$, respectively, then one easily verifies that the Weyl function $\widetilde{M}(\cdot)$ and γ-field $\widetilde{\gamma}(\cdot)$ corresponding to the boundary triplet $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ are given by

$$\widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) - \Re\mathrm{e}\,(D) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{3.8}$$

and

$$\widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \nu(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{3.9}$$

respectively. Observe that

$$K_0 := K^* \upharpoonright \ker(\widetilde{\Gamma}_0) = \begin{pmatrix} A_0 & 0 \\ 0 & G_0 \end{pmatrix} \tag{3.10}$$

holds. With respect to the decomposition

$$\widetilde{\mathcal{H}} = \ker(\Im\mathrm{m}\,(D)) \oplus \mathcal{H}_D \oplus \mathcal{H}_D$$

of $\widetilde{\mathcal{H}}$ (cf. (3.2)) we define the linear relation $\widetilde{\Theta}$ in $\widetilde{\mathcal{H}}$ by

$$\widetilde{\Theta} := \left\{ \begin{pmatrix} (u, v, v)^\top \\ (0, -w, w)^\top \end{pmatrix} : u \in \ker(\Im\mathrm{m}\,(D),\ v, w \in \mathcal{H}_D \right\}. \tag{3.11}$$

We leave it to the reader to check that $\widetilde{\Theta}$ is selfadjoint. Hence by Proposition 2.2 the operator

$$\begin{aligned} \widetilde{K} &:= K_{\widetilde{\Theta}} = K^* \upharpoonright \widetilde{\Gamma}^{-1}\widetilde{\Theta} \\ &= \left\{ f \oplus g \in \mathrm{dom}\,(A^*) \oplus \mathrm{dom}\,(G^*) : \left(\widetilde{\Gamma}_0(f \oplus g), \widetilde{\Gamma}_1(f \oplus g)\right)^\top \in \widetilde{\Theta} \right\} \end{aligned}$$

is a selfadjoint extension of the symmetric operator K in $\mathfrak{K} = \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$. The following theorem was proved in [10], see also [45, 46] for a special case involving Sturm-Liouville operators with dissipative boundary conditions.

Theorem 3.1 *Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $A_D = A^* \upharpoonright \ker(\Gamma_1 - D\Gamma_0)$ be as above. Furthermore, let G and $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$ be given by* (3.4) *and* (3.5)*, respectively, and let $K = A \oplus G$. Then the selfadjoint extension $\widetilde{K}$ of K has the form*

$$\widetilde{K} = K^* \upharpoonright \left\{ f \oplus g \in \operatorname{dom}(K^*) : \begin{array}{r} P_D\Gamma_0 f - \Upsilon_0 g = 0, \\ (I - P_D)(\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0) f = 0, \\ P_D(\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0) f + \Upsilon_1 g = 0 \end{array} \right\} \tag{3.12}$$

and $\widetilde{K}$ is a minimal selfadjoint dilation of the maximal dissipative operator A_D, that is, for all $\lambda \in \mathbb{C}_+$

$$P_{\mathfrak{H}}\big(\widetilde{K} - \lambda\big)^{-1} \upharpoonright_{\mathfrak{H}} = (A_D - \lambda)^{-1}$$

holds and the minimality condition $\mathfrak{K} = \operatorname{clospan}\{(\widetilde{K} - \lambda)^{-1}\mathfrak{H} : \lambda \in \mathbb{C}\backslash\mathbb{R}\}$ is satisfied. Moreover $\sigma(\widetilde{K}) = \mathbb{R}$.

We note that also in the case where the parameter D is not a dissipative matrix but a maximal dissipative relation in $\mathcal{H}$ a minimal selfadjoint dilation of A_D can be constructed in a similar way as in Theorem 3.1, see [10, Remark 3.3]

3.2 Spectral shift function and trace formula

In order to calculate the spectral shift function of the pair $\{\widetilde{K}, K_0\}$ from (3.10) and (3.12) we write the selfadjoint relation $\widetilde{\Theta}$ from (3.11) in the form $\widetilde{\Theta} = \widetilde{\Theta}_{\rm op} \oplus \widetilde{\Theta}_\infty$, where

$$\widetilde{\Theta}_{\rm op} := \left\{ \begin{pmatrix} (u, v, v)^\top \\ (0, 0, 0)^\top \end{pmatrix} : \ u \in \ker(\Im\mathrm{m}\,(D)), \ v \in \mathcal{H}_D \right\} \tag{3.13}$$

is the zero operator in the space

$$\widetilde{\mathcal{H}}_{\rm op} := \left\{ \begin{pmatrix} u \\ v \\ v \end{pmatrix} : \ u \in \ker(\Im\mathrm{m}\,(D)), \ v \in \mathcal{H}_D \right\} \tag{3.14}$$

and

$$\widetilde{\Theta}_\infty := \left\{ \begin{pmatrix} (0, 0, 0)^\top \\ (0, -w, w)^\top \end{pmatrix} : \ w \in \mathcal{H}_D \right\} \tag{3.15}$$

is the purely multivalued relation in the space

$$\widetilde{\mathcal{H}}_\infty = \widetilde{\mathcal{H}} \ominus \widetilde{\mathcal{H}}_{\rm op} = \left\{ \begin{pmatrix} 0 \\ -w \\ w \end{pmatrix} : \ w \in \mathcal{H}_D \right\}. \tag{3.16}$$

The orthogonal projection from $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}_{\rm op}$ will be denoted by $\widetilde{P}_{\rm op}$ and the canonical embedding of $\widetilde{\mathcal{H}}_{\rm op}$ in $\widetilde{\mathcal{H}}$ is denoted by $\widetilde{\iota}_{\rm op}$. As an immediate consequence of Theorem 2.4 we find the following representation of a spectral shift function for the pair $\{\widetilde{K}, K_0\}$.

Corollary 3.2 *Let A and G be the symmetric operators from Section 3.1 and let $K = A \oplus G$. Furthermore, let $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be the boundary triplet for K^* from* (3.7) *with Weyl function $\widetilde{M}(\cdot)$ given by* (3.8) *and define the $[\widetilde{\mathcal{H}}_{\rm op}]$-valued Nevanlinna function by*

$$\widetilde{M}_{\rm op}(\lambda) := \widetilde{P}_{\rm op}\, \widetilde{M}(\lambda)\, \widetilde{\iota}_{\rm op}\,.$$

Then the limit $\lim_{\epsilon\to+0} \widetilde{M}_{\rm op}(\lambda+i\epsilon)$ *exists for a.e.* $\lambda\in\mathbb{R}$ *and the function*

$$\xi_{\widetilde{\Theta}}(\lambda) := \frac{1}{\pi}\Im m\left(\mathrm{tr}\big(\log(\widetilde{M}_{\rm op}(\lambda+i0))\big)\right) \tag{3.17}$$

is a spectral shift function for the pair $\{\widetilde{K},K_0\}$ *with* $0\le \xi_{\widetilde{\Theta}}(\lambda)\le \dim\widetilde{\mathcal{H}}_{\rm op} = n$.

Observe that the spectral shift function in (3.17) satisfies the trace formula

$$\mathrm{tr}\big((\widetilde{K}-\lambda)^{-1}-(K_0-\lambda)^{-1}\big) = -\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\xi_{\widetilde{\Theta}}(t)\,dt \tag{3.18}$$

for $\lambda\in\mathbb{C}\backslash\mathbb{R}$. In the following theorem we calculate the spectral shift function of $\{\widetilde{K},K_0\}$ in a more explicit form up to a constant $2k$, $k\in\mathbb{Z}$. We mention that the spectral shift function in (3.19) below can be regarded as the spectral shift function of the dissipative scattering system $\{A_D,A_0\}$, cf. [40, 41, 42].

Theorem 3.3 *Let* A *and* G *be the symmetric operators from Section 3.1 and let* $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ *be a boundary triplet for* A^* *with corresponding Weyl function* $M(\cdot)$. *Let* $D\in[\mathcal{H}]$ *be a dissipative* $n\times n$*-matrix and let* $A_D = A^*\restriction\ker(\Gamma_1-D\Gamma_0)$ *be the corresponding maximal dissipative extension of* A. *Furthermore, let* K_0 *be as in* (3.10) *and let* $\widetilde{K}$ *be the minimal selfadjoint dilation of* A_D *from* (3.12).

Then the spectral shift function $\xi_{\widetilde{\Theta}}(\cdot)$ *of the pair* $\{\widetilde{K},K_0\}$ *admits the representation* $\xi_{\widetilde{\Theta}}(\cdot)=\eta_D(\cdot)+2k$ *for some* $k\in\mathbb{Z}$, *where*

$$\eta_D(\lambda) := \frac{1}{\pi}\Im m\left(\mathrm{tr}\big(\log(M(\lambda+i0)-D)\big)\right) \tag{3.19}$$

for a.e. $\lambda\in\mathbb{R}$, *and the modified trace formulas*

$$\mathrm{tr}\big((A_D-\lambda)^{-1}-(A_0-\lambda)^{-1}\big) = -\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\eta_D(t)\,dt,\quad \lambda\in\mathbb{C}_+, \tag{3.20}$$

and

$$\mathrm{tr}\big((A_D^*-\lambda)^{-1}-(A_0-\lambda)^{-1}\big) = -\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\eta_D(t)\,dt,\quad \lambda\in\mathbb{C}_-, \tag{3.21}$$

are valid.

Proof. With the help of the operator

$$V:\mathcal{H}\longrightarrow\mathcal{H},\quad x\mapsto\begin{pmatrix}(I-P_D)x\\ \frac{1}{\sqrt{2}}P_Dx\end{pmatrix}$$

and the unitary operator

$$\widetilde{V}:\mathcal{H}\longrightarrow\widetilde{\mathcal{H}}_{\rm op},\quad x\mapsto\begin{pmatrix}(I-P_D)x\\ \frac{1}{\sqrt{2}}P_Dx\\ \frac{1}{\sqrt{2}}P_Dx\end{pmatrix}$$

one easily verifies that

$$\begin{aligned}\widetilde{V}^*\widetilde{M}_{\rm op}(\lambda)\widetilde{V} &= V\left(M(\lambda) - \Re\mathrm{e}\,(D) + \begin{pmatrix}0 & 0\\ 0 & \tau(\lambda)\end{pmatrix}\right)V\\ &= V(M(\lambda)-D)V\end{aligned}\tag{3.22}$$

holds for all $\lambda\in\mathbb{C}_+$. Using this relation and the definition of $\log(\cdot)$ in (2.12) we get

$$\mathrm{tr}\big(\log\big(\widetilde{M}_{\rm op}(\lambda)\big)\big) = \mathrm{tr}\big(\log\big(\widetilde{V}^*\widetilde{M}_{\rm op}(\lambda)\widetilde{V}\big)\big) = \mathrm{tr}\big(\log\big(V(M(\lambda)-D)V\big)\big)$$

and therefore (2.14) (see also [29]) implies

$$\begin{aligned}&\frac{d}{d\lambda}\mathrm{tr}\big(\log\big(\widetilde{M}_{\rm op}(\lambda)\big)\big) = \frac{d}{d\lambda}\log\big(\det\big(V(M(\lambda)-D)V\big)\big)\\ &\quad= \frac{d}{d\lambda}\log\big(\det(M(\lambda)-D)\big) + \frac{d}{d\lambda}\log\big(\det V^2\big) = \frac{d}{d\lambda}\mathrm{tr}\big(\log(M(\lambda)-D)\big).\end{aligned}$$

Hence $\mathrm{tr}(\log(\widetilde{M}_{\rm op}(\cdot)))$ and $\mathrm{tr}(\log(M(\cdot)-D))$ differ by a constant. From

$$\exp\big(\mathrm{tr}\big(\log\big(\widetilde{M}_{\rm op}(\lambda)\big)\big)\big) = \exp\big(\mathrm{tr}\big(\log(M(\lambda)-D)\big)\big)\det V^2$$

we conclude that there exists $k\in\mathbb{Z}$ such that

$$\Im\mathrm{m}\,\big(\mathrm{tr}\big(\log\big(\widetilde{M}_{\rm op}(\lambda)\big)\big)\big) = \Im\mathrm{m}\,\big(\mathrm{tr}\big(\log(M(\lambda)-D)\big)\big) + 2k\pi$$

holds. Hence it follows that the spectral shift function $\xi_{\widetilde{\Theta}}$ of the pair $\{\widetilde{K}, K_0\}$ in (3.17) and the function $\eta_D(\cdot)$ in (3.19) differ by $2k$ for some $k\in\mathbb{Z}$.

Next we verify that the trace formulas (3.20) and (3.21) hold. From (2.7) we obtain

$$\begin{aligned}\mathrm{tr}\big((\widetilde{K}-\lambda)^{-1} - (K_0-\lambda)^{-1}\big) &= \mathrm{tr}\big(\widetilde{\gamma}(\lambda)\big(\widetilde{\Theta}-\widetilde{M}(\lambda)\big)^{-1}\widetilde{\gamma}(\bar{\lambda})^*\big)\\ &= \mathrm{tr}\big(\big(\widetilde{\Theta}-\widetilde{M}(\lambda)\big)^{-1}\widetilde{\gamma}(\bar{\lambda})^*\widetilde{\gamma}(\lambda)\big)\end{aligned}\tag{3.23}$$

for $\lambda\in\mathbb{C}\backslash\mathbb{R}$. As in (2.23) and (2.24) we find

$$\mathrm{tr}\big((\widetilde{K}-\lambda)^{-1} - (K_0-\lambda)^{-1}\big) = \mathrm{tr}\left(\big(\widetilde{\Theta}-\widetilde{M}(\lambda)\big)^{-1}\frac{d}{d\lambda}\widetilde{M}(\lambda)\right).\tag{3.24}$$

With the same argument as in the proof of Theorem 2.4 we then conclude

$$\mathrm{tr}\big((\widetilde{K}-\lambda)^{-1} - (K_0-\lambda)^{-1}\big) = \mathrm{tr}\left(\big(\widetilde{\Theta}_{\rm op}-\widetilde{M}_{\rm op}(\lambda)\big)^{-1}\frac{d}{d\lambda}\widetilde{M}_{\rm op}(\lambda)\right).\tag{3.25}$$

Since $\widetilde{\Theta}_{\rm op}=0$ and $\widetilde{V}$ is unitary it follows from (3.22) that

$$\big(\widetilde{\Theta}_{\rm op}-\widetilde{M}_{\rm op}(\lambda)\big)^{-1} = -\widetilde{M}_{\rm op}(\lambda)^{-1} = -\widetilde{V}V^{-1}\big(M(\lambda)-D\big)^{-1}V^{-1}\widetilde{V}^*$$

and

$$\frac{d}{d\lambda}\widetilde{M}_{\rm op}(\lambda) = \widetilde{V}V\frac{d}{d\lambda}M(\lambda)V\widetilde{V}^*\tag{3.26}$$

holds. This together with (3.25) implies

$$\operatorname{tr}\big((\widetilde{K}-\lambda)^{-1}-(K_0-\lambda)^{-1}\big)=\operatorname{tr}\left(-(M(\lambda)-D)^{-1}\frac{d}{d\lambda}M(\lambda)\right)$$

for all $\lambda\in\mathbb{C}_+$ and with (2.23) we get

$$\operatorname{tr}\big((\widetilde{K}-\lambda)^{-1}-(K_0-\lambda)^{-1}\big)=\operatorname{tr}\big(\gamma(\lambda)(D-M(\lambda))^{-1}\gamma(\bar\lambda)^*\big)$$

as in (2.24). Using (3.1) we obtain

$$\operatorname{tr}\big((\widetilde{K}-\lambda)^{-1}-(K_0-\lambda)^{-1}\big)=\operatorname{tr}\big((A_D-\lambda)^{-1}-(A_0-\lambda)^{-1}\big)\tag{3.27}$$

for $\lambda\in\mathbb{C}_+$. Taking into account (3.18) we prove (3.20) and (3.21) follows by taking adjoints. □

3.3 Scattering matrices of dissipative and Lax-Phillips scattering systems

In this section we recall some results from [10] on the interpretation of the diagonal entries of the scattering matrix of $\{\widetilde{K},K_0\}$ as scattering matrices of a dissipative and a Lax-Phillips scattering system. For this, let again A and G be the symmetric operators from Section 3.1 and let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^* with Weyl function $M(\cdot)$. Let $D\in[\mathcal{H}]$ be a dissipative $n\times n$-matrix and let $A_D=A^*\upharpoonright\ker(\Gamma_1-D\Gamma_0)$ be the corresponding maximal dissipative extension of A. Furthermore, let $\widetilde{\Pi}=\{\widetilde{\mathcal{H}},\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$ be the boundary triplet for $K^*=A^*\oplus G^*$ from (3.7) with Weyl function $\widetilde{M}(\cdot)$ given by (3.8), let $\widetilde{\Theta}$ be as in (3.11) and let $\widetilde{K}$ be the minimal selfadjoint dilation of A_D given by (3.12). It follows immediately from Theorem 2.5 that the scattering matrix $\{\widetilde{S}(\lambda)\}_{\lambda\in\mathbb{R}}$ of the complete scattering system $\{\widetilde{K},K_0\}$ is given by

$$\widetilde{S}(\lambda)=I_{\widetilde{\mathcal{H}}_{\widetilde{M}(\lambda)}}+2i\sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))}\big(\widetilde{\Theta}-\widetilde{M}(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))}$$

in the spectral representation $L^2(\mathbb{R},d\lambda,\mathcal{H}_{\widetilde{M}(\lambda)})$ of K_0^{ac}. Here the spaces

$$\widetilde{\mathcal{H}}_{\widetilde{M}(\lambda)}:=\operatorname{ran}\big(\Im\mathrm{m}\,(\widetilde{M}(\lambda+i0))\big)$$

for a.e. $\lambda\in\mathbb{R}$ are defined in analogy to (2.26). This representation can be made more explicit, cf. [10, Theorem 3.6].

Theorem 3.4 *Let A, $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$, $M(\cdot)$ and A_D be as above, let $K_0=A_0\oplus G_0$ and let $\widetilde{K}$ be the minimal selfadjoint dilation of A_D from Theorem* 3.1. *Then the following holds:*

(i) *$K_0^{ac}=A_0^{ac}\oplus G_0$ is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R},d\lambda,\mathcal{H}_{M(\lambda)}\oplus\mathcal{H}_D)$.*

(ii) *In $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_D)$ the scattering matrix $\{\widetilde{S}(\lambda)\}_{\lambda\in\mathbb{R}}$ of the complete scattering system $\{\widetilde{K}, K_0\}$ is given by*

$$\widetilde{S}(\lambda) = \begin{pmatrix} I_{\mathcal{H}_{M(\lambda)}} & 0 \\ 0 & I_{\mathcal{H}_D} \end{pmatrix} + 2i \begin{pmatrix} \widetilde{T}_{11}(\lambda) & \widetilde{T}_{12}(\lambda) \\ \widetilde{T}_{21}(\lambda) & \widetilde{T}_{22}(\lambda) \end{pmatrix} \in [\mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_D],$$

for a.e. $\lambda \in \mathbb{R}$, where

$$\begin{aligned} \widetilde{T}_{11}(\lambda) &= \sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}, \\ \widetilde{T}_{12}(\lambda) &= \sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(D - M(\lambda)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)}, \\ \widetilde{T}_{21}(\lambda) &= \sqrt{-\Im\mathrm{m}\,(D)}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}, \\ \widetilde{T}_{22}(\lambda) &= \sqrt{-\Im\mathrm{m}\,(D)}\big(D - M(\lambda)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)} \end{aligned}$$

and $M(\lambda) = M(\lambda + i0)$.

Observe that the scattering matrix $\{\widetilde{S}(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{\widetilde{K}, K_0\}$ depends only on the dissipative matrix D and the Weyl function $M(\cdot)$ of the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*, i.e., $\{\widetilde{S}(\lambda)\}_{\lambda\in\mathbb{R}}$ is completely determined by objects corresponding to the operators A, A_0 and A_D in $\mathfrak{H}$.

In the following we will focus on the so-called *dissipative scattering system* $\{A_D, A_0\}$ and we refer the reader to [15, 16, 36, 37, 38, 39, 40, 41, 42] for a detailed investigation of such scattering systems. We recall that the wave operators $W_\pm(A_D, A_0)$ of the dissipative scattering system $\{A_D, A_0\}$ are defined by

$$W_+(A_D, A_0) = \operatorname{s-}\lim_{t\to+\infty} e^{itA_D^*} e^{-itA_0} P^{ac}(A_0)$$

and

$$W_-(A_D, A_0) = \operatorname{s-}\lim_{t\to+\infty} e^{-itA_D} e^{itA_0} P^{ac}(A_0).$$

The scattering operator

$$S_D := W_+(A_D, A_0)^* W_-(A_D, A_0)$$

of the dissipative scattering system $\{A_D, A_0\}$ will be regarded as an operator in $\mathfrak{H}^{ac}(A_0)$. Then S_D is a contraction which in general is not unitary. Since S_D and A_0^{ac} commute it follows that S_D is unitarily equivalent to a multiplication operator induced by a family $\{S_D(\lambda)\}_{\lambda\in\mathbb{R}}$ of contractive operators in a spectral representation of A_0^{ac}.

With the help of Theorem 3.4 we obtain a representation of the scattering matrix of the dissipative scattering system $\{A_D, A_0\}$ in terms of the Weyl function $M(\cdot)$ of $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in the following corollary, cf. [10, Corollary 3.8].

Corollary 3.5 *Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and $M(\cdot)$ be as above and let $A_D = A^* \upharpoonright \ker(\Gamma_1 - D\Gamma_0)$, $D \in [\mathcal{H}]$, be maximal dissipative. Then the following holds:*

(i) *A_0^{ac} is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$.*

(ii) *The scattering matrix* $\{S_D(\lambda)\}$ *of the dissipative scattering system* $\{A_D, A_0\}$ *is given by the left upper corner of the scattering matrix* $\{\widetilde{S}(\lambda)\}$ *in Theorem 3.4, i.e.*

$$S_D(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}$$

for all a.e. $\lambda \in \mathbb{R}$, *where* $M(\lambda) = M(\lambda + i0)$.

In the following we are going to interpret the right lower corner of the scattering matrix $\{\widetilde{S}(\lambda)\}$ of $\{\widetilde{K}, K_0\}$ as the scattering matrix corresponding to a Lax-Phillips scattering system, see e.g. [8, 35] for further details. To this end we decompose the space $L^2(\mathbb{R}, \mathcal{H}_D)$ into the orthogonal sum of the subspaces

$$\mathcal{D}_- := L^2(\mathbb{R}_-, \mathcal{H}_D) \quad \text{and} \quad \mathcal{D}_+ := L^2(\mathbb{R}_+, \mathcal{H}_D). \tag{3.28}$$

Then clearly

$$\mathfrak{K} = \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D) = \mathfrak{H} \oplus \mathcal{D}_- \oplus \mathcal{D}_+$$

and we agree to denote the elements in $\mathfrak{K}$ in the form $f \oplus g_- \oplus g_+$, $f \in \mathfrak{H}$, $g_\pm \in \mathcal{D}_\pm$ and $g = g_- \oplus g_+ \in L^2(\mathbb{R}, \mathcal{H}_D)$. By J_+ and J_- we denote the operators

$$J_+ : L^2(\mathbb{R}, \mathcal{H}_D) \to \mathfrak{K}, \quad g \mapsto 0 \oplus 0 \oplus g_+,$$

and

$$J_- : L^2(\mathbb{R}, \mathcal{H}_D) \to \mathfrak{K}, \quad g \mapsto 0 \oplus g_- \oplus 0,$$

respectively. Observe that $J_+ + J_-$ is the embedding of $L^2(\mathbb{R}, \mathcal{H}_D)$ into $\mathfrak{K}$. The subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ are so-called *outgoing* and *incoming subspaces* for the selfadjoint dilation $\widetilde{K}$ in $\mathfrak{K}$, that is, one has

$$e^{-it\widetilde{K}}\mathcal{D}_\pm \subseteq \mathcal{D}_\pm, \quad t \in \mathbb{R}_\pm, \quad \text{and} \quad \bigcap_{t\in\mathbb{R}} e^{-it\widetilde{K}}\mathcal{D}_\pm = \{0\}.$$

If, in addition, $\sigma(A_0)$ is singular, then

$$\overline{\bigcup_{t\in\mathbb{R}} e^{-it\widetilde{K}}\mathcal{D}_+} = \overline{\bigcup_{t\in\mathbb{R}} e^{-it\widetilde{K}}\mathcal{D}_-} = \mathfrak{K}^{ac}(\widetilde{K}) \tag{3.29}$$

holds. Hence $\{\widetilde{K}, \mathcal{D}_-, \mathcal{D}_+\}$ is a Lax-Phillips scattering system and, in particular, the *Lax-Phillips wave operators*

$$\Omega_\pm := \operatorname{s-}\lim_{t\to\pm\infty} e^{it\widetilde{K}} J_\pm e^{-itG_0} : L^2(\mathbb{R}, \mathcal{H}_D) \to \mathfrak{K} \tag{3.30}$$

exist, cf. [8]. Since $\operatorname{s-lim}_{t\to\pm\infty} J_\mp e^{-itG_0} = 0$ the restrictions of the wave operators $W_\pm(\widetilde{K}, K_0)$ of the scattering system $\{\widetilde{K}, K_0\}$ onto $L^2(\mathbb{R}, \mathcal{H}_D)$ coincide with the Lax-Phillips wave operators $\Omega_\pm$,

$$W_\pm(\widetilde{K}, K_0)\iota_{L^2} = \operatorname{s-}\lim_{t\to\pm\infty} e^{it\widetilde{K}}(J_+ + J_-)e^{-itG_0} = \Omega_\pm.$$

Here ι_{L^2} is the canonical embedding of $L^2(\mathbb{R}, \mathcal{H}_D)$ into $\mathfrak{K}$. Hence the *Lax-Phillips scattering operator* $S^{LP} := \Omega_+^*\Omega_-$ admits the representation

$$S^{LP} = P_{L^2} S(\widetilde{K}, K_0)\, \iota_{L^2}$$

where $S(\widetilde{K}, K_0) = W_+(\widetilde{K}, K_0)^* W_-(\widetilde{K}, K_0)$ is the scattering operator of the scattering system $\{\widetilde{K}, K_0\}$ and P_{L^2} is the orthogonal projection from $\mathfrak{K}$ onto $L^2(\mathbb{R}, \mathcal{H}_D)$. Hence the Lax-Phillips scattering operator S^{LP} is a contraction in $L^2(\mathbb{R}, \mathcal{H}_D)$ and commutes with the selfadjoint differential operator G_0. Therefore S^{LP} is unitarily equivalent to a multiplication operator induced by a family $\{S^{LP}(\lambda)\}_{\lambda\in\mathbb{R}}$ of contractive operators in $L^2(\mathbb{R}, \mathcal{H}_D)$; this family is called the *Lax-Phillips scattering matrix.*

The above considerations together with Theorem 3.4 immediately imply the following corollary on the representation of the Lax-Phillips scattering matrix, cf. [10, Corollary 3.10].

Corollary 3.6 *Let $\{\widetilde{K}, \mathcal{D}_-, \mathcal{D}_+\}$ be the Lax-Phillips scattering system considered above and let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, A_D, $M(\cdot)$ and G_0 be as in the beginning of this section. Then the following holds:*

(i) *$G_0 = G_0^{ac}$ is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R}, \mathcal{H}_D) = L^2(\mathbb{R}, d\lambda, \mathcal{H}_D)$.*

(ii) *In $L^2(\mathbb{R}, d\lambda, \mathcal{H}_D)$ the Lax-Phillips scattering matrix $\{S^{LP}(\lambda)\}_{\lambda\in\mathbb{R}}$ admits the representation*

$$S^{LP}(\lambda) = I_{\mathcal{H}_D} + 2i\sqrt{-\Im\mathrm{m}\,(D)}\big(D - M(\lambda)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)} \tag{3.31}$$

for a.e. $\lambda \in \mathbb{R}$, where $M(\lambda) = M(\lambda + i0)$.

Let again A_D be the maximal dissipative extension of A corresponding to the maximal dissipative matrix $D \in [\mathcal{H}]$ and let $\mathcal{H}_D = \mathrm{ran}\,(\Im\mathrm{m}\,(D))$. By [21] the characteristic function $W_{A_D}(\cdot)$ of the A_D is given by

$$\begin{aligned} &W_{A_D} : \mathbb{C}_- \to [\mathcal{H}_D] \\ &\mu \mapsto I_{\mathcal{H}_D} - 2i\sqrt{-\Im\mathrm{m}\,(D)}\big(D^* - M(\mu)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)}. \end{aligned} \tag{3.32}$$

It determines a completely non-selfadjoint part of A_D uniquely up to unitary equivalence.

Comparing (3.31) and (3.32) we obtain the famous relation between the Lax-Phillips scattering matrix and the characteristic function discovered originally by Adamyan and Arov in [1, 2, 3, 4], cf. [10, Corollary 3.11] for another proof and further development.

Corollary 3.7 *Let the assumption be as in Corollary 3.6. Then the Lax-Phillips scattering matrix $\{S^{LP}(\lambda)\}_{\lambda\in\mathbb{R}}$ and the characteristic function $W_{A_D}(\cdot)$ of the maximal dissipative operator A_D are related by*

$$S^{LP}(\lambda) = W_{A_D}(\lambda - i0)^*$$

for a.e $\lambda \in \mathbb{R}$.

3.4 A modified Birman-Krein formula for dissipative scattering systems

Let $\{\widetilde{K}, K_0\}$ be the complete scattering system from the previous subsections and let $\{\widetilde{S}(\lambda)\}_{\lambda\in\mathbb{R}}$ be the corresponding scattering matrix. If $\xi_{\widetilde{\Theta}}(\cdot)$ is the spectral shift function in (3.17), then the Birman-Krein formula

$$\det(S_{\widetilde{\Theta}}(\lambda)) = \exp\big(-2\pi i \xi_{\widetilde{\Theta}}(\lambda)\big) \tag{3.33}$$

holds for a.e. $\lambda \in \mathbb{R}$, see Theorem 2.7. In the next theorem we prove a variant of the Birman-Krein formula for dissipative scattering systems.

Theorem 3.8 *Let A and G be the symmetric operators from Section 3.1 and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with Weyl function $M(\cdot)$. Let $D \in [\mathcal{H}]$ be dissipative and let $A_D = A^* \upharpoonright \ker(\Gamma_1 - D\Gamma_0)$ be the corresponding maximal dissipative extension of A Then the spectral shift function $\eta_D(\cdot)$ of the pair $\{A_D, A_0\}$ given by (3.19) and the scattering matrices $\{S_D(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{S^{LP}(\lambda)\}_{\lambda\in\mathbb{R}}$ from Corollary 3.5 and Corollary 3.6 are related via*

$$\det(S_D(\lambda)) = \overline{\det(S^{LP}(\lambda))}\exp\bigl(-2\pi i\eta_D(\lambda)\bigr) \tag{3.34}$$

and

$$\det(S^{LP}(\lambda)) = \overline{\det(S_D(\lambda))}\exp\bigl(-2\pi i\eta_D(\lambda)\bigr) \tag{3.35}$$

for a.e. $\lambda \in \mathbb{R}$.

Proof. Let $\widetilde{K}$ be the minimal selfadjoint dilation of A_D from (3.12) corresponding to the selfadjoint parameter $\widetilde{\Theta}$ in (3.11) via the boundary triplet $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$. Taking into account Corollary 2.6 it follows that the scattering matrix $\{\widetilde{S}(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{\widetilde{K}, K_0\}$ satisfies

$$\det(\widetilde{S}(\lambda)) = \det(\widetilde{S}_{\widetilde{\Theta}_{\rm op}}(\lambda)), \tag{3.36}$$

where $\widetilde{\Theta}_{\rm op}$ is the operator part of $\widetilde{\Theta}$ from (3.13) and

$$\widetilde{S}_{\widetilde{\Theta}_{\rm op}}(\lambda) = I_{\widetilde{\mathcal{H}}_{\widetilde{M}_{\rm op}(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(\widetilde{M}_{\rm op}(\lambda))}\bigl(\widetilde{\Theta}_{\rm op} - \widetilde{M}_{\rm op}(\lambda)\bigr)^{-1}\sqrt{\Im\mathrm{m}\,(\widetilde{M}_{\rm op}(\lambda))}$$

for a.e. $\lambda \in \mathbb{R}$. Making use of $\widetilde{\Theta}_{\rm op} = 0$ (see (3.13)) and formula (3.22) we obtain

$$\begin{aligned}\det\bigl(\widetilde{S}_{\widetilde{\Theta}_{\rm op}}(\lambda)\bigr) &= \det\Bigl(I_{\widetilde{\mathcal{H}}_{\widetilde{M}_{\rm op}(\lambda)}} + 2i\Im\mathrm{m}\,\bigl(\widetilde{M}_{\rm op}(\lambda)\bigr)\bigl(\widetilde{\Theta}_{\rm op} - \widetilde{M}_{\rm op}(\lambda)\bigr)^{-1}\Bigr)\\ &= \det\bigl(I_{\mathcal{H}} - 2i\Im\mathrm{m}\,(M(\lambda) - D)(M(\lambda) - D)^{-1}\bigr)\\ &= \frac{\det(M(\lambda)^* - D^*)}{\det(M(\lambda) - D)}.\end{aligned}$$

Hence

$$\frac{\det(M(\lambda)^* - D)}{\det(M(\lambda)^* - D^*)}\det\bigl(\widetilde{S}_{\widetilde{\Theta}_{\rm op}}(\lambda)\bigr) = \frac{\det(M(\lambda)^* - D)}{\det(M(\lambda) - D)}.$$

Obviously we have

$$\frac{\det(M(\lambda)^* - D)}{\det(M(\lambda)^* - D^*)} = \det\bigl(I_{\mathcal{H}} - 2i\Im\mathrm{m}\,(D)(M(\lambda)^* - D^*)^{-1}\bigr)$$

and since

$$\begin{aligned}&\det\bigl(I_{\mathcal{H}} - 2i\Im\mathrm{m}\,(D)(M(\lambda)^* - D^*)^{-1}\bigr)\\ &\quad= \det\bigl(I_{\mathcal{H}} - 2i\sqrt{-\Im\mathrm{m}\,(D)}(D^* - M(\lambda)^*)^{-1}\sqrt{-\Im\mathrm{m}\,(D)}\bigr)\\ &\quad= \overline{\det(S_{LP}(\lambda))}\end{aligned}$$

we get

$$\frac{\det(M(\lambda)^* - D)}{\det(M(\lambda)^* - D^*)} \det\bigl(\widetilde{S}_{\widetilde{\Theta}_{\rm op}}(\lambda)\bigr) = \overline{\det(S_{LP}(\lambda))} \det\bigl(\widetilde{S}_{\widetilde{\Theta}_{\rm op}}(\lambda)\bigr).$$

Similarly, we find

$$\begin{aligned}&\frac{\det(M(\lambda)^* - D)}{\det(M(\lambda) - D)}\\ &\qquad = \det\bigl(I_{\mathcal{H}} + 2i\sqrt{\Im\mathrm{m}\,(M(\lambda)}(D - M(\lambda))^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda)}\,\bigr)\\ &\qquad = \det(S_D(\lambda)),\end{aligned}$$

so that the relation

$$\overline{\det(S_{LP}(\lambda))} \det\bigl(\widetilde{S}_{\widetilde{\Theta}_{\rm op}}(\lambda)\bigr) = \det(S_D(\lambda))$$

holds for a.e. $\lambda \in \mathbb{R}$. Hence the Birman-Krein formula

$$\det\bigl(\widetilde{S}(\lambda)\bigr) = \exp\bigl(-2\pi i \xi_{\widetilde{\Theta}}(\lambda)\bigr),$$

which connects the scattering matrix of $\{\widetilde{K}, K_0\}$ and the spectral shift function $\xi_{\widetilde{\Theta}}(\cdot)$ in (3.17), Theorem 3.3 and (3.36) immediately imply (3.34) and (3.35) for a.e. $\lambda \in \mathbb{R}$. □

4 Coupled scattering systems

In the following we investigate so-called coupled scattering systems in a similar form as in [10], where, roughly speaking, the fixed dissipative scattering system in the previous section is replaced by a family of dissipative scattering systems which can be regarded as an open quantum system. These maximal dissipative operators form a Štraus family of extensions of a symmetric operator and their resolvents coincide pointwise with the resolvent of a certain selfadjoint operator in a bigger Hilbert space. The spectral shift functions of the dissipative scattering systems are explored and a variant of the Birman-Krein formula is proved.

4.1 Štraus family and coupling of symmetric operators

Let A be a densely defined closed simple symmetric operator with equal finite deficiency indices $n_\pm(A)$ in the separable Hilbert space $\mathfrak{H}$ and let $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with γ-field $\gamma(\cdot)$ and Weyl function $M(\cdot)$. Furthermore, let T be a densely defined closed simple symmetric operator with equal finite deficiency indices $n_\pm(T) = n_\pm(A)$ in the separable Hilbert space $\mathfrak{G}$ and let $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ be a boundary triplet of T^* with γ-field $\nu(\cdot)$ and Weyl function $\tau(\cdot)$.

Observe that $-\tau(\lambda) \in [\mathcal{H}]$ is a dissipative matrix for each $\lambda \in \mathbb{C}_+$ and therefore by Proposition 2.2

$$A_{-\tau(\lambda)} := A^* \upharpoonright \ker\bigl(\Gamma_1 + \tau(\lambda)\Gamma_0\bigr), \quad \lambda \in \mathbb{C}_+, \tag{4.1}$$

is a family of maximal dissipative extensions of A in $\mathfrak{H}$. This family is called the *Štraus family of A associated with τ*. Since the limit $\tau(\lambda) := \tau(\lambda + i0)$ exists for a.e. $\lambda \in \mathbb{R}$ the

Štraus family admits an extension to the real axis for a.e. $\lambda \in \mathbb{R}$. Analogously the Štraus family

$$T_{-M(\lambda)} := T^* \upharpoonright \ker\big(\Upsilon_1 + M(\lambda)\Upsilon_0\big), \quad \lambda \in \mathbb{C}_+, \tag{4.2}$$

of T associated with M consists of maximal dissipative extensions of T in $\mathfrak{G}$ and admits an extension to the real axis for a.e. $\lambda \in \mathbb{R}$. Sometimes it is convenient to define the Štraus family also on $\mathbb{C}_-$, in this case the extensions $A_{-\tau(\lambda)}$ and $T_{-M(\lambda)}$ are maximal accumulative for $\lambda \in \mathbb{C}_-$, cf. Proposition 2.2.

In a similar way as in Section 3.1 we consider the densely defined closed simple symmetric operator

$$L := \begin{pmatrix} A & 0 \\ 0 & T \end{pmatrix}$$

with equal finite deficiency indices $n_\pm(L) = 2n_\pm(A) = 2n_\pm(T)$ in the separable Hilbert space $\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{G}$. Then obviously $\Pi_L = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}$

$$\widetilde{\Gamma}_0(f \oplus g) := \begin{pmatrix} \Gamma_0 f \\ \Upsilon_0 g \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1(f \oplus g) := \begin{pmatrix} \Gamma_1 f \\ \Upsilon_1 g \end{pmatrix}, \tag{4.3}$$

$f \in \mathrm{dom}\,(A^*)$, $g \in \mathrm{dom}\,(T^*)$, is a boundary triplet for the adjoint

$$L^* = \begin{pmatrix} A^* & 0 \\ 0 & T^* \end{pmatrix}.$$

The γ-field $\widetilde{\gamma}(\cdot)$ and Weyl function $\widetilde{M}(\cdot)$ corresponding to the boundary triplet $\Pi_L = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ are given by

$$\widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \nu(\lambda) \end{pmatrix} \quad \text{and} \quad \widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

cf. (3.8) and (3.9). In the sequel we investigate the scattering system consisting of the selfadjoint operator

$$L_0 := L^* \upharpoonright \ker(\widetilde{\Gamma}_0) = \begin{pmatrix} A_0 & 0 \\ 0 & G_0 \end{pmatrix}, \tag{4.4}$$

where $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and $T_0 = T^* \upharpoonright \ker(\Upsilon_0)$, and the selfadjoint operator $\widetilde{L} = L^* \upharpoonright \widetilde{\Gamma}^{-1}\Theta$ which corresponds to the selfadjoint relation

$$\Theta := \left\{ \begin{pmatrix} (v, v)^\top \\ (w, -w)^\top \end{pmatrix} : v, w \in \mathcal{H} \right\} \tag{4.5}$$

in $\widetilde{\mathcal{H}}$. The selfadjoint extension $\widetilde{L}$ of L is sometimes called a coupling of the subsystems $\{\mathfrak{H}, A\}$ and $\{\mathfrak{G}, T\}$, cf. [17]. In the following theorem $\widetilde{L}$ and its connection to the Štraus family in (4.1) is made explicit, cf. [10, 17].

Theorem 4.1 *Let A, $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$, T, $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, $\tau(\cdot)$ and L be as above. Then the selfadjoint extension $\widetilde{L}$ of L in $\mathfrak{L}$ is given by*

$$\widetilde{L} = L^* \upharpoonright \left\{ f \oplus g \in \mathrm{dom}\,(L^*) : \begin{array}{c} \Gamma_0 f - \Upsilon_0 g = 0 \\ \Gamma_1 f + \Upsilon_1 g = 0 \end{array} \right\} \tag{4.6}$$

and satisfies

$$P_{\mathfrak{H}}\big(\widetilde{L}-\lambda\big)^{-1}\restriction_{\mathfrak{H}}=\big(A_{-\tau(\lambda)}-\lambda\big)^{-1}\quad\textit{and}\quad P_{\mathfrak{G}}\big(\widetilde{L}-\lambda\big)^{-1}\restriction_{\mathfrak{G}}=\big(T_{-M(\lambda)}-\lambda\big)^{-1}$$

for all $\lambda\in\mathbb{C}\backslash\mathbb{R}$. *Moreover, the following minimality conditions hold:*

$$\mathfrak{L}=\operatorname{clospan}\big\{\big(\widetilde{L}-\lambda\big)^{-1}\mathfrak{H}:\lambda\in\mathbb{C}\backslash\mathbb{R}\big\}=\operatorname{clospan}\big\{\big(\widetilde{L}-\lambda\big)^{-1}\mathfrak{K}:\lambda\in\mathbb{C}\backslash\mathbb{R}\big\}.$$

4.2 Spectral shift function and trace formula for a coupled scattering system

Next we calculate the spectral shift function of the complete scattering system $\{\widetilde{L},L_0\}$. By Theorem 2.4 a spectral shift function $\widetilde{\xi}_\Theta(\cdot)$ is given by

$$\widetilde{\xi}_\Theta(\lambda)=\frac{1}{\pi}\Im\mathrm{m}\,\big(\mathrm{tr}\big(\log(\widetilde{M}_{\mathrm{op}}(\lambda+i0)-\Theta_{\mathrm{op}})\big)\big)\tag{4.7}$$

for a.e. $\lambda\in\mathbb{R}$, where

$$\Theta_{\mathrm{op}}:=\left\{\begin{pmatrix}(v,v)^\top\\(0,0)^\top\end{pmatrix}:v\in\mathcal{H}\right\}\tag{4.8}$$

is the operator part of Θ in the space

$$\widetilde{\mathcal{H}}_{\mathrm{op}}:=\left\{\begin{pmatrix}v\\v\end{pmatrix}:v\in\mathcal{H}\right\}\subset\widetilde{\mathcal{H}}\tag{4.9}$$

and $\widetilde{M}_{\mathrm{op}}(\cdot)=\widetilde{P}_{\mathrm{op}}\widetilde{M}(\cdot)\widetilde{\iota}_{\mathrm{op}}$ denotes compression of the Weyl function $\widetilde{M}(\cdot)$ in $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}_{\mathrm{op}}$. Observe that $\Theta_{\mathrm{op}}=0$ so that the spectral shift function $\widetilde{\xi}_\Theta(\cdot)$ in (4.7) has the form

$$\widetilde{\xi}_\Theta(\lambda)=\frac{1}{\pi}\Im\mathrm{m}\,\big(\mathrm{tr}\big(\log(\widetilde{M}_{\mathrm{op}}(\lambda+i0))\big)\big)\tag{4.10}$$

for a.e. $\lambda\in\mathbb{R}$. Furthermore, the trace formula

$$\mathrm{tr}\big((\widetilde{L}-\lambda)^{-1}-(L_0-\lambda)^{-1}\big)=-\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\widetilde{\xi}_\Theta(t)\,dt\tag{4.11}$$

holds for all $\lambda\in\mathbb{C}\backslash\mathbb{R}$.

Theorem 4.2 *Let* A, $\Pi_A=\{\mathcal{H},\Gamma_0,\Gamma_1\}$, $M(\cdot)$ *and* T, $\Pi_T=\{\mathcal{H},\Upsilon_0,\Upsilon_1\}$, $\tau(\cdot)$ *be as in the beginning of Section 4.1. Then the spectral shift function* $\widetilde{\xi}_\Theta(\cdot)$ *of the pair* $\{\widetilde{L},L_0\}$ *admits the representation*

$$\widetilde{\xi}_\Theta(\lambda)=\frac{1}{\pi}\Im\mathrm{m}\,\big(\mathrm{tr}\big(\log(M(\lambda+i0)+\tau(\lambda+i0))\big)\big)+2k\tag{4.12}$$

for some $k\in\mathbb{Z}$ *and a.e.* $\lambda\in\mathbb{R}$. *Moreover, the modified trace formula*

$$\begin{aligned}&\mathrm{tr}\,\big((A_{-\tau(\lambda)}-\lambda)^{-1}-(A_0-\lambda)^{-1}\big)+\\&\quad\mathrm{tr}\,\big((T_{-M(\lambda)}-\lambda)^{-1}-(T_0-\lambda)^{-1}\big)=-\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\widetilde{\xi}_\Theta(t)\,dt\end{aligned}\tag{4.13}$$

holds for all $\lambda\in\mathbb{C}\backslash\mathbb{R}$.

Proof. With the help of the unitary operator

$$\widetilde{V} : \mathcal{H} \longrightarrow \widetilde{\mathcal{H}}_{\mathrm{op}}, \qquad x \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} x \\ x \end{pmatrix}, \tag{4.14}$$

we obtain

$$\widetilde{V}^* \widetilde{M}_{\mathrm{op}}(\lambda) \widetilde{V} = \frac{1}{2}\left(M(\lambda) + \tau(\lambda)\right). \tag{4.15}$$

We conclude in the same way as in the proof of Theorem 3.3 that the functions $\mathrm{tr}(\log(\widetilde{M}_{\mathrm{op}}(\cdot)))$ and $\mathrm{tr}(\log(M(\cdot) + \tau(\cdot)))$ differ by a constant and

$$\exp\bigl(\mathrm{tr}\bigl(\log(\widetilde{M}_{\mathrm{op}}(\lambda))\bigr)\bigr) = \exp\bigl(\mathrm{tr}\bigl(\log(M(\lambda) + \tau(\lambda))\bigr)\bigr) \det \tfrac{1}{2} I_{\mathcal{H}}$$

implies that there exists $k \in \mathbb{Z}$ such that

$$\Im\mathrm{m}\left(\mathrm{tr}\bigl(\log(\widetilde{M}_{\mathrm{op}}(\lambda))\bigr)\right) = \Im\mathrm{m}\left(\mathrm{tr}\bigl(\log(M(\lambda) + \tau(\lambda))\bigr)\right) + 2k\pi$$

holds. This together with (4.10) implies (4.12).

In order to verify the trace formula (4.13) note that by (2.7) we have

$$(\widetilde{L} - \lambda)^{-1} - (L_0 - \lambda)^{-1} = \widetilde{\gamma}(\lambda)\bigl(\widetilde{\Theta} - \widetilde{M}(\lambda)\bigr)^{-1}\widetilde{\gamma}(\bar{\lambda})^* \tag{4.16}$$

for all $\lambda \in \rho(\widetilde{L}) \cap \rho(L_0)$. Taking into account (2.19) we get

$$(\widetilde{L} - \lambda)^{-1} - (L_0 - \lambda)^{-1} = -\widetilde{\gamma}(\lambda)\widetilde{\iota}_{\mathrm{op}}\left(\widetilde{M}_{\mathrm{op}}(\lambda)\right)^{-1}\widetilde{P}_{\mathrm{op}}\widetilde{\gamma}(\bar{\lambda})^* \tag{4.17}$$

and by using

$$\left(\widetilde{M}_{\mathrm{op}}(\lambda)\right)^{-1} = 2\widetilde{V}\bigl((M(\lambda) + \tau(\lambda)\bigr)^{-1}\widetilde{V}^*,$$

cf. (4.15), we obtain

$$(\widetilde{L} - \lambda)^{-1} - (L_0 - \lambda)^{-1} = -2\widetilde{\gamma}(\lambda)\widetilde{\iota}_{\mathrm{op}}\widetilde{V}\bigl(M(\lambda) + \tau(\lambda)\bigr)^{-1}\widetilde{V}^*\widetilde{P}_{\mathrm{op}}\widetilde{\gamma}(\bar{\lambda})^* \tag{4.18}$$

which yields

$$\mathrm{tr}\bigl((\widetilde{L} - \lambda)^{-1} - (L_0 - \lambda)^{-1}\bigr) = -2\mathrm{tr}\bigl(\bigl(M(\lambda) + \tau(\lambda)\bigr)^{-1}\widetilde{V}^*\widetilde{P}_{\mathrm{op}}\widetilde{\gamma}(\bar{\lambda})^*\widetilde{\gamma}(\lambda)\widetilde{\iota}_{\mathrm{op}}\widetilde{V}\bigr) \tag{4.19}$$

for all $\lambda \in \rho(\widetilde{L}) \cap \rho(L_0)$. As in (2.6) we find

$$\widetilde{P}_{\mathrm{op}}\widetilde{\gamma}(\bar{\lambda})^*\widetilde{\gamma}(\lambda)\widetilde{\iota}_{\mathrm{op}} = \widetilde{P}_{\mathrm{op}}\frac{d}{d\lambda}\widetilde{M}(\lambda)\widetilde{\iota}_{\mathrm{op}} = \frac{d}{d\lambda}\widetilde{M}_{\mathrm{op}}(\lambda) \tag{4.20}$$

and with the help of (4.15) we conclude

$$\widetilde{V}^*\widetilde{P}_{\mathrm{op}}\widetilde{\gamma}(\bar{\lambda})^*\widetilde{\gamma}(\lambda)\widetilde{\iota}_{\mathrm{op}}\widetilde{V} = \frac{1}{2}\left(\frac{d}{d\lambda}M(\lambda) + \frac{d}{d\lambda}\tau(\lambda)\right). \tag{4.21}$$

Hence

$$\begin{aligned} &\mathrm{tr}\bigl((\widetilde{L} - \lambda)^{-1} - (L_0 - \lambda)^{-1}\bigr) \\ &\quad = -\mathrm{tr}\left(\bigl(M(\lambda) + \tau(\lambda)\bigr)^{-1}\left(\frac{d}{d\lambda}M(\lambda) + \frac{d}{d\lambda}\tau(\lambda)\right)\right). \end{aligned}$$

Using again (2.6) we find

$$\begin{aligned}&\mathrm{tr}\big((\widetilde{L}-\lambda)^{-1}-(L_0-\lambda)^{-1}\big)\\&\quad=-\mathrm{tr}\left(\gamma(\lambda)(M(\lambda)+\tau(\lambda))^{-1}\gamma(\bar\lambda)^*\right)-\mathrm{tr}\left(\nu(\lambda)(M(\lambda)+\tau(\lambda))^{-1}\nu(\bar\lambda)^*\right).\end{aligned}$$

By (2.7) the resolvents of the Štraus family of A associated with τ and the Štraus family of T associated with M are given by

$$\big(A_{-\tau(\lambda)}-\lambda\big)^{-1}-(A_0-\lambda)^{-1}=-\gamma(\lambda)\big(M(\lambda)+\tau(\lambda)\big)^{-1}\gamma(\bar\lambda)^* \tag{4.22}$$

and

$$\big(T_{-M(\lambda)}-\lambda\big)^{-1}-(T_0-\lambda)^{-1}=-\nu(\lambda)\big(M(\lambda)+\tau(\lambda)\big)^{-1}\nu(\bar\lambda)^*, \tag{4.23}$$

respectively. Taking into account (4.22), (4.23) and (4.11) we prove (4.13). □

Let us consider the the spectral shift function $\eta_{-\tau(\mu)}(\cdot)$ of the dissipative scattering system $\{A_{-\tau(\mu)},A_0\}$ for those $\mu\in\mathbb{R}$ for which the limit $\tau(\mu):=\tau(\mu+i0)$ exists. By Theorem 3.3 the function $\eta_{-\tau(\mu)}(\cdot)$ admits the representation

$$\eta_{-\tau(\mu)}(\lambda)=\frac{1}{\pi}\Im\mathrm{m}\left(\mathrm{tr}\big(\log(M(\lambda+i0)+\tau(\mu))\big)\right) \tag{4.24}$$

for a.e. $\lambda\in\mathbb{R}$. Moreover, we have

$$\mathrm{tr}\left((A_{-\tau(\mu)}-\lambda)^{-1}-(A_0-\lambda)^{-1}\right)=-\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\eta_{-\tau(\mu)}(t)\,dt$$

for all $\lambda\in\mathbb{C}_+$, cf. Theorem 3.3. Similarly, we introduce the spectral shift function $\eta_{-M(\mu)}(\cdot)$ of the dissipative scattering system $\{T_{-M(\mu)},T_0\}$ for those $\mu\in\mathbb{R}$ for which the limit $M(\mu)=M(\mu+i0)$ exists. It follows that

$$\eta_{-M(\mu)}(\lambda)=\frac{1}{\pi}\Im\mathrm{m}\left(\mathrm{tr}\big(\log(M(\mu)+\tau(\lambda+i0))\big)\right) \tag{4.25}$$

holds for a.e. $\lambda\in\mathbb{R}$ and

$$\mathrm{tr}\left((T_{-M(\mu)}-\lambda)^{-1}-(T_0-\lambda)^{-1}\right)=-\int_{\mathbb{R}}\frac{1}{(t-\lambda)^2}\,\eta_{-M(\mu)}(t)\,dt$$

is valid for $\lambda\in\mathbb{C}_+$. Hence we get immediately the following corollary.

Corollary 4.3 *Let the assumptions be as in Theorem 4.2, let L_0, $\widetilde{L}$ be as in* (4.4), (4.6) *and let $\eta_{-\tau(\mu)}(\cdot)$ and $\eta_{-M(\mu)}(\cdot)$ be the spectral shift functions in* (4.24) *and* (4.25), *respectively. Then the spectral shift function $\widetilde{\xi}_\Theta(\cdot)$ of the pair $\{\widetilde{L},L_0\}$ admits the representation*

$$\widetilde{\xi}_\Theta(\lambda)=\eta_{-\tau(\lambda)}(\lambda)+2k=\eta_{-M(\lambda)}(\lambda)+2l \tag{4.26}$$

for a.e. $\lambda\in\mathbb{R}$ and some $k,l\in\mathbb{Z}$.

4.3 Scattering matrices of coupled systems

We investigate the scattering matrix of the scattering system $\{\widetilde{L}, L_0\}$, where $\widetilde{L}$ and L_0 are the selfadjoint operators in $\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{G}$ from (4.6) and (4.4), respectively. By Theorem 2.5 the scattering matrix $\{\widetilde{S}_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ of $\{\widetilde{L}, L_0\}$ admits the representation

$$\widetilde{S}_\Theta(\lambda) = I_{\widetilde{\mathcal{H}}_{\widetilde{M}(\lambda)}} + 2i\sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))}\big(\Theta - \widetilde{M}(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))}. \tag{4.27}$$

Here $\widetilde{M}(\cdot)$ is the Weyl function of the boundary triplet $\Pi_L = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ from (4.3) and

$$\widetilde{\mathcal{H}}_{\widetilde{M}(\lambda)} := \mathrm{ran}\,\big(\Im\mathrm{m}\,\big(\widetilde{M}(\lambda + i0)\big)\big)$$

for a.e. $\lambda \in \mathbb{R}$. In [10] the scattering matrix of $\{\widetilde{L}, L_0\}$ was expressed in terms of the Weyl functions $M(\cdot)$ and $\tau(\cdot)$ of the boundary triplets $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, respectively. The following representation for $\{\widetilde{S}_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ can be deduced from Corollary 2.6.

Theorem 4.4 *Let A, $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$ and T, $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, $\tau(\cdot)$ be as above. Then the following holds:*

(i) *$L_0^{ac} = A_0^{ac} \oplus T_0^{ac}$ is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)})$.*

(ii) *In $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)})$ the scattering matrix $\{\widetilde{S}_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ of the complete scattering system $\{\widetilde{L}, L_0\}$ is given by*

$$\widetilde{S}_\Theta(\lambda) = I_{\mathcal{H}_{M(\lambda)}\oplus\mathcal{H}_{\tau(\lambda)}} - 2i\begin{pmatrix}\widetilde{T}_{11}(\lambda) & \widetilde{T}_{12}(\lambda)\\ \widetilde{T}_{21}(\lambda) & \widetilde{T}_{22}(\lambda)\end{pmatrix} \in [\mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)}], \tag{4.28}$$

for a.e. $\lambda \in \mathbb{R}$ where

$$\begin{aligned}
\widetilde{T}_{11}(\lambda) &= \sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))},\\
\widetilde{T}_{12}(\lambda) &= \sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))},\\
\widetilde{T}_{21}(\lambda) &= \sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))},\\
\widetilde{T}_{22}(\lambda) &= \sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}
\end{aligned}$$

and $M(\lambda) = M(\lambda + i0)$, $\tau(\lambda) = \tau(\lambda + i0)$.

Let $J_\mathfrak{H} : \mathfrak{H} \longrightarrow \mathfrak{L}$ and $J_\mathfrak{G} : \mathfrak{G} \longrightarrow \mathfrak{L}$ the natural embedding operators of the subspaces $\mathfrak{H}$ and $\mathfrak{G}$ into $\mathfrak{L}$, respectively. The wave operators

$$W_\pm(\widetilde{L}, A_0) := \operatorname*{s-\lim}_{t\to\pm\infty} e^{it\widetilde{L}} J_\mathfrak{H} e^{-itA_0} P^{ac}(A_0) \tag{4.29}$$

and

$$W_\pm(\widetilde{L}, T_0) := \operatorname*{s-\lim}_{t\to\pm\infty} e^{it\widetilde{L}} J_\mathfrak{G} e^{-itT_0} P^{ac}(T_0) \tag{4.30}$$

are called the *channel wave operators* or *partial wave operators.* The *channel scattering operators* $S_{\mathfrak{H}}$ and $S_{\mathfrak{G}}$ are defined by

$$S_{\mathfrak{H}} := W_+(\widetilde{L}, A_0)^* W_-(\widetilde{L}, A_0) \quad \text{and} \quad S_{\mathfrak{G}} := W_+(\widetilde{L}, T_0)^* W_-(\widetilde{L}, T_0). \tag{4.31}$$

The channel scattering operators $S_{\mathfrak{H}}$ and $S_{\mathfrak{G}}$ are contractions in $\mathfrak{H}^{ac}(A_0)$ and $\mathfrak{G}^{ac}(T_0)$ and commute with A_0 and T_0, respectively. Hence, there are measurable families of contractions

$$\{S_{\mathfrak{H}}(\lambda)\}_{\lambda\in\mathbb{R}} \quad \text{and} \quad \{S_{\mathfrak{G}}(\lambda)\}_{\lambda\in\mathbb{R}} \tag{4.32}$$

such that the multiplication operators induced by these families in the spectral representations $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ and $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\lambda)})$ of A_0^{ac} and T_0^{ac}, respectively, are unitarily equivalent to the channel scattering operators $S_{\mathfrak{H}}$ and $S_{\mathfrak{G}}$. The multiplication operators in (4.32) are called *channel scattering matrices.*

Corollary 4.5 *Let A, $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$ and T, $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, $\tau(\lambda)$ be as above. Then the following holds:*

(i) *A_0^{ac} and T_0^{ac} are unitarily equivalent to the multiplication operators with the free variable in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)}$ and $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\lambda)})$, respectively.*

(ii) *In $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ and $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\lambda)})$ the channel scattering matrices $\{S_{\mathfrak{H}}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{S_{\mathfrak{G}}(\lambda)\}_{\lambda\in\mathbb{R}}$ are given by*

$$S_{\mathfrak{H}}(\lambda) = I_{\mathcal{H}_{M(\lambda)}} - 2i\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(M(\lambda)+\tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}$$

and

$$S_{\mathfrak{G}}(\lambda) = I_{\mathcal{H}_{\tau(\lambda)}} - 2i\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(M(\lambda)+\tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}$$

for a.e. $\lambda \in \mathbb{R}$.

4.4 A modified Birman-Krein formula for coupled scattering systems

In a similar way as in Section 3.4 we prove a variant of the Birman-Krein formula for the coupled scattering system $\{\widetilde{L}, L_0\}$, where $\widetilde{L}$ and L_0 are as in (4.6) and (4.4), respectively. First of all it is clear that the scattering matrix $\{\widetilde{S}_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ of $\{\widetilde{L}, L_0\}$ and the spectral shift function $\widetilde{\xi}_\Theta(\cdot)$ from (4.10) are connected via the usual Birman-Krein formula

$$\det\big(\widetilde{S}_\Theta(\lambda)\big) = \exp\big(-2\pi i\widetilde{\xi}_\Theta(\lambda)\big) \tag{4.33}$$

for a.e. $\lambda \in \mathbb{R}$, cf. Theorem 2.7. With the help of the channel scattering matrices from (4.32) and Corollary 4.5 we find the following modified Birman-Krein formula.

Theorem 4.6 *Let A and T be as in Section 4.1 and let $\{\widetilde{L}, L_0\}$ be the complete scattering system from above. Then the spectral shift function $\widetilde{\xi}_\Theta(\cdot)$ of the pair $\{\widetilde{L}, L_0\}$ in (4.10) is related with the channel scattering matrices $\{S_{\mathfrak{H}}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{S_{\mathfrak{G}}(\lambda)\}_{\lambda\in\mathbb{R}}$ in (4.32) via*

$$\det(S_{\mathfrak{H}}(\lambda)) = \overline{\det(S_{\mathfrak{G}}(\lambda))}\exp\big(-2\pi i\widetilde{\xi}_\Theta(\lambda)\big) \tag{4.34}$$

and

$$\det(S_{\mathfrak{G}}(\lambda)) = \overline{\det(S_{\mathfrak{H}}(\lambda))}\exp\big(-2\pi i\widetilde{\xi}_\Theta(\lambda)\big) \tag{4.35}$$

for a.e. $\lambda \in \mathbb{R}$.

Proof. Let $\{\widetilde{S}_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ be the scattering matrix of $\{\widetilde{L}, L_0\}$ from (4.27). Making use of Corollary 2.6 we obtain

$$\det\bigl(\widetilde{S}_\Theta(\lambda)\bigr) = \det\bigl(\widetilde{S}_{\Theta_{\rm op}}(\lambda)\bigr) \tag{4.36}$$

where $\Theta_{\rm op} = 0 \in [\widetilde{\mathcal{H}}_{\rm op}]$ is the operator part of Θ in $\widetilde{\mathcal{H}}_{\rm op}$, cf. (4.8), (4.9), and $\widetilde{S}_{\Theta_{\rm op}}(\lambda)$ is given by

$$\widetilde{S}_{\Theta_{\rm op}}(\lambda) = I_{\mathcal{H}_{\widetilde{M}_{\rm op}(\lambda)}} - 2i\sqrt{\Im\mathrm{m}\,(\widetilde{M}_{\rm op}(\lambda))}\bigl(\widetilde{M}_{\rm op}(\lambda)\bigr)^{-1}\sqrt{\Im\mathrm{m}\,(\widetilde{M}_{\rm op}(\lambda))} \tag{4.37}$$

for a.e. $\lambda \in \mathbb{R}$. Here $\widetilde{M}_{\rm op}(\cdot) = \widetilde{P}_{\rm op}\widetilde{M}(\cdot)\widetilde{\iota}_{\rm op}$ is the compression of the Weyl function corresponding to the boundary triplet $\Pi_L = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ onto the space $\widetilde{\mathcal{H}}_{\rm op}$. Let $\widetilde{V}$ be as in (4.14). Then we have

$$\widetilde{M}_{\rm op}(\lambda) = \frac{1}{2}\widetilde{V}(M(\lambda)+\tau(\lambda))\widetilde{V}^* \quad \text{and} \quad \widetilde{M}_{\rm op}(\lambda)^{-1} = 2\widetilde{V}(M(\lambda)+\tau(\lambda))^{-1}\widetilde{V}^*,$$

cf. (4.15), and therefore we get

$$\det\bigl(\widetilde{S}_{\Theta_{\rm op}}(\lambda)\bigr) = \det\bigl(I_{\mathcal{H}} - 2i\Im\mathrm{m}\,\bigl(M(\lambda)+\tau(\lambda)\bigr)\bigl(M(\lambda)+\tau(\lambda)\bigr)^{-1}\bigr).$$

This yields

$$\det\bigl(\widetilde{S}_{\Theta_{\rm op}}(\lambda)\bigr) = \frac{\overline{\det(M(\lambda)+\tau(\lambda))}}{\det(M(\lambda)+\tau(\lambda))} \tag{4.38}$$

for a.e. $\lambda\in\mathbb{R}$ and hence

$$\frac{\overline{\det(M(\lambda)+\tau(\lambda)^*)}}{\overline{\det(M(\lambda)+\tau(\lambda))}}\det\bigl(\widetilde{S}_{\Theta_{\rm op}}(\lambda)\bigr) = \frac{\det(M(\lambda)^*+\tau(\lambda))}{\det(M(\lambda)+\tau(\lambda))} \tag{4.39}$$

for a.e. $\lambda\in\mathbb{R}$. On the other hand, as a consequence of Corollary 4.5 we obtain

$$\det(S_{\mathfrak{H}}(\lambda)) = \frac{\det(M(\lambda)^*+\tau(\lambda))}{\det(M(\lambda)+\tau(\lambda))} \tag{4.40}$$

and

$$\det(S_{\mathfrak{G}}(\lambda)) = \frac{\det(M(\lambda)+\tau(\lambda)^*)}{\det(M(\lambda)+\tau(\lambda))} \tag{4.41}$$

for a.e. $\lambda\in\mathbb{R}$ and therefore we find

$$\overline{\det(S_{\mathfrak{G}}(\lambda))}\det\bigl(\widetilde{S}_{\Theta_{\rm op}}(\lambda)\bigr) = \det(S_{\mathfrak{H}}(\lambda)) \tag{4.42}$$

for a.e. $\lambda\in\mathbb{R}$. Taking into account (4.33) and (4.36) we obtain (4.34). The relation (4.35) follows from (4.34). □

Making use of Corollary 4.3 we obtain the following form for the relations (4.34) and (4.35).

Corollary 4.7 *Let the assumptions be as in Theorem 4.6 and let $\eta_{-\tau(\mu)}(\cdot)$ and $\eta_{-M(\mu)}(\cdot)$ be as in (4.24) and (4.25), respectively. Then the channel scattering matrices $\{S_{\mathfrak{H}}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{S_{\mathfrak{G}}(\lambda)\}_{\lambda\in\mathbb{R}}$ are connected with the functions $\lambda\mapsto\eta_{-\tau(\lambda)}(\lambda)$ and $\lambda\mapsto\eta_{-M(\lambda)}(\lambda)$ via*

$$\det(S_{\mathfrak{H}}(\lambda)) = \overline{\det(S_{\mathfrak{G}}(\lambda))}\exp\bigl(-2\pi i\eta_{-\tau(\lambda)}(\lambda)\bigr) \tag{4.43}$$

and

$$\det(S_{\mathfrak{G}}(\lambda)) = \overline{\det(S_{\mathfrak{H}}(\lambda))}\exp\bigl(-2\pi i\eta_{-M(\lambda)}(\lambda)\bigr) \tag{4.44}$$

for a.e. $\lambda\in\mathbb{R}$.

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