ITERATIVE SOLUTION OF A CONSTRAINED TOTAL VARIATION REGULARIZED MODEL PROBLEM

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Abstract. The discretization of a bilaterally constrained total variation minimization problem with conforming low order finite elements is analyzed and three iterative schemes are proposed which differ in the treatment of the non-differentiable terms. Unconditional stability and convergence of the algorithms is addressed, an application to piecewise constant image segmentation is presented and numerical experiments are shown.

1. Introduction

In this paper we investigate iterative schemes for the solution of the *bilaterally constrained ROF problem* which consists in the minimization of the functional

$$E(u) := |Du|(\Omega) + \frac{\alpha}{2}\|u-g\|^2_{L^2(\Omega)} + I_K(u)$$

in the set of functions with bounded total variation $|Du|(\Omega)$, where I_K is the indicator functional of a convex set $K \subset L^2(\Omega)$ characterized by two obstacle functions. While existence and uniqueness of a minimizer of E can be established using the direct method in the calculus of variations, the non-differentiability of the total variation and the indicator functional are challenging from a numerical point of view. The minimization problem serves as a model problem for a wide class of functionals involving the total variation and the indicator functional which comprises a bilateral constraint on the variable u. For example, in [5, 11] the above minimization problem is considered in the context of image denoising and the bilateral constraint is imposed in order for the restored image to have pixel values lying in a certain range, e.g., in the interval $[0, 255]$. In [10] Chan et al. show the equivalence of a minimization problem occuring in image segmentation to the minimization of a functional that resembles E with the quadratic fidelity term replaced by a linear functional in u. The application of a regularization term to a bilaterally constrained variable also occurs in the modelling of damage processes in continuum mechanics, where the bilaterally constrained

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variable is the damage variable and the regularization term consists of a differentiable L^p-norm ($p \geq 2$) of the gradient of the damage variable (cf. [18]) or the total variation of the damage variable (cf. [20]) while the constraint ensures that the damage variable takes on values only in the interval $[0, 1]$.

In the recent years many PDE-based algorithms have been proposed for the classical ROF functional proposed by Rudin, Osher and Fatemi in [19], where the bilateral constraint is omitted, reaching from primal-dual methods, combinations of regularization and standard minimization techniques and dual methods to the alternate direction method of multipliers, augmented Lagrangian methods and split Brègman methods, see, e.g., [3, 22] for an overview of many of those algorithms. Regarding the bilaterally constrained ROF problem Casas et al. proposed in [7] an active set strategy using the anisotropic BV-seminorm. This ansatz works with the primal formulation of the problem and does not use regularization. However, a penalization approach is used to solve auxiliary subproblems. In [5], motivated by the dual approach of Chambolle in [8], Beck and Teboulle suggest a monotone fast iterative shrinkage/thresholding algorithm (MFISTA) for the dual of the bilaterally constrained ROF problem for which the convergence rate $\mathcal{O}(1/k^2)$ with k being the iteration counter is proven. Yet, as in [8], the constant entering the convergence rate grows like $\mathcal{O}(1/h^2)$ with h being the mesh size of the underlying grid. Chan et al. proposed in [11] an augmented Lagrangian method, which has already been successfully applied to the ROF problem by Wu and Tai in [21].

The aim of this paper is to compare three different approaches for the iterative minimization of E and to identify the most appropriate method with particular view to applications in materials science. To this extent, we introduce a finite element discretization with globally continuous piecewise affine finite elements. The iterative algorithms differ in the treatment of the two challenging terms appearing in E. The first approach uses splitting techniques for both the total variation and the bilateral constraint and is a variant of the *augmented Lagrangian method* proposed in [11], where the involved norms are properly weighted in order to ensure unconditional stability of the method, see also [3] where the authors proposed a properly weighted augmented Lagrangian method for the classical ROF problem. The next ansatz uses regularization to handle the non-differentiability of the total variation and the indicator functional and employs a semi-implicit subgradient flow which turns out to be unconditionally stable and will be referred to as the *Heron-penalty method*. For the classical ROF problem the *Heron method* has already been introduced in [3]. The third algorithm uses a combination of the regularization and the splitting approach, i.e., using the observation made in [3] that a regularization approach seems to be the most appropriate to deal with the total variation, we employ a regularization for the total variation and a splitting technique for the bilateral constraint, which we call the *Heron-split method*.

The outline of the paper is as follows. In Section 2 we give some basic notation and introduce the space of functions with bounded variation as well as the globally continuous piecewise affine and the piecewise constant finite element space. We present the minimization problem in Section 3 and discuss existence and uniqueness of a minimizer. In Section 4 we consider the finite element discretization and prove convergence of discrete minimizers to the continuous solution before we present the three algorithms and address their stability in Section 5. In Section 6 we address the minimization of the functional

$$E^{seg}(u) := \int_\Omega |Du| + \alpha \int_\Omega f u \,\mathrm{d}x + I_K(u),$$

prove convergence of discrete energies to the minimal energy and recall a minimization problem arising in an image segmentation model discussed by Chan et al. in [10]. We finally provide results of numerical experiments for the three iterative schemes applied to the bilaterally constrained ROF problem and for a particular choice of g and two different values of α as well as for the image segmentation problem in Section 7.

2. Preliminaries

2.1. **Notation.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain. We denote by $\|\cdot\|$ the L^2-norm on Ω induced by the scalar product

$$(v, w) := \int_\Omega v \cdot w \,\mathrm{d}x$$

for scalar functions or vector fields $v, w \in L^2(\Omega; \mathbb{R}^\ell)$, $\ell \in \{1, d\}$. The Euclidean norm will be denoted by $|\cdot|$. We will work with the space of functions of bounded variation $BV(\Omega) \subset L^1(\Omega)$ which contains all functions $v \in L^1(\Omega)$ for which the total variation given by

$$|Du|(\Omega) = \int_\Omega |Du| := \sup\{-\int_\Omega u \operatorname{div} q \,\mathrm{d}x : \; q \in C_c^1(\Omega; \mathbb{R}^d),\; |q| \le 1 \text{ a.e.}\}$$

is finite. The space $BV(\Omega)$ equipped with the norm $\|u\|_{BV} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$ is a Banach space. For more details concerning the space $BV(\Omega)$ we refer, e.g., to [1].
For a sequence $(a^j)_{j\in\mathbb{N}}$ and a step size $\tau > 0$ we let

$$d_t a^{j+1} := \frac{a^{j+1} - a^j}{\tau}$$

be the backward difference quotient.

2.2. **Finite element spaces.** Throughout the paper we let c be a generic constant. We denote by $(\mathcal{T}_h)_{h>0}$ a family of uniform triangulations of Ω with mesh sizes $h = \max_{T\in\mathcal{T}_h} h_T$ where h_T is the diameter of the simplex T. For a given triangulation $\mathcal{T}_h$ the set $\mathcal{N}_h$ contains the corresponding nodes and we consider the finite element spaces of continuous, piecewise affine functions

$$\mathcal{S}^1(\mathcal{T}_h) := \big\{v_h \in C(\overline{\Omega}) : \ v_h|_T \text{ affine for all } T \in \mathcal{T}_h\big\}$$

and of elementwise constant functions ($r = 1$) or vector fields ($r = d$)

$$\mathcal{L}^0(\mathcal{T}_h)^r := \big\{q_h \in L^\infty(\Omega;\mathbb{R}^r) : \ q_h|_T \text{ constant for all } T \in \mathcal{T}_h\big\}.$$

We also denote by $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$, a triangulation of Ω generated from an initial triangulation $\mathcal{T}_0$ by ℓ uniform refinements and the integer ℓ being related to the mesh size h by $h = c2^{-\ell}$. The set of nodes $\mathcal{N}_\ell$ is defined correspondigly.

We introduce on $\mathcal{S}^1(\mathcal{T}_h)$ the discrete norm $\|\cdot\|_h$ induced by the discrete scalar product

$$(v,w)_h := \sum_{z\in\mathcal{N}_h} \beta_z v(z)w(z)$$

for $v, w \in \mathcal{S}^1(\mathcal{T}_h)$, where $\beta_z = \int_\Omega \varphi_z \,\mathrm{d}x$ and $\varphi_z \in \mathcal{S}^1(\mathcal{T}_h)$ is the nodal basis function associated with the node $z \in \mathcal{N}_h$. The norm $\|\cdot\|_h$ and the L^2-norm are related on $\mathcal{S}^1(\mathcal{T}_h)$ as follows.

Lemma 2.1. *For $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ we have*

$$\|v_h\| \le \|v_h\|_h \le (d+2)^{1/2}\|v_h\|.$$

The proof of Lemma 2.1 can be found, e.g., in [2, Lemma 3.9].

3. Model problem

For a function $g \in L^\infty(\Omega)$ and a parameter $\alpha > 0$ we consider the constrained minimization problem defined by the functional

$$E(u) := \int_\Omega |Du| + \frac{\alpha}{2}\|u-g\|^2 + I_K(u)$$

in the set of functions $BV(\Omega)\cap L^2(\Omega)$. The convex set K is given by

$$K := \{v \in L^2(\Omega) : \ \chi \le v \le \psi \ \text{ a.e.}\}$$

with $\chi, \psi \in BV(\Omega)\cap L^\infty(\Omega)$ and $\chi < \psi$ almost everywhere. We will refer to this minimization problem as the *bilaterally constrained ROF problem.*

Proposition 3.1 (Existence and Uniqueness)**.** *There exists a unique minimizer $u \in BV(\Omega)\cap L^2(\Omega)$ of E.*

Proof. The set of feasible functions is nonempty and the functional E is bounded from below. Hence, there exists an infimizing sequence $(u_j)_j$ which is bounded in $BV(\Omega)\cap L^2(\Omega)$ due to the coercivity of E. The compact embedding of $BV(\Omega)$ into $L^1(\Omega)$ (cf. [1]) and the boundedness of the sequence imply the existence of a - not relabelled - weakly convergent subsequence

$u_j \rightharpoonup u$ in $BV(\Omega)$ and by passing to a further subsequence $u_j \rightharpoonup u$ in $BV(\Omega) \cap L^2(\Omega)$. Since $u_j \to u$ in $L^1(\Omega)$ there exists a further subsequence such that $u_j \to u$ almost everywhere in Ω. This implies that $\chi \le u \le \psi$ almost everywhere in Ω since $\chi \le u_j \le \psi$ for all $j \in \mathbb{N}$. The weak lower semi-continuity of E (cf. [1]) then yields that u is a minimizer of E. Uniqueness follows by the strict convexity of E. □

Remark 3.2. *In fact, we have* $u \in BV(\Omega) \cap L^\infty(\Omega)$ *since* $\chi, \psi \in L^\infty(\Omega)$.

We define the functionals $G : L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$, $H : L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ and $F : L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ by $G(v) := \frac{\alpha}{2}\|v - g\|^2$, $H(v) := I_K(v)$ and

$$F(v) := \begin{cases} \int_\Omega |Dv|, & \text{if } v \in BV(\Omega) \cap L^2(\Omega), \\ \infty, & \text{if } v \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

Lemma 3.3. *Let u be the unique minimizer of E in $BV(\Omega) \cap L^2(\Omega)$. Then for any $v \in BV(\Omega) \cap L^2(\Omega)$ we have*

$$\frac{\alpha}{2}\|u - v\|^2 \le E(v) - E(u).$$

Proof. We note that G is Fréchet-differentiable, i.e., we have $\partial G(v) = \{\delta G(v)\}$ with $\delta G(v)[w] = \alpha(v - g, w)$ for $v, w \in L^2(\Omega)$. Since u is a minimizer of E, we have

$$0 \in \partial E(u) = \partial(F + G + H)(u).$$

Standard arguments from convex analysis (cf. [1, Thm. 9.5.4]) yield

$$-\delta G(u) \in \partial(F + H)(u).$$

The strong convexity of G and the subgradient inequality then imply that for arbitrary $v \in BV(\Omega) \cap L^2(\Omega)$

$$\begin{aligned} \frac{\alpha}{2}\|u - v\|^2 &= -\delta G(u)[v - u] + G(v) - G(u) \\ &\le F(v) + G(v) + H(v) - F(u) - G(u) - H(u) \\ &= E(v) - E(u), \end{aligned}$$

which proves the lemma. □

4. Finite element discretization

For simplicity, we assume throughout the paper that for a family of triangulations $(\mathcal{T}_h)_{h>0}$ the obstacle functions χ and ψ are such that there exists $h_0 > 0$ and $\chi, \psi \in \mathcal{S}^1(\mathcal{T}_h)$ for all $h \le h_0$ and we assume $h \le h_0$ in the sequel. Let us then remark that $I_K(v_h)$ is finite for $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ if and only if $\chi(z) \le v_h(z) \le \psi(z)$ at all nodes $z \in \mathcal{N}_h$.

We then seek a discrete minimizer $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ of

$$E(u_h) = \int_\Omega |\nabla u_h| \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2 + I_K(u_h)$$

in $\mathcal{S}^1(\mathcal{T}_h)$, where we used that $|Dv|(\Omega) = \|\nabla v\|_{L^1(\Omega)}$ for $v \in W^{1,1}(\Omega)$. Existence and uniqueness can be proven following the arguments of the proof of Proposition 3.1.

Proposition 4.1. *Let u and u_h be the unique minimizers of E in $BV(\Omega) \cap L^2(\Omega)$ and $\mathcal{S}^1(\mathcal{T}_h)$, respectively. Then we have*

$$\frac{\alpha}{2}\|u - u_h\|^2 \le ch^{1/2}.$$

Proof. Following the arguments in the proof of [2, Thm. 10.7] we note that by Lemma 3.3 we have for $v = u_h$ and for arbitrary $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ that

$$\begin{aligned} \frac{\alpha}{2}\|u - u_h\|^2 &\le E(u_h) - E(u) \\ &\le E(v_h) - E(u). \end{aligned} \tag{1}$$

Let $u_{\varepsilon,h} \in \mathcal{S}^1(\mathcal{T}_h)$ be as in [2, Lemma 10.1], i.e., such that $u_{\varepsilon,h} \to u$ with respect to intermediate convergence. We define the truncation $v \mapsto v^K$ for $v \in L^1(\Omega)$ by

$$v^K := \max\{\min\{\psi, v\}, \chi\} \quad \text{pointwise.}$$

One can check that the mapping $v \mapsto v^K$ is Lipschitz continuous with respect to the L^1-norm. Let $\mathcal{J}_h : L^1(\Omega) \to \mathcal{S}^1(\mathcal{T}_h)$ be the *Chen-Nochetto* quasi-interpolation operator introduced in [12] by Chen and Nochetto. We then have $\mathcal{J}_h v \in K$ if $v \in K$. Choosing $v_h = \mathcal{J}_h u^K_{\varepsilon,h}$ in (1) and using $\|\nabla \mathcal{J}_h u^K_{\varepsilon,h}\|_{L^1(\Omega)} \le \|\nabla u^K_{\varepsilon,h}\|_{L^1(\Omega)} \le \|\nabla u_{\varepsilon,h}\|_{L^1(\Omega)}$, the binomial formula, the identity $u^K = u$ and the Lipschitz continuity of the truncation operator we obtain

$$\begin{aligned} &E(\mathcal{J}_h u^K_{\varepsilon,h}) - E(u) \\ &= \int_\Omega |\nabla \mathcal{J}_h u^K_{\varepsilon,h}| \,\mathrm{d}x + \frac{\alpha}{2}\|\mathcal{J}_h u^K_{\varepsilon,h} - g\|^2 - \int_\Omega |Du| + \frac{\alpha}{2}\|u - g\|^2 \\ &\le \int_\Omega |\nabla u_{\varepsilon,h}| \,\mathrm{d}x - \int_\Omega |Du| + \frac{\alpha}{2}\int_\Omega (\mathcal{J}_h u^K_{\varepsilon,h} - u)(\mathcal{J}_h u^K_{\varepsilon,h} + u - 2g) \,\mathrm{d}x \\ &\le c(h\varepsilon^{-1} + \varepsilon)|Du|(\Omega) + \frac{\alpha}{2}\|\mathcal{J}_h u^K_{\varepsilon,h} - u^K\|_{L^1(\Omega)}\|\mathcal{J}_h u^K_{\varepsilon,h} + u - 2g\|_{L^\infty(\Omega)} \\ &\le c(h\varepsilon^{-1} + \varepsilon)|Du|(\Omega) + c\frac{\alpha}{2}\big(\|\mathcal{J}_h u^K_{\varepsilon,h} - u^K_{\varepsilon,h}\|_{L^1(\Omega)} + \|u^K_{\varepsilon,h} - u^K\|_{L^1(\Omega)}\big) \\ &\le c(h\varepsilon^{-1} + \varepsilon)|Du|(\Omega) + c\frac{\alpha}{2}\big(h\|\nabla u_{\varepsilon,h}\|_{L^1(\Omega)} + \|u_{\varepsilon,h} - u\|_{L^1(\Omega)}\big) \\ &\le c(h\varepsilon^{-1} + \varepsilon)|Du|(\Omega) + c(h + h^2\varepsilon^{-1} + h\varepsilon + \varepsilon)|Du|(\Omega), \end{aligned}$$

where we have used the quasi-interpolation estimate $\|v - \mathcal{J}_h v\|_{L^1(\Omega)} \le ch\|\nabla v\|_{L^1(\Omega)}$ for all $v \in W^{1,1}(\Omega)$ (cf. [12, 16]). The choice $\varepsilon = h^{1/2}$ concludes the proof. $\square$

Remark 4.2. *The error estimate is suboptimal in the sense that for* $u \in BV(\Omega) \cap L^\infty(\Omega)$ *we have the approximation property*

$$\inf_{v_h \in \mathcal{S}^1(\mathcal{T}_h)} \|u - v_h\| \le ch^{1/2},$$

cf. [4, Thm. 6.5]*. On highly symmetric triangulations and for an anisotropic version of the total variation this quasioptimal convergence rate can be proved for related minimization problems, cf.* [4]*.*

5. Iterative solution

The iterative minimization of E is difficult due to the non-differentiability of the seminorm and the occurrence of the constraints. In this section we will discuss three approaches to deal with these difficulties.

5.1. **Splitting.** We introduce for $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ the auxiliary variables $p_h = \nabla u_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ and $s_h = u_h \in \mathcal{S}^1(\mathcal{T}_h)$ and incorporate Lagrange multipliers $\lambda_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ and $\eta_h \in \mathcal{S}^1(\mathcal{T}_h)$ to enforce these relations. Note that since the sequence $(\nabla u_h)_{h>0}$ of discrete minimizers u_h of E may not be bounded in $L^2(\Omega)$ but only in $L^1(\Omega)$ we have to measure these quantities in a weaker norm than the L^2-norm. We therefore equip the space $\mathcal{L}^0(\mathcal{T}_h)^d$ with the weighted L^2-scalar product

$$(p_h, q_h)_w := h^d \int_\Omega p_h \cdot q_h \, \mathrm{d}x$$

and induced norm $\|\cdot\|_w$. An inverse estimate shows that $(\nabla u_h)_{h>0}$ is bounded with respect to $\|\cdot\|_w$, cf. [3].
We then consider the consistently stabilized Lagrange functional

$$\begin{aligned} L_h(u_h, p_h, s_h; \lambda_h, \eta_h) &:= \int_\Omega |p_h| \, \mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2 + I_K(s_h) \\ &+ (\lambda_h, p_h - \nabla u_h)_w + \frac{\sigma_1}{2}\|p_h - \nabla u_h\|_w^2 + (\eta_h, s_h - u_h)_h + \frac{\sigma_2}{2}\|s_h - u_h\|_h^2. \end{aligned}$$

The parameters σ_1 and σ_2 are assumed to be positive. We then have that $(u_h, p_h, s_h; \lambda_h, \eta_h)$ is a saddle point for L_h if and only if u_h minimizes E (cf. [21, Thm. 4.1] for the same result for the ROF functional without bilateral constraint). Indeed, since

$$L_h(u_h, p_h, s_h; \mu_h, \nu_h) \le L_h(u_h, p_h, s_h; \lambda_h, \eta_h)$$

for all (μ_h, ν_h) we obtain $p_h = \nabla u_h$ and $s_h = u_h$. Then we have

$$\begin{aligned} E(u_h) = L_h(u_h, p_h, s_h; \lambda_h, \mu_h) &\le L_h(v_h, \nabla v_h, v_h; \lambda_h, \mu_h) \\ &= \int_\Omega |\nabla v_h| \, \mathrm{d}x + \frac{\alpha}{2}\|v_h - g\|^2 + I_K(v_h) = E(v_h) \end{aligned}$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$, i.e., u_h minmizes E. Conversely, if u_h minimizes E we choose $p_h = \nabla u_h$ and $s_h = u_h$. Since u_h is the unique minimizer of E we have

$$0 \in \partial E(u_h) = \partial F(u_h) + \partial G(u_h) + \partial H(u_h),$$

cf. [17, Thm. 23.8]. In particular, there exist $-\lambda_h \in \partial\widetilde{F}(\nabla u_h)$ and $-\eta_h \in \partial H(u_h)$ with

$$-\alpha(u_h - g) = -\operatorname{div}_h \lambda_h + \eta_h, \tag{2}$$

cf. [17, Thm. 23.9], where the operator $\operatorname{div}_h : \mathcal{L}^0(\mathcal{T}_h) \to \mathcal{S}^1(\mathcal{T}_h)$ is defined by $(-\operatorname{div}_h \mu_h, v_h) = (\mu_h, \nabla v_h)$ for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$, $\mu_h \in \mathcal{L}^0(\mathcal{T}_h)$, and $\widetilde{F}(q_h) := \|q_h\|_{L^1(\Omega)}$. With our particular choice of p_h and s_h we have

$$L_h(u_h, p_h, s_h; \mu_h, \nu_h) = L_h(u_h, p_h, s_h; \lambda_h, \eta_h)$$

for all (μ_h, ν_h). One can show that the function

$$(v_h, q_h, r_h) \mapsto L_h(v_h, q_h, r_h; \lambda_h, \eta_h)$$

has a minimizer (v_h^*, q_h^*, r_h^*). The optimality conditions read

$$\begin{aligned} \sigma_1(q_h^* - \nabla v_h^*, \nabla v_h)_w + \sigma_2(r_h^* - v_h^*, v_h)_h& \\ +(\lambda_h, \nabla v_h)_w + (\eta_h, v_h^*)_h &= \alpha(v_h^* - g, v_h), \\ -\lambda_h + \sigma_1(\nabla v_h^* - q_h^*) &\in \partial\widetilde{F}(q_h^*), \\ -\eta_h + \sigma_2(v_h^* - r_h^*) &\in \partial H(r_h^*). \end{aligned}$$

Using (2) and the particular choice of functions λ_h and η_h we observe that the triple (u_h, p_h, s_h) satisfies these optimality conditions. Altogether, we have

$$\min_{u_h} E(u_h) = \min_{(u_h, p_h, s_h)} \max_{(\lambda_h, \eta_h)} L_h(u_h, p_h, s_h; \lambda_h, \eta_h).$$

Due to the fact that p_h is elementwise constant and that mass lumping is used for the Lagrange multiplier η_h and the auxiliary variable s_h the minimization with respect to p_h and s_h can be done explicitly on each element and in each node, respectively. We approximate a saddle-point with the following iterative scheme (cf. [13]).

Algorithm 5.1 (Split-split method)**.** *Choose* $(p_h^0, s_h^0) \in \mathcal{L}^0(\mathcal{T}_h)^d \times \mathcal{S}^1(\mathcal{T}_h)$ *and* $(\lambda_h^0, \eta_h^0) \in \mathcal{L}^0(\mathcal{T}_h)^d \times \mathcal{S}^1(\mathcal{T}_h)$ *and set* $j = 0$.
(1) Compute the minimizer u_h^{j+1} *of*

$$u_h \mapsto L_h(u_h, p_h^j, s_h^j; \lambda_h^j, \eta_h^j),$$

i.e., find $u_h^{j+1} \in \mathcal{S}^1(\mathcal{T}_h)$ *such that*

$$\begin{aligned} \alpha(u_h^{j+1} - g, v_h) - (\lambda_h^j, \nabla v_h)_w - \sigma_1(p_h^j - \nabla u_h^{j+1}, \nabla v_h)_w& \\ - (\eta_h^j, v_h)_h - \sigma_2(s_h^j - u_h^{j+1}, v_h)_h &= 0 \end{aligned}$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$.
(2) Compute the minimizing (p_h^{j+1}, s_h^{j+1}) *of*

$$(p_h, s_h) \mapsto L_h(u_h^{j+1}, p_h, s_h; \lambda_h^j, \eta_h^j),$$

i.e., define $(p_h^{j+1}, s_h^{j+1}) \in \mathcal{L}^0(\mathcal{T}_h)^d \times \mathcal{S}^1(\mathcal{T}_h)$ *via*

$$p_h^{j+1} = \frac{1}{\sigma_1}\big(|\sigma_1 \nabla u_h^{j+1} - \lambda_h^j| - h^{-d}\big)_+ \frac{\sigma_1 \nabla u_h^{j+1} - \lambda_h^j}{|\sigma_1 \nabla u_h^{j+1} - \lambda_h^j|},$$

where $(t)_+ = \max\{t, 0\}$, *and*

$$s_h(z) = \max\Big\{\chi(z), \min\Big\{u_h^{j+1}(z) - \frac{\eta_h^j(z)}{\sigma_2}, \psi(z)\Big\}\Big\}.$$

for all $z \in \mathcal{N}_h$.
(3) Update

$$\begin{aligned}\lambda_h^{j+1} &= \lambda_h^j + \sigma_1(p_h^{j+1} - \nabla u_h^{j+1}),\\ \eta_h^{j+1} &= \eta_h^j + \sigma_2(s_h^{j+1} - u_h^{j+1}),\end{aligned}$$

and stop if $\frac{1}{\sigma_1}\|\lambda_h^{j+1} - \lambda_h^j\|_w + \|p_h^{j+1} - p_h^j\|_w \le \varepsilon_{stop}$ *and* $\frac{1}{\sigma_2}\|\eta_h^{j+1} - \eta_h^j\|_h + \|s_h^{j+1} - s_h^j\|_h \le \varepsilon_{stop}$. *Increase* $j \to j+1$ *and continue with (1) otherwise.*

The following stability estimate for the split-split method is a consequence of general results for splitting methods, see, e.g., [13, 14], and arguments exploiting only the (strong) convexity of the involved functionals, which have been used by Wu and Tai in [21] for the classical ROF problem. It implies the convergence $\|\lambda_h^{j+1} - \lambda_h^j\|_w + \|p_h^{j+1} - p_h^j\|_w \to 0$ and $\|\eta_h^{j+1} - \eta_h^j\|_h + \|s_h^{j+1} - s_h^j\|_h \to 0$, i.e., Algorithm 5.1 terminates.

Proposition 5.2. *Let* $(u_h, p_h, s_h; \lambda_h, \eta_h)$ *be a saddle-point of* L_h *with* $p_h = \nabla u_h$ *and* $s_h = u_h$. *For arbitrary initializations of Algorithm 5.1 we have for any* $J \ge 0$

$$\begin{aligned}&\frac{1}{2}\big(\|\eta_h - \eta_h^{J+1}\|_h^2 + \|\lambda_h - \lambda_h^{J+1}\|_w^2 + \sigma_1^2\|p_h - p_h^{J+1}\|_w^2 + \sigma_2^2\|s_h - s_h^{J+1}\|_h^2\big)\\ &+ \frac{1}{2}\sum_{j=0}^{J}\big(\sigma_1\|p_h^{j+1} - p_h^j\|_w^2 + \sigma_2\|s_h^{j+1} - s_h^j\|_h^2\\ &\quad + \frac{1}{\sigma_1}\|\lambda_h^{j+1} - \lambda_h^j\|_w^2 + \frac{1}{\sigma_2}\|\eta_h^{j+1} - \eta_h^j\|_h^2 + \alpha\|u_h - u_h^{j+1}\|^2\big)\\ &\quad \le \frac{1}{2}\big(\|\eta_h - \eta_h^0\|_h^2 + \|\lambda_h - \lambda_h^0\|_w^2 + \sigma_1^2\|\nabla u_h - p_h^0\|_w^2 + \sigma_2^2\|u_h - s_h^0\|_h^2\big).\end{aligned}$$

5.2. **Regularization and penalization.** Another way to approximate the discrete minimizer u_h of E is to regularize the BV-seminorm, introduce an auxiliary variable $s_h = u_h$ and to penalize the difference $s_h - u_h$ with a penalization parameter $\delta > 0$ while allowing u_h to penetrate the obstacles. We obtain the functional

$$E_{h,\varepsilon,\delta}(v_h, r_h) := \int_\Omega |\nabla v_h|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|v_h - g\|^2 + \frac{1}{2\delta}\|r_h - v_h\|_h^2 + I_K(r_h),$$

where $|\nabla v_h|_\varepsilon = (|\nabla v_h|^2 + \varepsilon^2)^{1/2}$ and $\varepsilon, \delta > 0$. The introduction of s_h decouples the two nonlinearities. A similar approach has been used by Hintermüller et al. in [15] for the predual of the classical ROF problem and an alternate minimization technique has been employed for the numerical solution.

Remark 5.3. *Note that in contrast to the other two methods, the constraint $u_h \in K$ is enforced only in the limit as $\delta \to 0$.*

We can establish the existence of a discrete minimizer $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ of $E_{h,\varepsilon,\delta}$ using the direct method in the calculus of variations. Let $\Pi_{K,h}$ be the projection operator from $\mathcal{S}^1(\mathcal{T}_h)$ onto K with respect to $(\cdot,\cdot)_h$, i.e., $\Pi_{K,h}$ is defined by

$$\Pi_{K,h} v_h(z) := \begin{cases} v_h(z), & \text{if } \chi(z) \le v_h(z) \le \psi(z), \\ \psi(z), & \text{if } v_h(z) > \psi(z), \\ \chi(z), & \text{if } v_h(z) < \chi(z), \end{cases}$$

for $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ and all $z \in \mathcal{N}_h$. The optimality conditions for a minimizing pair $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta})$ then imply the relation $s_{h,\varepsilon,\delta} = \Pi_{K,h} u_{h,\varepsilon,\delta}$.

Proposition 5.4. *Let $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ and $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ be the unique minimizer of E and a minimizer of $E_{h,\varepsilon,\delta}$, respectively, and let $(v_h, r_h) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ be arbitrary. Then we have*

$$\frac{\alpha}{2}\|v_h - u_{h,\varepsilon,\delta}\|^2 \le E_{h,\varepsilon,\delta}(v_h, r_h) - E_{h,\varepsilon,\delta}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}), \tag{3}$$

and in particular

$$\begin{aligned} \frac{\alpha}{2}\|u_h - u_{h,\varepsilon,\delta}\|^2 + \frac{1}{4\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 &\le c(\varepsilon + \delta), \\ \frac{\alpha}{2}\|u_h - \Pi_{K,h} u_{h,\varepsilon,\delta}\|^2 &\le c(\varepsilon + \delta). \end{aligned}$$

Proof. The proof of (3) is analogous to the proof of Lemma 3.3. Since $u_h \in K$ we can choose $(v_h, r_h) = (u_h, u_h)$ in (3). Observing that $|x| \le |x|_\varepsilon \le |x| + \varepsilon$ for $x \in \mathbb{R}^d$, using Lemma 2.1, the minimality property of u_h, the identity $s_{h,\varepsilon,\delta} = \Pi_{K,h} u_{h,\varepsilon,\delta}$, the estimate $\|\nabla \Pi_{K,h} u_{h,\varepsilon,\delta}\|_{L^1(\Omega)} \le \|\nabla u_{h,\varepsilon,\delta}\|_{L^1(\Omega)}$, a binomial formula and Young's inequality we deduce

$$\begin{aligned} \frac{\alpha}{2}\|u_h - u_{h,\varepsilon,\delta}\|^2 &\le E_{h,\varepsilon,\delta}(u_h, u_h) - E_{h,\varepsilon,\delta}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \\ &= \int_\Omega |\nabla u_h|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2 \\ &\qquad - \int_\Omega |\nabla u_{h,\varepsilon,\delta}|_\varepsilon \,\mathrm{d}x - \frac{\alpha}{2}\|u_{h,\varepsilon,\delta} - g\|^2 - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2 \\ &\le \varepsilon|\Omega| + \int_\Omega |\nabla u_h| \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_\Omega |\nabla u_{h,\varepsilon,\delta}| \,\mathrm{d}x - \frac{\alpha}{2}\|u_{h,\varepsilon,\delta} - g\|^2 - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 \\
&\le \varepsilon|\Omega| + \int_\Omega |\nabla \Pi_{K,h} u_{h,\varepsilon,\delta}| \,\mathrm{d}x + \frac{\alpha}{2}\|\Pi_{K,h} u_{h,\varepsilon,\delta} - g\|^2 \\
&\quad - \int_\Omega |\nabla u_{h,\varepsilon,\delta}| \,\mathrm{d}x - \frac{\alpha}{2}\|u_{h,\varepsilon,\delta} - g\|^2 - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 \\
&\le \varepsilon|\Omega| - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 + \frac{\alpha}{2}\int_\Omega (s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta})(s_{h,\varepsilon,\delta} + u_{h,\varepsilon,\delta} - 2g) \,\mathrm{d}x \\
&\le \varepsilon|\Omega| - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 + \frac{1}{4\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 \\
&\quad + \frac{\alpha^2}{4}\delta\|s_{h,\varepsilon,\delta} + u_{h,\varepsilon,\delta} - 2g\|^2 \\
&= \varepsilon|\Omega| - \frac{1}{4\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 + \frac{\alpha^2}{4}\delta\|s_{h,\varepsilon,\delta} + u_{h,\varepsilon,\delta} - 2g\|^2,
\end{aligned}
$$

which implies the second estimate. To prove the third estimate we note that similarly to Lemma 3.3 we have

$$
\frac{\alpha}{2}\|v_h - u_h\|^2 \le E(v_h) - E(u_h)
$$

for arbitrary $v_h \in \mathcal{S}^1(\mathcal{T}_h) \cap K$. Choosing $v_h = s_{h,\varepsilon,\delta}$ we deduce

$$
\begin{aligned}
\frac{\alpha}{2}\|u_h - s_{h,\varepsilon,\delta}\|^2 &\le \int_\Omega |\nabla s_{h,\varepsilon,\delta}| \,\mathrm{d}x + \frac{\alpha}{2}\|s_{h,\varepsilon,\delta} - g\|^2 \\
&\qquad - \int_\Omega |\nabla u_h| \,\mathrm{d}x - \frac{\alpha}{2}\|u_h - g\|^2 \\
&\le \int_\Omega |\nabla s_{h,\varepsilon,\delta}|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|s_{h,\varepsilon,\delta} - g\|^2 \\
&\qquad - \int_\Omega |\nabla u_h|_\varepsilon \,\mathrm{d}x - \frac{\alpha}{2}\|u_h - g\|^2 + \varepsilon|\Omega| \\
&\le \int_\Omega |\nabla s_{h,\varepsilon,\delta}|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|s_{h,\varepsilon,\delta} - g\|^2 \\
&\qquad - \int_\Omega |\nabla u_{h,\varepsilon,\delta}|_\varepsilon \,\mathrm{d}x - \frac{\alpha}{2}\|u_{h,\varepsilon,\delta} - g\|^2 + \varepsilon|\Omega| \\
&\le \varepsilon|\Omega| + \frac{\alpha}{2}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|\|s_{h,\varepsilon,\delta} + u_{h,\varepsilon,\delta} - 2g\|,
\end{aligned}
$$

where the third inequality is due to the fact that

$$
\begin{aligned}
\int_\Omega |\nabla u_{h,\varepsilon,\delta}|_\varepsilon + &\frac{\alpha}{2}\|u_{h,\varepsilon,\delta} - g\|^2 \\
&\le \int_\Omega |\nabla u_{h,\varepsilon,\delta}|_\varepsilon + \frac{\alpha}{2}\|u_{h,\varepsilon,\delta} - g\|^2 + \frac{1}{2\delta}\|u_{h,\varepsilon,\delta} - s_{h,\varepsilon,\delta}\|_h^2 \\
&\le \int_\Omega |\nabla u_h|_\varepsilon + \frac{\alpha}{2}\|u_h - g\|^2.
\end{aligned}
$$

The second estimate then implies the third estimate. □

Remark 5.5. *With a view to recovering the convergence rate from Proposition 4.1 we choose $\varepsilon = h^{1/2}/|\Omega|$ and $\delta = h^{1/2}/\alpha$ or $\varepsilon = h/|\Omega|$ and $\delta = h/\alpha$ to recover the optimal convergence rate $\mathcal{O}(h^{1/2})$, cf.* [2, Rmk. 10.9 (ii)] *and* [4, Thm. 6.5].

In order to define a stable numerical method for the approximation of a minimizer $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ of $E_{h,\varepsilon,\delta}$ we consider for an auxiliary variable $p_h = |\nabla u_{h,\varepsilon,\delta}|_\varepsilon^{1/2}$ the functional

$$\begin{aligned}\widetilde{E}_{h,\varepsilon,\delta}(v_h, q_h, r_h) := \int_\Omega \frac{|\nabla v_h|^2 + \varepsilon^2}{2q_h^2} + \frac{q_h^2}{2}\,\mathrm{d}x + \frac{\alpha}{2}\|v_h - g\|^2 \\ + \frac{1}{2\delta}\|r_h - v_h\|_h^2 + I_K(r_h).\end{aligned}$$

Note that for fixed (v_h, r_h) it follows that

$$\min_{q_h \in \mathcal{L}^0(\mathcal{T}_h), q_h > 0} \widetilde{E}_{h,\varepsilon,\delta}(v_h, q_h, r_h) = E_{h,\varepsilon,\delta}(v_h, r_h)$$

with optimal q_h given by

$$q_h = (|\nabla v_h|^2 + \varepsilon^2)^{1/4}.$$

Due to the seperate convexity of the functional $\widetilde{E}_{h,\varepsilon,\delta}$ with respect to v_h, q_h and r_h we use a decoupled semi-implicit subgradient flow whose iterates can be determined explicitly.

Algorithm 5.6 (Heron-penalty method)**.** *Choose $(u_h^0, p_h^0, s_h^0) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ such that $\chi \le s_h^0 \le \psi$ and $p_h^0 \ge \sqrt{\varepsilon}$. Set $j = 0$.*
(1) Compute the minimizer u_h^{j+1} of

$$u_h \mapsto \widetilde{E}_{h,\varepsilon,\delta}(u_h, p_h^j, s_h^j) + \frac{1}{2\tau}\|u_h - u_h^j\|^2,$$

i.e., $u_h^{j+1} \in \mathcal{S}^1(\mathcal{T}_h)$ such that

$$\begin{aligned}(-d_t u_h^{j+1}, v_h) = \int_\Omega \frac{\nabla u_h^{j+1} \cdot \nabla v_h}{(p_h^j)^2}\,\mathrm{d}x \\ + \alpha(u_h^{j+1} - g, v_h) - \frac{1}{\delta}(s_h^j - u_h^{j+1}, v_h)_h\end{aligned}$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$.
(2) Compute the minimizing (p_h^{j+1}, s_h^{j+1}) of

$$(p_h, s_h) \mapsto I_{h,\varepsilon,\delta}(u_h^{j+1}, p_h, s_h) + \frac{1}{2\tau}\|p_h - p_h^j\|^2 + \frac{1}{2\tau}\|s_h - s_h^j\|_h^2,$$

i.e., define $(p_h^{j+1}, s_h^{j+1}) \in \mathcal{L}^0(\mathcal{T}_h)^d \times \mathcal{S}^1(\mathcal{T}_h)$ via

$$(-d_t p_h^{j+1}, q_h) = \int_\Omega \left(p_h^{j+1} - \frac{|\nabla u_h^{j+1}|^2 + \varepsilon^2}{(p_h^{j+1})^3}\right) q_h\,\mathrm{d}x$$

for all $q_h \in \mathcal{L}^0(\mathcal{T}_h)$*, and*

$$s_h^{j+1}(z) = \max\Big\{\chi(z), \min\Big\{\frac{\delta s_h^j(z) + \tau u_h^{j+1}(z)}{\tau+\delta}, \psi(z)\Big\}\Big\}$$

for all $z \in \mathcal{N}_h$*.*
(3) Stop if $\|d_t u_h^{j+1}\| + \|d_t p_h^{j+1}\| + \|d_t s_h^{j+1}\|_h \le \varepsilon_{stop}$*. Otherwise, increase* $j \to j+1$ *and continue with (1).*

The following proposition guarantees stability and convergence of the algorithm.

Proposition 5.7. *Let* $\tau > 0$*,* $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)$*,* $s_h^0 \in \mathcal{S}^1(\mathcal{T}_h)$ *with* $\chi \le s_h^0 \le \psi$ *and* $p_h^0 \in \mathcal{L}^0(\mathcal{T}_h)$ *with* $p_h^0 > 0$ *be arbitrary initializations of Algorithm 5.6 and let the sequences* $(u_h^j)_{j\in\mathbb{N}}$*,* $(s_h^j)_{j\in\mathbb{N}}$ *and* $(p_h^j)_{j\in\mathbb{N}}$ *be generated by Algorithm 5.6. Then for arbitrary* $J \in \mathbb{N}$ *the stability estimate*

$$\tau \sum_{j=0}^{J} \|d_t u_h^{j+1}\|^2 + \|d_t p_h^{j+1}\|^2 + \|d_t s_h^{j+1}\|_h^2 + \widetilde{E}_{h,\varepsilon,\delta}(u_h^J, p_h^J, s_h^J) \le \widetilde{E}_{h,\varepsilon,\delta}(u_h^0, p_h^0, s_h^0), \tag{4}$$

holds. Particularly, Algorithm 5.6 terminates and $\big(\widetilde{E}_{h,\varepsilon,\delta}(u_h^j, p_h^j, s_h^j)\big)_{j\in\mathbb{N}}$ *is monotonically decreasing. Furthermore, every convergent subsequence* $(u_h^{j_\ell}, s_h^{j_\ell})_{\ell\in\mathbb{N}} \subset \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ *converges to a minimizer* $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ *of* $E_{h,\varepsilon,\delta}$*.*

Proof. The proof of the stability estimate follows as in the proof of [3, Theorem III.4] by choosing $-d_t u_h^{j+1}$, $-d_t p_h^{j+1}$ and $-d_t s_h^{j+1}$ in the optimality conditions defining the iterates in Algorithm 5.6 and using the seperate convexity of $\widetilde{E}_{h,\varepsilon,\delta}$. The stability estimate (4) implies that the sequence $(u_h^j)_j$ is bounded. We choose a subsequence $(u_h^{j_\ell})_\ell$ which converges to some $u_{h,\varepsilon,\delta} \in \mathcal{S}^1(\mathcal{T}_h)$. Since $d_t u_h^j \to 0$ it follows that $u_h^{j_\ell+1} \to u_{h,\varepsilon,\delta}$ as well and by the equivalence of norms on finite-dimensional spaces we also have $\nabla u_h^{j_\ell} \to \nabla u_{h,\varepsilon,\delta}$ and $\nabla u_h^{j_\ell+1} \to \nabla u_{h,\varepsilon,\delta}$. Furthermore, since $d_t p_h^{j_\ell+1} \to 0$ the optimality condition for the iterates $(p_h^{j_\ell+1})_\ell$ implies

$$\frac{|\nabla u_h^{j_\ell+1}|^2 + \varepsilon^2}{(p_h^{j_\ell+1})^3} - p_h^{j_\ell+1} \to 0,$$

which means that $p_h^{j_\ell} \to (|\nabla u_{h,\varepsilon,\delta}|^2 + \varepsilon^2)^{1/4}$. The optimality condition for the iterate $u_h^{j_\ell+1}$ and $d_t u_h^{j_\ell+1} \to 0$ then imply that

$$\frac{1}{\delta}(s_h^{j_\ell} - u_h^{j_\ell+1}, v_h)_h \to \int_\Omega \frac{\nabla u_{h,\varepsilon,\delta}\cdot\nabla v_h}{(|\nabla u_{h,\varepsilon,\delta}|^2+\varepsilon^2)^{1/2}}\,\mathrm{d}x + \alpha\int_\Omega (u_{h,\varepsilon,\delta} - g)v_h\,\mathrm{d}x$$

as $\ell \to \infty$ for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$, i.e., $(s_h^{j_\ell})_\ell$ converges to some function $s_{h,\varepsilon,\delta} \in \mathcal{S}^1(\mathcal{T}_h)$. The optimality condition for the iterate $s_h^{j_\ell+1}$ reads

$$(-d_t s_h^{j_\ell+1}, r_h - s_h^{j_\ell+1})_h + \frac{1}{\delta}(u_h^{j_\ell+1} - s_h^{j_\ell+1}, r_h - s_h^{j_\ell+1})_h + I_K(s_h^{j_\ell+1}) \le I_K(r_h)$$

for all $r_h \in K \cap \mathcal{S}^1(\mathcal{T}_h)$. This means that for $\ell \to \infty$ we have

$$\frac{1}{\delta}(u_{h,\varepsilon,\delta} - s_{h,\varepsilon,\delta}, r_h - s_{h,\varepsilon,\delta})_h + I_K(s_{h,\varepsilon,\delta}) \le I_K(r_h)$$

for all $r_h \in K \cap \mathcal{S}^1(\mathcal{T}_h)$, i.e., $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta})$ satisfies the necessary and sufficient optimality conditions for a minimizer of $E_{h,\varepsilon,\delta}$. □

Remark 5.8. *Note that due to Proposition 5.4 all minimizers $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta})$ of $E_{h,\varepsilon,\delta}$ are within the ball in $L^2(\Omega)$ around the unique minimizer u_h of E with radius $c(\varepsilon+\delta)$. Therefore, by Proposition 5.7 the iterates (u_h^j, s_h^j) generated by Algorithm 5.6 get arbitrarily close to this ball around u_h.*

5.3. **Regularization and splitting.** Alternatively, we may consider only a regularization of the BV-seminorm and seek a minimizer $u_{h,\varepsilon} \in \mathcal{S}^1(\mathcal{T}_h)$ of the functional

$$E_\varepsilon(v_h) := \int_\Omega |\nabla v_h|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|v_h - g\|^2 + I_K(v_h).$$

In order to enforce the constraint strictly we can use an augmented Lagrangian ansatz, i.e., work with the regularized augmented Lagrangian functional

$$\begin{aligned} L_{h,\varepsilon}(v_h, r_h; \nu_h) := &\int_\Omega |\nabla v_h|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|v_h - g\|^2 \\ &+ (\nu_h, r_h - v_h)_h + \frac{\sigma}{2}\|r_h - v_h\|_h^2 + I_K(r_h) \end{aligned}$$

to approximate the unique minimizer $u_{h,\varepsilon} \in \mathcal{S}^1(\mathcal{T}_h)$ of E_ε. We can use the following iteration to approximate the unique minimizer $u_{h,\varepsilon}$.

Algorithm 5.9 (Regularized splitting method)**.** *Choose $\eta_h^0 \in \mathcal{S}^1(\mathcal{T}_h)$ and set $j = 0$.*
(1) Compute a minimizer $(u_h^{j+1}, s_h^{j+1}) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ of

$$(u_h, s_h) \mapsto L_{h,\varepsilon}(u_h, s_h; \eta_h^j).$$

(2) Update

$$\eta_h^{j+1} = \eta_h^j + \sigma(s_h^{j+1} - u_h^{j+1}).$$

and stop if $\frac{1}{\sigma}\|\eta_h^{j+1} - \eta_h^j\|_h \le \varepsilon_{stop}$.

Remark 5.10. *The stability (and convergence) of Algorithm 5.9 follows from the theory of augmented Lagrangian methods (cf.* [13, 14]*) using arguments from* [21] *which avoid differentiability assumptions on the involved functionals but exploit their (strong) convexity.*

Since the minimizer in step (1) of Algorithm 5.9 is not directly accessible we consider for $j \in \mathbb{N}$ the augmented functional

$$\widetilde{L}_{h,\varepsilon}(u_h, p_h, s_h; \eta_h^j) := \int_\Omega \frac{|\nabla u_h|^2 + \varepsilon^2}{2p_h^2} + \frac{p_h^2}{2}\,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2 \\ + (\eta_h^j, s_h - u_h)_h + \frac{\sigma}{2}\|s_h - u_h\|_h^2 + I_K(s_h).$$

Noting that $\widetilde{L}_{h,\varepsilon}$ is separately convex in u_h, p_h and s_h we use as in the Heron-penalty method a semi-implicit subgradient flow to approximate a minimizer (u_h^{j+1}, s_h^{j+1}) of $L_{h,\varepsilon}(u_h, s_h; \eta_h^j)$ in the $(j+1)$-th iteration.

Algorithm 5.11 (Heron-split method)**.** *Choose* $(u_h^0, p_h^0, s_h^0) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ *and* $\eta_h^0 \in \mathcal{S}^1(\mathcal{T}_h)$ *such that* $\chi_h \le s_h^0 \le \psi_h$ *and* $p_h^0 \ge \sqrt{\varepsilon}$. *Set* $j = 0$.
(1) Set $u_h^{j+1,0} = u_h^j$, $p_h^{j+1,0} = p_h^j$ *and* $s_h^{j+1,0} = s_h^j$.
(2) For $\ell = 0, \dots, M-1$*: first compute the minimizer* $u_h^{j+1,\ell+1}$ *of*

$$u_h \mapsto \widetilde{L}_{h,\varepsilon}(u_h, p_h^{j+1,\ell}, s_h^{j+1,\ell}; \eta_h^j) + \frac{1}{2\tau}\|u_h - u_h^{j+1,\ell}\|^2,$$

and then compute the minimizing $(p_h^{j+1,\ell+1}, s_h^{j+1,\ell+1})$ *of*

$$(p_h, s_h) \mapsto \widetilde{L}_{h,\varepsilon}(u_h^{j+1,\ell+1}, p_h, s_h; \eta_h^j) + \frac{1}{2\tau}\|p_h - p_h^{j+1,\ell}\|^2 + \frac{1}{2\tau}\|s_h - s_h^{j+1,\ell}\|_h^2.$$

(3) Set $u_h^{j+1} = u_h^{j+1,M}$, $p_h^{j+1} = p_h^{j+1,M}$ *and* $s_h^{j+1} = s_h^{j+1,M}$.
(4) Update

$$\eta_h^{j+1} = \eta_h^j + \sigma(s_h^{j+1} - u_h^{j+1}),$$

and stop if $\frac{1}{\sigma}\|\eta_h^{j+1} - \eta_h^j\|_h \le \varepsilon_{stop}$. *Otherwise, increase* $j \to j+1$ *and continue with (1).*

Remark 5.12. *(i) Instead of the fixed number* M *of inner iterations in step (2) of Algorithm 5.11 one can also stop the inner iteration using the stopping criterion*

$$\|d_t u_h^{j+1,\ell+1}\| + \|d_t p_h^{j+1,\ell+1}\| + \|d_t s_h^{j+1,\ell+1}\|_h \le \widetilde{\varepsilon}_{stop}.$$

(ii) In our experiments we will set $M = 1$ *and use the stopping criterion*

$$\|d_t u_h^{j+1}\| + \|d_t p_h^{j+1}\| + \|d_t s_h^{j+1}\|_h + \frac{1}{\sigma}\|\eta_h^{j+1} - \eta_h^j\|_h \le \varepsilon_{stop}$$

in step (4) of Algorithm 5.11. We will refer to this setting as the Heron-split method.

6. Application to piecewise constant segmentation

In this section we investigate the constrained minimization problem

$$\min_{u \in BV(\Omega)} E^{seg}(u) := \min_{u \in BV(\Omega)} \int_\Omega |Du| + \alpha \int_\Omega f u\,\mathrm{d}x + I_K(u)$$

with $f \in L^\infty(\Omega)$ and $K := \{v \in L^2(\Omega) : \, 0 \le v \le 1 \text{ a.e.}\}$. This constrained minimization problem occurs, e.g., in the piecewise constant segmentation of images discussed in [10] where one aims at minimizing for a given image $g \in L^\infty(\Omega)$ the nonconvex functional

$$MS(\Sigma, c_1, c_2) := \int_\Omega |D\chi_\Sigma| + \alpha \int_\Sigma (c_1 - g)^2 \,\mathrm{d}x + \alpha \int_{\Omega\setminus\Sigma} (c_2 - g)^2 \,\mathrm{d}x$$

with χ_Σ being the characteristic function of the unknown set Σ. The authors propose an alternating minimization with respect to the unknown variables Σ, c_1 and c_2. For a fixed set Σ the optimal c_1 and c_2 are given by

$$c_1 = \frac{1}{|\Sigma|} \int_\Sigma g \,\mathrm{d}x \quad \text{and} \quad c_2 = \frac{1}{|\Omega \setminus \Sigma|} \int_{\Omega\setminus\Sigma} g \,\mathrm{d}x.$$

However, the major difficulty amounts to solving the minimization problem defined by the functional $MS(\cdot, c_1, c_2)$ for fixed c_1 and c_2. Berkels et al. considered a relaxed functional of $MS(\cdot, c_1, c_2)$ in [6], employed a finite element discretization and used the primal-dual method proposed by Chambolle and Pock in [9] using adaptive refinement techniques as well.
The crucial observation is that one may solve this global optimization problem defined by $MS(\cdot, c_1, c_2)$ by computing a minimizer u of E^{seg} with $\chi \equiv 0$, $\psi \equiv 1$ and

$$f := (c_1 - g)^2 - (c_2 - g)^2.$$

Then, for almost every $\gamma \in [0, 1]$, the set

$$\Sigma := \{x \in \mathbb{R}^d : u(x) \ge \gamma\}$$

is a minimizer of $MS(\cdot, c_1, c_2)$, cf. [10, Thm. 2].

One can establish the existence of a minimizer $u \in BV(\Omega) \cap K$ of E^{seg}, however, the minimizer may not be unique. Restricting the minimization of E^{seg} to $\mathcal{S}^1(\mathcal{T}_h)$ then also provides a minimizer $u_h \in \mathcal{S}^1(\mathcal{T}_h)$. Analogously to Lemma 4.1 we obtain the following estimate.

Lemma 6.1. *Let u and u_h be minimizers of E^{seg} on $BV(\Omega)$ and on $\mathcal{S}^1(\mathcal{T}_h)$, respectively. We then have*

$$E^{seg}(u_h) - E^{seg}(u) \le ch^{1/2}.$$

Proof. The proof is analogous to that of Lemma 4.1 and we therefore omit it. □

The authors in [10] furthermore observed that the functional

$$\widetilde{E}^{seg}(u) := \int_\Omega |Du| + \int_\Omega \beta\theta(u) + \alpha f u \,\mathrm{d}x,$$

where $\theta(\xi) := \max\{0, 2|\xi - \frac{1}{2}| - 1\}$, has the same minimizers as E^{seg} provided that $\beta > \frac{\alpha}{2}\|f\|_{L^\infty(\Omega)}$, cf. [10, Claim 1]. They employ a regularization of the BV-seminorm and of the function θ and use an explicit gradient descent scheme to approximate a minimizer of $\widetilde{E}^{seg}$ and E^{seg}, respectively.

We remark that this procedure may be practical in this specific situation, however, the construction of the functional $\widetilde{E}^{seg}$ only enables one to deal with this particular constrained minimization problem defined by E^{seg}. Since we can adapt the iterative schemes presented in Section 5 to the minimization of E^{seg} after making minor adjustments we will also present numerical experiments for the piecewise constant segmentation problem in the next section. We conclude this section with a result which is analogous to Proposition 5.4.

Proposition 6.2. *Let $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ and $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{S}^1(\mathcal{T}_h)$ be minimizers of E^{seg} and of the functional*

$$E^{seg}_{h,\varepsilon,\delta}(v_h, r_h) := \int_\Omega |\nabla v_h|_\varepsilon \,\mathrm{d}x + \alpha \int_\Omega f u_h \,\mathrm{d}x + \frac{1}{2\delta}\|v_h - r_h\|_h^2 + I_K(r_h),$$

respectively. Then we have

$$E^{seg}(u_h) - E^{seg}_{h,\varepsilon,\delta}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) + \frac{1}{4\delta}\|u_{h,\varepsilon,\delta} - s_{h,\varepsilon,\delta}\|^2 \le c(\varepsilon + \delta).$$

Proof. Observing that for $a \in \mathbb{R}^d$ we have $|a| \le |a|_\varepsilon \le |a| + \varepsilon$ and that $\Pi_K u_{h,\varepsilon,\delta} = s_{h,\varepsilon,\delta}$, and using the arguments in the proof of Proposition 5.4 we obtain

$$\begin{aligned}
E^{seg}(u_h) - E_{h,\varepsilon,\delta}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) &\le E_{h,\varepsilon,\delta}(u_h, u_h) - E_{h,\varepsilon,\delta}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \\
&= \int_\Omega |\nabla u_h|_\varepsilon \,\mathrm{d}x + \alpha \int_\Omega f u_h \,\mathrm{d}x \\
&\quad - \int_\Omega |\nabla u_{h,\varepsilon,\delta}|_\varepsilon \,\mathrm{d}x - \alpha \int_\Omega f u_{h,\varepsilon,\delta} \,\mathrm{d}x - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2 \\
&\le \varepsilon|\Omega| + \int_\Omega |\nabla u_h| \,\mathrm{d}x + \alpha \int_\Omega f u_h \,\mathrm{d}x \\
&\quad - \int_\Omega |\nabla u_{h,\varepsilon,\delta}| \,\mathrm{d}x - \alpha \int_\Omega f u_{h,\varepsilon,\delta} \,\mathrm{d}x - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 \\
&\le \varepsilon|\Omega| + \int_\Omega |\nabla \Pi_K u_{h,\varepsilon,\delta}| \,\mathrm{d}x + \alpha \int_\Omega f \Pi_K u_{h,\varepsilon,\delta} \,\mathrm{d}x \\
&\quad - \int_\Omega |\nabla u_{h,\varepsilon,\delta}| \,\mathrm{d}x - \alpha \int_\Omega f u_{h,\varepsilon,\delta} \,\mathrm{d}x - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 \\
&\le \varepsilon|\Omega| - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 + \alpha\|f\|\|u_{h,\varepsilon,\delta} - s_{h,\varepsilon,\delta}\| \\
&\le \varepsilon|\Omega| - \frac{1}{2\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 + \frac{1}{4\delta}\|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 + \alpha^2\delta\|f\|^2.
\end{aligned}$$

This proves the assertion. □

Remark 6.3. *In order to recover the convergence rate from Lemma 6.1 we choose $\delta = \frac{h^{1/2}}{\alpha\|f\|}$.*

7. Numerical Experiments

7.1. **Constrained ROF.** We investigate two examples which differ only in the choice of the parameter $\alpha > 0$. In both cases the algorithms are initialized as follows: we set $p_h^0 = \lambda_h^0 = 0$, $\eta_h^0 = 0$, $s_h^0 = (\chi + \psi)/2$ and $\sigma_1 = h^{-3/2}$ (see [3]) in Algorithm 5.1. We start Algorithm 5.6 with $\varepsilon = h$, $\tau = 1$, $u_h^0 = (\chi+\psi)/2$, $p_h^0 = \varepsilon^{1/2}$ and $s_h^0 = u_h^0$, and Algorithm 5.11 with $\varepsilon = h$, $\tau = 1$, $u_h^0 = (\chi+\psi)/2$, $p_h^0 = \varepsilon^{1/2}$, $s_h^0 = u_h^0$ and $\eta_h^0 = 0$. To compare the accuracy of the algorithms we use precomputed approximate minimizers $\widetilde{u}_h$ of E generated by Algorithm 5.1 with $\sigma_2 = \alpha$, $(\widetilde{u}_{h,\varepsilon,\delta}, \widetilde{s}_{h,\varepsilon,\delta})$ of $E_{h,\varepsilon,\delta}$ generated by Algorithm 5.6 with both $\delta = h/\alpha$ and $\delta = h$, and $\widetilde{u}_{h,\varepsilon}$ of E_ε generated by Algorithm 5.11 with $\sigma = \alpha$ after 10^4 iterations. The stopping criteria have been chosen as follows:

- Split-split method:

$$\|\widetilde{u}_h - u_h^j\| \le 10^{-3} \tag{5}$$

- Heron-penalty method:

$$\max\{\|\widetilde{u}_{h,\varepsilon,\delta} - u_h^j\|, \|\widetilde{s}_{h,\varepsilon,\delta} - s_h^j\|\} \le 10^{-3} \tag{6}$$

- Heron-split method:

$$\|\widetilde{u}_{h,\varepsilon} - u_h^j\| \le 10^{-3} \tag{7}$$

Example 7.1. *We let $d = 2$, $\Omega = (0,1)^2$, $\chi \equiv 1/5$, $\psi \equiv 1/2$ and $\alpha = 50$. Moreover, given a uniform triangulation $\mathcal{T}_5$ of Ω with mesh size $h = \sqrt{2}2^{-5}$, we let $\widetilde{g} \in \mathcal{S}^1(\mathcal{T}_5)$ to be the continuous, piecewise affine approximation of the characteristic function $\chi_{B_{1/5}(x_\Omega)}$ of the closed ball with radius $1/5$ around the center x_Ω of Ω, i.e., $\widetilde{g}$ is defined by*

$$\widetilde{g}(z) := \begin{cases} 1, & \text{if } z \in B_{1/5}(x_\Omega) \\ 0, & \text{else} \end{cases}$$

for $z \in \mathcal{N}_5$. We then set $g := \widetilde{g} + \xi_5$ where $\xi_5 \in \mathcal{S}^1(\mathcal{T}_5)$ whose coefficient vector is a sample of a random variable uniformly distributed in the interval $[-1/2, 1/2]$.

In Table 1 we displayed the iteration numbers needed by the algorithms to satisfy the stopping criteria (5), (6) and (7). Since the optimal choice of the step sizes σ_2 and σ for the split-split method and the Heron-split method, respectively, are not clear, we ran the algorithms for three different choices of step sizes. As one can observe the choice $\sigma_2 = \alpha$ and $\sigma = \alpha$, respectively, lead to the smallest iteration numbers which is due to the balance of fit-to-data and fit-to-constraint, i.e., when $\sigma_2 = \sigma = \alpha$ is chosen the split-split algorithm and the Heron-split method try to recover the datum g and to satisfy the bilateral constraint simultaneously. Regarding the Heron-penalty method, we ran the algorithm for the choices $\delta = h/\alpha$ and $\delta = h$. Obviously, due to a worse conditioning, the iteration numbers for the choice

	Split-split			Heron-penalty		Heron-split		
	σ_2			δ		σ		
h	1	α	$1/h$	h/α	h	1	α	$1/h$
$\sqrt{2}/2^5$	344	39	37	89	42	892	52	57
$\sqrt{2}/2^6$	289	79	79	143	55	576	77	77
$\sqrt{2}/2^7$	283	77	78	211	69	473	94	94
$\sqrt{2}/2^8$	287	115	120	426	101	373	148	151

TABLE 1. Iteration numbers with (5), (6), (7) for Example 7.1 and $\sigma_1 = h^{-3/2}$ for the split-split method and $\tau = 1$ for the Heron-penalty and the Heron-split method.

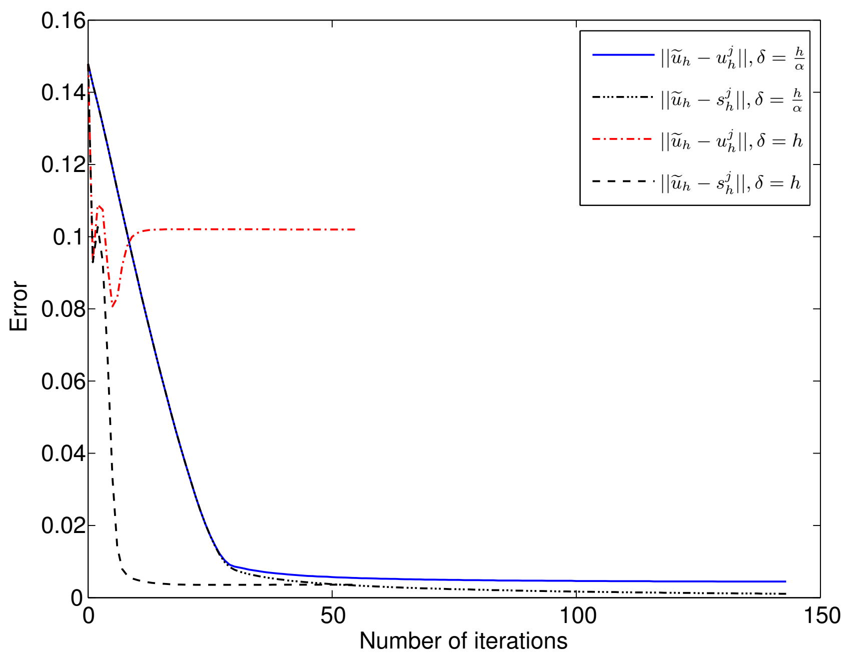


FIGURE 1. L^2-error between approximate minimizer $\widetilde{u}_h$ of E (generated by the split-split method) and iterates u_h^j, s_h^j of the Heron-penalty method, $h = \sqrt{2}2^{-6}$, $\varepsilon = h$, for Example 7.1. Note that the iterates u_h^j serve as good approximations of $\widetilde{u}_h$ when $\delta = h/\alpha$.

$\delta = h/\alpha$ are much larger than for $\delta = h$. However, for $\delta = h$ the iterates u_h^j generated by the Heron-penalty method stay far from the minimizer u_h of E and the distance corresponds to the magnitude of α as predicted by Proposition 5.4. This effect is also illustrated in Figure 1 where the L^2-error between the approximate minimizer $\widetilde{u}_h$ of E generated by the split-split method and the iterates u_h^j and s_h^j, respectively, generated by the Heron-penalty method for both $\delta = h/\alpha$ and $\delta = h$ is plotted against the number of iterations. For $\delta = h/\alpha$ we see that the final iterates u_h^N and s_h^N serve as good approximations of $\widetilde{u}_h$ and the approximation error is of order h. Yet, for $\delta = h$, the iterates u_h^j do not have practical approximation properties and one would have to decrease the mesh size significantly to obtain a similar accuracy as for $\delta = h/\alpha$. Still, one can also observe that the different choices of δ do not considerably affect the approximation properties of the iterates s_h^j, i.e., one may set $\delta = h$ and choose the final iterate s_h^N as an approximation of $\widetilde{u}_h$.

Example 7.2. *The setting is as in Example 7.1 except for the parameter α, which is set to $\alpha = 500$ here.*

In Table 2 the iteration numbers for Example 7.2 are displayed. We see that the effect of higher iteration numbers for the split-split method and the Heron-split method for the choice $\sigma_2 = \sigma = 1$ and for the choice $\delta = h/\alpha$ in the Heron-penalty method are now even more pronounced due to the choice of a large α. Moreover, Figure 2 shows once again that the iterates u_h^j generated by the Heron-penalty method for $\delta = h$ may not be used as a good approximation of $\widetilde{u}_h$ since the constant α affects the L^2-distance between u_h^j and $\widetilde{u}_h$ while the iterates s_h^j seem to define accurate approximations of $\widetilde{u}_h$ almost independently of the choice of δ.
The overall conclusion of the experiments is that the split-split method with $\sigma_2 = \alpha$, Heron-penalty method with $\delta = h$ and the Heron-split method with $\sigma = \alpha$ lead to the smallest iteration numbers and have a similar performance. An advantage of the Heron-penalty method is that the choice of parameters is clear while the choice of the step sizes σ_2 and σ in the split-split method and the Heron-split method, respectively, remain unclear in general. However, the penalization parameter δ in the Heron-penalty method has to scale as h/α for the iterates u_h^j to be close to u_h.

	Split-split			Heron-penalty		Heron-split		
	σ_2			δ		σ		
h	1	α	$1/h$	h/α	h	1	α	$1/h$
$\sqrt{2}/2^5$	2745	10	124	74	24	4592	15	189
$\sqrt{2}/2^6$	2808	15	65	156	27	3586	21	72
$\sqrt{2}/2^7$	2847	23	35	298	35	3133	25	42
$\sqrt{2}/2^8$	–	29	30	572	43	–	33	37

TABLE 2. Iteration numbers with (5), (6), (7) for Example 7.2 and $\sigma_1 = h^{-3/2}$ for the split-split method and $\tau = 1$ for the Heron-penalty and the Heron-split method.

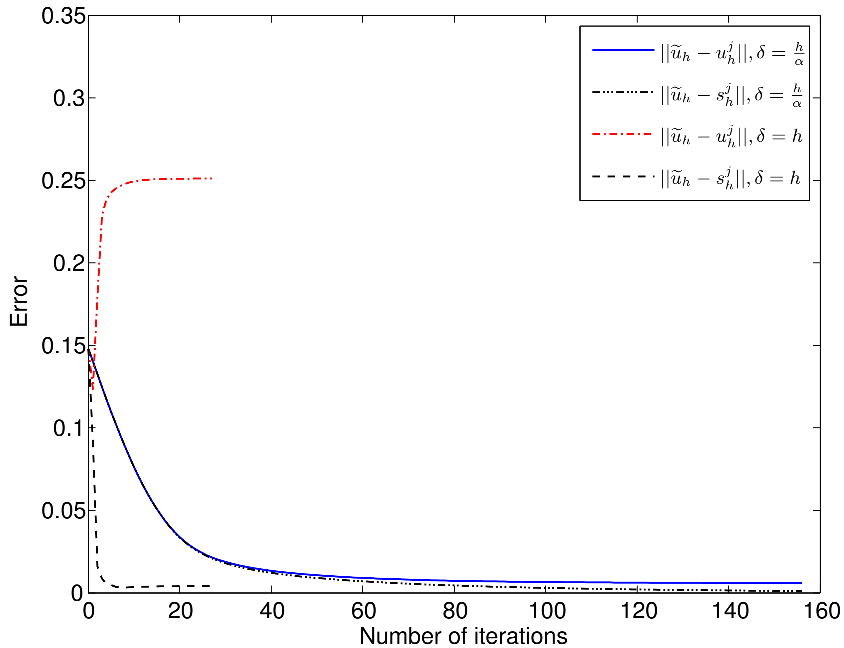


FIGURE 2. L^2-error between approximate minimizer $\widetilde{u}_h$ of E (generated by the split-split method) and iterates u_h^j, s_h^j of the Heron-penalty method, $h = \sqrt{2}2^{-6}$, $\varepsilon = h$, for Example 7.2. Again, the iterates u_h^j approximate $\widetilde{u}_h$ properly when $\delta = h/\alpha$.

7.2. **Piecewise constant segmentation.** In this section we report numerical experiments for the binary image segmentation model presented in Section 6. We consider the two-phase image segmentation of the image "cameraman". To this extent, we let g be the continuous piecewise affine function with coefficient vector whose entries consist of the (scaled) gray-values of the image ranging from 0 to 1. We further let $\alpha = 1500$, $\chi \equiv 0$, $\psi \equiv 1$, and consider the following algorithm of [10] which involves the solution of a constrained TV-minimization problem.

Algorithm 7.3. *Choose* $c_1^0, c_2^0 \in \mathbb{R}$ *and* $\gamma \in [0,1]$. *Set* $k = 0$.
(1) Compute for $f = (c_1^k - g)^2 - (c_2^k - g)^2$ *an approximate minimizer* $\widetilde{u}_h^{k+1} \in \mathcal{S}^1(\mathcal{T}_h)$ *of the functional*

$$E^{seg}(v_h) = \int_\Omega |\nabla v_h| + \alpha \int_\Omega f v_h \,\mathrm{d}x + I_K(v_h)$$

using either the split-split method or the Heron-penalty method or the Heron-split method.

(2) Define u_h^{k+1} by

$$u_h^{k+1}(z) = \begin{cases} 1, & \text{if } \widetilde{u}_h^{k+1}(z) \geq \gamma, \\ 0, & \text{if } \widetilde{u}_h^{k+1}(z) < \gamma, \end{cases}$$

for $z \in \mathcal{N}_h$. Set

$$\Sigma^{k+1} = \bigcup \{T \in \mathcal{T}_h : u_h^{k+1}|_T \equiv 1\}.$$

(3) Set

$$c_1^{k+1} = \frac{1}{|\Sigma^{k+1}|} \int_{\Sigma^{k+1}} g \, \mathrm{d}x \quad \text{and} \quad c_2^{k+1} = \frac{1}{|\Omega \setminus \Sigma^{k+1}|} \int_{\Omega \setminus \Sigma^{k+1}} g \, \mathrm{d}x.$$

(4) Stop if

$$\frac{|MS(\Sigma^{k+1}, c_1^{k+1}, c_2^{k+1}) - MS(\Sigma^k, c_1^k, c_2^k)|}{|MS(\Sigma^{k+1}, c_1^{k+1}, c_2^{k+1})|} \leq \varepsilon_{stop}.$$

Otherwise, increase $k \to k+1$ and continue with (1).

Using a uniform triangulation with mesh size $h \sim 2^{-8}$ we chose $c_1^0 = 1$ and $c_2^0 = 0$ and set the thresholding parameter to $\gamma = 1/2$. We set $\varepsilon_{stop} = 10^{-3}$ in step (4). In step (2), we initialize the algorithms as in the preceding subsection and used $\varepsilon = h^{1/2}$ for the Heron-penalty and the Heron-split method. We set $\delta = h^{1/2}$ in the Heron-penalty method and used $\sigma_2 = \sigma = \alpha\|f\|$ in the split-split and the Heron-split method and $\tau = 1$ in the Heron-penalty and the Heron-split method. The stopping criteria in step (2) were chosen as follows:

- Split-split method:

$$\begin{aligned} &\big(\|\lambda_h^{j+1} - \lambda_h^j\|_w^2 + h^{-3}\|p_h^{j+1} - p_h^j\|_w^2\big)^{1/2} \leq 10^{-3} \quad \text{and} \\ &\Big(\frac{1}{\sigma_2^2}\|\eta_h^{j+1} - \eta_h^j\|_h^2 + \|s_h^{j+1} - s_h^j\|_h^2\Big)^{1/2} \leq 10^{-3} \end{aligned} \tag{8}$$

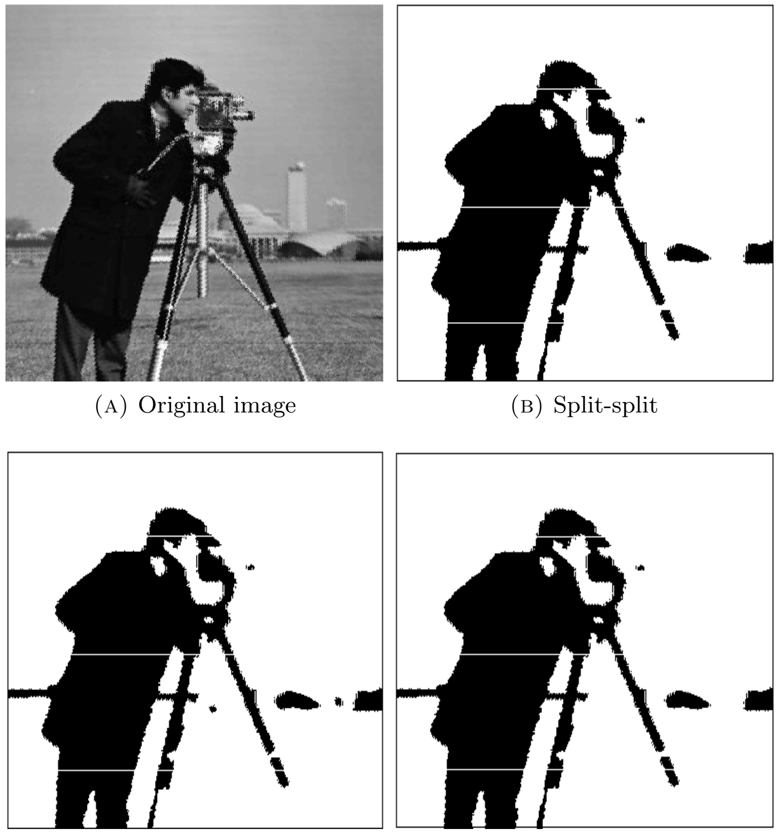
- Heron-penalty method:

$$\big(\|d_t u_h^{j+1}\|^2 + \|d_t p_h^{j+1}\|^2 + \|d_t s_h^{j+1}\|_h^2\big)^{1/2} \leq 10^{-3} \tag{9}$$

- Heron-split method:

$$\begin{aligned} &\big(\|d_t u_h^{j+1}\|^2 + \|d_t p_h^{j+1}\|^2\big)^{1/2} \leq 10^{-3} \quad \text{and} \\ &\Big(\frac{1}{\sigma^2}\|\eta_h^{j+1} - \eta_h^j\|_h^2 + \|s_h^{j+1} - s_h^j\|_h^2\Big)^{1/2} \leq 10^{-3} \end{aligned} \tag{10}$$

Figure 3 shows the original image and the outputs of Algorithm 7.3 using the split-split method, the Heron-penalty method and the Heron-split method in step (2), respectively, where we used the iterates s_h^j as approximations of a minimizer in step (2) in the Heron-penalty method. The three white horizontal lines in the outputs are due to image conversion. One can observe that all outputs are visually almost the same. The output of the split-split method and the Heron-split method visually do not differ at all, whereas



(A) Original image (B) Split-split

(C) Heron-penalize (D) Heron-split

FIGURE 3. Original image and outputs of Algorithm 7.3 using the split-split, Heron-penalty and Heron-split method in step (2), respectively (horizontal white lines are due to image conversion).

the image generated by the Heron-penalty method has two black dots which do not appear in the other two images. These dots are not contained in the output of the Heron-penalty method when using $\delta = \frac{h^{1/2}}{\alpha\|f\|}$. The final values of c_1 and c_2 were $c_1 = 0.5946$ and $c_2 = 0.1144$ for both the split-split method and the Heron-split method and $c_1 = 0.5951$, $c_2 = 0.1147$ for the Heron-penalty method. The final values of the energy $MS(\Sigma, c_1, c_2)$ were 25.3745, 25.3863 and 25.3711 for the split-split method, the Heron-penalty method and the Heron-split method, respectively. The split-split method needed 5000 iterations in total for termination (6 outer iterations), while the Heron-penalty method needed 1382 iterations in total (7 outer iterations) and the Heron-split method needed 857 iterations in total (6 outer iterations). The Heron-penalty method with $\delta = \frac{h^{1/2}}{\alpha\|f\|}$ needed 1675 iterations in total (6 outer iterations). Using u_h^j generated by the Heron-penalty method as approximations for the minimizer in step (2) led to the same results as for the case using s_h^j.

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