Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Stress-driven local-solution approach to quasistatic brittle delamination

Tomáš Roubíček[1], Marita Thomas[2], Christos Panagiotopoulos[3]

submitted: December 10, 2013

[1] Mathematical Institute
Charles University
Sokolovská 83
CZ-186 75 Praha 8, Czech Republic
and
Institute of Thermomechanics of the ASCR
Dolejškova 5
CZ-182 00 Praha 8, Czech Republic
E-Mail: tomas.roubicek@mff.cuni.cz

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin, Germany
E-Mail: Marita.Thomas@wias-berlin.de

[3] Group of Elasticity and Strength of Materials
Dept. of Continuum Mech.
School of Engineering
University of Seville
Camino de los Descubrimientos s/n
E-41092 Seville, Spain
E-Mail: cpanagiotopoulos@us.es, pchr76@gmail.com

No. 1889

Berlin 2013

2010 *Mathematics Subject Classification.* 35K86, 35R35, 47J20, 49J45, 49J40, 49S05, 65M38, 65Z05, 74M15, 74R10.

Key words and phrases. Unilateral adhesive contact, brittle limit, rate-independent processes, semi-implicit time discretisation, finite perimeter, property $\mathfrak{a}$, $(d-1)$-thick set, lower density estimate, Hardy's inequality, computational simulations.

This research has been partially funded by DFG MATHEON Project C18 and the Spanish Ministry of Economy and Competitiveness (Proyecto MAT2012-37387) as well as by the Czech Republic through the grants 201/10/0357 and 105/13/18652S (GA ČR) together with the institutional support RVO: 61388998 (ČR).

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

A unilateral contact problem between elastic bodies at small strains glued by a brittle adhesive is addressed in the quasistatic rate-independent setting. The delamination process is modelled as governed by stresses rather than by energies. This results in a specific scaling of an approximating elastic adhesive contact problem, discretised by a semi-implicit scheme and regularized by a BV-type gradient term. An analytical zero-dimensional example motivates the model and a specific local-solution concept. Two-dimensional numerical simulations performed on an engineering benchmark problem of debonding a fiber in an elastic matrix further illustrate the validity of the model, convergence, and algorithmical efficiency even for very rigid adhesives with high elastic moduli.

1 Introduction

Both fracture mechanics and mathematical theory of rate-independent processes have achieved great progress during the past decades. A lot of models have been developed in engineering and in mathematics accounting for different features of the materials. Even more, particular models admit various concepts of solutions which, in combination with the specific model, can describe certain specific aspects of the process under consideration. However, it is well recognized that solutions to rate-independent systems governed by non-convex potentials, as it is the case in fracture models, may exhibit sudden jumps, i.e., sudden rupture. Various concepts of weak solutions – in this rate-independent case also called *local solutions* – have been devised, ranging from *energetic solutions*, which *conserve energy*, to approximable, vanishing-viscosity, BV-, ε-sliding, or maximally-dissipative solutions, cf. in particular [14, 16, 25, 29, 38, 39, 52]. In convex situations, all these concepts essentially coincide with each other but in general nonconvex situations they are very different. This is related with the conceptual quastion whether rather energy or rather stress governs the inelastic process in question and it also has to do with the issue of global versus local minimization, cf. the discussion in mathematical literature [5, 46] and in engineering [19], and also the examples [2, Sect. 9], [15, Sect. 6], or [26, Example 7.1]. In particular, *energetic solutions*, which form a sub-class of the *local solutions*, are known to exhibit a tendency to unphysically early jumps. Therefore, in this article, we will focus on another type of local solutions.

In this work we will restrict the fracture process to a prescribed interface, and thus confine ourselves to a so-called *delamination* problem, also called *adhesive contact problem*, cf. [44, 45] for a survey of various models. In this context, it was already observed in [41] that the local solutions obtained by semi-implicit time discretisation nicely coincide numerically with the vanishing-viscosity solutions in all investigated examples; of course, energy conservation is lost for such local but non-energetic solutions. Mathematical justification of such a, in fact, *stress-driven evolution* has been scrutinized in [38] for the setting of delamination with *adhesive contact*, where it reveals a certain connection with the maximal-dissipation principle, and then numerically in [40], for a slightly more general adhesive model. One of the motivations for the stress-driven local solutions is to avoid the undesired delamination due to big stored energy in a large bulk even under small stress – also called a "long-bar paradox" in [16], cf. also [38, 39].

In view of their good performance in the setting of adhesive contact, cf. [40, 41], it is the aim of this paper to establish the notion of stress-driven local solutions also for the setting of brittle delamination. For this, we will valorize the method of an *adhesive contact approximation of brittle delamination* studied in [42] in the context of energetic solutions. More specifically, in this paper we address a *delamination* problem of two *elastic bodies at small strains* glued along a contact boundary $\Gamma_{\rm C}$ by a *brittle* adhesive with a prescribed fracture toughness. The interface $\Gamma_{\rm C}$ separates the body located in the domain $\Omega \subset \mathbb{R}^d$, with $2 \le d \in \mathbb{N}$, into two parts, Ω_- and Ω_+. In the spirit of Generalized Standard Materials, in particular Frémond's concept of adhesion [10], the degradation state of the adhesive during the time span $[0,T]$ is captured by an additional internal variable $z : [0,T] \times \Gamma_{\rm C} \to [0,1]$, where $z(t,x) = 1$ stands for the fully intact state, whereas $z(t,x) = 0$ models complete rupture (= debonding) of the adhesive in the material point $x \in \Gamma_{\rm C}$ at time $t \in [0,T]$. This approach essentially admits arbitrarily shaped $(d-1)$-dimensional cracks evolving along the interface $\Gamma_{\rm C}$. Moreover it allows to display both adhesive contact and brittle delamination in a unified way: The brittle delamination model describes the crack growth in a brittle adhesive. Expressed in terms of the displacement field $u : [0,T] \times (\Omega\backslash\Gamma_{\rm C}) \to \mathbb{R}^d$ and the delamination variable $z : [0,T] \times \Gamma_{\rm C} \to [0,1]$, this means that the displacements must not jump on $\operatorname{supp} z(t) \subset \overline{\Gamma}_{\rm C}$, the spatial support of z at time t, while on the crack set $\Gamma_{\rm C} \backslash \operatorname{supp} z(t)$ they may jump. This so-called *brittle constraint* can be expressed with the aid of the indicator function

$$J_\infty(\llbracket u \rrbracket, z) := \begin{cases} 0 & \text{if } \big|\llbracket u \rrbracket\big| = 0 \text{ in } x \in \operatorname{supp} z, \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

In the adhesive contact model, due to the more viscous properties of the adhesive, the two parts of the body can be slightly detached from each other without that the adhesive has necessarily experienced degradation. In other words, here, the displacements u are allowed to jump on supp z at a current time, but the jump is penalized by the *adhesive contact term*

$$J_k(\llbracket u \rrbracket, z) = \frac{k}{2}z)\big|\llbracket u \rrbracket\big|^2. \tag{1.2}$$

Thus, (1.2) can be used to relax the non-convex and nonsmooth constraint (1.1). The contact between the two components of the body will be considered *unilateral* but frictionless, which is encoded in the *non-penetration condition*

$$I_{\mathrm{C}}(x, \llbracket u \rrbracket) := \begin{cases} 0 & \text{if } \llbracket u \rrbracket{\cdot}\mathrm{n}(x) \geq 0, \\ \infty & \text{otherwise,} \end{cases} \tag{1.3}$$

where $\mathrm{n}(x)$ denotes the unit normal vector pointing from Ω_- to Ω_+ at $x \in \Gamma_{\mathrm{C}}$. Any rate effects, such as viscosity, inertia or temperature dependence, are neglected and the problem is thus completely *rate-independent*. The time-continuous brittle problem, involving (1.1), will be approximated by adhesive contact problems, involving (1.2) with $k \to \infty$, and discretised in time by a semi-implicit scheme scaled in such a way that *stress-driven nucleation of the crack* will be correctly modelled in the limit. As a side effect, an efficient robust numerical strategy will be devised. The convergence proof will require a BV-type gradient regularizing term scaled to zero in the limit model. However, by compactness, the BV-property of the approximating solutions is passed on to the (approximable) solutions of the limit model. It can be understood in a similar way that also the information on the stress-driven nature of the delamination process is handed down from the approximating time-discrete adhesive problems to the time-continuous brittle limit.

The scaling used here in the context of stress-driven local solutions, cf. (2.6) and (3.1c) below, differs significantly from the scaling applied in [42] for the setting of energetic solutions. On the first glance, the new scaling even looks rather surprising because asymptotically the fracture toughness tends to 0, and thus, the dissipated energy due to delamination vanishes. But this scaling has already been investigated numerically in engineering literature for static problems close to the onset of rupture, cf. [21, Formula (16)] or [47, Formula (7)]. It is recognised that this scaling has the capacity to predict correctly crack nucleations. On the other hand, due to the typical stress concentration on the crack tips of already existing cracks, this scaling usually leads to the effect of too easy crack and therefore this simple model must be combined with some plasticity mechanism (usually called a ductile fracture, cf. e.g. [7, 8, 18, 43]), which is however far beyond of the scope of this paper.

The plan of the paper is the following: In Section 2, we introduce the problem, the notion of local solution, and motivate the new scaling for the adhesive models towards the brittle limit on a simple explicit example. Then, in Section 3, we construct an approximate problem in a general multi-dimensional situation by using a suitable time discretisation combined with a gradient-type regularization of the delamination parameter, while the limit passage to a continuum problem is carried out in Section 4. Eventually, in Section 5, by using also spatial discretisation by the boundary-element method, we demonstrate the efficiency, convergence, and applicability of the proposed model and the solution concept on a two-dimensional engineering benchmark problem, namely at the debonding of a cylindrical inclusion of a fiber in an elastic matrix under a transverse tension.

2 The delamination model, local solutions, and the scaling towards the brittle limit

The quasistatic rate-independent evolution considered in this paper will always be governed by two functionals, the stored energy $\mathscr{E} : [0,T]\times\mathscr{U}\times\mathscr{Z} \to \mathbb{R}\cup\{\infty\}$ and the dissipation energy $\mathscr{R} : \mathscr{X} \to \mathbb{R}\cup\{\infty\}$. We consider $\mathscr{U}$ and $\mathscr{Z} \subset \mathscr{X}$, three Banach spaces specified below, and we will always assume that $\mathscr{E}(t, \cdot, z)$ is convex and that $\mathscr{R}$ is convex and 1-homogeneous.

Rather heuristically, we can say that the rate-independent evolution we have in mind is governed by the following system of *doubly nonlinear* degenerate *parabolic/elliptic variational inclusions*:

$$\partial_u \mathscr{E}(t,u,z) \ni 0 \quad \text{and} \quad \partial\mathscr{R}\big(\dot z\big) + \partial_z\mathscr{E}(t,u,z) \ni 0, \tag{2.1}$$

where $\dot z$ denotes the time derivative of z and the symbol "∂" refers to a (partial) subdifferential of the involved functional, which has to be convex with respect to the respective variables in order to well-define the subdifferental. We will now introduce the notion of local solution, which is in fact a weaker concept of solution than (2.1), as it does not involve the derivatives or subdifferentials of the functionals.

We will use the notation $\mathrm{B}([0,T];\cdot)$, resp. $\mathrm{BV}([0,T];\cdot)$, for Banach-space-valued, Bochner-measurable functions which are everywhere defined in $[0,T]$, and which are bounded, resp. of bounded variation. The general framework, which any "reasonable" solution to (2.1) should comply with, is the following:

Definition 2.1 (Local solutions)**.** *We call the pair (u,z) with $u\in B([0,T];\mathscr{U})$ and $z\in B([0,T];\mathscr{Z})\cap BV([0,T];\mathscr{X})$ a local solution to the initial-boundary-value problem* (2.1) *if*

$$\forall_{\text{a.a.}} t\in[0,T]\ \forall \tilde u\in\mathscr{U}: \quad \mathscr{E}(t,u(t),z(t)) \le \mathscr{E}(t,\tilde u,z(t)), \tag{2.2a}$$

$$\forall_{\text{a.a.}} t\in[0,T]\ \forall \tilde z\in\mathscr{Z}: \quad \mathscr{E}(t,u(t),z(t)) \le \mathscr{E}(t,u(t),\tilde z) + \mathscr{R}(\tilde z - z(t)), \quad \textit{and} \tag{2.2b}$$

$$\begin{aligned}\forall\, 0\le t_1<t_2\le T: \quad & \mathscr{E}(t_2,u(t_2),z(t_2)) + \mathrm{Diss}_{\mathscr{R}}(z;[t_1,t_2]) \\ & \le \mathscr{E}(t_1,u(t_1),z(t_2)) + \int_{t_1}^{t_2} \partial_t\mathscr{E}(t,u(t),z(t))\,\mathrm{d}t,\end{aligned} \tag{2.2c}$$

where $\mathrm{Diss}_{\mathscr{R}}(z;[t_1,t_2]) = \sup\sum_{j=1}^N \mathscr{R}(z(s_j)-z(s_{j-1}))$ *with the supremum taken over all partitions* $t_1 \le s_0 < s_1 < \ldots < s_{N-1} \le s_N \le t_2$ *of* $[t_1,t_2]$.

The concept of local solutions has been invented for the context of delamination in [52], cf. also [25]; here we combine it with the concept of semi-stability [36]. In fact, the local solutions are, under mild assumptions, in particular the existence of $\partial_z\mathscr{E}$, the conventional weak solutions, cf. [38, Prop. 2.3]. But note that Definition 2.1 does not require the existence of $\partial_z\mathscr{E}$ and, in fact, $\mathscr{E}(t,u,\cdot)$ does not need to be convex, which we will exploit below. It is well recognized that the local-solution concept is rather a (wide) basic framework and some additional attributes of the solutions would be desirable in particular situations.

Stress-driven local solutions Such an additional attribute is to impose that the evolution of a local solution is stress-driven. This feature can be retrieved from the *maximum-dissipation principle*, i.e.,

$$\big\langle \dot z(t), \mathfrak{f}(t)\big\rangle = \max_{f\in\partial\mathscr{R}(0)} \big\langle \dot z(t), f(t)\big\rangle \qquad \text{with the driving force } \ \mathfrak{f}(t)\in -\partial_z\mathscr{E}(t,u(t),z(t)); \tag{2.3}$$

here $\dot z$ can be a measure, so (2.3) is written rather formally. If $\mathfrak{f}(t)$ is in the interior of the convex "elastic" domain $\partial\mathscr{R}(0)$, then inevitably $\dot z(t)=0$. Therefore, delamination can evolve only if the driving force reaches the boundary of $\partial\mathscr{R}(0)$, but not earlier. Intentionally, this type of solutions typically does not conserve energy.

For comparison, let us now also address *energetic solutions*, introduced in [29]; they conserve energy. In addition to (2.2) they require the *full* stability $\mathscr{E}(t,u(t),z(t)) \le \mathscr{E}(t,\tilde u,\tilde z) + \mathscr{R}(\tilde z - z(t))$ to be valid for all $(\tilde u,\tilde z)\in\mathscr{U}\times\mathscr{Z}$ and for all (a.a.) $t\in[0,T]$. Then (2.2c) holds as an equality.

Motivation of the scaling with a 0-dimensional example Let us first start with revisiting a motivating zero-dimensional (i.e., lumped paramater) example from [39], consisting of two springs in series, the left one undergoing degradation, hence representing the adhesive interface $\Gamma_{\rm C}$, which is zero-dimensional, i.e., located at one single point. Thus u and z are just scalar variables, the whole problem has only two degrees of freedom, and the solutions can be calculated explicitely. Let us make a simple gedankenexperiment by considering the Dirichlet load starting from zero and growing in time with a constant speed $v_{\rm D}>0$, i.e. $u_{\rm D}(t)=v_{\rm D}t$. We deal with the functionals $\mathscr{E}_k:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ and $\mathscr{R}_k:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ given by

$$\mathscr{E}_k(t,u,z) = \begin{cases} \frac12 kzu^2 + \frac12 c|u-v_{\rm D}t|^2 & \text{if } 0\le z\le 1,\\ +\infty & \text{otherwise,}\end{cases} \qquad \mathscr{R}_k(\dot z) = \begin{cases} \alpha_k|\dot z| & \text{if } \dot z\le 0,\\ +\infty & \text{otherwise,}\end{cases} \tag{2.4}$$

with the elastic modulus of the adhesive $k > 0$, resp. of the bulk $c > 0$, just scalars, and $\alpha_k > 0$ the fracture toughness of the adhesive. As we pull the springs, we do not need to consider a possible Signorini-type constraint $u \geq 0$ that would not be active in this regime anyhow. Our goal is to calculate the time when the damageable spring, i.e., the adhesive, ruptures. We start from the unbroken initial state $z(0) = z_0 \equiv 1$.

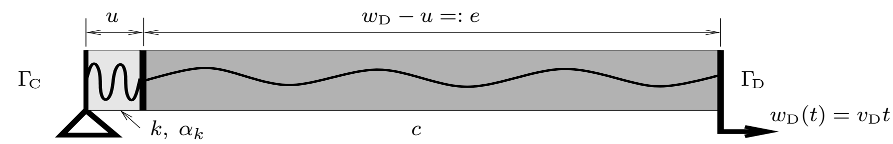


Fig. 1. Zero-dimensional delamination described by (2.4); a damageable spring (= adhesive) in series with an elastic bulk loaded by time-dependent Dirichlet condition (= hard-device load); e denotes the strain provided by the specimen of unit length.

To comply with (2.2), any local solution (u, z) must satisfy $z(t) = 1$ for $t < t_{ES,k} := \frac{1}{v_D}\sqrt{2\alpha_k(k+c)/(kc)}$ and $z(t) = 0$ for $t > t_{MD,k} := \frac{k+c}{v_D c}\sqrt{2\alpha_k/k}$, otherwise either (2.2c) or (2.2b) would be violated, respectively. The energetic solution completely delaminates at time $t_{ES,k}$, while the maximally-dissipative local solution delaminates at $t_{MD,k}$.

Let us consider $c > 0$ fixed while k will vary. In particular, let us investigate the asymptotics towards the brittle limit, i.e. for $k \to \infty$. Obviously,

$$\alpha_k = \text{const.} = \alpha \quad \Rightarrow \quad t_{ES,k} := \sqrt{\frac{2\alpha k + 2\alpha c}{v_D^2 kc}} \to \frac{1}{v_D}\sqrt{\frac{2\alpha}{c}} =: t_{ES}, \quad \text{while} \quad t_{MD,k} := \frac{k+c}{v_D c}\sqrt{\frac{2\alpha_k}{k}} \to \infty \tag{2.5}$$

as $k \to \infty$. In other words, as expected, the rupture occurs when the bulk energy $\frac{1}{2}c(tv_D)^2$ reaches the threshold α, i.e. when $t = t_{ES}$. On the other hand, cf. (2.5), $t_{MD,k}$ clearly blows up for $k \to \infty$ if $\alpha_k = \alpha$ is kept constant because the stress needed for rupture blows up in this scaling, cf. Fig. 2(left). This indicates that stress-driven delamination would never occur in brittle adhesive if $\alpha_k = \alpha$ is kept fixed.

This paradox ultimately calls for a different scaling. In view of (2.5), obviously, $\alpha_k = \mathscr{O}(1/k)$ will lead to a finite rupture time; more specifically, we choose $\sigma_{RUP} > 0$ having the meaning of the desired stress under which brittle rupture happens and then

$$\alpha_k := \frac{\sigma_{RUP}^2}{2k} \qquad \Rightarrow \qquad t_{MD,k} := \frac{k+c}{v_D c}\sqrt{\frac{2\alpha_k}{k}} \to \frac{\sigma_{RUP}}{v_D c} =: t_{MD} \qquad \text{for} \quad k \to \infty. \tag{2.6}$$

Hence, rupture occurs when the stress $c(tv_D)$ reaches a prescribed threshold σ_{RUP}, as desired. Notably, $\alpha_k \to 0$ in (2.6) so that, in the brittle limit, the dissipation in the adhesive vanishes, cf. Fig. 2(right); this rather paradoxical phenomenon seems to be in agreement with the observations in, e.g., [6, 17] and this scaling is also used in engineering literature [21, 47], but in the static setting.

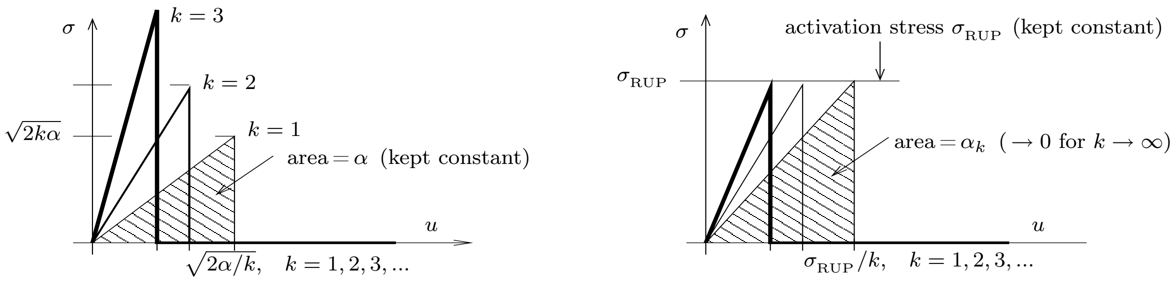


Fig. 2. Schematic illustration of the response of the stress σ on the adhesive stretch u during the pulling experiment as in Figure 1 for different elastic moduli k of the adhesive.
Left: the scaling towards brittle delamination for energy-driven evolution from (2.5).
Right: the scaling towards brittle delamination for stress-driven evolution from (2.6).

The semi-stability (2.2b) under this "brittle" scaling (2.6) reads

$$\forall\, 0 \leq \tilde{z} \leq z_k(t): \qquad \frac{1}{2}kz_k(t)u_k(t)^2 \leq \frac{1}{2}k\tilde{z}u_k(t)^2 + \frac{1}{2k}\sigma_{RUP}^2(z_k(t) - \tilde{z}). \tag{2.7}$$

We know that $kz_k(t)u_k^2 \to 0$ by the usual penalty-function argument, cf. also (4.22) below, and therefore passing to the limit in (2.7) for $k \to \infty$ gives just $0 \le 0$. To see some information in the limit, we multiply (2.7) by k, i.e. we consider $k^2 z_k(t) u_k(t)^2 \le k^2 \tilde{z} u_k(t)^2 + \sigma_{\text{RUP}}^2(z_k(t)-\tilde{z})$. By using the equilibrium (2.2a), we can see that always $kz_k(t)u_k(t) = \sigma_k = c(w_{\text{D}}-u_k(t))$ is the stress in the springs. As we will do also in Sections 3–4 below, let us now confine our arguments to the setting where the delamination variable distinguishes between two states of the adhesive, only: the unbroken and the completely broken one, i.e. for all $t \in [0,T]$, for all $x \in \Gamma_{\text{C}}$ it is $z(t,x), \tilde{z}(x) \in \{0,1\}$. Then we arrive at

$$\forall\, 0 \le \tilde{z} \le z_k(t): \qquad z_k(t)\sigma_k^2(t) \le \tilde{z}\sigma_k^2(t) + \sigma_{\text{RUP}}^2(z_k(t)-\tilde{z}) \quad \text{with} \quad \sigma_k(t) = c\big(w_{\text{D}}(t)-u_k(t)\big), \tag{2.8}$$

which preserves certain nontrivial information in the "brittle limit" for $k \to \infty$. Yet, it only says that such local solutions cannot rupture later than at time $t_{\text{MD},k}$. To select the solution which indeed ruptures at $t = t_{\text{MD},k}$ when the stress $\sigma_k(t)$ reaches the threshold σ_{RUP}, one may invoce the maximum-dissipation principle (2.3) which, in a certain variant devised in [38] survives the limit for $k \to \infty$. Here, in the general, multidimensional setting, if z is constrained to be valued in $\{0,1\}$, there is the problem that the driving force $-\partial_z \mathscr{E}_k$ is not well defined. Anyhow, in this particular example, one can substitute $\partial_z \mathscr{E}_k(t,u,z) = \frac{1}{2}ku^2$ from the equilibrium equation $\sigma_k(t) = kz_k(t)u_k(t)$ if $z_k(t) = 1$ to consider $\partial_z \mathscr{E}_k(t,u,z) = \frac{1}{2k}\sigma_k^2$ while if $z_k(t) = 0$ it holds $\sigma_k(t) = 0$ so that possibly $\frac{1}{2}ku^2 \ne \frac{1}{2k}\sigma_k^2$ but then anyhow $\dot{z}_k(t) = 0$. Thus we can formulate the maximum-dissipation principle rather as

$$\big\langle \dot{z}_k(t), \mathfrak{f}_k(t) \big\rangle = \max_{-f \le \alpha_k} \big\langle \dot{z}_k(t), f \big\rangle \qquad \text{for} \qquad \mathfrak{f}_k(t) = -\frac{1}{2k}\sigma_k^2(t). \tag{2.9}$$

Let us note that, under the scaling of α_k as in (2.6), the maximum-dissipation principle (2.9) yields

$$\big\langle \dot{z}_k(t), \sigma_k^2(t) - \sigma_{\text{RUP}}^2 \big\rangle = 0 \tag{2.10}$$

and thus rupture cannot occur any sooner than the stress σ_k achieves the prescribed threshold σ_{RUP}. Therefore, we can see that (2.2) with $\mathscr{R} = \mathscr{R}_k$ from (2.4) together with the scaling $\alpha_k \to 0$ from (2.6) smears out any information about the limit stress σ_{RUP} and even the rescaled semistability (2.7) gives only a one-sided restiction on the rupture time where σ_{RUP} explicitly occurs.

The indeed determining information is contained in (2.9) or (2.10). Again, like (2.3), both (2.9) and (2.10) are only very formal because during rupture, $\dot{z}_k$ is a measure while σ_k jumps and thus the duality is not well defined. In the multidimensional case, which we will focus on later, there is also a spatial discrepancy because, in general, $\dot{z}_k \in L^1(\Gamma_{\text{C}})$ while σ_k is not in $L^\infty(\Gamma_{\text{C}})$. The rigorous formulation of (2.9) and (2.10) which would survive the limit passage is currently out of reach for the multi-dimensional brittle delamination problem, as the above mentioned obstructions reflect the natural difficulty of fracture problems. Instead, we will have to rely only on a rather implicit concept of *approximable local solutions* as in [14, 52].

The brittle model in the d-dimensional setting Relying on the above observations, in the general d-dimensional situation, we consider $\mathscr{U} = \{u \in W^{1,p}(\Omega\backslash\Gamma_{\text{C}};\mathbb{R}^d);\ u|_{\Gamma_{\text{D}}} = 0\}$, for some $p \in (1,\infty)$, $\mathscr{Z} = L^\infty(\Gamma_{\text{C}})$, and $\mathscr{X} = L^1(\Gamma_{\text{C}})$. The target brittle delamination problem is governed by the energy functional $\mathscr{E} : [0,T] \times \mathscr{U} \times \mathscr{Z} \to \mathbb{R}_\infty$ and the dissipation potential $\mathscr{R} : \mathscr{Z} \to \mathbb{R}_\infty$:

$$\mathscr{E}(t,u,z) := \mathscr{E}^{\text{bulk}}(t,u) + \mathscr{E}^{\text{surf}}(u,z), \quad \text{where} \tag{2.11a}$$

$$\mathscr{E}^{\text{bulk}}(t,u) := \int_{\Omega\backslash\Gamma_{\text{C}}} W\big(e(u{+}g(t)\big)\,\mathrm{d}x - \big\langle f(t), u \big\rangle \quad \text{and} \tag{2.11b}$$

$$\mathscr{E}^{\text{surf}}(u,z) := \int_{\Gamma_{\text{C}}} I_{\text{C}}(\llbracket u \rrbracket) + I_{[0,1]}(z) + J_\infty(\llbracket u \rrbracket, z)\,\mathrm{d}\mathscr{H}^{d-1}, \tag{2.11c}$$

$$\mathscr{R}(\dot{z}) := \int_{\Gamma_{\text{C}}} R(\dot{z})\,\mathrm{d}\mathscr{H}^{d-1} \qquad \text{with} \qquad R(\dot{z}) := \begin{cases} 0 & \text{if } \dot{z} \le 0, \\ \infty & \text{otherwise}, \end{cases} \tag{2.11d}$$

where $e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u$ is the small-strain tensor and $W : \mathbb{R}^{d\times d} \to \mathbb{R}^+$ is a convex possibly non-quadratic specific stored energy, while g describes the nonhomogenous time-varying Dirichlet boundary conditions on the part of the boundary $\Gamma_{\rm D}$ after the standard shift so that the solution is searched among functions with zero traces on $\Gamma_{\rm D}$. Moreover, f comprises both bulk forces and surface forces on $\Gamma_{\rm N}$. With Assumption (3.9) below, we will make sure that $\Gamma_{\rm D}$ is far from $\Gamma_{\rm C}$ so that one can assume $g|_{\Gamma_{\rm C}} = 0$ and indeed work with $\mathscr{E}^{\rm surf}$ not involving g. The surface energy $\mathscr{E}^{\rm surf}(u,z)$ in (2.11c) accounts for several constraints: $I_{\rm C}(\cdot)$ as in (1.3) ensures the Signorini unilateral contact along the interface, $I_{[0,1]}$ is the indicator function of the interval $[0,1]$, i.e. $I_{[0,1]}(z)=0$ if $z\in[0,1]$ and $I_{[0,1]}(z)=\infty$ otherwise. Finally, J_∞ from (1.1) accounts for the brittle constraint.

Note that, in view of (2.11d), here simply always $\mathrm{Diss}_{\mathscr{R}}(z;[t_1,t_2])=0$ in (2.2c). Taking into account the definition of the energy and the dissipation functionals (2.11), the semistability inequality (2.2b) for z does not seem to contain any non-trivial information apart from the preservation of unidirectionality. As shown in the 0-dimensional example, the stress-driven local solutions are understood to feature an extra attribute involving the given stress $\sigma_{\rm RUP}$. This will be incorporated to the solutions of the local, brittle model via the mentioned concept of approximable local solutions, by combining the time-discrete scheme with an adhesive contact approximation, whose solutions (are close to) satisfy the property (3.8) in analogy to the maximum-dissipation condition (2.9).

3 The adhesive-type regularized time-discrete scheme

In this section we discuss the time-discrete scheme to obtain local solutions (2.2) for the brittle system (2.11). For the approximation of a time-continous solution for the brittle system (2.11) we will introduce suitably regularized time-discrete systems. These are motivated by the scaling (2.6), such that they indeed reward the evolution of delamination with a decrease of mechanical energy, i.e., such that crack growth is captured by the time-discrete models. Our aim is to introduce a time-discrete scheme for the regularized systems (3.1), which links the time-step size with the parameter k, i.e., the elastic modulus of the adhesive. In this section we will develop the existence of piecewise constant interpolants solving a time-discrete version of the local formulation (2.2) and satisfying uniform a-priori bounds. This will allow us in Section 4 to carry out the limit passage $k\to\infty$ and thus to approximate with the time-discrete solutions of the regularized systems a time-contiuous solution of the brittle system. In other words, as motivated in the example, by this approximation procedure the non-trivial information on the driving forces for crack growth contained in the respective semistability inequalities of the approximating systems is passed over to the limit and hence encoded in (u,z) satisfying (2.2) for the brittle system.

3.1 The time-discrete, approximating problem

For the time-discrete scheme we will resort to the following regularized functionals:

$$\mathscr{E}_k(t,u,z) := \mathscr{E}^{\rm bulk}(t,u) + \mathscr{E}^{\rm surf}_k(u,z) \quad \text{with } \mathscr{E}^{\rm bulk} \text{ again from (2.11b) and} \tag{3.1a}$$

$$\mathscr{E}^{\rm surf}_k(u,z) := \begin{cases} \displaystyle\int_{\Gamma_{\rm C}} J_k(\llbracket u\rrbracket,z) - \frac{\alpha_0}{k}z + I_{\rm C}(\llbracket u\rrbracket) + I_{[0,1]}(z)\,\mathrm{d}\mathscr{H}^{d-1} + \frac{\beta}{k}\mathscr{P}(Z,\Gamma_{\rm C}) & \text{if } (u,z)\in\mathscr{U}\times\mathscr{Z}_{\rm SBV},\\ \infty & \text{otherwise,}\end{cases} \tag{3.1b}$$

$$\mathscr{R}_k(\dot z) := \int_{\Gamma_{\rm C}} R_k(\dot z)\,\mathrm{d}\mathscr{H}^{d-1} \qquad \text{with} \quad R_k(\dot z) := \begin{cases} \dfrac{\alpha_1}{k}|\dot z| & \text{if } \dot z\le 0,\\ \infty & \text{otherwise.}\end{cases} \tag{3.1c}$$

The scaling of the terms in (3.1b) and (3.1c) is motivated by the 0-dimensional example. Moreover, in (3.1b) the expression $\mathscr{P}(Z,\Gamma_{\rm C})$ denotes the perimeter of the set Z in $\Gamma_{\rm C}$, which is the total variation of the corresponding characteristic function z, i.e. $z(x)=1$ if $x\in Z$ and $z(x)=0$ otherwise. The total variation of a general function $v\in \mathrm{BV}(\Gamma_{\rm C})$ is defined as $|\mathrm{D}v|(\Gamma_{\rm C}) := \sup\big\{\int_{\Gamma_{\rm C}} z\,\mathrm{div}\,\varphi\,\mathrm{d}x \,|\, \varphi\in C^1_0(\Gamma_{\rm C})^d\big\}$. The weight $\beta>0$, premultiplying the perimeter in (3.1b) shall keep the influence of $\mathscr{P}(Z,\Gamma_{\rm C})$ to the model as small as possible. This type of regularization of the adhesive energy functional is based on the results obtained in [35], where the need for the use of characteristic functions of finite-perimeter sets became evident for an adhesive-contact approximation of brittle delamination in the setting of thermo-viscoelastic materials. Due to this regularization the functional $\mathscr{E}^{\rm surf}_k(u,\cdot)$ attains finite values only on

$$\mathscr{Z}_{\rm SBV} := \big\{z:\Gamma_{\rm C}\to\{0,1\},\ z \text{ is the characteristic function of the set } Z \text{ with } \mathscr{P}(Z,\Gamma_{\rm C})<\infty\big\}, \tag{3.2}$$

while $\mathscr{E}^{\mathrm{surf}}(u,\cdot)$ in (2.11c) is defined on $\mathscr{Z} := L^\infty(\Gamma_{\mathrm{C}})$. Furthermore, the material constants α_0 from (3.1b) and α_1 from (3.1c), or rather their sum $\alpha_0+\alpha_1 =: \alpha > 0$ defines the activation energy to trigger delamination for the case that the elastic modulus of the adhesive is $k = 1$. Later, we will link it with the stress needed for delamination $\sigma_{_{\mathrm{RUP}}}$ by $\alpha = \frac{1}{2}\sigma_{_{\mathrm{RUP}}}^2$, cf. (2.6) for $k = 1$.

For the time interval $[0,T]$, $T > 0$ fixed, we introduce a sequence of equidistant partitions $(\Pi_N)_N$ getting finer and finer as $N \to \infty$. Since $N \to \infty$ shall be linked with $k \to \infty$ we now suitably define N as a function of k. With a closer look to (2.2c), where each $0 \le t_1 < t_2 \le T$ are to be approximated, it is convenient if the partitions are nested in the sense that

$$\forall\, N_1, N_2 \in \mathbb{N},\ N_1 < N_2: \qquad \Pi_{N_1} \subset \Pi_{N_2}. \tag{3.3}$$

This can be achieved, e.g., by the particular choice $N(k) = 2^k$. With such a choice of $N(k)$, the partitions are given by $\Pi_{N(k)} = \big(t_i^{N(k)}\big)_{i=0}^{N(k)}$ with

$$t_0^{N(k)} = 0 \quad \text{and} \quad \forall\, i \in \{1,\dots,N(k)\}: \quad t_i^{N(k)} = t_0^{N(k)} + i/N(k), \quad \text{hence} \quad t_{N(k)}^{N(k)} = T. \tag{3.4}$$

Given the initial datum $z_0 = z(0)$, we introduce the recursive time-discrete scheme, for all $i \in \{1,\dots,N(k)\}$, for all $k \in \mathbb{N}$:

$$u_i^{N(k)} \;=\; \operatorname*{argmin}_{u\in\mathscr{U}} \mathscr{E}_k(t_i^{N(k)}, u, z_{i-1}^{N(k)})\,, \tag{3.5a}$$

$$z_i^{N(k)} \;\in\; \operatorname*{Argmin}_{z\in\mathscr{Z}} \big(\mathscr{E}_k(t_i^{N(k)}, u_i^{N(k)}, z) + \mathscr{R}_k(z - z_{i-1}^{N(k)})\big)\,, \tag{3.5b}$$

starting for $k = 1$ with $z_0^{N(k)} = z_0$. Note that, in fact, (3.5b) does not explicitly depend on $t_i^{N(k)}$, since the bulk terms do not contribute to the minimization problem.

Let us emphasize that, in particular, the decoupled semi-implicit scheme (3.5) intentionally excludes the global minimization and, by this way, also the mentioned long-bar paradox. Moreover, it can be understood as the fractional-step method, cf. [37, Remark 8.25].

Relying on the existence of $(u_i^{N(k)}, z_i^{N(k)})$, which is given by Proposition 3.1 below, we introduce the piecewise-constant interpolants

$$\left.\begin{array}{l} \overline{\mathscr{E}}_k(t,u,z) := \mathscr{E}_k(t_i^{N(k)}, u, z), \\ \overline{w}_k(t) := w_i^{N(k)} \quad \& \quad \underline{w}_k(t) := w_{i-1}^{N(k)}, \end{array}\right\} \text{ for } t \in [t_{i-1}^{N(k)}, t_i^{N(k)}), \tag{3.6}$$

where w generically stands for u, z, g or f. With Proposition 3.2 below, we verify that the discrete solution obtained by (3.5) and (3.6) satisfies the following discrete version of the local formulation (2.2)

$$\forall\, t\in[0,T]: \qquad \partial_u \overline{\mathscr{E}}_k(t, \overline{u}_k(t), \underline{z}_k(t)) \ni 0, \tag{3.7a}$$

$$\forall\, t\in[0,T]\ \forall\, \tilde{z}\in Z: \quad \overline{\mathscr{E}}_k(t, \overline{u}_k(t), \overline{z}_k(t)) \le \overline{\mathscr{E}}_k(t, \overline{u}_k(t), \tilde{z}) + \mathscr{R}_k\big(\tilde{z} - \overline{z}_k(t)\big), \tag{3.7b}$$

$$\overline{\mathscr{E}}_k(t_2, \overline{u}_k(t_2), \overline{z}_k(t_2)) + \mathrm{Diss}_{\mathscr{R}_k}(\overline{z}_k; [t_1,t_2]) \le \overline{\mathscr{E}}_k(t_1, \overline{u}_k(t_1), \overline{z}_k(t_1)) + \int_{t_l^{N(k)}}^{t_m^{N(k)}} \partial_t \mathscr{E}(t, \underline{u}_k(t), \underline{z}_k(t))\,\mathrm{d}t \tag{3.7c}$$

for all $t_1 \in (t_{l-1}^{N(k)}, t_l^{N(k)}]$ and $t_2 \in (t_{m-1}^{N(k)}, t_m^{N(k)}]$ with $l < m \in \{1,\dots,N(k)\}$. Moreover, if the coefficient β were zero, the following discrete version of the maximum-dissipation principle (2.10) would hold:

$$\big\langle \dot{z}_k, \overline{\sigma}_k - \sigma_{_{\mathrm{RUP}}} \big\rangle = 0 \qquad \text{with} \quad \overline{\sigma}_k = k\overline{z}_k [\![\overline{u}_k]\!] \ \text{ on } \ [0,T]\times\Gamma_{\mathrm{C}}. \tag{3.8}$$

In particular, it indeed holds for the example from Section 2. For $\beta > 0$, (3.8) holds only approximately, however. But nevertheless, this argument underlines that the character of the evolution is rather stress-driven, instead of energy-driven.

Before establishing the existence of the time-discrete solutions satisfying (3.7) in Section 3.3 and recalling some additional fine regularity properties of these solutions in Section 3.4, we first collect the assumptions on the domain and the given data used to carry out the analysis.

3.2 Data qualification and function spaces

Assumptions on the domain Ω. We suppose that

$$\Omega \subset \mathbb{R}^d,\ d \geq 2, \text{ is bounded, } \Omega_-,\ \Omega_+,\ \Omega \text{ are Lipschitz domains, } \Omega_+ \cap \Omega_- = \emptyset\,, \tag{3.9a}$$

$$\partial\Omega = \Gamma_{\rm D} \cup \Gamma_{\rm N},\ \Gamma_{\rm D},\ \Gamma_{\rm N} \text{ open subsets in } \partial\Omega, \tag{3.9b}$$

$$\Gamma_{\rm D} \cap \Gamma_{\rm N} = \emptyset,\ \overline{\Gamma}_{\rm D} \cap \overline{\Gamma}_{\rm C} = \emptyset,\ \mathscr{H}^{d-1}(\Gamma_{\rm D} \cap \overline{\Omega}_+) > 0\,,\ \mathscr{H}^{d-1}(\Gamma_{\rm D} \cap \overline{\Omega}_-) > 0\,, \tag{3.9c}$$

$$\begin{array}{c}\Gamma_{\rm C} = \overline{\Omega}_+ \cap \overline{\Omega}_- \subset \mathbb{R}^{d-1} \text{ is a convex “flat” surface, i.e. contained in a hyperplane of } \mathbb{R}^d, \\ \text{such that, in particular, } \mathscr{H}^{d-1}(\Gamma_{\rm C}) = \mathscr{L}^{d-1}(\Gamma_{\rm C}) > 0\,,\end{array} \tag{3.9d}$$

where $\mathscr{H}^{d-1}$, resp. $\mathscr{L}^{d-1}$, denotes the $(d-1)$-dimensional Hausdorff-, resp. Lebesgue measure.
Assumptions on the bulk stored energy density. For the density W we assume that

$$W : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R} \text{ is continuous, strictly convex, and} \tag{3.10a}$$

$$\exists\, p \in (1,\infty),\ C_1, C_2 > 0 :\quad C_1|e|^p \leq W(e) \leq C_2(|e|^p{+}1) \tag{3.10b}$$

In accordance to the growth/coercivity condition (3.10b), we set

$$\mathscr{U} \;:=\; \big\{\tilde{u} \in W^{1,p}(\Omega\backslash\Gamma_{\rm C}, \mathbb{R}^d),\ \tilde{u} = 0 \text{ on } \Gamma_{\rm D}\big\}\,, \tag{3.11a}$$

$$\mathscr{U}_z \;:=\; \big\{\tilde{u} \in \mathscr{U},\ J_\infty(\llbracket u \rrbracket, z) = 0 \text{ a.e. on } \Gamma_{\rm C}\big\}\,. \tag{3.11b}$$

The latter set of functions will become relevant for the brittle model.
Assumptions on the given data. We prescribe

$$g \in C^1([0,T], W^{1,p}(\Omega,\mathbb{R}^d)) \text{ and } \exists\, \varepsilon > 0\ \forall x \in \Gamma_{\rm C}{+}B_\varepsilon(0) :\ g(x) = 0, \tag{3.12a}$$

$$f \in C^1([0,T]; (W^{1,p}(\Omega\backslash\Gamma_{\rm C},\mathbb{R}^d))^*) \text{ comprising bulk loads in } \Omega\backslash\Gamma_{\rm C} \text{ and surface tractions on } \Gamma_{\rm N}, \tag{3.12b}$$

$$(u_0, z_0) \in \mathscr{U} \times \mathscr{Z}_{\rm SBV} \text{ s.t. (2.2b) holds and } \forall\, k \in \mathbb{N} :\ J_\infty(\llbracket u_0 \rrbracket, z_0) = J_k(\llbracket u_0 \rrbracket, z_0) = 0 \text{ a.e. in } \Gamma_{\rm C}. \tag{3.12c}$$

3.3 Existence of time-discrete local solutions

Proposition 3.1 (Existence of minimizers at each time step)**.** *Let $(\mathscr{U} \times \mathscr{Z}_{\rm SBV}, \mathscr{E}_k, \mathscr{R}_k)$ be given by* (3.1), *such that the assumptions* (3.9)–(3.12) *are satisfied. Keep $k \in \mathbb{N}$ fixed. Then, for every $i \in \{1, \ldots, N(k)\}$, for all $k \in \mathbb{N}$, there exists a pair $(u_i^{N(k)}, z_i^{N(k)}) \in \mathscr{U} \times \mathscr{Z}_{\rm SBV}$ satisfying* (3.5).

Proof. Let $k \in \mathbb{N}$ fixed. Observe that $\mathscr{E}_k(t_i^{N(k)}, \cdot, z_{i-1}^{N(k)}) : \mathscr{U} \to \mathbb{R}$ is coercive and strictly convex on $\mathscr{U}$. Thus, for each $(t_i^{N(k)}, z_{i-1}^{N(k)}) \in \Pi_{N(k)} \times \mathscr{Z}_{\rm SBV}$ there exists a unique minimizer $u_i^{N(k)} \in \mathscr{U}$. Similarly, for $\mathscr{E}_k(t_i^{N(k)}, u_i^{N(k)}, \cdot) : \mathscr{Z}_{\rm SBV} \to \mathbb{R}$ the perimeter gradient together with the L^∞-constraint $I_{[0,1]}$ ensures weak* compactness in $\mathscr{Z}_{\rm SBV}$ and it is lower semicontinuous with respect to the strong L^1-convergence. Hence, for each $(t_i^{N(k)}, u_i^{N(k)}) \in [0,T] \times \mathscr{U}$ the classical arguments from calculus of variations yield the existence of a minimizer $z_i^{N(k)} \in \mathscr{Z}_{\rm SBV}$. □

Proposition 3.2 (Time-discrete local solutions and a-priori bounds)**.** *Let the assumptions of Proposition 3.1 be satisfied. Then, for all $k \in \mathbb{N}$, for any $i \in \{1, \ldots, N(k)\}$ the minimizer $z_i^{N(k)}$, resp. the associated finite-perimeter set $Z_i^{N(k)}$, obtained by* (3.5b) *satisfies the following stability inequality*

$$\forall \tilde{z} \in \mathscr{Z}_{\rm SBV} \textit{ with } \tilde{z} \leq z_i^{N(k)} :\ \ \beta\mathscr{P}(Z_i^{N(k)}, \Gamma_{\rm C}) \leq \beta\mathscr{P}(\widetilde{Z}, \Gamma_{\rm C}) + (\alpha_0{+}\alpha_1)\mathscr{L}^{d-1}(Z_i^{N(k)}\backslash\widetilde{Z})\,, \tag{3.13}$$

where $\tilde{z}$ denotes the characteristic function of the finite-perimeter set $\widetilde{Z}$. Moreover, the piecewise constant interpolants $(\overline{u}_k, \underline{u}_k, \overline{z}_k, \underline{z}_k)$ obtained by (3.5) *and* (3.6) *satisfy the time-discrete version of the local formulation* (3.7)*. Furthermore, there is a constant $C > 0$ such that the following a-priori bounds hold uniformly for all $k \in \mathbb{N}$:*

$$\forall\, t \in [0,T] :\qquad \big\|u_k(t)\big\|_{\mathscr{U}} \leq C \quad \textit{and} \quad \mathscr{P}\big(Z_k(t), \Gamma_{\rm C}\big) \leq C, \tag{3.14a}$$

$$\forall\, t < t_* \in [0,T] :\qquad \mathscr{R}_1\big(z_k(t_*) - z_k(t)\big) \leq C \tag{3.14b}$$

where $Z_k(t)$ is the set underlying the characteristic function $z_k(t)$, and u_k and z_k here stand both for the forward and for the backward piecewise constant interpolants.

Proof. **Ad** (3.7a)**:** The minimality property (3.5a) directly implies for all $t \in [0,T]$, for all $\tilde{u} \in \mathscr{U}$ that $\overline{\mathscr{E}}_k(t, \overline{u}_k(t), \underline{z}_k(t)) \leq \overline{\mathscr{E}}_k(t, \tilde{u}, \underline{z}_k(t))$. Moreover, due to the strict convexity of $\overline{\mathscr{E}}_k(t, \cdot, \underline{z}_k(t)) : \mathscr{U} \to \mathbb{R}$, this is equivalent to $\overline{u}_k(t)$ being a weak solution to the associated Euler-Lagrange inclusion (3.7a).

Ad (3.7b) **&** (3.13)**:** It is $\mathscr{E}_k(t_i^{N(k)}, u_i^{N(k)}, z_i^{N(k)}) + \mathscr{R}_k(z_i^{N(k)} - z_{i-1}^{N(k)}) \leq \mathscr{E}_k(t_i^{N(k)}, u_i^{N(k)}, \tilde{z}) + \mathscr{R}_k(\tilde{z} - z_{i-1}^{N(k)})$ for any $\tilde{z} \in \mathscr{Z}_{\mathrm{SBV}}$ by minimality (3.5b). Since $\mathscr{R}_k$ satisfies the triangle inequality we find $\mathscr{R}_k(\tilde{z} - z_{i-1}^{N(k)}) \leq \mathscr{R}_k(\tilde{z} - z_i^{N(k)}) + \mathscr{R}_k(z_i^{N(k)} - z_{i-1}^{N(k)})$ above on the right-hand side. This yields

$$\mathscr{E}_k(t_i^{N(k)}, u_i^{N(k)}, z_i^{N(k)}) \leq \mathscr{E}_k(t_i^{N(k)}, u_i^{N(k)}, \tilde{z}) + \mathscr{R}_k(\tilde{z} - z_i^{N(k)}). \tag{3.15}$$

By the very definition of the piecewise constant interpolants (3.6) the resulting inequality (3.15) is equivalent to (3.7b). Moreover, (3.13) can be concluded from (3.15) by exploiting that $\widetilde{Z} \subset Z_i^{N(k)}$ together with the monotonicity of $J_k([\![u_i^{N(k)}]\!], \cdot)$.

Ad (3.7c)**:** Testing the minimality property (3.5a) by $u_{i-1}^{N(k)}$ and using that terms solely depending on $z_{i-1}^{N(k)}$ cancel out, yields

$$\begin{aligned} &\mathscr{E}^{\mathrm{bulk}}\big(t_i^{N(k)}, u_i^{N(k)}\big) + \int_{\Gamma_{\mathrm{C}}} I_{\mathrm{C}}\big([\![u_i^{N(k)}]\!]\big) + J_k\big([\![u_i^{N(k)}]\!], z_{i-1}^{N(k)}\big)\,\mathrm{d}\mathscr{H}^{d-1} \\ &\quad\leq \mathscr{E}^{\mathrm{bulk}}\big(t_i^{N(k)}, u_{i-1}^{N(k)}\big) + \int_{\Gamma_{\mathrm{C}}} I_{\mathrm{C}}\big([\![u_{i-1}^{N(k)}]\!]\big) + J_k\big([\![u_{i-1}^{N(k)}]\!], z_{i-1}^{N(k)}\big)\,\mathrm{d}\mathscr{H}^{d-1} \\ &\quad\leq \mathscr{E}^{\mathrm{bulk}}\big(t_{i-1}^{N(k)}, u_{i-1}^{N(k)}\big) + \int_{\Gamma_{\mathrm{C}}} I_{\mathrm{C}}\big([\![u_{i-1}^{N(k)}]\!]\big) + J_k\big([\![u_{i-1}^{N(k)}]\!], z_{i-1}^{N(k)}\big)\,\mathrm{d}\mathscr{H}^{d-1} - \int_{t_{i-1}^{N(k)}}^{t_i^{N(k)}} \big\langle \dot{f}(t), u_{i-1}^{N(k)}\big\rangle\,\mathrm{d}\mathscr{H}^{d-1}. \end{aligned} \tag{3.16}$$

Moreover, testing inequality (3.5b) by $z_{i-1}^{N(k)}$ and exploiting the cancellation of the terms solely depending on $u_i^{N(k)}$, results in the following estimate

$$\begin{aligned} &\int_{\Gamma_{\mathrm{C}}} J_k\big([\![u_i^{N(k)}]\!], z_i^{N(k)}\big) - \frac{\alpha_0}{k} z_i^{N(k)}\,\mathrm{d}\mathscr{H}^{d-1} + \frac{\beta}{k}\mathscr{P}\big(Z_i^{N(k)}, \Gamma_{\mathrm{C}}\big) + \mathscr{R}_k\big(z_i^{N(k)} - z_{i-1}^{N(k)}\big) \\ &\qquad\leq \int_{\Gamma_{\mathrm{C}}} J_k\big([\![u_i^{N(k)}]\!], z_{i-1}^{N(k)}\big) - \frac{\alpha_0}{k} z_{i-1}^{N(k)}\,\mathrm{d}\mathscr{H}^{d-1} + \frac{\beta}{k}\mathscr{P}\big(Z_{i-1}^{N(k)}, \Gamma_{\mathrm{C}}\big). \end{aligned} \tag{3.17}$$

Adding (3.16) and (3.17) yields the discrete energy estimate (3.7c) for $t_1 = t_{i-1}^{N(k)}$ and $t_2 = t_i^{N(k)}$:

$$\begin{aligned} &\mathscr{E}_k\big(t_i^{N(k)}, u_i^{N(k)}, z_i^{N(k)}\big) + \mathscr{R}_k\big(z_i^{N(k)} - z_{i-1}^{N(k)}\big) \\ &\quad= \mathscr{E}^{\mathrm{bulk}}\big(t_i^{N(k)}, u_i^{N(k)}\big) + \int_{\Gamma_{\mathrm{C}}} I_{\mathrm{C}}\big([\![u_i^{N(k)}]\!]\big) + J_k\big([\![u_i^{N(k)}]\!], z_i^{N(k)}\big) - \frac{\alpha_0}{k} z_i^{N(k)}\,\mathrm{d}\mathscr{H}^{d-1} \\ &\qquad\qquad + \frac{\beta}{k}\mathscr{P}\big(Z_i^{N(k)}, \Gamma_{\mathrm{C}}\big) + \mathscr{R}_k\big(z_i^{N(k)} - z_{i-1}^{N(k)}\big) \\ &\quad\leq \mathscr{E}^{\mathrm{bulk}}\big(t_{i-1}^{N(k)}, u_{i-1}^{N(k)}\big) + \int_{\Gamma_{\mathrm{C}}} I_{\mathrm{C}}\big([\![u_{i-1}^{N(k)}]\!]\big) + J_k\big([\![u_{i-1}^{N(k)}]\!], z_{i-1}^{N(k)}\big) - \frac{\alpha_0}{k} z_{i-1}^{N(k)}\,\mathrm{d}\mathscr{H}^{d-1} \\ &\qquad\qquad + \frac{\beta}{k}\mathscr{P}\big(Z_{i-1}^{N(k)}, \Gamma_{\mathrm{C}}\big) - \int_{t_{i-1}^{N(k)}}^{t_i^{N(k)}} \big\langle \dot{f}(t), u_{i-1}^{N(k)}\big\rangle\,\mathrm{d}s \\ &\quad= \mathscr{E}_k\big(t_{i-1}^{N(k)}, u_{i-1}^{N(k)}, z_{i-1}^{N(k)}\big) - \int_{t_{i-1}^{N(k)}}^{t_i^{N(k)}} \big\langle \dot{f}(t), u_{i-1}^{N(k)}\big\rangle\,\mathrm{d}s. \end{aligned} \tag{3.18}$$

Estimate (3.7c) for $t_1 = t_l^{N(k)}$ and $t_2 = t_m^{N(k)}$ for any $t_l^{N(k)} < t_m^{N(k)} \in \Pi_{N(k)}$ is obtained by summing (3.18) up over i and exploiting the very definition (3.6) of the piecewise constant interpolants

Ad (3.14)**:** Observe that there are constants $c_0, c_1 > 0$, such that for all $(t,u,z) \in [0,T] \times \mathscr{U} \times \mathscr{Z}_{\mathrm{SBV}}$ with $\mathscr{E}_k(t,u,z) < \infty$ it holds $|\partial_t \mathscr{E}_k(t,u,z)| \leq c_1(c_0 + \mathscr{E}_k(t,u,z))$. This allows to apply a Gronwall estimate under the time-integral in (3.18). Following the classical arguments for energy inequalities in the rate-independent setting, see e.g. the lines along with [9, Thm. 3.2], results in the estimates

$$c_0 + \mathscr{E}_k(t_m^{N(k)}, u_m^{N(k)}, z_m^{N(k)}) \le \big(c_0 + \mathscr{E}_k(0, u_0, z_0)\big)\exp(c_1 T) \le C, \tag{3.19a}$$

$$\mathscr{R}_k\big(z_m^{N(k)} - z_0\big) \le \big(c_0 + \mathscr{E}_k(0, u_0, z_0)\big)\exp(c_1 T) \le C\,, \tag{3.19b}$$

where the uniform boundedness by a constant $C > 0$ is due to assumption (3.12c).

The first estimate in (3.14a) is standardly obtained from the bound (3.19a), exploiting (3.10b), Korn's and Young's inequality.

The bound (3.19b) is equivalent to $\mathscr{R}_k(\underline{z}_k(t_m^{N(k)}) - z_0) \le \mathscr{R}_k(\overline{z}(t_m^{N(k)}) - z_0) \le C$. On the first glance, this does not provide the uniform $\mathrm{BV}((0,T); L^1(\Gamma_\mathrm{C}))$-bound as $k \to \infty$. Nevertheless, for all $k \in \mathbb{N}$, the functional $\mathscr{R}_k$ ensures the temporal monotonicity of $\underline{z}_k$ and $\overline{z}_k$, which means that they belong to $\mathrm{BV}((0,T); L^1(\Gamma_\mathrm{C}))$. Thus, we have

$$\begin{aligned} \mathscr{R}_1\big(\underline{z}_k(t_m^{N(k)}) - z_0\big) &= \mathrm{Diss}_{\mathscr{R}_1}\big(\underline{z}_k; [0, t_m^{N(k)}]\big) \le \alpha_1 \mathscr{L}^{d-1}(\Gamma_\mathrm{C}),\\ \mathscr{R}_1\big(\overline{z}_k(t_m^{N(k)}) - z_0\big) &= \mathrm{Diss}_{\mathscr{R}_1}\big(\overline{z}_k; [0, t_m^{N(k)}]\big) \le \alpha_1 \mathscr{L}^{d-1}(\Gamma_\mathrm{C}), \end{aligned} \tag{3.20}$$

which is (3.14b). The second bound in (3.14a) is no consequence of (3.19a). It is rather obtained by testing (3.15) with $\tilde z = 0$, the indicator function of $\widetilde{Z} = \emptyset$, by cancelling redundant terms and by multiplying this resulting inequality with k. □

3.4 Fine properties of the solutions

In the following we collect some additional regularity properties of functions being semistable in the sense of (2.2b) or (3.7b). These fine properties were deduced in [35, Section 6] in order to pass from adhesive contact to brittle delamination in thermo-viscoelastic materials. Also here in the rate-independent setting they will play a crucial role when passing from time-discrete to continuous in the weak force balance (3.7a). The mentioned fine properties are essentially based on the semistability with respect to the perimeter-regularization and a volume term, which is in our problem induced by the dissipation distance, cf. (3.13). In particular, due to the definition of the interpolants (3.6), semistability inequality (3.13) also holds true for the forward, resp. backward, interpolants $\overline{Z}_k(t)$, resp. $\underline{Z}_k(t)$, for any $t \in [0,T]$ fixed, since for $t \in [t_{i-1}^{N(k)}, t_i^{N(k)})$ and $\underline{t} \in [t_i^{N(k)}, t_{i+1}^{N(k)})$ it is $\overline{Z}_k(t) = z_i^{N(k)} = \underline{Z}_k(\underline{t})$. Moreover, for a.a. $t \in (0,T)$ they also hold true for the limit of these sequences, since the mentioned minimality property carries over from the time-discrete adhesive problems to the time-continuous brittle limit, cf. Theorem 4.1. Hence the above mentioned property can be viewed as a general static minimality property and therefore we drop the time-dependence of the functions in the presentation below.

Lemma 3.3 (Consequence of semistability)**.** *Let $t \in [0,T]$, $k \in \mathbb{N}\cup\{\infty\}$ be fixed and $\overline{z}_k(t)$ be semistable for $\mathscr{E}_k(t, \overline{u}_k(t), \cdot)$ in the sense of* (3.7b)*. Then the finite-perimeter set $Z_k \in \{\overline{Z}_k(t), \underline{Z}_k(t)\}$ with characteristic function $z_k \in \{\overline{z}_k(t), \underline{z}_k(t)\}$ also satisfies the following inequality for all $\widetilde{Z} \subset Z_k$:*

$$\beta \mathscr{P}(Z_k, \Gamma_\mathrm{C}) \le \beta \mathscr{P}(\widetilde{Z}, \Gamma_\mathrm{C}) + (\alpha_0 + \alpha_1)\mathscr{L}^{d-1}(Z_k \backslash \widetilde{Z})\,. \tag{3.21}$$

It was deduced in [35, Thm. 6.3] that finite-perimeter sets satisfying semistability (3.21) have an additional regularity property, the so-called Property $\mathfrak{a}$; this notion goes back on Campanato, see e.g. [3, 4, 11, 12]:

Proposition 3.4 (Property $\mathfrak{a}$ of semistable sets)**.** *Let the assumptions of Lemma 3.3 hold and let the interface $\Gamma_\mathrm{C} \subset \mathbb{R}^{d-1}$ be convex. Keep $t \in [0,T]$ fixed and assume that the finite-perimeter set $Z_k \subset \Gamma_\mathrm{C}$ satisfies* (3.21)*. Then Z_k has the Property $\mathfrak{a}$, i.e. there are constants $R, \mathfrak{a}(\Gamma_\mathrm{C}) > 0$ depending solely on $\Gamma_\mathrm{C} \subset \mathbb{R}^{d-1}$, on d, and on the parameters $\alpha_0, \alpha_1 > 0$, such that*

$$\forall\, y \in \operatorname{supp} z_k\ \forall\, \rho_\star > 0: \quad \mathscr{L}^{d-1}(Z_k \cap B_{\rho_\star}(y)) \ge \begin{cases} \mathfrak{a}(\Gamma_\mathrm{C})\rho_\star^{d-1} & \text{if } \rho_\star < R,\\ \mathfrak{a}(\Gamma_\mathrm{C})R^{d-1} & \text{if } \rho_\star \ge R. \end{cases} \tag{3.22}$$

Here, $B_{\rho_\star}(y)$ denotes the open ball of radius $\rho_\star$ with center in y and the support of the SBV*-function z_k is defined by* $\operatorname{supp} z_k := \bigcap\{A \subset \mathbb{R}^{d-1};\ A \text{ closed},\ \mathscr{L}^{d-1}(Z_k\backslash A) = 0\}$.

Let us point out that sets with the Property $\mathfrak{a}$, i.e. (3.22), are sometimes also called $(d-1)$-thick, see e.g. [20, 13]. The proof of Proposition 3.4 is carried out by contradiction to (3.21). The lower bound $\mathfrak{a}(\Gamma_{\rm C})\rho_\star^{d-1}$, which holds uniformly for all radii $\rho_\star$, in every point of $\operatorname{supp} z_k$, is obtained with the aid of a uniform relative isoperimetric inequality deduced in [51, Thm. 3.2].

Let now $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{SBV}(\Gamma_{\rm C},\{0,1\})$. By compactness, this implies $\|z_k - z\|_{L^1(\Gamma_{\rm C})} \to 0$, hence $\mathscr{L}^{d-1}(\operatorname{supp} z_k \cap (\Gamma_{\rm C}\backslash \operatorname{supp} z)) \to 0$, and thus one obtains so-called support convergence as a consequence of Property $\mathfrak{a}$.

Proposition 3.5 (Support convergence [35, Thm. 6.1]). *Let $\Gamma_{\rm C}$ be convex. For all $k\in\mathbb{N}$ assume that the finite-perimeter sets $Z_k\subset\Gamma_{\rm C}$ satisfy (3.21) and that the associated characteristic functions $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{SBV}(\Gamma_{\rm C},\{0,1\})$, where z is the characteristic function of a finite-perimeter set $Z\subset\Gamma_{\rm C}$. For all $k\in\mathbb{N}$ set*

$$\rho(k) := \inf\big\{\rho>0:\ \operatorname{supp} z_k \subset \operatorname{supp} z + B_\rho(0)\big\}. \tag{3.23}$$

Then support convergence holds true, i.e.

$$\operatorname{supp} z_k \subset \operatorname{supp} z + B_{\rho(k)}(0) \quad \textit{and} \quad \rho(k)\to 0 \ \textit{as}\ k\to\infty. \tag{3.24}$$

In particular, if $\operatorname{supp} z=\emptyset$, then also $\operatorname{supp} z_k=\emptyset$ for all $k\geq k_0$ from a particular index $k_0\in\mathbb{N}$ on.

Note that (3.24) states one part of Hausdorff convergence of the supports. Indeed the missing part for Hausdorff convergence can be obtained directly from the strong $L^1(\Gamma_{\rm C})$-convergence of the sequence, see [35, Cor. 6.8].

4 Passage from the adhesive time-discrete to the brittle time-continuous problem

In this section we carry out the limit passage $k\to\infty$. By construction, simultaneously, the time step size will tend to zero, the elastic modulus of the adhesive will blow up, while the dissipation functional will tend to the indicator function of the unidirectionality constraint and the regularizing perimeter term will disappear from the model. However, by the a priori bound (3.14b) the BV-property of the solutions of the time-discrete adhesive models is passed over to the time-continous limit. In a similar way we also understand that the information on the driving forces for the crack evolution, encoded in the semistability (3.7b) of the time-discrete adhesive solutions may be carried over to the time-continuous brittle limit; however, a rigorous proof of this property is currently out of reach.

Theorem 4.1 (Convergence of the time-discrete local solutions). *For $k\in\mathbb{N}\cup\{\infty\}$ let $(\mathscr{U}\times\mathscr{Z},\mathscr{E}_k,\mathscr{R}_k)$ be given by (3.1), resp. (2.11), such that the assumptions (3.9)–(3.12) hold true. Let $(\overline{u}_k,\overline{z}_k,\underline{u}_k,\underline{z}_k)_k\subset L^\infty(0,T;\mathscr{U}\times\mathscr{Z}_{\rm SBV}\times\mathscr{U}\times\mathscr{Z}_{\rm SBV})$ satisfy (3.7). Then there is a (not relabeled) subsequence $(\overline{u}_k,\overline{z}_k,\underline{u}_k,\underline{z}_k)_k$ and $(u,z,\underline{u},\underline{z})\in L^\infty(0,T;\mathscr{U}\times\mathscr{Z}_{\rm SBV}\times\mathscr{U}\times\mathscr{Z}_{\rm SBV})$ such that:*

$$\forall\, t\in[0,T]: \qquad \overline{u}_k(t)\rightharpoonup u(t)\ \textit{ in } \mathscr{U} \quad \textit{and} \quad \overline{z}_k(t)\overset{*}{\rightharpoonup} z(t),\ \ \underline{z}_k(t)\overset{*}{\rightharpoonup}\underline{z}(t)\ \textit{ in } \mathscr{Z}_{\rm SBV}, \tag{4.1a}$$

$$\forall\, t\in[0,T]\backslash J: \qquad \underline{u}_k(t)\rightharpoonup u(t)\ \textit{ in } \mathscr{U} \quad \textit{and} \quad z(t)=\underline{z}(t), \tag{4.1b}$$

where J denotes the union of the jump times of $z,\underline{z}\in \mathrm{BV}(0,T;L^1(\Gamma_{\rm C}))$. Moreover, any (u,z) obtained by this way is a local solution (2.2) for the brittle system (1.1).

Proof. In what follows we verify the convergences (4.1); the local solution property of the limit (u,z) will be shown in separate Sections. The reader is referred to Section 4.1 for the proof of the force balance and the energy inequality, and to Section 4.2 for the deduction of the semistability inequality. To obtain the convergence result for the delamination variables in (4.1a) we make use of the uniform bound in (3.14b). This, together with the fact that $\mathscr{R}_1: L^1(\Gamma_{\rm C})\times L^1(\Gamma_{\rm C})\to[0,\infty]$ is a weakly sequentially lower semicontinuous dissipation distance, allows us to apply a generalized version of Helly's selection principle, see e.g. [29, Thm. 6.1], and hence to find a (not relabeled) subsequence as well as limit functions $z,\underline{z}\in \mathrm{BV}([0,T],L^1(\Gamma_{\rm C}))$, such that

$$\forall\, t\in[0,T]: \quad \overline{z}_k(t)\rightharpoonup z(t) \quad \text{and} \quad \underline{z}_k(t)\rightharpoonup\underline{z}(t) \quad \text{ in } \mathscr{Z}_{\rm SBV}. \tag{4.2}$$

For some $t \in [0,T]$ fixed, select a further subsequence such that $\overline{u}_k(t) \rightharpoonup u(t)$ in $\mathscr{U}$. Exploiting the minimality (3.7a) of $\overline{u}_k(t)$ for $\overline{\mathscr{E}}_k(t,\cdot,\underline{z}_k(t))$ as well as cancellations and the weak sequential lower semicontinuity properties, we find

$$0 \le \limsup_{k\to\infty} \left(\overline{\mathscr{E}}_k(t,\tilde{u},\underline{z}_k(t)) - \overline{\mathscr{E}}_k(t,\overline{u}_k(t),\underline{z}_k(t))\right) \le \mathscr{E}(t,\tilde{u},\underline{z}(t)) - \mathscr{E}(t,u(t),\underline{z}(t))\big) \tag{4.3}$$

for all $\tilde{u} \in \mathscr{U}$. In other words, $u(t)$ is the unique minimizer of the strictly convex functional $\mathscr{E}(t,\cdot,\underline{z}(t)) : \mathscr{U} \to [0,\infty]$. Thus, the above selection of a subsequence of $(\overline{u}_k(t))_k$ was unnecessary. This observation holds for all $t \in [0,T]$. Moreover, since $\underline{z}$ and f are measurable with respect to time, we also have that $u : [0,T] \to \mathscr{U}$ is measurable.

Let $J \subset [0,T]$ denote the union of the jump times of $z, \underline{z} \in \mathrm{BV}(0,T;L^1(\Gamma_{\mathrm{C}}))$. By the properties of BV-functions, J is at most countable. Consider $t \in [0,T]\backslash J$ and a sequence $t_{l_k}^{N(k)} \to t$ as $k \to \infty$ with $t_{l_k}^{N(k)} \in \Pi_{N(k)}$ for all $k \in \mathbb{N}$. With $z_{l_k}^{N(k)}$ obtained by (3.5b), it holds that $z_{l_k}^{N(k)} = \underline{z}_k(t_{l_k}^1) = \overline{z}_k(t_{l_k}^1)$ for all $t_{l_k-1}^{N(k)} \le t_{l_k}^1 \le t_{l_k}^2 \le t_{l_k}^{N(k)}$. For $t_{l_k}^1 \to t$ and $t_{l_k}^2 \to t$ as $k \to \infty$ we thus conclude $z(t) = \underline{z}(t)$ for all $t \in [0,T]\backslash J$. Since $u_{l_k}^{N(k)}$ is the unique minimizer of $\mathscr{E}_k(t_{l_k}^{N(k)}, \cdot, z_{l_k-1}^{N(k)})$, we find the convergence result in (4.1b) with similar arguments. □

4.1 Limit passage in the mechanical force balance via recovery sequences and energy inequality

In the following we pass to the limit $k \to \infty$ in the time-discrete versions of the mechanical force balance (3.7a) and the energy inequality (3.7c). We first carry out the limit $k \to \infty$ in (3.7a), which reads in detail for every $k \in \mathbb{N}$:

$$\begin{aligned}&\forall\, t \in [0,T]: \qquad \overline{u}_k(t) \in \mathscr{U},\ [\![\overline{u}_k(t)]\!]\cdot \mathrm{n} \ge 0 \quad \text{and for all } v \in \mathscr{U} \text{ with } [\![v]\!]\cdot\mathrm{n} \ge 0:\\ &\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathrm{D}_e W(e(\overline{u}_k(t){+}\overline{g}_k(t))){:}(v{-}\overline{u}_k(t))\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} k\underline{z}_k(t)[\![\overline{u}_k(t)]\!]\cdot[\![v{-}\overline{u}_k(t)]\!]\,\mathrm{d}\mathscr{H}^{d-1} \ge \langle \overline{f}_k(t), v{-}\overline{u}_k(t)\rangle .\end{aligned} \tag{4.4}$$

In contrast, owing to the second statement of (4.1b) the weak mechanical force balance for the brittle limit system and the pair $(u(t), z(t))$ will be valid for all $t \in [0,T]\backslash J$ only, i.e.,

$$\forall\, t \in [0,T]\backslash J: \quad u(t) \in \mathscr{U}_{z(t)},\ [\![u(t)]\!]\cdot\mathrm{n} \ge 0 \quad \text{and } \forall\, v \in \mathscr{U}_{z(t)} \text{ with } [\![v]\!]\cdot\mathrm{n} \ge 0: \tag{4.5a}$$

$$\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathrm{D}_e W(e(u(t){+}g(t))){:}(v{-}u(t))\,\mathrm{d}x \ge \langle f(t), v{-}u(t)\rangle . \tag{4.5b}$$

In what follows, however, we will show that the pair $(u(t), \underline{z}(t))$ satisfies the weak mechanical force balance for the brittle limit system even for every $t \in [0,T]$, i.e.

$$\forall\, t \in [0,T]: \quad u(t) \in \mathscr{U}_{\underline{z}(t)},\ [\![u(t)]\!]\cdot\mathrm{n} \ge 0 \quad \text{and } \forall\, v \in \mathscr{U}_{\underline{z}(t)} \text{ with } [\![v]\!]\cdot\mathrm{n} \ge 0: \quad \text{(4.5b) is true.} \tag{4.6}$$

In view of the second relation in (4.1b), this implies (4.5). For testing (4.4) with functions $v \in \mathscr{U}_{\underline{z}(t)}$ suited for the brittle limit (4.6), we would need to have

$$\forall\, t \in [0,T]\ \forall\, v \in \mathscr{U}_{\underline{z}(t)}: \qquad \int_{\Gamma_{\mathrm{C}}} k\underline{z}_k(t)[\![\overline{u}_k(t)]\!]\cdot[\![v]\!]\,\mathrm{d}\mathscr{H}^{d-1} \overset{\text{desired!}}{\longrightarrow} 0 \quad \text{as} \quad k \to \infty \tag{4.7}$$

with $\overline{u}_k(t)$ being a solution of (4.4) and $\underline{z}_k(t)$ given by (3.5b), converging suitably to $(u(t), \underline{z}(t))$. However, we only have that $\int_{\Gamma_{\mathrm{C}}} k\underline{z}_k(t)|[\![\overline{u}_k(t)]\!]|^2\,\mathrm{d}\mathscr{H}^{d-1} \le C$ by (3.7b), whereas $\int_{\Gamma_{\mathrm{C}}} \underline{z}_k(t)|[\![v]\!]|^2\,\mathrm{d}\mathscr{H}^{d-1} \to 0$ only without the prefactor k. Therefore the integral in (4.7) might even blow up as $k \to \infty$. This is why we have to avoid dealing directly with (4.7), i.e. passing to the limit in (4.4) with fixed test functions $v \in \mathscr{U}_{\underline{z}(t)}$. Instead, we are going to device a recovery sequence $(v_k)_k \subset \mathscr{U}$ for the test functions $v \in \mathscr{U}_{\underline{z}(t)}$, which satisfies

$$\forall\, k \in \mathbb{N},\ \forall\, t \in [0,T]: \qquad \mathscr{J}_k(v_k, \underline{z}_k(t)) := \int_{\Gamma_{\mathrm{C}}} \frac{k}{2}\,\underline{z}_k(t)\big|[\![v_k]\!]\big|^2\,\mathrm{d}\mathscr{H}^{d-1} = 0. \tag{4.8}$$

In addition to (4.8), the sequence $(v_k)_k$ has to display a convergence suited to recover the bulk terms. In other words, for every $k \in \mathbb{N}$, v has to be modified in such a way that the support of $[\![v_k]\!]$ matches the null set of $\underline{z}_k(t)$ and, as $k \to \infty$, $v_k \to v$ even strongly in $\mathscr{U}$. For obvious reasons this convergence first of all necessitates that the supports of $\underline{z}_k(t)$ converge for all $t \in [0,T]$ to the support of $\underline{z}(t)$ in the sense of support convergence (3.24); this feature is ensured by the second statement of Lemma 3.3 combined with Proposition 3.5. To construct the recovery sequence $(v_k)_k$ with the desired strong convergence properties we proceed as follows: Any function $v \in \mathscr{U}_{\underline{z}(t)}$ can be written in terms of its symmetric v_{sym} and its antisymmetric part v_{anti}; rewriting any $x \in \Omega$ as $x = (x_1, y)$ for $y = (x_2, \ldots, x_d) \in \mathbb{R}^{d-2}$, this is

$$\begin{aligned} v(x_1,y) &:= v_{\text{sym}}(x_1,y) + v_{\text{anti}}(x_1,y) \in \mathscr{U}_{\underline{z}(t)}, \quad \text{with} \\ v_{\text{sym}}(x_1,y) &:= \tfrac{1}{2}(v(x_1,y) + v(-x_1,y)) \quad \in W^{1,p}(\Omega) \quad \text{and} \\ v_{\text{anti}}(x_1,y) &:= \tfrac{1}{2}(v(x_1,y) - v(-x_1,y)) \quad \in W^{1,p}((\Omega\backslash\Gamma_{\text{C}}) \cup \operatorname{supp} z(t)), \end{aligned} \tag{4.9}$$

where we assume here and in what follows that the domain Ω is oriented in a coorinate system such that the origin is contained in Γ_{C} and the normal n to Γ_{C} points in x_1-direction. Let us once more stress that $v_{\text{anti}} = 0$ on $\operatorname{supp} \underline{z}(t)$. Hence, in view of (3.9a), $v_{\text{anti}}|_{\Omega_\pm} =: v^\pm_{\text{anti}} \in W^{1,p}(\Omega_\pm, \mathbb{R}^d)$ satisfies homogeneous Dirichlet conditions on the closed set $M := \operatorname{supp} \underline{z}(t) \subset \Gamma_{\text{C}}$, i.e. $v \in W^{1,p}_M(\Omega_\pm, \mathbb{R}^d)$. This observation is essential, because it enables us to apply a Hardy's inequality, stating the existence of a constant $C_M > 0$ such that for all $v \in W^{1,p}_M(\Omega_\pm, \mathbb{R}^d)$:

$$\big\|v/d_M\big\|_{L^p(\Omega_\pm,\mathbb{R}^d)} \le C_M \big\|\nabla v\big\|_{L^p(\Omega_\pm,\mathbb{R}^{d\times d})}\,, \tag{4.10}$$

where $d_M(x) := \min_{\hat{x}\in M} |x - \hat{x}|$ for all $x \in \overline{\Omega}_\pm$. Such type of Hardy's inequality is the crucial tool allowing it verify the strong $W^{1,p}(\Omega_\pm, \mathbb{R}^d)$-convergence of the recovery sequence under construction. But it has to be stressed that, to our knowledge, the above Hardy's inequality for closed sets M of arbitrarily low regularity was proved by now only under the additional assumption of $p > 0$, see [22, p. 190]. This is essentially the reason for the assumption $p > d$ in the works [28, 35]. So much the better that very recently Hardy's inequality (4.10) was proved in [13, Thm. 3.4] under much weaker integrability assumptions on the displacements, with only slightly strengthened regularity assumptions on the closed set M. The result [13, Thm. 3.4] implies the following

Proposition 4.2 (Hardy's inequality for $p \in (1,\infty)$)**.** *Let $\Omega_\pm$ satisfy* (3.9a)*. Suppose that the closed set $M \subset \partial\Omega_\pm$ has Property $\mathfrak{a}$, i.e.* (3.22) *holds, and $W^{1,p}_M(\Omega_\pm, \mathbb{R}^d) := \{\tilde{u} \in W^{1,p}(\Omega_\pm, \mathbb{R}^d),\ \tilde{u} = 0 \text{ on } M\}$, i.e. Poincaré's inequality holds on $W^{1,p}_M(\Omega_\pm, \mathbb{R}^d)$. Then, for all $p \in (1,\infty)$ there exists a constant $C_M = C(M,p)$ such that Hardy's inequality* (4.10) *is fulfilled in $W^{1,p}_M(\Omega_\pm, \mathbb{R}^d)$.*

Even more inspiring that the additional regularity imposed on M in Proposition 4.2 for Hardy's inequality to hold, is Property $\mathfrak{a}$; exactly the fine regularity property deduced in Proposition 3.4 for finite-perimeter sets being semistable in the sense of (3.21)! Thus, due to these recent results, [13, Thm. 3.4] in combination with [35, Thm. 6.1], we are now able to perform the limit passage from adhesive to brittle without an additional $W^{1,p}$-regularization, where $p > d$, for the displacements. The results presented in what follows can also be applied in [35], so that the assumption $p > d$ becomes unnecessary also there.

Observe though, that for $p > d$ it is $[\![v]\!] \in C(\overline{\Gamma}_{\text{C}}, \mathbb{R}^d)$ for any $v \in W^{1,p}(\Omega\backslash\Gamma_{\text{C}}, \mathbb{R}^d)$. Thus, if $z|[\![v]\!]| = 0$ a.e. on Γ_{C} for a given function $z \in L^\infty(\Gamma_{\text{C}})$, then in particular $[\![v]\!] \equiv 0$ on $\operatorname{supp} z$. This conclusion is no longer valid for $p \le d$ and therefore the above property is directly incorporated in the definition of $\mathscr{U}_z$ in (3.11b). This is essential, because we will exploit the support convergence (3.24) for the construction of the recovery sequence. Using the *closed* set $\operatorname{supp} z$ is important, as can be seen from Example 4.3 below.

Example 4.3. Let $Z := \Gamma_{\text{C}} \cap \mathbb{Q}^{d-1}$ and z its characteristic function. Thus, $\operatorname{supp} z = \Gamma_{\text{C}}$ and hence, $\Gamma_{\text{C}} \subset Z + B_\rho(0) \not\to Z$ as $\rho \to 0$, since Z is dense and not closed in Γ_{C}. If we dealt with $v \in \mathscr{U}$ with $z|[\![v]\!]| = 0$ a.e. on Γ_{C} and if we constructed a recovery sequence $(v_\rho)_\rho$ such that $v_\rho|_{\Omega_\pm} = 0$ in $Z + B_\rho$, then $v_\rho \not\to v$ in $W^{1,p}(\Omega_\pm, \mathbb{R}^d)$ as $\rho \to 0$. But for the strengthened assumption $[\![v]\!] = 0$ $\mathscr{L}^{d-1}$-a.e. on the *closed* set $\operatorname{supp} z$, a construction $(v_\rho)_\rho$ with $v_\rho \to v$ is possible.

With the above observations in mind, we may now invoce the recovery sequence taylored in [28, Cor. 2], see also [35, Prop. 5.2]. The proof of Proposition 4.4 is the same as the one of [28, Cor. 2], since the only point where the assumption $p > d$ was used for [28, Cor. 2] is for the validity of Hardy's inequality [22, p. 190], which can now be replaced by the one for $p \in (1,\infty)$ stated in Proposition 4.2.

Proposition 4.4 (Recovery sequence for the test functions, [28]). *Keep $t \in [0,T]$ fixed. Consider $\underline{z}(t) \in L^\infty(\Gamma_\mathrm{C})$ and let $M(t) := \operatorname{supp} \underline{z}(t)$. Let $d_M(t,x) := \min_{\hat{x}\in M(t)} |x-\hat{x}|$ for all $x \in \overline{\Omega}_\pm$. For $p \in (1,\infty)$, let $v \in W^{1,p}(\Omega_- \cup M(t) \cup \Omega_+, \mathbb{R}^d)$, such that $v = 0$ on Γ_D in the trace sense. With $\xi^M_\rho(t,x) := \min\{\frac{1}{\rho}(d_M(t,x)-\rho)^+, 1\}$ set*

$$v^\rho(x_1,y) := v_{\mathrm{sym}}(x_1,y) + \xi^M_\rho(x_1,y)\, v_{\mathrm{anti}}(x_1,y)\,, \tag{4.11}$$

with v_{sym} and v_{anti} as in (4.9)*. Then, the following statements hold:*

(i) $v^\rho \to v$ strongly in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ as $\rho \to 0$,

(ii) $v \in W^{1,p}(\Omega_- \cup M(t) \cup \Omega_+, \mathbb{R}^d) \Rightarrow v^\rho \in W^{1,p}(\Omega_- \cup (\hat{M}(t){+}B_{\rho(t)}(0)) \cup \Omega_+, \mathbb{R}^d)$,

(iii) $[\![v]\!] \cdot \mathrm{n} \geq 0$ on $\Gamma_\mathrm{C} \Rightarrow [\![v^\rho]\!] \cdot \mathrm{n} \geq 0$ on Γ_C.

The above construction (4.11) is used to prove the Mosco-convergence of the following functionals:

$$\mathscr{J}_k(\cdot,\underline{z}_k(t)) : \mathscr{U} \to [0,\infty), \qquad \mathscr{J}_k(v,\underline{z}_k(t)) := \int_{\Gamma_\mathrm{C}} \frac{k}{2}\underline{z}_k(t)|[\![v]\!]|^2\,\mathrm{d}\mathscr{H}^{d-1}\,, \tag{4.12a}$$

$$\mathscr{J}_\infty(\cdot,\underline{z}(t)) : \mathscr{U} \to [0,\infty], \qquad \mathscr{J}_\infty(v,\underline{z}(t)) := \int_{\Gamma_\mathrm{C}} J_\infty(v,\underline{z}(t))\,\mathrm{d}\mathscr{H}^{d-1}, \tag{4.12b}$$

$$\mathscr{F}_k(\cdot,\underline{z}_k(t)) : \mathscr{U} \to [0,\infty), \qquad \mathscr{F}_k(v,\underline{z}_k(t)) := \int_{\Gamma_\mathrm{C}} I_\mathrm{C}([\![v]\!]) + \frac{k}{2}\underline{z}_k(t)|[\![v]\!]|^2\,\mathrm{d}\mathscr{H}^{d-1}\,, \tag{4.12c}$$

$$\mathscr{F}_\infty(\cdot,\underline{z}(t)) : \mathscr{U} \to [0,\infty], \qquad \mathscr{F}_\infty(v,\underline{z}(t)) := \int_{\Gamma_\mathrm{C}} I_\mathrm{C}([\![v]\!]) + J_\infty(v,\operatorname{supp}\underline{z}(t))\,\mathrm{d}\mathscr{H}^{d-1}, \tag{4.12d}$$

$$\widetilde{\mathscr{F}}_k(\cdot,\underline{z}_k(t)) : \mathscr{U} \to [0,\infty], \qquad \widetilde{\mathscr{F}}_k(v,\underline{z}_k(t)) := \int_{\Omega\backslash\Gamma_\mathrm{C}} W(e(v(t){+}\overline{g}_k(t)))\,\mathrm{d}\mathscr{H}^{d-1} {+} \mathscr{F}_k(v(t),\underline{z}_k(t))\,, \tag{4.12e}$$

$$\widetilde{\mathscr{F}}_\infty(\cdot,\underline{z}(t)) : \mathscr{U} \to [0,\infty], \qquad \widetilde{\mathscr{F}}_\infty(v,\underline{z}(t)) := \int_{\Omega\backslash\Gamma_\mathrm{C}} W(e(v(t){+}g(t)))\,\mathrm{d}\mathscr{H}^{d-1} {+} \mathscr{F}_\infty(v(t),\underline{z}(t))\,. \tag{4.12f}$$

Hereby, the Mosco-convergence of functionals means their Γ-convergence, such that the recovery sequence for the lim sup-estimate converges even strongly in $\mathscr{U}$, see [30] or e.g. also [1, Sect. 3.3, p. 295].

Proposition 4.5 (Mosco convergence of the functionals (4.12)). *Assume* (3.9) *and let $t \in [0,T]$. Then:*

(1) For $(\underline{z}_k(t))_k \subset \mathrm{SBV}(\Gamma_\mathrm{C};\{0,1\})$ with $\underline{z}_k(t) \overset{*}{\rightharpoonup} \underline{z}(t)$ in $\mathrm{SBV}(\Gamma_\mathrm{C};\{0,1\})$ as $k\to\infty$, the sequence of the functionals $\mathscr{J}_k(\cdot,\underline{z}_k(t))$ from (4.12a) *Mosco-converge in $\mathscr{U}$ as $k\to\infty$ to $\mathscr{J}_\infty(\cdot,\underline{z}(t))$ from* (4.12b)*.*

(2) For $(\underline{z}_k(t))_k$ as in (1), also the sequence of $\mathscr{F}_k(\cdot,\underline{z}_k(t))$ from (4.12c) *Mosco-converges in $\mathscr{U}$ as $k\to\infty$ to $\mathscr{F}_\infty(\cdot,\underline{z}(t))$ from* (4.12d)*.*

(3) For $(\underline{z}_k(t))_k \subset \mathrm{SBV}(\Gamma_\mathrm{C};\{0,1\})$ such that $\underline{z}_k(t) \overset{*}{\rightharpoonup} \underline{z}(t)$ in $\mathrm{SBV}(\Gamma_\mathrm{C};\{0,1\})$, the sequence of the functional $\widetilde{\mathscr{F}}_k(\cdot,\underline{z}_k(t))$ from (4.12e) *Mosco-converges in $\mathscr{U}$ as $k\to\infty$ to $\widetilde{\mathscr{F}}_\infty(\cdot,\underline{z}(t))$ from* (4.12f)*.*

Proof. For any of the above sequences of functionals the lim inf inequality for a sequence $v_k \rightharpoonup v$ in $\mathscr{U}$ is a direct consequence of their positivity and convexity, hence lower semicontinuity, combined with the fact that $\mathscr{J}_\infty(v,\underline{z}(t)) < \infty$ means that $\mathscr{J}_\infty(v,\underline{z}(t)) = 0$. The respective lim sup condition is proved by associating with each $v \in \mathscr{U}_{\underline{z}(t)}$ the recovery sequence $v_k(x_1,y) := v_{\mathrm{sym}}(x_1,y) + \xi^M_{\rho(k)} v_{\mathrm{anti}}(x_1,y)$ if $\operatorname{supp}\underline{z}(t) \neq \emptyset$ and $\operatorname{supp}\underline{z}_k(t) \not\subset \operatorname{supp}\underline{z}(t)$, and $v_k(x_1,y) := v(x_1,y)$ if $\operatorname{supp}\underline{z}_k(t) \subset \operatorname{supp}\underline{z}(t)$ or if $\operatorname{supp}\underline{z}(t) = \emptyset$. Here, the radius $\rho(k)$ is given by (3.23) and Proposition 3.5 guarantees that $\rho(k) \to 0$ as $k\to\infty$ by the semistability of $\underline{z}_k(t)$ according to Lemma 3.3 and the convergence $\underline{z}_k(t) \overset{*}{\rightharpoonup} \underline{z}(t)$ in $\mathscr{Z}_{\mathrm{SBV}}$, cf. (4.1a). Thus, Proposition 4.4 is applicable and provides $\mathscr{J}_k(v_k,\underline{z}_k(t)) = 0$ for all k as well as the strong convergence $v_k \to v$ in $\mathscr{U}$. For more details about the proof we refer to [35, Prop. 5.3]. □

To pass to the limit in the weak momentum balance (4.4) we equivalently resort to the corresponding subdifferential inclusion:

$$\forall\, t \in [0,T],\ v \in \mathscr{U}: \quad \int_{\Omega\backslash\Gamma_{\rm C}} \mathrm{D}_e W(e(\overline{u}_k(t)+\overline{g}_k(t)){:}e(v)\,\mathrm{d}x + \big\langle \overline{\lambda}_k(t), v\big\rangle = \big\langle \overline{f}_k(t), v\big\rangle \\ \text{with } \overline{\lambda}_k \in \partial_u \mathscr{F}_k(\overline{u}_k(t), \underline{z}_k(t))\,, \tag{4.13}$$

where $\partial_u\mathscr{F}_k(\overline{u}_k(t),\underline{z}_k(t)) : \mathscr{U} \rightrightarrows \mathscr{U}^*$ denotes the convex-analysis subdifferential of the functional $\mathscr{F}_k$ from (4.12c). Since $\mathscr{J}_k(\cdot,\underline{z}_k) : \mathscr{U} \to (0,\infty)$ is Fréchet-differentiable, the sum rule holds and hence $\partial_u\mathscr{F}_k(\overline{u}_k(t),\underline{z}_k(t)) = \partial\mathscr{J}_{\rm C}(\overline{u}_k(t))+\mathrm{D}_u\mathscr{J}_k(\overline{u}_k(t),\underline{z}_k(t))$. Again $\partial\mathscr{J}_{\rm C}$ is the convex-analysis subdifferential of the functional $\mathscr{J}_{\rm C} : \mathscr{U} \to [0,\infty]$ defined by $\mathscr{J}_{\rm C}(u) := \int_{\Gamma_{\rm C}} I_{\rm C}([\![u]\!])\,\mathrm{d}\mathscr{H}^{d-1}$. When passing to the limit in (4.13), we of course set $v = v_k$ from Proposition (4.4), but first of all we have to identify the limits of the nonlinear term $\mathrm{D}_eW(e(\overline{u}_k(t)+\overline{g}_k(t)))$ and the elements $\overline{\lambda}_k(t) \in \partial_u\mathscr{F}_k(\overline{u}_k(t),\underline{z}_k(t))$. In view of the p-growth assumption (3.10b) and the uniform bound (3.14a) we conclude that also $(\mathrm{D}_eW(e(\overline{u}_k(t)+\overline{g}_k(t))))_k$ is uniformly bounded in $L^{p'}(\Omega\backslash\Gamma_{\rm C},\mathbb{R}^{d\times d})$ and thus, the existence of a (not relabeled) subsequence and an element $\mu(t) \in L^{p'}(\Omega\backslash\Gamma_{\rm C},\mathbb{R}^d)$ such that

$$\mathrm{D}_eW(e(\overline{u}_k(t)+\,\overline{g}_k(t))) \rightharpoonup \mu(t) \quad \text{ in }\ L^{p'}(\Omega\backslash\Gamma_{\rm C},\mathbb{R}^d). \tag{4.14}$$

Recall that $\overline{g}_k$, resp. $\overline{f}_k$, denotes the piecewise constant interpolant of $g \in C^1([0,T],W^{1,p}(\Omega,\mathbb{R}^d))$, resp. $f \in C^1([0,T];\mathscr{U}^*)$. Hence, $\overline{g}_k$, resp. $\overline{f}_k$, is uniformly bounded in $C^1([0,T],W^{1,p}(\Omega,\mathbb{R}^d))$, resp. $C^1([0,T];\mathscr{U}^*)$, and in particular

$$\forall\, t\in[0,T]: \quad \overline{g}_k(t) \to g(t) \quad \text{ strongly in }\ W^{1,p}(\Omega,\mathbb{R}^d) \ \text{ and }\ \overline{f}_k(t) \to f(t) \quad \text{ strongly in }\ \mathscr{U}^*, \tag{4.15}$$

where $\mathscr{U}^*$ is the dual of $\mathscr{U} := \{\tilde{u} \in W^{1,p}(\Omega\backslash\Gamma_{\rm C},\mathbb{R}^d),\ \tilde{u} = 0 \text{ on } \Gamma_{\rm D}\}$. Therefore, a comparison argument in (4.13) provides that also $\overline{\lambda}_k(t)$ is uniformly bounded in $\mathscr{U}^*$ and hence a further subsequence and an element $\lambda(t) \in \mathscr{U}^*$ such that

$$\overline{\lambda}_k(t) \rightharpoonup \lambda(t) \quad \text{ in }\ \mathscr{U}^*. \tag{4.16}$$

By choosing a further subsequence we thus may verify that the limit $(u(t),\underline{z}(t),\mu(t),\lambda(t))$ satisfies

$$\forall\, t{\in}[0,T]\ \ \forall\, v{\in}\mathscr{U}: \qquad \int_{\Omega\backslash\Gamma_{\rm C}} \mu(t){:}e(v)\,\mathrm{d}x + \big\langle \lambda(t), v\big\rangle = \big\langle f(t), v\big\rangle. \tag{4.17}$$

In order to conclude that (4.17) coincides with the time-continuous brittle force balance (2.2a) we have to identify that

$$\forall\, t\in[0,T]: \quad \mu(t) = \mathrm{D}_eW(e(u(t)+g(t))) \quad \text{and} \quad \lambda(t) \in \partial_u\mathscr{F}_\infty(u(t),z(t)). \tag{4.18}$$

For this, we will apply well-known results from maximal monotone operator theory on the operators induced by convex potentials $\widetilde{\mathscr{F}}_k(\cdot,\underline{z}_k(t)) : \mathscr{U} \to [0,\infty]$. First of all, we conclude that the Mosco-convergence of $(\widetilde{\mathscr{F}}_k(\cdot,\underline{z}_k(t)))_k$ in $\mathscr{U}$ to $\widetilde{\mathscr{F}}_\infty(\cdot,\underline{z}(t))$, stated in Proposition 4.5, yields that the maximal monotone subdifferentials

$$\partial_u\widetilde{\mathscr{F}}_k(\cdot,\underline{z}_k(t)) : \mathscr{U} \rightrightarrows \mathscr{U}^* \quad \text{G-converge to }\ \partial_u\widetilde{\mathscr{F}}_\infty(\cdot,z) : \mathscr{U} \rightrightarrows \mathscr{U}^*. \tag{4.19}$$

Furthermore, we have

$$(\overline{u}_k(t),u_k^*(t)) \in \mathscr{U}\times\mathscr{U}^* \text{ with } \overline{u}_k(t) \rightharpoonup u(t) \text{ in } \mathscr{U} \text{ and } u_k^*(t) \rightharpoonup u^*(t) \text{ in } \mathscr{U}^* \tag{4.20}$$

for $u_k^*(t){\in}\mathscr{U}^*$ given by $\langle u_k^*(t),v\rangle_{\mathscr{U}} := \int_{\Omega\backslash\Gamma_{\rm C}} \mathrm{D}_eW(e(\overline{u}_k(t){+}\overline{g}_k(t))){:}e(v(t))\,\mathrm{d}x + \langle\overline{\lambda}_k(t),v(t)\rangle$ and $u^*(t) \in \mathscr{U}^*$ by $\langle u^*(t),v\rangle_{\mathscr{U}} := \int_{\Omega\backslash\Gamma_{\rm C}} \mu(t){:}e(v(t))\,\mathrm{d}x + \langle\lambda(t),v(t)\rangle$. Furthermore, testing (4.13) by $\overline{u}_k(t)$ yields

$$\lim_{k\to\infty} \int_{\Omega\backslash\Gamma_{\rm C}} \mathrm{D}_eW(e(\overline{u}_k(t)+\overline{g}_k(t)){:}e(\overline{u}_k(t))\,\mathrm{d}x + \langle\overline{\lambda}_k(t),\overline{u}_k(t)\rangle \\ = \lim_{k\to\infty}\langle\overline{f}_k(t),\overline{u}_k(t)\rangle = \langle f(t),u(t)\rangle = \int_{\Omega\backslash\Gamma_{\rm C}} \mu(t){:}e(u(t))\,\mathrm{d}x + \langle\lambda(t),u(t)\rangle \tag{4.21}$$

by convergences (4.15) and (4.1a), and by relation (4.17). Thus, relation (4.21) states in particular that $\limsup_{k\to\infty}\langle u_k^*(t),\overline{u}_k(t)\rangle_{\mathscr{U}} \le \langle u^*(t),u(t)\rangle_{\mathscr{U}}$. Hence, observations (4.19), (4.20) and (4.21) allow us to conclude that $u^*(t)\in\partial_u\widetilde{\mathscr{F}}(u(t),\underline{z}(t))$ via e.g. [1, p. 361, Prop. 3.59] and consequently $\overline{u}_k(t)\to u(t)$ even strongly in $W^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d)$ for any $t\in[0,T]$.

Observe that the piecewise continuous interpolants are defined for every $t\in[0,T]$ and satisfy semistability (3.7b) for every $t\in[0,T]$. Hence also Property $\mathfrak{a}$ and support convergence, see Propositions 3.4 and 3.5, and consequently, the construction of the recovery sequence (4.11) as well as the results on Mosco-convergence in Proposition 4.5, Items (1) & (2), are valid for every $t\in[0,T]$. Thus, for any $t\in[0,T]$ fixed, testing the time-discrete variational inequality (4.4) with the recovery sequence $(v_k)_k$ constructed via (4.11) for the time-continuous limit solution $u(t)$, rearranging the terms in (4.4) and exploiting that $\underline{z}_k(t)[\![v_k]\!]=0$ a.e. on Γ_{C} together with the fact that $\overline{z}_k(t)\le\underline{z}_k(t)$, cf. (3.6), yields

$$\begin{aligned}0&\le\int_{\Gamma_{\mathrm{C}}}k\overline{z}_k(t)\big|[\![\overline{u}_k(t)]\!]\big|^2\mathrm{d}\mathscr{H}^{d-1}\le\int_{\Gamma_{\mathrm{C}}}k\underline{z}_k(t)\big|[\![\overline{u}_k(t)]\!]\big|^2\mathrm{d}\mathscr{H}^{d-1}\\&\le\int_{\Omega\backslash\Gamma_{\mathrm{C}}}\mathrm{D}_eW(e(\overline{u}_k(t)+\overline{g}_k(t))):e(v_k-\overline{u}_k(t))\,\mathrm{d}x-\big\langle\overline{f}_k(t),v_k-\overline{u}_k(t)\big\rangle\overset{k\to\infty}{\longrightarrow}0,\end{aligned}\tag{4.22}$$

since both $v_k\to u(t)$ and $\overline{u}_k(t)\to u(t)$ strongly in $W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}})$ for all $t\in[0,T]$.

The above conclusions can be summarized in the following result.

Proposition 4.6 (Passage to the limit in the weak force balance (3.7a))**.** *Let the assumptions of Theorem 4.1 be satisfied and assume convergences* (4.1)*. Then, for all $t\in[0,T]$ the limit $(u,\underline{z}):[0,T]\to\mathscr{U}\times\mathscr{Z}$ satisfies* (4.6)*, whereas the limit $(u,z):[0,T]\to\mathscr{U}\times\mathscr{Z}$ satisfies the brittle force balance* (2.2a) *for all $t\in[0,T]\backslash J$. In addition, for all $t\in[0,T]$, there holds*

$$\overline{u}_k(t)\to u(t)\ \textit{strongly in}\ W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d)\quad\textit{and}\quad 0\le\mathscr{J}_k(\overline{u}_k(t),\overline{z}_k(t))\le\mathscr{J}_k(\overline{u}_k(t),\underline{z}_k(t))\to0.\tag{4.23}$$

Let now $0\le t_1<t_2\le T$ and in (3.7c) consider sequences $(t_l^{N(k)})_k,(t_m^{N(k)})_k\subset\Pi_{N(k)}$ with $t_l^{N(k)}\to t_1$ and $t_m^{N(k)}\to t_2$ as $k\to\infty$. By exploiting convergences (4.1), the strong convergences (4.23) and the uniform boundedness of both $\mathscr{P}(Z_k(t),\Gamma_{\mathrm{C}})$ and $\mathscr{R}_1(\overline{z}_k(t)-z(0))$ as well as by the temporal continuity of f by (3.12a), we may conclude the convergence of the energy inequalities (3.7c) to (2.2c).

Corollary 4.7 (Limit passage in the energy inequality (3.7c))**.** *Let the assumptions of Proposition 4.6 be satisfied, such that* (4.23) *is fulfilled. Then, the limit $(u,z):[0,T]\to\mathscr{U}\times\mathscr{Z}$ satisfies the brittle energy inequality* (2.2c) *for all $t\in[0,T]$.*

4.2 Closedness of semistable sets

It remains to show that (u,z) satisfies the semistability inequality (2.2b) for a.a. $t\in(0,T)$. As developed in [27] this can be done by verifying the so-called *mutual-recovery-sequence condition* (again we drop indicating the time-dependence of the functions): *Let $(z_k)_k\subset\mathscr{Z}_{\mathrm{SBV}}$, such that for all $k\in\mathbb{N}$, z_k satisfies* (3.7b) *with respect to $\mathscr{E}_k(t,u_k,\cdot)$ and $z_k\overset{*}{\rightharpoonup}z$ in $\mathscr{Z}_{\mathrm{SBV}}$. Then, for every $\tilde{z}\in\mathscr{Z}=L^\infty(\Gamma_{\mathrm{C}})$ there exists a sequence $(\tilde{z}_k)_k$ with $\tilde{z}_k\overset{*}{\rightharpoonup}\tilde{z}$ in $L^\infty(\Gamma_{\mathrm{C}})$, such that:*

$$0\le\limsup_{k\to\infty}\big(\mathscr{E}_k(t,u_k,\tilde{z}_k)+\mathscr{R}_k(\tilde{z}_k-z_k)-\mathscr{E}_k(t,u_k,z_k)\big)\le\mathscr{E}(t,u,\tilde{z})+\mathscr{R}(\tilde{z}-z)-\mathscr{E}(t,u,z).\tag{4.24}$$

Thanks to (3.7b) and (4.1a), the above relation (4.24), and hence the stability of the limit, can be deduced even for all $t\in[0,T]$. The proof of (4.24), in particular the construction of the mutual recovery sequence can be directly taken from [35, Sect. 7], see (4.25) in Proposition 4.8 below. The construction of recovery operators as in (4.25) was deviced in [50, Sect. 2] for rate-independent damage processes with general BV-regularization and adapted in [35, Sect. 5.2] to the special case of characteristic functions for a different scaling of the adhesive functionals. The full construction (4.25) is carried out in [35, Sect. 7] for the adhesive functionals from (3.1) in a rate-dependent setting.

Proposition 4.8 (Mutual-recovery-sequence condition)**.** *Let the assumptions of Theorem 4.1 be satisfied. Moreover, let* $(z_k)_k \subset \mathscr{Z}_{\mathrm{SBV}}$*, such that for all* $k \in \mathbb{N}$*,* z_k *satisfies* (3.7b) *with respect to* $\mathscr{E}_k(t, u_k, \cdot)$ *and* $z_k \overset{*}{\rightharpoonup} z$ *in* $\mathscr{Z}_{\mathrm{SBV}}$*. Then, for every* $\tilde{z} \in \mathscr{Z}$*, the sequence* $(\tilde{z}_k)_k$ *given by*

$$\begin{aligned} &\tilde{z}_k := \begin{cases} r_k(\tilde{z}) & \text{if } \tilde{z} \in \mathscr{Z}_{\mathrm{SBV}}, \\ r_k(z) & \text{if } \tilde{z} \in \mathscr{Z} \backslash \mathscr{Z}_{\mathrm{SBV}}, \end{cases} \\ &\text{with} \quad r_k(\zeta) := \zeta\chi_{A_k} + z_k(1-\chi_{A_k}) \quad \text{and} \quad A_k := \big\{x \in \Gamma_{\mathrm{C}} : 0 \le \zeta(x) \le z_k(x)\big\}, \end{aligned} \tag{4.25}$$

is a mutual recovery sequence in the sense of (4.24)*. Here,* χ_{A_k} *is the characteristic function of* A_k*.*

Proof. First, let $\tilde{z} \in \mathscr{Z}$ such that there exists an $\mathscr{L}^{d-1}$-measurable set $B \subset \Gamma_{\mathrm{C}}$ with $\mathscr{L}^{d-1}(B) > 0$ and $\tilde{z} > z$ on B. Then $\mathscr{R}(\tilde{z} - z) = \infty$ and thus (4.24) trivially holds.

Let now $\tilde{z} \le z$ a.e. in Γ_{C}. Then, $\mathscr{R}(\tilde{z} - z) < \infty$ and also $J_\infty([\![u]\!], \tilde{z}) = 0$ a.e. in Γ_{C}. According to (4.25) we now show the validity of (4.24) for $r_k(\zeta)$ with $\zeta \in \mathscr{Z}_{\mathrm{SBV}}$, either $\zeta = \tilde{z} \le z$ or $\zeta = z$. For this, first note that $r_k(\zeta) \le z_k$ for every $k \in \mathbb{N}$ by construction. Moreover, observe that the bulk terms cancel out in (4.24), and $I_{\mathrm{C}}([\![u_k]\!]) = 0$ a.e. in Γ_{C} for all $k \in \mathbb{N}$. In addition,

$$\int_{\Gamma_{\mathrm{C}}} J_k([\![u_k]\!], r_k(\zeta)) - J_k([\![u_k]\!], z_k) \,\mathrm{d}\mathscr{H}^{d-1} < 0 = \int_{\Gamma_{\mathrm{C}}} J_\infty([\![u]\!], \tilde{z}) - J_\infty([\![u]\!], z) \,\mathrm{d}\mathscr{H}^{d-1}, \tag{4.26}$$

so that also the adhesive energy terms on the left-hand side of (4.24) for every $k \in \mathbb{N}$ can be estimated from above by 0.

To fix notation for the treatment of the remaining energy terms we say that $\zeta \in \mathscr{Z}_{\mathrm{SBV}}$ is the characteristic function of the finite-perimeter set $K \in \{Z, \widetilde{Z}\}$ and observe that $r_k(\zeta)$ is the characteristic function of the set A_k. We have to ensure that A_k has a uniformly bounded perimeter in Γ_{C}. It is $\Gamma_{\mathrm{C}} \backslash A_k = [z_k < \zeta] = [z_k = 0] \cap [\zeta = 1]$ and both $\mathscr{P}(K, \Gamma_{\mathrm{C}}) < \infty$ and $\mathscr{P}(Z_k, \Gamma_{\mathrm{C}}) \le C$ uniformly for all $k \in \mathbb{N}$ by (3.14a). Hence also $\mathscr{P}(A_k, \Gamma_{\mathrm{C}})$ is uniformly bounded for all $k \in \mathbb{N}$. Therefore, $\frac{\beta}{k}(\mathscr{P}(A_k, \Gamma_{\mathrm{C}}) - \mathscr{P}(Z_k, \Gamma_{\mathrm{C}})) \to 0$. Similarly, due to $r_k(\zeta) \le z_k$ a.e. in Γ_{C}, we have $\mathscr{R}_1(r_k(\zeta) - z_k)$ uniformly bounded for all $k \in \mathbb{N}$ and thus $\mathscr{R}_k(r_k(\zeta) - z_k) \to 0$ as $k \to \infty$. This completes the proof of (4.24). □

5 Computational experiments: cylindrical inclusion under a transverse tension

At the end, we want to demonstrate computationally the efficiency (especially for very large adhesive elastic moduli k) and convergence (proved rigorously in Theorem 4.1 only without spatial discretisation) of the proposed algorithm as well as applicability of the proposed model together with the stress-driven local solution concept. However, compared to the previous theoretical part, we make several shortcuts. First, the regularizing small parameter $\beta > 0$ is now considered simply zero, like in Sect. 2, and also $z(\cdot)$ is constrained to be valued in $[0, 1]$ instead of $\{0, 1\}$; this facilitates considerably the implementation of the problem and pressumably does not change considerably our simulations as the regularizing role of the finite perimeter is now likewise taken by the spatial discretisation. A further shortcut is in considering a special hard-device loading, so that, in particular, the bulk load f is zero, and we can afford (3.9c) not to be satisfied without destroying the a-priori estimate; in fact, we can use Korn inequality only on a factor space up to rigid-body motions.

The semi-implicit discretisation (3.5) leads, after a spatial discretisation, to an efficient numerical procedure which is easy to implement by using non-iterative linear-quadratic-programming solvers and which is robust under the limit towards brittle delamination, in contrast to the energetic-solution approach which needs complicated global optimization strategies usually with very large number of iterations especially if the adhesive is closer to be brittle.

We implemented the standard engineering test: an elastic cylindrical inclusion embedded (and initially bonded along its whole interface) in an elastic "matrix" exposed to a gradually increasing transverse tension on two opposite sides. The problem has been studied in depth during the past years, see e.g. [23, 24, 31, 33, 34, 48, 49] and references therein. In the present study, a two-dimensional cross-section of an infinitely long cylindrical inclusion of radius a=7.5μm is considered inside a finite square matrix of the size 30×30 μm; cf. Figure 4-A. The above configuration leads to the assumption of plane strain.

A typical bi-material system among the fiber reinforced composite materials has been chosen, that is composed by glass fiber and epoxy matrix. The elastic properties of these materials are given in Table 1.

Table 1: Elastic properties of the isotropic bi-material (f=fiber; m=matrix).

Material	Young's modulus	Poisson's ratio
Glass	E_f=70.8 GPa	ν_f=0.22
Epoxy	E_m=2.79 GPa	ν_m=0.33

In accordance with standard engineering models, we distinguish the normal stiffness $\kappa_{\rm n}$ and the tangential stiffness $\kappa_{\rm t}$ of the adhesive and we consider

$$\text{the specific elastic energy of the adhesive} = \frac{k}{2}z\Big(\kappa_{\rm n}\big|[\![u]\!]_{\rm n}\big|^2+\kappa_{\rm t}\big|[\![u]\!]_{\rm t}\big|^2\Big) \tag{5.1a}$$

with $[\![u]\!]_{\rm n}$ and $[\![u]\!]_{\rm t}$ the normal and the tangential components of $[\![u]\!]$, respectively. This obviously gives the original term $\frac{k}{2}z|[\![u]\!]|^2$ in (3.1b) if $\kappa_{\rm n}=\kappa_{\rm t}=1$. In spite of a lack of well-established values, we have adopted one option from [48, Table 2], namely

$$\kappa_{\rm n}\text{=2025 TPa/m} \quad \text{and} \quad \kappa_{\rm n}\text{=}\kappa_{\rm t}/3 = 675\,\text{TPa/m}; \tag{5.1b}$$

here we slightly generalized the theory presented in Sections 2–4 in the spirit of typical engineering applications. Furthermore, we took the critical tension of the interface $\sigma_{\rm RUP}$=90.7 MPa.

Here, one can also benefit from the fact that the problem is linear in the bulk and the only inelasticity occurs on the boundary $\Gamma_{\rm C}$. This suggests to use the boundary-element method (BEM). Here, the outer boundary of the square domain was discretized by 32 elements of equal length, that means 8 elements for each square side. For the embedded circular inclusion, the equidistant partition 64 (and for Figure 5-left also 40 and 128) linear boundary elements (=segments) have been used for the conctact boundary each subdomain (the fiber and the epoxy matrix) consistently so that the non-penetration conditions can be prescribed just in the boundary nodes, cf. [32].

The loading experiment was devised so that the specimen was pulled on two opposite sides of the rectangle by gradually increasing the Dirichlet boundary condition with the speed 0.5 μm/s prescribed for the normal displacement. The tangential displacement on these sides and the displacement on the other two sides have been kept free. For $t=T=1$s, the original size 30 μm is thus pulled to 31 μm. The force response on this loading is depicted on Figure 3 for different values of the elastic moduli of the adhesive and, of course, the correspondingly scaled fracture toughness from (3.1b,c), considering $\alpha_1+\alpha_0=\sigma_{\rm RUP}^2/2$.

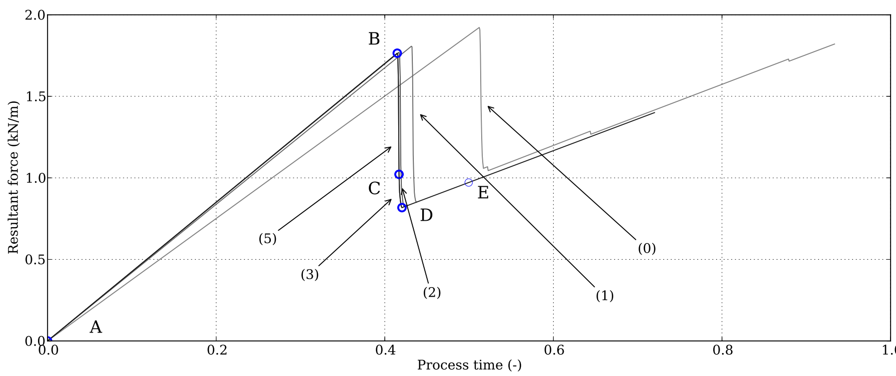


Fig. 3. Force response on the Dirichlet loading gradually increasing in time for five different adhesive stiffnesses, namely $k=10^i$ with with $i=0,1,2,3,$ and 5 in (5.1). The difference between the last two is practically invisible, and the convergence of the rupture nucleation under the scaling (3.1b,c), cf. also Figure 2 (right), is thus clearly demonstrated. The discretisation is kept fixed; namely $\tau=35\times10^{-5}$, and 64 boundary elements on $\Gamma_{\rm C}$.

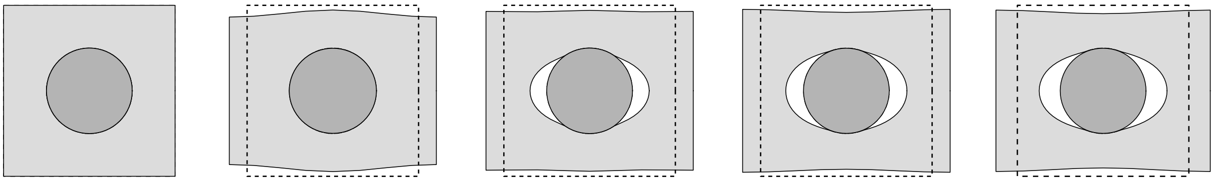
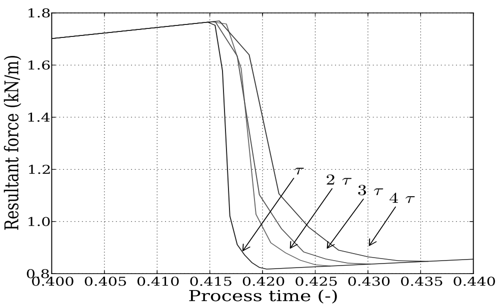


Fig. 4. Geometry of the specimen deformed during loading: A) unloaded, B) loaded just before the rupture starts, C) during the rupture proceeds, D) just after the rupture is (nearly) completed, E) after further loading; the letters A)–E) refer to Figure 3. Displacement calculated for $k = 10^5$ in (5.1). $\tau = 35 \times 10^{-5}$, and 64 boundary elements on Γ_C are depicted as $15\times$ magnified.

As there is no uniqueness of the solution, one can only say that the set of all solutions will inherit the symmetry of our problem but not necesarily the particular solutions. In general, it is even not clear whether there is at least one solution which has the symmetry. Here, however, when considering symmetric space discretisation and when analysing the corresponding discretisation of the semi-implicit scheme (3.5) in detail, one can see that, generically for a.a. time discretisations, the discrete solutions are symmetric. This is then reflected by Figure 4. Nevertheless, the computation simulations have been performed on the full domain disregarding its symmetry. The convergence of the time/space discretisation has been tested (little inconsistently with Section 4) separately for the elastic modulus of the adhesive fixed, the results being depicted on Figure 5.



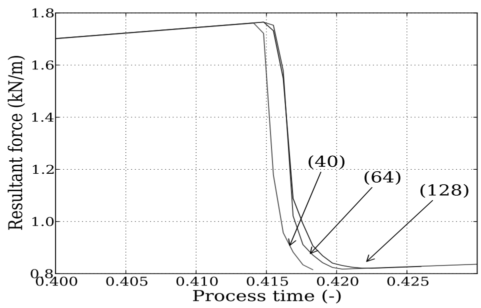


Fig. 5. Convergence with respect to discretisation: a detailed force response on the Dirichlet loading on a small time-interval around the rupture with fixed $k = 10^5$ in (5.1) and with varying time or space discretisation:
Left: τ=35×10^{-5}, 70×10^{-5}, 105×10^{-5}, and 140×10^{-5}, while spacial discretisation is fixed (64 boundary elements on Γ_C).
Right: 40, 64, and 128 boundary elements on Γ_C while time discretisation is fixed (τ=35×10^{-5}).

We can observe an expected and well understood effect occuring for brittle delamination: when rupture arises, it can propagate very easily because there in no dissipation related to it in the limit, cf. Figure 2 (right), and simultaneously there typically occurs a stress concentration on the crack tip. Therefore the rupture occurs essentially at one moment nearly completely and symmetrically. In fact, the stresses in real materials are bounded and always there is some (even small) dissipation related even with brittle crack propagation, which goes however beyond this merely elastic/brittle model, as already said in the introduction.

On the other hand, this model with the scaling (3.1b,c) correctly (and in a convergent way) describes the crack nucleation, which was already documented on the 0-dimensional example (2.4) and is now also documented on this 2-dimensional example. This last example even exhibits a good qualitative agreement with an everyday experience that, in very brittle materials, like, e.g., glass or, here, the brittle adhesive, the crack propagates very easily if once nucleated. Obviously, the scaling keeping the dissipation fixed as on Figure 2 (left) and as used for the energetic solutions in [42] would give completely wrong response without any delamination in the limit. Conversely, energetic solutions with the scaling as on Figure 2 (right) (3.1c) would give also completely wrong response with delamination at the very beginning.

Acknowledgements: The authors are indebted to Vladislav Mantič, Alexander Mielke, and Riccarda Rossi for inspiring discussions about modeling issues. This research has been covered partially from DFG MATHEON Project C18 and the Spanish Ministry of Economy and Competitiveness (Proyecto MAT2012-37387) as well as from the Czech Republic through the grants 201/10/0357 and 105/13/18652S (GA ČR) together with the institutional support RVO: 61388998 (ČR).

References:

References

[1] H. Attouch. *Variational convergence for functions and operators.* Pitman, Boston, MA, 1984.

[2] F. Cagnetti. A vanishing viscosity approach to fracture growth in a cohesive zone model with prescribed crack path. *Math. Models Meth. Appl. Sci*, 18:1027–1071, 2009.

[3] S. Campanato. Proprietà di hölderianità di alcune classi di funzioni. *Annali della Scuola Normale Superiore di Pisa*, 17:175–188, 1963.

[4] S. Campanato. Proprietà di una famiglia di spazi funzionali. *Annali della Scuola Normale Superiore di Pisa*, 18:137–160, 1964.

[5] M. Charlotte, G.A. Francfort, J.-J. Marigo, and L. Truskinovsky. Revisting brittle fracture as an energy minimization problem: comparison of Griffith and Barenblatt surface energy models. In A.B. Cachan, editor, *Continuous Damage and Fracture*, pages 7–12, Paris, 2000. Elsevier.

[6] G. Dal Maso and C. Larsen. Existence for wave equations on domains with arbitrary growing cracks. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 22:387–408, 2011.

[7] D.C. Drucker and J.R. Rice. Plastic deformation in brittle and ductile fracture. *Engineering Fracture Mechanics*, 1:577–602, 1970.

[8] D.S. Dugdale. Yielding of steel sheets containing slits. *Mech. Phys. Solid.*, 8:100–104, 1960.

[9] G. Francfort and A. Mielke. Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. reine angew. Math.*, 595:55–91, 2006.

[10] M. Frémond. Contact with adhesion. In J.J. Moreau and G. Panagiotopoulos, editors, *Topics in Nonsmooth Mechanics*. Birkhäuser, 1988.

[11] M. Giaquinta. *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Priceton University Press, 1983.

[12] J.A. Griepentrog. Linear elliptic boundary value problems with non-smooth data: Campanato spaces of functionals. *Mathematische Nachrichten*, 243:19–42, 2002.

[13] R. Haller-Dintelmann, H.-C. Kaiser, and J. Rehberg. Hardy's inequality for functions vanishing on a part of the boundary. *in preparation.*

[14] D. Knees, A. Mielke, and C. Zanini. On the inviscid limit of a model for crack propagation. *Math. Models Meth. Appl. Sci.*, 18:1529–1569, 2008.

[15] D. Knees and A. Schröder. Computational aspects of quasi-static crack propagation. *Disc. Cont. Dynam. Syst. - S*, 6:63–99, 2013.

[16] C.J. Larsen. Epsilon-stable quasi-static brittle fracture evolution. *Comm. Pure Appl. Math.*, 63:630–654, 2010.

[17] C.J. Larsen, C. Ortner, and E. Süli. Existence of solution to a regularized model of dynamic fracture. *Math. Models Meth. Appl. Sci.*, 20:1021–1048, 2010.

[18] J.-B. Leblond. *Brittle fracture and ductile fracture. (in French: Mecanique de la rupture fragile et ductile).* Hermes Sci. Publ., Paris, 2003.

[19] D. Leguillon. Strength or toughness? A criterion for crack onset at a notch. *European J. of Mechanics A/Solids*, 21:61–72, 2002.

[20] J. Lehrbäck. Pointwise Hardy inequalities and uniformly fat sets. *Proc. Amer. Math. Soc.*, 136:2193–2200, 2008.

[21] S. Lenci. Analysis of a crack at a weak interface. *Intl J. of Fracture*, 108:275–290, 2001.

[22] J.L. Lewis. Uniformly fat sets. *Trans. Amer. Math. Soc.*, 308:177–196, July 1988.

[23] V. Mantič. Interface crack onset at a circular cylindrical inclusion under a remote transverse tension. application of a coupled stress and energy criterion. *Intl. J. Solids Struct.*, 46:1287–1304, 2009.

[24] V. Mantič and I.G. García. Crack onset and growth at the fibre-matrix interface under a remote biaxial transverse load. Application of a coupled stress and energy criterion. *Intl. J. Solids Struct.*, 49:2273–2290, 2012.

[25] A. Mielke. Differential, energetic and metric formulations for rate-independent processes. Lecture Series given at "C.I.M.E. Summer School on Nonlinear PDEs and Applications", Cetraro, June 2008.

[26] A. Mielke, R. Rossi, and G. Savaré. Modeling solutions with jumps for rate-independent systems on metric spaces. *Discr. Cont. Dynam. Systems Ser. A*, 25:585–615, 2009.

[27] A. Mielke, T. Roubíček, and U. Stefanelli. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. Partial Differ. Equ.*, 31:387–416, 2008.

[28] A. Mielke, T. Roubíček, and M. Thomas. From damage to delamination in nonlinearly elastic materials at small strains. *J. Elasticity*, 109:235–273, 2012.

[29] A. Mielke and F. Theil. On rate-independent hysteresis models. *Nonl. Diff. Eqns. Appl.*, 11:151–189, 2004.

[30] U. Mosco. Approximation of the solutions of some variational inequalities. *Ann. Scuola Norm. Sup. Pisa*, 21:373–394, 1967. Erratum, ibid. (3) 765.

[31] C.G. Panagiotopoulos and V. Mantič. Symmetric and non-symmetric debonds at fiber-matrix interface under transverse loads. an application of energetic approaches using collocation BEM. *Anales de Mecánica de la Fractura*, 1:125–130, 2013.

[32] C.G. Panagiotopoulos, V. Mantič, and T. Roubíček. BEM solution of delamination problems using an interface damage and plasticity model. *Computational Mechanics*, 51:505–521, 2013.

[33] F. París, E. Correa, and V. Mantič. Kinking of transverse interface cracks between fiber and matrix. *Journal of Applied Mechanics*, 74:703–716, 2007.

[34] S. Parvanova and S. Delijski. On the calculation of the effective elastic properties of composites accounting for particle-matrix debonding by the boundary element method. *Annual of UACEG, Sofia*, 45:203–219, 2010.

[35] R. Rossi and M. Thomas. From an adhesive to a brittle delamination model in thermo-visco-elasticity. *ESAIM-COCV*, submitted. (Preprint no.1692, WIAS, Berlin, 2012).

[36] T. Roubíček. Rate independent processes in viscous solids at small strains. *Math. Methods Appl. Sci.*, 32:825–862, 2009. Erratum Vol. 32(16) p. 2176.

[37] T. Roubíček. *Nonlinear Partial Differential Equations with Applications.* Birkhäuser, Basel, 2nd edition, 2013.

[38] T. Roubíček. Maximally-dissipative local solutions to rate-independent systems and application to damage and delamination problems. *Nonlin. Anal, Th. Meth. Appl., submitted.* (Preprint No.2013-021, Nečas center, Prague).

[39] T. Roubíček. Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity. *SIAM J. Math. Anal.*, 45:101–126, 2013.

[40] T. Roubíček, C.G. Panagiotopoulos, and V. Mantič. Local-solution approach to quasistatic rate-independent mixed-mode delamination. *Math. Models Methods Appl. Sci.* submitted.

[41] T. Roubíček, C.G. Panagiotopoulos, and V. Mantič. Quasistatic adhesive contact of visco-elastic bodies and its numerical treatment for very small viscosity. *Zeitschrift angew. Math. Mech.*, 93:823–840, 2013.

[42] T. Roubíček, L. Scardia, and C. Zanini. Quasistatic delamination problem. *Contin. Mech. Thermodyn.*, 21:223–235, 2009.

[43] I. Scheider and J. Mosler. Novel approach for the treatment of cyclic loading using a potential-based cohesive zone model. *Procedia Engineering*, 10:2164–2169, 2011.

[44] M. Shillor, M. Sofonea, and J.J. Telega. *Models and Analysis of Quasistatic Contact.* Springer, Berlin, 2004.

[45] M. Sofonea, W. Han, and M. Shillor. *Analysis and Approximation of Contact Problems and Adhesion or Damage.* Chapman & Hall, New York, 2006.

[46] U. Stefanelli. A variational characterization of rate-independent evolution. *Mathem. Nach.*, 282:1492–1512, 2009.

[47] L. Távara, V. Mantič, E. Graciani, J. Cañas, and F. París. Analysis of a crack in a thin adhesive layer between orthotropic materials: an application to composite interlaminar fracture toughness test. *CMES*, 58:247–270, 2010.

[48] L. Távara, V. Mantič, E. Graciani, and E. París. BEM analysis of crack onset and propagation along fiber–matrix interface under transverse tension using a linear elastic–brittle interface model. *Engineering Analysis with Boundary Elements*, 35:207–222, 2011.

[49] L. Távara, V. Mantič, E. Graciani, and F. París. BEM analysis of crack onset and propagation along fiber-matrix interface under transverse tension using a linear elastic-brittle interface model. *Eng. Anal. Bound. Elem.*, 35:207–222, 2011.

[50] M. Thomas. Quasistatic damage evolution with spatial BV-regularization. *Discrete Continuous Dynam. Syst. - S*, 6:235–255, 2013.

[51] M. Thomas. Uniform Poincaré-Sobolev and relative isoperimetric inequalities for classes of domains. *Preprint no.1797, WIAS, Berlin*, 2013.

[52] R. Toader and C. Zanini. An artificial viscosity approach to quasistatic crack growth. *Boll. Unione Matem. Ital.*, 2:1–36, 2009.