Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Transition between Airy$_1$ and Airy$_2$ processes and TASEP fluctuations

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submitted: March 12, 2007

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No. 1214
Berlin 2007

2000 *Mathematics Subject Classification.* 82C22, 60K35, 15A52.

Key words and phrases. Simple exclusion process, universality, KPZ class, Airy process, random matrices.

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Abstract

We consider the totally asymmetric simple exclusion process, a model in the KPZ universality class. We focus on the fluctuations of particle positions starting with certain deterministic initial conditions. For large time t, one has regions with constant and linearly decreasing density. The fluctuations on these two regions are given by the Airy_1 and Airy_2 processes, whose one-point distributions are the GOE and GUE Tracy-Widom distributions of random matrix theory. In this paper we analyze the transition region between these two regimes and obtain the transition process. Its one-point distribution is a new interpolation between GOE and GUE edge distributions.

1 Introduction

In the search of universal limit distribution functions and limit processes, we consider the KPZ universality class (KPZ for Kardar-Parisi-Zhang) originally introduced for stochastic growth models [13]. For growth in $1+1$ dimensions the scaling exponents of fluctuations, $1/3$, and correlations, $2/3$, can be (non rigorously) determined by some involved arguments, see e.g. [16] for an extended discussion. However, to get more insights into the limit laws and limit processes, one is led to consider solvable models in the universality class.

One such model is the polynuclear growth (PNG) model in discrete time, which has two interesting limits, where new processes have been discovered. The first limit is the continuous time PNG model, for which it has been shown that the surface growing in a droplet shape is, in the large time limit, governed by the Airy_2 process [20]. The second one is the totally asymmetric simple exclusion process (TASEP), in which quite recently the limit process of the particles positions starting from a periodic initial conditions has been unravelled and called Airy_1 process [2,21]. In the surface growth picture this corresponds to the flat initial conditions.

If $x_k(t)$ denotes the position of particle with label k, one of the usual geometric representation of the TASEP in terms of surface growth is obtained by the graph $\{(k, x_k(t)+k)\}$, see Figure 1 (right) and also e.g. [5]. By universality it is expected that the limit process in one-dimensional KPZ growth is the Airy_2 process for the curved regions of the limit shape, and the Airy_1 process for the flat parts. However initial conditions can easily generate limit shapes which have both curved and flat regions. Therefore there exist transition regions where the limit shape smoothly changes between curved and flat.

The novelty of this paper is the analysis of this transition region in the framework of the TASEP. The observables we consider are positions of several particles at time t. In [10], step-initial conditions (particles starting from $\mathbb{Z}_-$) have been considered from the perspective of a growth model and it was proved that the Airy_2 process

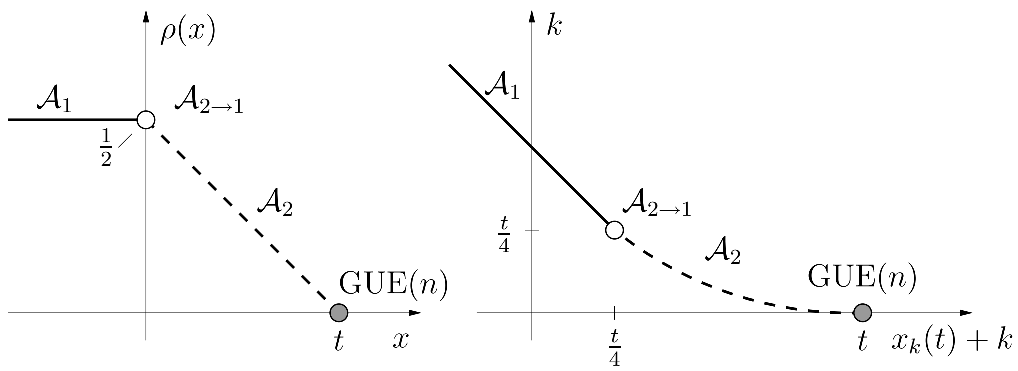

Figure 1: Left: the density ρ for large time t is linearly decreasing from $(0, 1/2)$ to $(t, 0)$. Right: The limit shape in an associated growth, obtained from the density ($k \in \mathbb{N}$ is the label of the particle which starts at $-2k$ and $x_k(t)$ is its position at time t).

appears in the large time limit. In [1, 2] we considered periodic initial conditions (particles starting from $d\mathbb{Z}$, $d = 2, 3, \ldots$) and obtained the Airy$_1$ process as the limit process.

To obtain both regimes and the transition region, we consider in this paper particles starting from $2\mathbb{Z}_-$ as in [21]. There are four regions of interest as illustrated in Figure 1 (left).

(1) Constant density region. The limit process of particle positions is given by the Airy$_1$ process $\mathcal{A}_1$. In particular, the one-point distribution is $F_1(2^{2/3}s)$, with F_1 being the GOE Tracy-Widom distribution.

(2) Linearly decreasing density region. The limit process is the Airy$_2$ process $\mathcal{A}_2$, which has $F_2(s)$ as one-point distribution, with F_2 being the GUE Tracy-Widom distribution.

(3) Finite distance from the right-most particle. There the particle positions are described via the GUE-minor kernel [12]. In particular, the n-th right-most particle is distributed as the largest eigenvalue of the n-particle GUE ensemble.

(4) The transition region between (1) and (2). The fluctuations are governed by a new process obtained in this paper: the transition process Airy$_{2\to1}$, which we denote $\mathcal{A}_{2\to1}$. In particular, the one-point distribution interpolates between $F_2(s)$ and $F_1(2^{2/3}s)$ and the transition region has width which scales in time as $t^{2/3}$.

The analysis is done by using the framework of *signed* determinantal point processes introduced in [2]. This new approach allows us to analyze all four regions for our initial conditions. This is contrasted to the previously used determinantal point process issued by the RSK construction, by which only the step initial condition or its variants could be analyzed [3, 4, 6, 8–10, 20, 22]. We explain how the analysis has to be done for all 4 cases, but the complete asymptotic analysis is presented only for the transition region, the technically most difficult one, and the really new result of this paper. The result is a process, $\mathcal{A}_{2\to 1}$, interpolating between the Airy$_2$ and the Airy$_1$ processes. For more details about the Airy processes, see the review [5].

The transition we discovered is not the first one between some GUE and GOE type distributions, but it seems to be different from the one previously known for random matrices, non-colliding Brownian motions with open boundary condition and so on [7,14,19,22]. The main differences are the following. On the natural scale of the problems considered, the final distribution is $F_1(2^{2/3}s)$ for our case and $F_1(s)$ in the previous case. Secondly, in the previous case, the GOE-type distribution appears at a single point, while in our case, the GOE-type distribution is on an extended region. Moreover, our transition smoothly interpolates between $F_2(s)$ and $F_1(2^{2/3}s)$, which is not the case for the other transition. In principle, we can not however yet exclude that by a change of variable, with both the rescaling of fluctuations and spatial correlations, the two transitions map one to the other.

Outline. In Section 2 we define the model we analyze and state the results. In Section 3 we explain the finite time result and set the scaling limit. In Section 4 we do the complete asymptotic analysis for the transition region and in Section 5 we explain how to do the analysis for the other cases. Finally, we present an explicit form of the transition kernel in terms of Airy functions in Appendix A and we explain the correctness of the Fredholm determinants involved in B.

Acknowledgments

A. Borodin was partially supported by the NSF grant DMS-0402047 and the CRDF grant RIM1-2622-ST-04. P.L. Ferrari would like to thank for the support of the German-Japanese cooperation project of German National Foundation, DFG 446JAP113/325/0-1. The work of T. Sasamoto is supported by the Grant-in-Aid for Young Scientists (B), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

2 Model and results

In this paper we consider the continuous-time totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$. At any given time t, every site $j \in \mathbb{Z}$ can be occupied at most by one particle. Thus a configuration of the TASEP can be described by $\eta = \{\eta_j, j \in \mathbb{Z} | \eta_j \in \{0,1\}\} \in \Omega = \{0,1\}^{\mathbb{Z}}$. η_j is called the *occupation variable* of site j, which is defined by $\eta_j = 1$ if site j is occupied and $\eta_j = 0$ if site j is empty.

The dynamics of the TASEP is defined as follows. Particles jump on the neighboring right site with rate 1 provided that the site is empty. This means that jumps are independent of each other and are performed after an exponential waiting time with mean 1. More precisely, let $f\colon \Omega \to \mathbb{R}$ be a function depending only on a finite number of η_j's. Then the backward generator of the TASEP is given by

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} \eta_j(1-\eta_{j+1})\big(f(\eta^{j,j+1}) - f(\eta)\big). \tag{2.1}$$

Here $\eta^{j,j+1}$ denotes the configuration η with the occupations at sites j and $j+1$ interchanged. The semigroup e^{Lt} is well-defined as acting on bounded and continuous functions on Ω. e^{Lt} is the transition probability of the TASEP [18].

We denote by $x_k(t)$ the position of the particle number k at time t. As initial condition we consider particles starting from $2\mathbb{Z}_-$, i.e., $x_k(0) = -2k$ for $k = 1, 2, \ldots$. On the macroscopic level, the limit particle density $u(\xi)$ is given by

$$u(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big(\#(k : x_k(t) \geq \xi t)\big) = \begin{cases} 1/2, & \xi < 0, \\ 1/2 - \xi/2, & \xi \in [0,1], \\ 0, & \xi > 1. \end{cases} \tag{2.2}$$

Thus for large time t the expected number of particle at site x, $\rho(x)$, is close to $u(x/t)$, see Figure 1.

As observables we consider the positions of finite subsets of particles, $\{x_i(t), i \in I\}$ for some $I \subset \mathbb{N}$, $|I| < \infty$. The scaling limits we have to take depend on which of the four regions described in the Introduction we focus on, see also Figure 1. The main result of this paper is the description of the large time fluctuations in the transition region, which now we describe.

(4) Transition region: the $\mathcal{A}_{2\to 1}$ process

The transition region has width of order $t^{2/3}$, which is indicated by the fact that the index of the particles which at time t are around $x = 0$ fluctuates on the $t^{2/3}$ scale around the macroscopic value $t/4$. Therefore we set

$$n(\tau, t) = [t/4 + \tau(t/2)^{2/3}]. \tag{2.3}$$

The density (2.2) changes in the transition region. The limit density can be used to determine, on the macroscopic scale, the expected location at time t of a particle with index $n(a) = [t/4 + at]$ (of course $a \geq -1/4$). The result is then

$$\lim_{t\to\infty} \frac{x_{n(a)}}{t} = \begin{cases} 1 - \sqrt{1+4a}, & a \in [-1/4, 0], \\ -2a, & a \geq 0. \end{cases} \tag{2.4}$$

By using (2.4) with $at = \tau(t/2)^{2/3}$ we are led to define the rescaled process of particle positions by

$$\tau \mapsto X_t(\tau) = \frac{x_{n(\tau,t)}(t) - (-2\tau(t/2)^{2/3} + \min\{0,\tau\}^2 (t/2)^{1/3})}{-(t/2)^{1/3}}. \tag{2.5}$$

The main result of this paper is the convergence of $X_t(\tau)$ to the transition process $\mathcal{A}_{2\to 1}$ defined below.

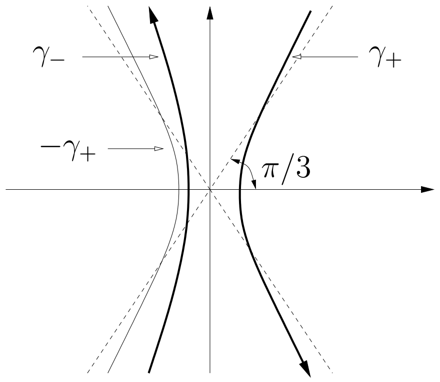

Figure 2: An illustration of the paths γ_+ and γ_- in Definition 1.

Definition 1 (The Airy$_{2\to 1}$ process)**.** *Let us set*

$$\tilde{s}_i = \begin{cases} s_i, & \tau_i \geq 0, \\ s_i - \tau_i^2, & \tau_i \leq 0. \end{cases} \tag{2.6}$$

and define the transition kernel

$$\begin{aligned} K_\infty(\tau_1, s_1; \tau_2, s_2) &= -\frac{1}{\sqrt{4\pi(\tau_2-\tau_1)}} \exp\left(-\frac{(\tilde{s}_2-\tilde{s}_1)^2}{4(\tau_2-\tau_1)}\right) \mathbb{1}(\tau_2 > \tau_1) \\ &+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_+} \mathrm{d}w \int_{\gamma_-} \mathrm{d}z \frac{e^{w^3/3+\tau_2 w^2 - \tilde{s}_2 w}}{e^{z^3/3+\tau_1 z^2-\tilde{s}_1 z}} \frac{2w}{(z-w)(z+w)} \end{aligned} \tag{2.7}$$

with the paths γ_+, γ_- satisfying $-\gamma_+ \subset \gamma_-$ with $\gamma_+ : e^{\mathrm{i}\phi_+}\infty \to e^{-\mathrm{i}\phi_+}\infty$, $\gamma_- : e^{-\mathrm{i}\phi_-}\infty \to e^{\mathrm{i}\phi_-}\infty$ for some $\phi_+ \in (\pi/3, \pi/2)$, $\phi_- \in (\pi/2, \pi - \phi_+)$, see Figure 2.

The Airy$_{2\to 1}$ process*, $\mathcal{A}_{2\to 1}$, is the process with m-point joint distributions at $\tau_1 < \tau_2 < \ldots < \tau_m$ given by the Fredholm determinant*

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_{2\to 1}(\tau_k) \leq s_k\}\right) = \det(\mathbb{1} - \chi_s K_\infty \chi_s)_{L^2(\{\tau_1,\ldots,\tau_m\}\times\mathbb{R})} \tag{2.8}$$

where $\chi_s(\tau_k, x) = \mathbb{1}(x > s_k)$. An explicit expression for K_∞ in terms of Airy functions can be found in Appendix A.

Remarks: $\mathcal{A}_{2\to 1}(t+\tau)$ becomes $2^{-1/3}\mathcal{A}_1(2^{2/3}\tau)$ as $t \to \infty$ and $\mathcal{A}_2(\tau)$ when $t \to -\infty$. The Fredholm determinant in (2.8) is well defined because, as proven in Proposition 9 of Appendix B, there exists a conjugate kernel of $\chi_s K_\infty \chi_s$ which is trace-class on $\mathcal{H} = L^2(\{\tau_1, \ldots, \tau_m\} \times \mathbb{R})$.

Now we can state precisely our main Theorem.

Theorem 2. *The convergence of X_t to the transition process $\mathcal{A}_{2\to 1}$,*

$$\lim_{t\to\infty} X_t(\tau) = \mathcal{A}_{2\to 1}(\tau), \tag{2.9}$$

holds in the sense of finite-dimensional distributions.

A remark on initial conditions. In this work as well as in many of the previous papers in the field, the situations analyzed with deterministic initial conditions might look quite peculiar: step-initial conditions [11], periodic with period 2 or more [1,2]. However, it is intuitively clear that small perturbations of the initial conditions do not affect the large time behavior. This is indeed the case by a coupling argument.

Consider two TASEP initial conditions of N particles, $X(0) = \{x_N(0) < \ldots < x_2(0) < x_1(0)\}$, $Z(0) = \{z_N(0) < \ldots < z_2(0) < z_1(0)\}$ with $X(0) \leq Z(0)$ meaning $x_k(0) \leq z_k(0)$, $k = 1, \ldots, N$. By a standard coupling argument, see e.g. [17], for any subset $I \subset \{1, \ldots, N\}$,

$$\mathbb{P}(\{x_i(t) \leq a_i, i \in I\}) \geq \mathbb{P}(\{z_i(t) \leq a_i, i \in I\}). \tag{2.10}$$

We can apply (2.10) to our case to show that the limit result is unchanged if we do any bounded perturbation of the initial condition. In Theorem 2 we started with initial conditions $x_i(0) = -2i$. Consider any other initial condition $Z = \{z_i(0)\}$ and define

$$M = \max\{|x_i(0) - z_i(0)|\}. \tag{2.11}$$

Then, by (2.10), we have

$$\begin{aligned} \mathbb{P}(\{x_i(t) \leq a_i + M, i \in I\}) &\geq \mathbb{P}(\{z_i(t) \leq a_i, i \in I\}) \\ &\geq \mathbb{P}(\{x_i(t) \leq a_i - M, i \in I\}). \end{aligned} \tag{2.12}$$

In the scaling limit (2.5), the first and last term in (2.12) have the same limit as $t \to \infty$ as long as $\lim_{t\to\infty} M/t^{1/3} = 0$. This holds in particular if Z is any bounded perturbation of X, i.e., if $M < \infty$ is independent of t.

For completeness we state the results in the other three regions. In Section 5 we outline how the asymptotic analysis for the transition region has to be modified in order to obtain the results. The scaling is obtained using (2.4).

(1) Fixed particle number: GUE(n) minors

Consider particles with index not rescaled in time, i.e.,

$$n \text{ of order one}, \tag{2.13}$$

and the rescaled random variables

$$X_t(n) = \frac{x_n(t) - t}{-\sqrt{2t}}. \tag{2.14}$$

Then, in the $t \to \infty$ limit, one gets the GUE-minors(n) given in [12],

$$\lim_{t\to\infty} X_t(n) = \text{GUE-minors}(n). \tag{2.15}$$

(2) Linearly decreasing density region: Airy$_2$ process, $\mathcal{A}_2$

For $0 < \alpha < 1$, define

$$n(\tau, t) = [\alpha t/4 + \tau(t/2)^{2/3}], \tag{2.16}$$

and the rescaled process

$$\tau \mapsto X_t(\tau) = \frac{x_{n(\tau,t)}(t) - ((1-\sqrt{\alpha})t - 2\tau\alpha^{-1/2}(t/2)^{2/3} + \tau^2\alpha^{-3/2}(t/2)^{1/3})}{-(t/2)^{1/3}}. \tag{2.17}$$

Then in the $t \to \infty$ limit, one gets

$$\lim_{t\to\infty} X_t(\tau) = \frac{(2-\sqrt{\alpha})^{2/3}}{\alpha^{1/6}} \mathcal{A}_2(\tau\alpha^{2/3}(2-\sqrt{\alpha})^{1/3}). \tag{2.18}$$

(3) Constant density region: Airy$_1$ process, $\mathcal{A}_1$

For $\alpha > 1$,

$$n(\tau, t) = [\alpha t/4 + \tau(t/2)^{2/3}], \tag{2.19}$$

and the rescaled process variables

$$\tau \mapsto X_t(\tau) = \frac{x_{n(\tau,t)}(t) - ((1-\alpha)t/2 - 2\tau(t/2)^{2/3})}{-(t/2)^{1/3}}. \tag{2.20}$$

Then in the $t \to \infty$ limit, one gets

$$\lim_{t\to\infty} X_t(\tau) = 2^{1/3}\mathcal{A}_1(2^{2/3}\tau). \tag{2.21}$$

3 Kernel and its scaling limit

In this section we derive the expression of the joint distributions of particle positions and then set the proper scaling limit.

Consider N particles starting at time $t = 0$ at positions $x_k(0) = -2k$, $k = 1, \ldots, N$. In Theorem 2.1 of [2] we proved that the joint distribution of the positions of the particles are given by a Fredholm determinant. The kernel is determined via a certain orthogonalization, which for our initial conditions has been made in Lemma 4.1 of [2] (with $z = x+2n-2N$ replaced by $z = x+2n$). Once the orthogonalization is made, one can compute the kernel which is (4.11) of [2] (with $z_i = x_i + 2n_i - 2N$ replaced by $z_i = x_i + 2n_i$). This is summarized in Proposition 3.

Proposition 3. *Let particle with label i start at $x_i(0) = -2i$, $i = 1, \ldots, N$. At time t, the particles are at positions x_i. Let $\sigma(1) < \sigma(2) < \ldots < \sigma(m)$ be the indices of m out of the N particles. The joint distribution of their positions $x_{\sigma(k)}(t)$ is given by*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m} \big\{x_{\sigma(k)}(t) \geq a_k\big\}\Big) = \det(\mathbb{1} - \chi_a K_t \chi_a)_{\ell^2(\{\sigma(1),\ldots,\sigma(m)\}\times\mathbb{Z})} \tag{3.1}$$

where $\chi_a(\sigma(k),x)=\mathbb{1}(x<a_k)$. *The kernel* K_t *is given by*

$$K_t(n_1,x_1;n_2,x_2)=-\binom{x_1-x_2-1}{n_2-n_1-1}\mathbb{1}_{[n_2>n_1]}+\widehat{K}_t(n_1,x_1;n_2,x_2), \tag{3.2}$$

where

$$\begin{aligned}\widehat{K}_t(n_1,x_1;n_2,x_2) &= \frac{(-1)^{n_1-n_2}}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}v\oint_{\Gamma_{-1}}\mathrm{d}u\frac{e^{-vt}(1+v)^{x_2+n_2}}{v^{n_2}} \\ & \frac{e^{ut}u^{n_1}}{(1+u)^{x_1+n_1+1}}\frac{1+2v}{(u-v)(1+u+v)}\end{aligned} \tag{3.3}$$

where Γ_0*, resp.* Γ_{-1}*, is any simple loop, anticlockwise oriented, which includes the pole at* $v=0$*, resp.* $u=-1$*, satisfying* $-1-\Gamma_0\subset\Gamma_{-1}$*, i.e., all the points of* $-1-\Gamma_0$ *lie inside the loop* Γ_{-1}*.*

In order to prove Theorem 2, we need to focus at particles with number n_i close to $t/4$ since these particles will be in the transition region at time t. The transition region has width which scales as $t^{2/3}$. The limit density is constant to the left of the transition region and it is decreasing linearly to the right of it. Therefore, the scaling limit used to prove the main theorem is

$$\begin{aligned} n_i &= [t/4+\tau_i(t/2)^{2/3}], \\ x_i &= [-2\tau_i(t/2)^{2/3}-\tilde{s}_i(t/2)^{1/3}], \end{aligned} \tag{3.4}$$

where

$$\tilde{s}_i=\begin{cases} s_i, & \tau_i\geq 0, \\ s_i-\tau_i^2, & \tau_i\leq 0. \end{cases} \tag{3.5}$$

As a consequence the rescaled kernel writes

$$K_t^{\mathrm{resc}}(\tau_1,s_1;\tau_2,s_2)=K_t(n_1,x_1;n_2,x_2)(t/2)^{1/3}2^{x_2-x_1} \tag{3.6}$$

where $2^{x_2-x_1}$ is just a conjugation so that the kernel has a proper limit. We denote by $\widehat{K}_t^{\mathrm{resc}}$ the term of the rescaled kernel without the binomial contribution, which then writes

$$\begin{aligned}\widehat{K}_t^{\mathrm{resc}}(\tau_1,s_1;\tau_2,s_2) &= (t/2)^{1/3}\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}v\oint_{\Gamma_{-1}}\mathrm{d}u\frac{1+2v}{(u-v)(1+u+v)} \\ &\times\frac{\exp(tf_0(v)+(t/2)^{2/3}\tau_2f_1(v)+(t/2)^{1/3}\tilde{s}_2f_2(v))}{\exp(tf_0(u)+(t/2)^{2/3}\tau_1f_1(u)+(t/2)^{1/3}\tilde{s}_1f_2(u)+f_3(u))},\end{aligned} \tag{3.7}$$

where the functions f_i are given by

$$\begin{aligned} f_0(v) &= -v+\tfrac{1}{4}\ln((1+v)/v), \\ f_1(v) &= -\ln(-4v(1+v)), \\ f_2(v) &= -\ln(2(1+v)), \\ f_3(v) &= \ln(1+v). \end{aligned} \tag{3.8}$$

From now on the τ_i's are some *fixed* values. With this preparation we can proceed to the asymptotic analysis needed to prove Theorem 2.

4 Asymptotic analysis

Proof of Theorem 2. The proof of Theorem 2 is identical to the one of Theorem 2.5 in [1], provided the following Propositions 4, 5, 6, 7, and 8 (convergence on bounded sets and large deviations bounds) hold. □

Proposition 4 (Uniform convergence on bounded sets)**.** *Fix any $L > 0$ and x_i, s_i with the scaling (3.4). Then, uniformly for $(s_1, s_2) \in [-L, L]^2$,*

$$\lim_{t\to\infty} \widehat{K}_t^{\mathrm{resc}}(n_1, x_1; n_2, x_2) = \widehat{K}_\infty^{\mathrm{resc}}(\tau_1, s_1; \tau_2, s_2) \tag{4.1}$$

where

$$\widehat{K}_\infty(\tau_1, s_1; \tau_2, s_2) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_+} \mathrm{d}w \int_{\gamma_-} \mathrm{d}z \frac{e^{w^3/3+\tau_2 w^2 - \tilde{s}_2 w}}{e^{z^3/3+\tau_1 z^2 - \tilde{s}_1 z}} \frac{2w}{(z-w)(z+w)} \tag{4.2}$$

with the paths γ_+, γ_- satisfying $-\gamma_+ \subset \gamma_-$ with $\gamma_+ : e^{\mathrm{i}\phi_+}\infty \to e^{-\mathrm{i}\phi_+}\infty$, $\gamma_- : e^{-\mathrm{i}\phi_-}\infty \to e^{\mathrm{i}\phi_-}\infty$ for some $\phi_+ \in (\pi/3, \pi/2)$, $\phi_- \in (\pi/2, \pi - \phi_+)$.

Proof. The first step is to control the contribution away from the critical point given by

$$\frac{\mathrm{d}f_0(v)}{\mathrm{d}v} = -\frac{(1+2v)^2}{4v(1+v)} = 0 \quad \Longleftrightarrow \quad v = -1/2. \tag{4.3}$$

If we write $v = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, then we can analyze

$$\mathrm{Re}(f_0(v) - f_0(-1/2)) = -(x + 1/2) + \tfrac{1}{8} \ln(((1+x)^2 + y^2)/(x^2 + y^2)). \tag{4.4}$$

This expression equals zero for
a) $x = -1/2$, $y \in \mathbb{R}$,
b) $y = \pm g(x)$, with $g(x) = \sqrt{\frac{1+2x+x^2(1-e^{8x+4})}{e^{8x+4}-1}}$.
If is easy to see that the solutions $\pm g(x)$ are symmetric with respect to $v = -1/2$ and they go around -1 and 0 once. Moreover, the loops leave the critical point $v = -1/2$ in the directions $e^{\pm \mathrm{i}\pi/6}$ and $e^{\pm \mathrm{i}5\pi/6}$, see Figure 3. We denote by $D_1, \ldots, D_4$ the following regions: D_1 is the region enclosed by $\pm g$ around -1, D_2 is the rest with real part less than $-1/2$, D_4 is the symmetric image w.r.t. $-1/2$ of D_1 and D_3 of D_2, see Figure 3. Then Γ_0 can be chosen to be any simple anticlockwise oriented finite length path staying in D_3 and, similarly, Γ_{-1} is chosen to stay in D_2 (except at $v = -1/2$). The constraint $-1 - \Gamma_0 \subset \Gamma_{-1}$ is easily satisfied except that for Γ_0 we have to go through D_4 too, very close to $v = -1/2$. Moreover, we can take Γ_0 leaving from $-1/2$ with an angle between $-\pi/6$ and $-\pi/3$. Similarly, Γ_{-1} leaves in the direction from $2\pi/3$ and $5\pi/6$. This will simplify the argument for moderate and large deviations.

Let us set $\Gamma_0^\delta = \{v \in \Gamma_0, |v + 1/2| \leq \delta\}$ and $\Gamma_{-1}^\delta = \{u \in \Gamma_{-1}, |u + 1/2| \leq \delta\}$. Then the integral is over $\Gamma_0 \cup \Gamma_{-1} = \Gamma_0^\delta \cup \Gamma_{-1}^\delta + \Sigma$, where Σ is the rest of the contours.

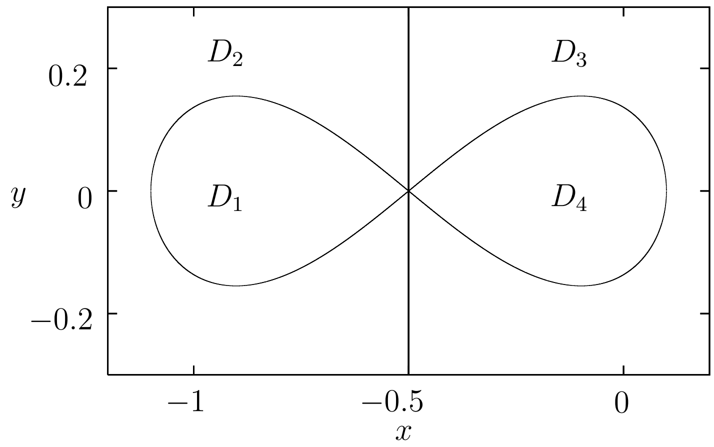

Figure 3: The signum of $\mathrm{Re}(f_0(x+\mathrm{i}y)-f_0(-1/2))$ is positive in D_2 and D_4 and negative in D_1 and D_3.

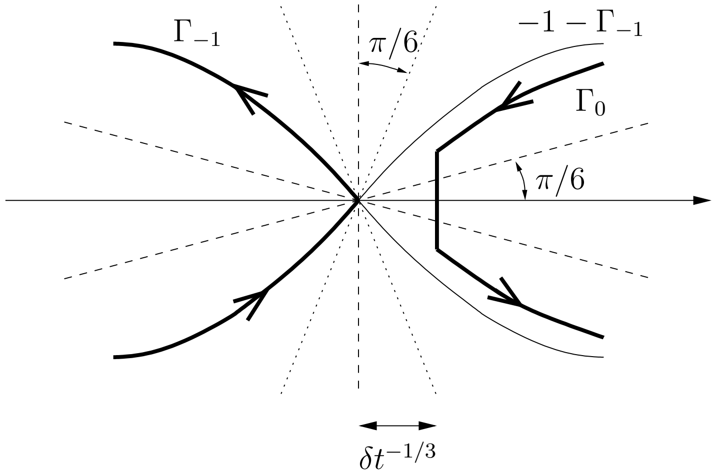

Figure 4: The paths Γ_0 and Γ_{-1} close to the critical point $-1/2$. The dashed lines are the zeros of $\mathrm{Re}(f_0(x+\mathrm{i}y)-f_0(-1/2))$.

The first step is to bound the integral over Σ. For $0<\delta\ll 1$, we can choose Γ_0 and Γ_{-1} such that, for $(u,v)\in\Sigma$, $|u-v|/\delta$ and $|1+u+v|/\delta$ are bounded away from 0. Then, on Σ, $\left|\frac{1+2v}{(u-v)(1+u+v)}\right| \le |u-v|^{-1}+|1+u+v|^{-1}=\mathcal{O}(1/\delta)$ and, for some $c_0=c_0(\delta)>0$, $\mathrm{Re}(f_0(v)-f_0(-1/2))\le -c_0$ and/or $-\mathrm{Re}(f_0(u)-f_0(-1/2))\le -c_0$. Thus, the integral over Σ can be bounded as

$$c_1\delta^{-1}t^{1/3}\exp(-c_0t+\mathcal{O}(t^{2/3})) \tag{4.5}$$

for some $c_1>0$. For t large enough, both $e^{\mathcal{O}(t^{2/3})}$ and $c_1t^{1/3}$ are bounded by $e^{-c_0t/4}$. Thus, for t large enough, we have the bound

$$\left|\iint_\Sigma \cdots\right| \le \delta^{-1}e^{-c_0t/2}. \tag{4.6}$$

The second step is to control the integral over $\Gamma_0^\delta\cup\Gamma_{-1}^\delta$. Since δ is small, we can apply Taylor series expansion on the functions f_i defined in (3.8). For this we change variables by setting

$$u=-1/2+U,\quad v=-1/2+V \tag{4.7}$$

and we denote $\gamma_+^\delta=\Gamma_0^\delta+1/2$, $\gamma_-^\delta=\Gamma_{-1}^\delta+1/2$. We have

$$\begin{aligned} f_0 &= \tfrac{1}{2}+\mathrm{i}\tfrac{\pi}{4}+\tfrac{4}{3}V^3+\mathcal{O}(V^4),\\ f_1 &= 4V^2+\mathcal{O}(V^4),\\ f_2 &= -2V+\mathcal{O}(V^2),\\ f_3 &= -\ln(2)+\mathcal{O}(V). \end{aligned}\tag{4.8}$$

Therefore the integral over $\Gamma_0^\delta\cup\Gamma_{-1}^\delta$ is given by

$$\begin{aligned} &\frac{(t/2)^{1/3}}{(2\pi\mathrm{i})^2}\int_{\gamma_+^\delta}\mathrm{d}V\int_{\gamma_-^\delta}\mathrm{d}U\frac{4V}{(U-V)(U+V)}\frac{e^{\frac{4}{3}tV^3+(t/2)^{2/3}\tau_2 4V^2-\tilde s_2(t/2)^{1/3}2V}}{e^{\frac{4}{3}tU^3+(t/2)^{2/3}\tau_1 4U^2-\tilde s_1(t/2)^{1/3}2U}}\\ \times\;& e^{\mathcal{O}(tV^4,t^{2/3}V^4,Lt^{1/3}V^2,tU^4,t^{2/3}U^4,Lt^{1/3}U^2,U)}\\ =\;&\frac{(t/2)^{1/3}}{(2\pi\mathrm{i})^2}\int_{\gamma_+^\delta}\mathrm{d}V\int_{\gamma_-^\delta}\mathrm{d}U\frac{4V}{(U-V)(U+V)}\frac{e^{\frac{4}{3}tV^3+(t/2)^{2/3}\tau_2 4V^2-\tilde s_2(t/2)^{1/3}2V}}{e^{\frac{4}{3}tU^3+(t/2)^{2/3}\tau_1 4U^2-\tilde s_1(t/2)^{1/3}2U}}\\ &+\mathrm{R}. \end{aligned}\tag{4.9}$$

To bound the remainder, R, we use $|e^x-1|\le|x|e^{|x|}$ applied to $x=\mathcal{O}(\cdots)$. Moreover, note that $\mathcal{O}(t^{2/3}V^4)$ is dominated by $\mathcal{O}(tV^4)$. Therefore,

$$\begin{aligned} |R| &\le c_2t^{1/3}\int_{\gamma_+^\delta}\mathrm{d}V\int_{\gamma_-^\delta}\mathrm{d}U\bigg|\frac{4V}{(U-V)(U+V)}\frac{e^{\frac{4}{3}tV^3+(t/2)^{2/3}\tau_2 4V^2-\tilde s_2(t/2)^{1/3}2V}}{e^{\frac{4}{3}tU^3+(t/2)^{2/3}\tau_1 4U^2-\tilde s_1(t/2)^{1/3}2U}}\\ &\times\; e^{\mathcal{O}(tV^4,Lt^{1/3}V^2,tU^4,Lt^{1/3}U^2,U)}\mathcal{O}(tV^4,Lt^{1/3}V^2,tU^4,Lt^{1/3}U^2,U)\bigg|. \end{aligned}\tag{4.10}$$

At this point we do the change of variables $V=w(4t)^{-1/3}$ and $U=z(4t)^{-1/3}$ and obtain

$$\begin{aligned} |R| &\le c_3t^{-1/3}\int_{(4t)^{1/3}\gamma_+^\delta}\mathrm{d}w\int_{(4t)^{1/3}\gamma_-^\delta}\mathrm{d}z\bigg|\frac{w}{(z-w)(z+w)}\frac{e^{w^3/3+\tau_2w^2-\tilde s_2w}}{e^{z^3/3+\tau_1z^2-\tilde s_1z}}\\ &\times\; e^{t^{-1/3}\mathcal{O}(w^4,Lw^2,z^4,Lz^2,z)}\mathcal{O}(w^4,Lw^2,z^4,Lz^2,z)\bigg|. \end{aligned}\tag{4.11}$$

By choosing δ small enough, we may assume that $\mathcal{O}(w^4t^{-1/3})\ll w^3$, $\mathcal{O}(zt^{-1/3})\ll 1$, and for t large enough $\mathcal{O}(Lt^{-1/3})\ll 1$. Therefore, the exponential in the integral in the w variable can be bounded by $|\exp(\chi_0w^3/3+\tau_2\chi_1w^2-\tilde s_2\chi_2w)|$ for some χ_0,χ_1,χ_2. By choosing δ small enough, the χ's can be made as close to 1 as desired. More importantly, for δ small, one has $\chi_0>0$. Similar for the variable z for some $\tilde\chi_k$. We have

$$\begin{aligned} |R| &\le c_3t^{-1/3}\int_{(4t)^{1/3}\gamma_+^\delta}\mathrm{d}w\int_{(4t)^{1/3}\gamma_-^\delta}\mathrm{d}z\bigg|\frac{w}{(z-w)(z+w)}\frac{e^{\chi_0w^3/3+\tau_2\chi_1w^2-\tilde s_2\chi_2w}}{e^{\tilde\chi_0z^3/3+\tau_1\tilde\chi_1z^2-\tilde s_1\tilde\chi_2z}}\\ &\times\;\mathcal{O}(w^4,Lw^2,z^4,Lz^2,z)\bigg|. \end{aligned}\tag{4.12}$$

The integral in (4.12), without the prefactor $t^{-1/3}$, is uniformly bounded in t. In fact, the only dependence on t is at the boundaries of the integrals, which are at $\delta e^{\pm i\theta_+}$ and $\delta e^{\pm i\theta_-}$ with $\theta_+ \in (\pi/6, \pi/3)$ and $\theta_- \in (2\pi/3, 5\pi/6)$. The convergence is ensured by the fact that $\mathrm{Re}(w^3) = \delta^3 t \cos(3\theta_+)$, with $\cos(3\theta_+) < 0$, and $\mathrm{Re}(-z^3) = -\delta^3 t \cos(3\theta_-)$, with $\cos(3\theta_-) > 0$. Thus, the w^3 and z^3 terms dominate the others at the boundary of the integrals and this domination becomes stronger while t increases. The final result is that, we can set $\delta > 0$ small enough and then for t large enough we have

$$|R| \leq c_4 t^{-1/3}. \tag{4.13}$$

The last step is to analyze the first term in r.h.s. of (4.9). One does the same change of variable as above and gets

$$\frac{1}{(2\pi i)^2} \int_{(4t)^{1/3}\gamma_+^\delta} \mathrm{d}w \int_{(4t)^{1/3}\gamma_-^\delta} \mathrm{d}z \frac{2w}{(z-w)(z+w)} \frac{e^{w^3/3+\tau_2 w^2 - \tilde{s}_2 w}}{e^{z^3/3+\tau_1 z^2 - \tilde{s}_1 z}}. \tag{4.14}$$

We can extend the paths to $t = \infty$ and by doing so we gain the error term of order $\mathcal{O}(e^{-c_5\delta^3 t})$ for some $c_5 > 0$. With this extension the paths satisfy the conditions of γ_+ and γ_- of the Proposition.

Just to summarize, the error term we have accumulated during the above procedure is

$$\mathcal{O}(\delta^{-1} e^{-c_0 t/2}, c_4 t^{-1/3}, e^{-c_5 \delta^3 t}). \tag{4.15}$$

$\square$

Proposition 5 (Moderate deviations). *For any L large enough, $\exists\, \varepsilon_0(L) > 0$ and $t_0(L) > 0$ such that, $\forall\, 0 < \varepsilon \leq \varepsilon_0$ and $t \geq t_0$, the estimate*

$$\left| \widehat{K}_t^{\mathrm{resc}}(\tau_1, s_1; \tau_2, s_2) \right| \leq e^{-(s_1+s_2)} \tag{4.16}$$

holds for $(s_1, s_2) \in [-L, \varepsilon t^{2/3}]^2 \setminus [-L, L]^2$.

Proof. In this proof we introduce the notation, $\sigma_i = \tilde{s}_i t^{-2/3} 2^{-1/3} \in (0, \varepsilon]$, $i = 1, 2$. We divide the analysis in the cases $\tilde{s}_1 \geq \tilde{s}_2$ and $\tilde{s}_1 \leq \tilde{s}_2$. The strategy is the following. First, for the case $\tilde{s}_1 \geq \tilde{s}_2$, we choose the same paths Γ_0 and Γ_{-1} as in Proposition 4 except for a small modification close to $v = u = -1/2$. We then see that in the unmodified part of the paths one has the same integral as for the case $\sigma_1 = \sigma_2 = 0$ times a factor which can be simply bounded and gives the needed decay. Then we consider the modified parts of the integration paths and see that the integral over these has also the required decay. Secondly, for the case $\tilde{s}_1 \leq \tilde{s}_2$, we first modify the condition of the integral since, otherwise, the optimal paths for the exponential can not be followed close to the critical points. The modification produces an extra ter m, a residue, which is a simple integral and it can be bounded in a similar way.

Case $\sigma_1 \leq \sigma_2$. The paths Γ_0 and Γ_{-1} as represented in Figure 5.

The modification with respect to the ones in Proposition 4 is just one vertical piece, given by $\Gamma_{\mathrm{vert}} = \{-1/2 + \sqrt{\sigma_2}(1 + i\xi)/2, \xi \in [-a, a]\}$ for some $a \in (1/\sqrt{3}, \sqrt{3})$.

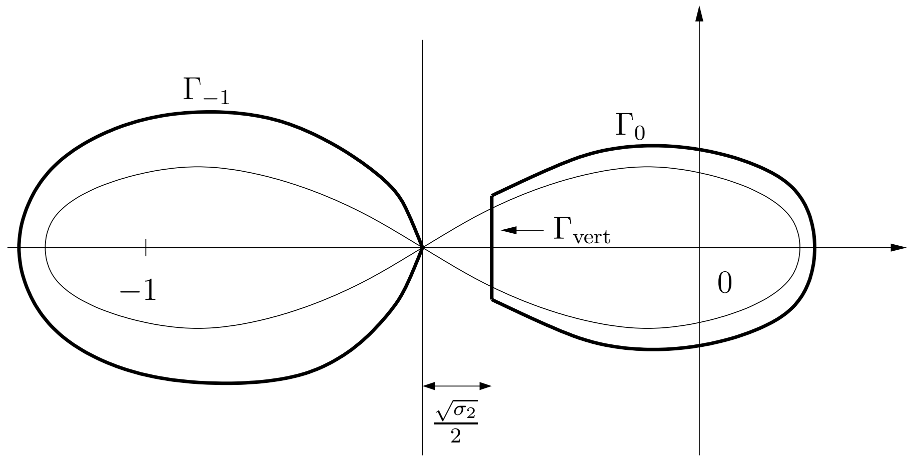

Figure 5: The paths Γ_0 and Γ_{-1} used to obtain the bound in the moderate deviations regime for $\sigma_1 \leq \sigma_2$.

With respect to the case $\sigma_1 = \sigma_2 = 0$, the integrand in the integral representation of the kernel $\widehat{K}_t^{\rm resc}$, see (3.7), has the extra factor

$$\exp(-t\sigma_2 \ln(2+2v)) \exp(t\sigma_1 \ln(2+2u)), \tag{4.17}$$

whose magnitude is given by

$$|(4.17)| = \exp(-t\sigma_2 \ln(2|1+v|)) \exp(t\sigma_1 \ln(2|1+u|)). \tag{4.18}$$

a) For the term $(1+2v)/((u-v)(1+u+v))$, we can choose $t \gg 1$ such that

$$\mathrm{dist}(\Gamma_0, -1-\Gamma_{-1}) = \frac{\sqrt{\sigma_2}}{2} \geq \frac{\sqrt{L}}{4t^{1/3}} \tag{4.19}$$

which is much better than in Proposition 4, where we had, see Figure 4 $\mathrm{dist}(\Gamma_0, -1-\Gamma_{-1}) = \delta t^{-1/3}$. Therefore the term $(1+2v)/((u-v)(1+u+v))$ does not create any problems.
b) Similarly, Γ_{-1} can be chosen such that the maximum of $|1+u|$, for $u \in \Gamma_{-1}$, is obtained at $u = -1/2$, thus

$$e^{t\sigma_1 \ln(2|1+u|)} \leq 1. \tag{4.20}$$

c) Γ_0 can be chosen such that the minimum of $|1+v|$ for $v \in \Gamma_0 \setminus \Gamma_{\rm vert}$, is obtained at $\Gamma_{\rm vert}$ for $\xi = \pm a$. A simple computation leads to

$$e^{-t\sigma_2 \ln(2|1+v|)} = e^{-t\sigma_2 \ln(1+\sqrt{\sigma_2}+\mathcal{O}(\sigma_2))} = e^{-\tilde{s}_2^{3/2}(1+\mathcal{O}(\varepsilon))/\sqrt{2}} \leq e^{-\tilde{s}_2^{3/2}/2} \tag{4.21}$$

for ε small enough.
d) Now we evaluate the integral over $\Gamma_{\rm vert}$. As considered in the ξ variable, the prefactor $t^{-1/3}$ cancels out and ξ varies over an interval of order one. Therefore, to estimate the integral it is enough to estimate the integrand. Since ε is small, σ_2 is small too. Thus, $\Gamma_{\rm vert}$ is very close to $-1/2$ and we can apply Taylor expansion

of the integrand. The term with the exponential in the v variable becomes ($v = -1/2 + \sqrt{\sigma_2}(1+\mathrm{i}\xi)/2$)

$$\exp\left(tf_0(-1/2)+\tfrac{1}{6}t\sigma_2^{3/2}(1+\mathrm{i}\xi)^3+\tau_2(t/2)^{2/3}\sigma_2(1+\mathrm{i}\xi)^2-\sigma_2^{3/2}t(1+\mathrm{i}\xi)+\mathcal{O}(t\sigma_2^2)\right). \tag{4.22}$$

By using $\sigma_2 = \tilde{s}_2 t^{-2/3}2^{-1/3}$ and computing the real values of the exponent, we get

$$|(4.22)| \leq \exp\left(tf_0(-1/2) + \tilde{s}_2^{3/2}2^{-1/2}(-\tfrac{5}{6} - \tfrac{1}{2}\xi^2 + \mathcal{O}(\sqrt{\varepsilon})) + \tfrac{1}{2}\tau_2\tilde{s}_2(1-\xi^2)\right). \tag{4.23}$$

Here we have $s_2 \geq L$, thus $\tilde{s}_2 \geq s_2/2$ for large L and $\tilde{s}_2^{3/2} \gg \tilde{s}_2$. Therefore, the integrand to be studied can be bounded by

$$e^{tf_0(-1/2)}e^{-\tilde{s}_2^{3/2}/2} \tag{4.24}$$

for L large enough and ε small enough. The factor $e^{tf_0(-1/2)}$ is cancelled exactly with the one coming from the integrand in the u variable.

For the case $\sigma_1 = \sigma_2 = 0$, the analysis of Proposition 4 leads to the bound on the kernel $\widehat{K}_t^{\mathrm{resc}}$

$$(4.15) + \frac{1}{(2\pi)^2}\int_{\gamma_+}\mathrm{d}w\int_{\gamma_-}\mathrm{d}z\left|\frac{e^{w^3/3+\tau_2 w^2}}{e^{z^3/3+\tau_1 z^2}}\frac{2w}{(z-w)(z+w)}\right| \leq c_6 \tag{4.25}$$

for some constant $c_6 > 0$, as soon as t is large enough.

Putting together the results of a)-d), the kernel is bounded by c_6 times the factor $e^{-\tilde{s}_2^{3/2}/2}$. For L large, $\tilde{s}_2 \geq L/2$ and $\tilde{s}_2 \geq s_2/\sqrt{2}$, therefore

$$c_6 e^{-\tilde{s}_2^{3/2}/2} \leq c_6 e^{-\frac{1}{4}\sqrt{L}s_2} \leq c_6 e^{-\frac{1}{8}\sqrt{L}(s_1+s_2)} \leq e^{-(s_1+s_2)} \tag{4.26}$$

where we used $s_2 \geq s_1$.

Case $\sigma_1 \geq \sigma_2$. To obtain the bound for this case, we use a different expression for the kernel $\widehat{K}_t^{\mathrm{resc}}$, namely

$$\begin{aligned}\widehat{K}_t^{\mathrm{resc}} &= (t/2)^{1/3}\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}v\oint_{\Gamma_{-1}}\mathrm{d}u\frac{1+2v}{(u-v)(1+u+v)} \\ &\times \frac{\exp(tf_0(v)+(t/2)^{2/3}\tau_2 f_1(v)+(t/2)^{1/3}\tilde{s}_2 f_2(v))}{\exp(tf_0(u)+(t/2)^{2/3}\tau_1 f_1(u)+(t/2)^{1/3}\tilde{s}_1 f_2(u)+f_3(u))} + I_2,\end{aligned} \tag{4.27}$$

where

$$\begin{aligned}I_2 &= (t/2)^{1/3}\frac{-1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}v e^{t(f_0(v)-f_0(-1-v))}e^{(t/2)^{2/3}(\tau_2 f_1(v)-\tau_1 f_1(-1-v))} \\ &\times e^{(t/2)^{1/3}(\tilde{s}_2 f_2(v)-\tilde{s}_1 f_2(-1-v))}e^{-f_3(-1-v)},\end{aligned} \tag{4.28}$$

with the constraint $\Gamma_{-1} \subset -\Gamma_0$ instead of $-\Gamma_0 \subset \Gamma_{-1}$. The term I_2 comes from the fact that, for any fixed v, the new constraint on the paths is obtained by deforming

Γ_{-1} and during this process one passes via a simple pole at $u = -1 - v$, whose residue is I_2.

The analysis of the double integral term in (4.27) is the same as in the previous case, where however (u, s_1, τ_1) play the role of (v, s_2, τ_2), so this time it is Γ_{-1} which is modified instead of Γ_0 (symmetrically w.r.t. $-1/2$). We can then get as in (4.26) the bound $\exp(-(s_1+s_2))/2$ and it remains to prove that I_2 is bounded by $\exp(-(s_1+s_2))/2$ too.

Denote $h_0(v) = f_0(v) - f_0(-1-v)$. It is given by $h_0(v) = -1 + 2f_0(v)$. Therefore the regions where sign of $\mathrm{Re}(h_0(v) - h_0(-1/2))$ is positive and negative are again the ones of Figure 3. In the case $\sigma_1 = \sigma_2 = 0$, one can do essentially the asymptotic analysis made to obtain the estimate on the integral over Γ_{-1} of Proposition 4 and we get that the integral is bounded in the $t \to \infty$ limit. The corrections to the limit expression are of just order $\mathcal{O}(t^{-1/3}, e^{-\mu t})$, for some $\mu > 0$. But here we are in the case $s_1 \in [L, \varepsilon t^{2/3}]$. The difference with respect to the case $\sigma_1 = \sigma_2 = 0$ is a factor of magnitude

$$\exp(t\sigma_1 \ln(2|v|) - t\sigma_2 \ln(2|1+v|)), \tag{4.29}$$

in the integrand. The Γ_0 used for the $\sigma_1 = \sigma_2 = 0$ asymptotic analysis can be chosen such that, while going away from the critical point $v = -1/2$,
a) $|v|$ decreases, thus $\ln(2|v|)$ decreases,
b) $|1+v|$ increases, thus $-\ln(2|1+v|)$ decreases,
take for example $-1 - \Gamma_{-1}$ of Figure 5.

Now we use the same trick as above, namely we modify Γ_0 only in the neighborhood of $v = -1/2$ as in Figure 5 (just this time the distance to $v = -1/2$ is $\sqrt{\sigma_1}/2$ instead of $\sqrt{\sigma_2}/2$). We denote Γ_{vert} the vertical piece here too. Then, the contribution on $\Gamma_0 \setminus \Gamma_{\mathrm{vert}}$ carries an extra term (as in (4.21))

$$e^{-t\sigma_1 \ln(2|1+v|)} \leq e^{-\tilde{s}_1^{3/2}/2}, \tag{4.30}$$

for ε small enough. Then, for L large enough, $|(4.30)| \leq -e^{-c_7\sqrt{L}(s_1+s_2)}$ for some $c_7 > 0$.

For the contribution of the integral over Γ_{vert}, we set $v = -1/2 + V$ and do Taylor expansion. Then set $V = \sqrt{\sigma_1}(1+\mathrm{i}\xi)/2$ with $\xi \in [-a, a]$, for some $a \in (1/\sqrt{3}, \sqrt{3})$. The integral over Γ_{vert} is an integral over $[-a, a]$, which writes

$$\begin{aligned} &(t/2)^{1/3}\frac{-1}{2\pi}\int_{-a}^{a} \mathrm{d}\xi \sqrt{\sigma_1} e^{t\sigma_1^{3/2}(1+\mathrm{i}\xi)^3(1+\mathcal{O}(\sqrt{\varepsilon}))/3} e^{-(t/2)^{2/3}(\tau_1-\tau_2)\sigma_1(1+\mathrm{i}\xi)^2(1+\mathcal{O}(\varepsilon))} \\ \times\ & e^{-(\sigma_1+\sigma_2)\sqrt{\sigma_1}(1+\mathrm{i}\xi)t(1+\mathcal{O}(\sqrt{\varepsilon}))} e^{\mathcal{O}(\sqrt{\varepsilon})}. \end{aligned} \tag{4.31}$$

We then use
a) $\mathrm{Re}((1+\mathrm{i}\xi)^3) = 1 - 3\xi^2$,
b) $\mathrm{Re}((1+\mathrm{i}\xi)^2) = 1 - \xi^2$,
c) $\sqrt{\sigma_1} t^{1/3} \geq \sqrt{L}$,
d) $2/3 \leq |1 + \mathcal{O}(\sqrt{\varepsilon})| \leq 2$, for ε small enough,

to obtain that $|(4.31)|$ is bounded by

$$\int_{-a}^{a} \mathrm{d}\xi c_8 \sqrt{\tilde s_1} \exp\Big(t\sigma_1^{3/2}\big(\tfrac{1}{3}-\xi^2-c_9(1-\xi^2)/\sqrt{L}\big)\Big) \exp\big(-\tfrac{2}{3}(\sigma_1+\sigma_2)t\sqrt{\sigma_1}\big). \quad (4.32)$$

The integral (4.32) is bounded and, for L large enough, the integrand is maximal at $\xi = 0$. Thus

$$\begin{aligned} (4.32) &\leq c_{10}\sqrt{\tilde s_1} \exp\big(\tfrac{1}{3}t\sigma_1^{3/2} - c_9/\sqrt{L} - \tfrac{2}{3}(\sigma_1+\sigma_2)\sqrt{\sigma_1}t\big) \\ &\leq \exp(-\tfrac{1}{6}(\sigma_1+\sigma_2)\sqrt{\sigma_1}t) \end{aligned} \quad (4.33)$$

for L large enough. Reinserting the expressions for σ_1 and σ_2, we have

$$|(4.32)| \leq \exp\big(-c_{10}(\tilde s_1+\tilde s_2)t\sqrt{\tilde s_1}\big) \leq \exp\big(-c_{11}(s_1+s_2)t\sqrt{L}\big) \quad (4.34)$$

for L large enough and some $c_{11} > 0$. This bound is good enough to get $\exp(-(s_1 + s_2))/2$ as bound for L large enough, ε small enough and t large enough. $\square$

Proposition 6 (Large deviations). *Set $\varepsilon > 0$, then for t large enough we have*

$$\Big|\widehat{K}_t^{\mathrm{resc}}(\tau_1, s_1; \tau_2, s_2)\Big| \leq e^{-(s_1+s_2)} \quad (4.35)$$

for $(s_1, s_2) \in [-L,\infty)^2 \setminus [-L, \varepsilon t^{2/3}]^2$.

Proof. One can do large deviations directly, but a shorter way is to use the result of the moderate deviations. As in the proof of Proposition 5 we use the notation, $\sigma_i = \tilde s_i t^{-2/3} 2^{-1/3}$, $i = 1, 2$.

Case $\sigma_1 \leq \sigma_2$. The term linear in t in the exponential is $\exp(tf_{0,\sigma_2}(v) - tf_{0,\sigma_1}(u))$, where $f_{0,\sigma}(v) = f_0(v) - \sigma \ln(2 + 2v)$. To obtain the bound we just remark that

$$f_{0,\sigma_2}(v) = f_{0,\varepsilon/2}(v) - (\sigma_2 - \varepsilon/2) \ln(2 + 2v). \quad (4.36)$$

We take Γ_0 to be the one used for moderate deviations with $\sigma_2 = \varepsilon/2$. Γ_0 satisfies $|1 + v| \geq 1/2 + \sqrt{\varepsilon/2}/2$. $\sigma_2 \geq \varepsilon$ implies $\sigma_2 - \varepsilon/2 \geq \sigma_2/2$. Therefore, for ε small enough and t large enough,

$$|\exp(-t(\sigma_2-\varepsilon/2)\ln(2+2v))| \leq \exp(-\tfrac{1}{2}t\sigma_2 \ln(1+\sqrt{\varepsilon/2})) \leq \exp(-c_{12}t^{1/3}s_2). \quad (4.37)$$

The integral (3.7) with $f_0(v) = f_{0,\varepsilon/2}(v)$ is finite by the same argument as for the moderate deviations. The extra factor (4.37) together with $\tilde s_2 \geq (\tilde s_1 + \tilde s_2)/2$ leads to the bound $\exp(-(s_1 + s_2))$ for t large enough.

Case $\sigma_1 \geq \sigma_2$. Using the representation as in the moderate deviation case, we have, with respect to $\sigma_1 = \varepsilon/2$, the extra factor

$$\exp(\tfrac{1}{2}t\sigma_1 \ln(1 - \sqrt{\varepsilon/2})) \leq \exp(-c_{13}t^{1/3}s_1), \quad (4.38)$$

from which we get the bound $\exp(-(s_1 + s_2))$ as before. $\square$

Proposition 7 (Uniform convergence on bounded sets)**.** *Fix any $L > 0$ and x_i, s_i with the above rescaling. Then, uniformly for $(s_1, s_2) \in [-L, L]^2$,*

$$\begin{aligned} &\lim_{t\to\infty} (t/2)^{1/3} 2^{x_2-x_1} \binom{x_1 - x_2 - 1}{n_2 - n_1 - 1} \\ = \;& \frac{1}{\sqrt{4\pi(\tau_2 - \tau_1)}} \exp\left(-\frac{(\tilde{s}_2 - \tilde{s}_1)^2}{4(\tau_2 - \tau_1)}\right) \mathbb{1}(\tau_2 > \tau_1). \end{aligned} \tag{4.39}$$

Proof. It is a special case of the first part of Proposition 5.1 of [1], where p is chosen such that $\kappa = 2^{-1/3}$ and (r_i, s_i) are replaced by $(\tau_i, \tilde{s}_i)$. □

Proposition 8. *For any $s_1, s_2 \in \mathbb{R}$ and $\tau_2 - \tau_1 > 0$ fixed, the bound*

$$(t/2)^{1/3} 2^{x_2-x_1} \binom{x_1 - x_2 - 1}{n_2 - n_1 - 1} \leq c_{12} e^{-|\tilde{s}_2 - \tilde{s}_1|} \tag{4.40}$$

holds for t large enough and c_{12} independent of t.

Proof. It is a special case of the first part of Proposition 5.5 of [1], where p is chosen such that $\kappa = 2^{-1/3}$ and (r_i, s_i) are replaced by $(\tau_i, \tilde{s}_i)$. □

5 About the other three regions

(1) Constant density region.

To obtain the result in the constant density region we consider the scaling

$$\begin{aligned} n_i &= [\alpha t/4 + \tau_i (t/2)^{2/3}], \\ x_i &= [(1-\alpha)t/2 - 2\tau_i (t/2)^{2/3} - s_i (t/2)^{1/3}] \end{aligned} \tag{5.1}$$

with $\alpha > 1$ fixed. The rescaled and conjugate kernel is as before

$$K_t^{\rm resc}(\tau_1, s_1; \tau_2, s_2) = K_t(n_1, x_1; n_2, x_2)(t/2)^{1/3} 2^{x_2-x_1}. \tag{5.2}$$

The binomial term is easily estimated and controlled. The main term $\widehat{K}_t^{\rm resc}$ is given by the formula (3.7), with $\tilde{s}_i = s_i$, f_1, f_2, f_3 as in (3.8), and the new f_0 is

$$f_0(v) = -v + \frac{2-\alpha}{4} \ln(1+v) - \frac{\alpha}{4} \ln(-v). \tag{5.3}$$

The two critical points v_-, v_+ of f_0 are now distinct, namely $v_- = -\alpha/2 < -1/2 = v_+$. The constraint between the integration paths $-1 - \Gamma_0 \subset \Gamma_{-1}$ can not be satisfied if we want to choose Γ_0 and Γ_{-1} optimally, i.e., passing by v_+ and v_- respectively. For the analysis, one considers another representation of $\widehat{K}_t^{\rm resc}$, the same used in (4.27). The first term is as before but with the constraint $\Gamma_{-1} \subset -1 - \Gamma_0$ and the second is the residue at $u = -1 - v$, namely equal to I_2 in (4.28).

The first term is now controlled by choosing optimal paths for $f_0(v)$ and $-f_0(u)$, which pass by v_+ and v_- respectively. $f_0(v_+) < f_0(v_-)$, thus the first term is of order $\mathcal{O}(\exp(tf_0(v_+) - tf_0(v_-))) = \mathcal{O}(e^{-at})$ for some $a > 0$. In particular, for $\alpha > 2$ the first term vanishes identically (for t large enough), and as $\alpha \searrow 1$, the first term is $\mathcal{O}(e^{-t(\alpha-1)^3/12})$.

The second term is just I_2, up to some $2^{1/3}$ factors due to the slightly different rescaling, the same kernel appearing in (5.5) of [2], where we already proved the pointwise convergence. The moderate and large deviations are the ones of I_2 in (4.28) analyzed in Propositions 5 and 6. In the $t \to \infty$ limit one then obtains

$$\lim_{t\to\infty} K_t^{\rm resc}(\tau_1, s_1; \tau_2, s_2) = 2^{-1/3} K_{\mathcal{A}_1}(2^{-2/3}\tau_1, 2^{-1/3}s_1; 2^{-2/3}\tau_2, 2^{-1/3}s_2) \tag{5.4}$$

with $K_{\mathcal{A}_1}$ is the kernel of the Airy$_1$ process.

(2) Linearly decreasing density region.

To obtain the result in the linearly decreasing density region we consider the scaling

$$\begin{aligned} n_i &= [\alpha t/4 + \tau_i (t/2)^{2/3}], \\ x_i &= \left[(1-\sqrt{\alpha})t - \frac{2\tau_i}{\sqrt{\alpha}}(t/2)^{2/3} + \frac{\tau_i^2}{\alpha^{3/2}}(t/2)^{1/3} - s_i (t/2)^{1/3}\right] \end{aligned} \tag{5.5}$$

with $0 < \alpha < 1$ fixed. The rescaled and conjugate kernel is

$$K_t^{\rm resc}(\tau_1, s_1; \tau_2, s_2) = K_t(n_1, x_1; n_2, x_2)(t/2)^{1/3} \frac{(\sqrt{\alpha}/2)^{n_2 - n_1}}{(1-\sqrt{\alpha}/2)^{x_2+n_2-x_1-n_1}}. \tag{5.6}$$

The main term of the kernel $\widehat{K}_t^{\rm resc}$ writes as (3.7) with $\tilde{s}_i = s_i - \tau_i^2/\alpha^{3/2}$, f_3 as in (3.8), and

$$\begin{aligned} f_0(v) &= -v + (1 - \sqrt{\alpha} + \alpha/4)\ln(1+v) - (\alpha/4)\ln(-v), \\ f_1(v) &= (1 - 2/\sqrt{\alpha})\ln(1+v) - \ln(-v) + \ln(\sqrt{\alpha}/2) \\ &\quad -(1-2/\sqrt{\alpha})\ln(1-\sqrt{\alpha}/2), \\ f_2(v) &= -\ln(1+v) + \ln(1-\sqrt{\alpha}/2). \end{aligned} \tag{5.7}$$

The function $f_0(v)$ has a double critical point at $v = -\sqrt{\alpha}/2$. The factor $1+u+v = 1-\sqrt{\alpha}$ at the critical point and the paths Γ_0 and Γ_{-1} can be chosen such that $1+u+v$ remains uniformly bounded away from 0. The leading term of the integral comes from the neighborhood of the critical point. There, one applies the following change of variables,

$$v = -\frac{\sqrt{\alpha}}{2} + \frac{\alpha^{1/6}(2-\sqrt{\alpha})^{1/3}}{2^{2/3}t^{1/3}}V, \quad u = -\frac{\sqrt{\alpha}}{2} + \frac{\alpha^{1/6}(2-\sqrt{\alpha})^{1/3}}{2^{2/3}t^{1/3}}U. \tag{5.8}$$

Set $S_h = \alpha^{-2/3}(2-\sqrt{\alpha})^{-1/3}$ and $S_v = \alpha^{1/6}(2-\sqrt{\alpha})^{-2/3}$. Then, the leading term in the main term of the kernel becomes

$$\widehat{K}_t^{\rm resc}(\tau_1,s_1;\tau_2,s_2) \simeq \frac{S_v}{(2\pi{\rm i})^2}\int {\rm d}V \int {\rm d}U \frac{1}{U-V}\frac{e^{V^3/3+\tau_2 S_h V^2 - \tilde{s}_2 S_v V}}{e^{U^3/3+\tau_1 S_h U^2-\tilde{s}_1 S_v U}}. \tag{5.9}$$

Thus

$$K_t^{\rm resc}(\tau_1,s_1;\tau_2,s_2) \to S_v K_{\mathcal{A}_2}(S_h\tau_1,S_v s_1;S_h\tau_2,S_v s_2) \tag{5.10}$$

as $t\to\infty$. By adequate control for moderate and large deviations, one proves (2.18).

(3) Finite distance from the right-most particle.

From the discussion on the initial condition, in particular from (2.12), it follows that the asymptotic result is unchanged if one considers step initial conditions instead of our initial conditions. In [12] the case of step initial conditions was analyzed in a closely related model (a kind of discrete time TASEP but from the growth point of view). For step initial conditions, we have

$$K_t(n_1,x_1;n_2,x_2) = -\binom{x_1-x_2-1}{n_2-n_1-1}\mathbb{1}_{[n_2>n_1]} + \widehat{K}_t(n_1,x_1;n_2,x_2) \tag{5.11}$$

with

$$\widehat{K}_t(n_1,x_1;n_2,x_2) = \sum_{k=0}^{n_2-1} \Psi^{n_1}_{n_1-n_2+k}(x_1)\Phi^{n_2}_k(x_2), \tag{5.12}$$

where

$$\Psi^n_k(x) = \frac{e^{-t}t^{x+2n}}{(x+2n)!}C_k(x+2n,t), \quad \Phi^n_l(y) = C_l(y+2n,t), \tag{5.13}$$

the C_k being the Charlier orthogonal polynomials [15]. This is obtained in the same way as in Appendix B of [2]. $\Psi^n_k(z)$ is the same as (B.7) of [2] with $z-k$ replaced by $z=x+2n$, and consequently the matrix $S_{k,l}$ becomes the identity matrix.

The Charlier polynomials converge to the Hermite polynomials H_k as follows

$$\lim_{t\to\infty}(2t)^{k/2}C_k(t-\sqrt{2t}\sigma,t) = (-1)^k H_k(-\sigma) = H_k(\sigma). \tag{5.14}$$

The scaling we have to use is

$$n_i, \quad x_i = [t-\sqrt{2t}s_i] \tag{5.15}$$

and the kernel rescaled as

$$K_t^{\rm resc}(n_1,s_1;n_2,s_2) = \sqrt{2t}\frac{e^{-s_2^2/2+s_1^2/2}}{t^{n_2/2-n_1/2}}K_t(n_1,x_1;n_2,x_2). \tag{5.16}$$

It is easy to see that the binomial contribution converges to

$$-\frac{e^{(s_1^2-s_2^2)/2}2^{(n_2-n_1)/2}}{(n_2-n_1-1)!}(s_2-s_1)^{n_2-n_1-1}\mathbb{1}_{[s_2>s_1]}. \tag{5.17}$$

Also, by (5.14), we have

$$\lim_{t\to\infty}\Psi_k^n(x_i)=\frac{e^{-s_i^2}}{(2t)^{k/2}\sqrt{2\pi t}}H_k(s_i),\quad \lim_{t\to\infty}\Phi_k^n(x_i)=\frac{(t/2)^{k/2}}{k!}H_k(s_i). \tag{5.18}$$

The kernel is a finite sum, thus

$$\lim_{t\to\infty}\widehat{K}_t^{\rm resc}(n_1,s_1;n_2,s_2)=e^{-(s_1^2+s_2^2)/2}\sum_{j=-n_2}^{-1}\sqrt{\frac{(n_1+j)!}{(n_2+j)!}}h_{n_2+j}(s_1)h_{n_1+j}(s_2) \tag{5.19}$$

where $h_k(s)=\pi^{-1/4}k!^{-1/2}2^{-k/2}H_k(s)$.

(5.17) plus (5.19) gives

$$\lim_{t\to\infty}K_t^{\rm resc}(n_1,s_1;n_2,s_2)=K^{\rm GUE}(n_2,s_2;n_1,s_1) \tag{5.20}$$

with $K^{\rm GUE}$ the kernel defined in Definition 1.2 of [12]. (Here we just order the entries differently).

A Explicit form of the limit kernel

Transition kernel in terms of Airy functions

Let us denote

$$\tilde{s}_i=s_i-\min\{0,\tau_i\}^2,\quad \hat{s}_i=s_i+\max\{0,\tau_i\}^2. \tag{A.1}$$

Then

$$K_\infty(\tau_1,s_1;\tau_2,s_2)=K_0(\tau_1,s_1;\tau_2,s_2)+K_1(\tau_1,s_1;\tau_2,s_2)+K_2(\tau_1,s_1;\tau_2,s_2) \tag{A.2}$$

where

$$K_0(\tau_1,s_1;\tau_2,s_2)=-\frac{e^{\frac{2}{3}\tau_2^3+\tau_2\tilde{s}_2}}{e^{\frac{2}{3}\tau_1^3+\tau_1\tilde{s}_1}}\frac{1}{\sqrt{4\pi(\tau_2-\tau_1)}}\exp\left(-\frac{(\tilde{s}_2-\tilde{s}_1)^2}{4(\tau_2-\tau_1)}\right)\mathbb{1}_{[\tau_2>\tau_1]}, \tag{A.3}$$

$$K_2(\tau_1,s_1;\tau_2,s_2)=\int_0^\infty \mathrm{d}\lambda e^{\lambda(\tau_2-\tau_1)}\mathrm{Ai}(\hat{s}_2+\lambda)\mathrm{Ai}(\hat{s}_1+\lambda), \tag{A.4}$$

and

$$K_1(\tau_1,s_1;\tau_2,s_2)=\int_0^\infty \mathrm{d}\lambda e^{\lambda(\tau_2+\tau_1)}\mathrm{Ai}(\hat{s}_2+\lambda)\mathrm{Ai}(\hat{s}_1-\lambda). \tag{A.5}$$

Equivalently, one can see that

$$\begin{aligned}K_1(\tau_1,s_1;\tau_2,s_2)=&-\int_{-\infty}^0 \mathrm{d}\lambda e^{\lambda(\tau_2+\tau_1)}\mathrm{Ai}(\hat{s}_2+\lambda)\mathrm{Ai}(\hat{s}_1-\lambda)\\ &+2^{-1/3}\mathrm{Ai}\left(2^{-1/3}(\tilde{s}_1+\tilde{s}_2+\tfrac{1}{2}(\tau_1-\tau_2)^2)\right)e^{-\frac{1}{2}(\tau_1+\tau_2)(\hat{s}_2-\hat{s}_1)}\end{aligned} \tag{A.6}$$

B Trace class and transition kernel

Proposition 9. *The Fredholm determinant*

$$\det(\mathbb{1} - \chi_s K_\infty \chi_s)_{\mathcal{H}}, \tag{B.1}$$

with K_∞ given in (2.7) or (A.2)-(A.5), is well defined, because there exists a conjugate kernel of $\chi_s K_\infty \chi_s$ which is trace-class on $\mathcal{H} = L^2(\{\tau_1, \ldots, \tau_m\} \times \mathbb{R})$.

Proof. In this proof, let us choose a T_0 such that $-T_0 < \tau_1 < \tau_2 < \ldots < \tau_m < T_0$. Denote by $K^{\rm conj}$ a conjugate of $\chi_s K_\infty \chi_s$. Let P_k be the projector onto the space $\{f \in \mathcal{H} | f(\tau_l, x) = 0 \text{ for } l \neq k\}$ and $K_{k,l}^{\rm conj} = P_k K^{\rm conj} P_l$. Then

$$\|K^{\rm conj}\|_1 \leq \sum_{k,l=1}^{m} \|K_{k,l}^{\rm conj}\|_1. \tag{B.2}$$

From (A.2) we have

$$K_{k,l}^{\rm conj}(\tau_k, x; \tau_l, y) = \mathbb{1}_{[x \geq s_k]} \mathbb{1}_{[y \geq s_l]} \frac{\rho(\tau_k, x)}{\rho(\tau_l, y)} \sum_{n=0,1,2} K_n(\tau_k, x; \tau_l, y) \tag{B.3}$$

where the conjugation function $\rho(\tau, x) \neq 0$ will be specified later.

The formula defining $K_0(\tau_k, x; \tau_l, y)$ is particularly nice in the $\tilde{s}_k = s_k - [\tau_k]_-^2$ variables (with $[x]_- = x$ for $x \leq 0$ and 0 otherwise). Thus we use the variables $\tilde{s}_k$ instead of s_k. This is just a shift of the coordinate at the corresponding "time" τ_k. Thus, if we prove that $\widetilde{K}_{k,l}^{\rm conj}(\tau_k, x; \tau_l, y) = K_{k,l}^{\rm conj}(\tau_k, x + [\tau_k]_-^2; \tau_l, y + [\tau_l]_-^2)$ is trace-class for all $\tilde{s}_k$, $k = 1, \ldots, m$, bounded from below, then $K_{k,l}^{\rm conj}$ will also be trace-class for all s_k bounded from below.

Therefore, we now work with the $\widetilde{K}^{\rm conj}$ kernels and choose the conjugation functions ($\tilde{\rho}(\tau_k, x) = \rho(\tau_k, x + [\tau_k]_-^2)$) to be

$$\tilde{\rho}(\tau_k, x) = (1 + x^2)^{2k} e^{\tau_k x + \frac{2}{3}\tau_k^3}. \tag{B.4}$$

We analyze separately the three parts of the kernel. Let $\widetilde{K}_n(\tau_k, x; \tau_l, y) = K_n(\tau_k, x + [\tau_k]_-^2; \tau_l, y + [\tau_l]_-^2)$, $n = 0, 1, 2$.

Part a) $\widetilde{K}_0(\tau_k, x; \tau_l, y)$**.** We have

$$\begin{aligned} & \widetilde{K}_0(\tau_k, x; \tau_l, y) \frac{\tilde{\rho}(\tau_k, x)}{\tilde{\rho}(\tau_l, y)} \mathbb{1}_{[x \geq \tilde{s}_k]} \mathbb{1}_{[y \geq \tilde{s}_l]} \\ = & -\frac{\mathbb{1}_{[\tau_l > \tau_k]}}{\sqrt{4\pi(\tau_l - \tau_k)}} \mathbb{1}_{[x \geq \tilde{s}_k]} \mathbb{1}_{[y \geq \tilde{s}_l]} \exp\left(-\frac{(y-x)^2}{4(\tau_l - \tau_k)}\right) \frac{(1+x^2)^{2k}}{(1+y^2)^{2l}}. \end{aligned} \tag{B.5}$$

In Lemma A.2 of [1], we proved that the operator with above kernel is trace-class on $L^2(\mathbb{R})$. (Recall that $\tau_l > \tau_k$ if and only if $l > k$).

Part b) $\widetilde{K}_2(\tau_k, x; \tau_l, y)$**.** We have

$$\widetilde{K}_2(\tau_k, x; \tau_l, y)\frac{\tilde{\rho}(\tau_k, x)}{\tilde{\rho}(\tau_l, y)}\mathbb{1}_{[x\geq\tilde{s}_k]}\mathbb{1}_{[y\geq\tilde{s}_l]} = \int_{\mathbb{R}} \mathrm{d}\lambda A_1(x,\lambda)A_2(\lambda,y) \tag{B.6}$$

with

$$A_1(x,\lambda) = \mathbb{1}_{[x\geq\tilde{s}_k]}\mathbb{1}_{[\lambda\geq 0]}\tilde{\rho}(\tau_k, x)e^{-\tau_k\lambda}\mathrm{Ai}(x+\lambda+\tau_k^2) \tag{B.7}$$

and

$$A_2(\lambda,y) = \mathbb{1}_{[\lambda\geq 0]}\mathbb{1}_{[y\geq\tilde{s}_l]}\frac{e^{\tau_l\lambda}}{\tilde{\rho}(\tau_l, y)}\mathrm{Ai}(y+\lambda+\tau_l^2). \tag{B.8}$$

Then we use $\|A_1A_2\|_1 \leq \|A_1\|_2\,\|A_2\|_2$. Thus we have just to prove that A_1 and A_2 are Hilbert-Schmidt operators. This is easy to see, since

$$\begin{aligned}
\|A_1\|_2^2 &= \int_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}\lambda|A_1(x,\lambda)|^2 \\
&= \int_{\tilde{s}_k}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}\lambda\tilde{\rho}(\tau_k,x)^2e^{-2\tau_k\lambda}|\mathrm{Ai}(x+\lambda+\tau_k^2)|^2 \\
&\leq C(T_0,\tilde{s}_k) < \infty
\end{aligned} \tag{B.9}$$

because the integrand is bounded, and for large x and λ the decay is super-exponential due to the Airy function ($\mathrm{Ai}(z) \simeq e^{-\frac{2}{3}z^{3/2}}$ for $z \gg 1$). Similarly one shows that $\|A_2\|_2 < \infty$.

Part c) $\widetilde{K}_1(\tau_k, x, \tau_l, y)$**.** We have

$$\widetilde{K}_1(\tau_k, x, \tau_l, y)\frac{\tilde{\rho}(\tau_k, x)}{\tilde{\rho}(\tau_l, y)}\mathbb{1}_{[x\geq\tilde{s}_k]}\mathbb{1}_{[y\geq\tilde{s}_l]} = \int_{\mathbb{R}} \mathrm{d}\lambda B_1(x,\lambda)B_2(\lambda,y) \tag{B.10}$$

with

$$B_1(x,\lambda) = \mathbb{1}_{[x\geq\tilde{s}_k]}\mathbb{1}_{[\lambda\geq 0]}e^{\frac{2}{3}\tau_k^3}(1+x^2)^{2k}e^{\tau_k x}e^{3T_0\lambda}\mathrm{Ai}(x+\lambda+\tau_k^2) \tag{B.11}$$

and

$$B_2(\lambda,y) = \mathbb{1}_{[\lambda\geq 0]}\mathbb{1}_{[y\geq\tilde{s}_l]}e^{-\frac{2}{3}\tau_l^3}\frac{1}{(1+y^2)^{2l}}f(\lambda)g(\lambda,y) \tag{B.12}$$

with $f(\lambda) = e^{(\tau_l+\tau_k-2T_0)\lambda}$ and $g(\lambda,y) = e^{-\tau_l y}e^{-T_0\lambda}\mathrm{Ai}(y-\lambda+\tau_l^2)$.

We need some estimates now. Since $\tau_l+\tau_k-2T_0 < 2\tau_m - 2T_0$, we have

$$|f(\lambda)| \leq e^{-\mu\lambda} \tag{B.13}$$

for $\mu = 2(T_0-\tau_m) > 0$. Moreover,

$$|g(\lambda,y)| = e^{-T_0\lambda}e^{-\tau_l y}|\mathrm{Ai}(y+\tau_l^2-\lambda)|. \tag{B.14}$$

Setting $z = y+\tau_l^2$ and $c_1 = e^{\tau_l^3}$, we get

$$|g(\lambda,y)| \leq c_1e^{-\tau_l z}e^{-T_0\lambda}|\mathrm{Ai}(z-\lambda)|. \tag{B.15}$$

The first case is $z \leq \lambda$. There, $|\mathrm{Ai}(z-\lambda)| \leq 1$, thus

$$|g(\lambda, y)| \leq c_2. \tag{B.16}$$

The second case is $z \geq \lambda$ (recall that $\lambda \geq 0$). There

$$|g(\lambda, y)| \leq c_1 e^{T_0(z-\lambda)} \mathrm{Ai}(z-\lambda) \leq c_3 \tag{B.17}$$

because $\max_{x\geq 0} e^{T_0 x}\mathrm{Ai}(x) = c_3 < \infty$ due to the super-exponential decay of $\mathrm{Ai}(x)$ for large x. Thus by (B.16) and (B.17) we conclude that, for all $\lambda \geq 0$ and $y \geq \tilde{s}_l$, there exists a constant c_4 such that $|g(\lambda, y)| \leq c_4$.

The inequality $\|B_1\|_2 < \infty$ is similar to the $\|A_1\|_2$ case (use the decay of the Airy function). To see that $\|B_2\|_2 < \infty$, we use the bound (B.13) to control the behavior in λ, $|g|$ is just bounded by a constant and the decay in y is controlled by the $(1+y^2)^{-2l}$ term.

In parts a), b) and c) we proved that all the kernel elements are trace-class on $L^2(\mathbb{R})$ and this ends the proof of Proposition 9. $\square$

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