The geometry of the space of branched Rough Paths

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Introduction

Rough paths were introduced by Terry Lyons near the end of the 90's to deal with stochastic integration (and SDEs) in a path-wise sense.

Some years later Massimiliano Gubinelli introduced controlled rough paths, and brached Rough Paths a decade after Lyons' work.

In 2014, Martin Hairer introduced Regularity Structures which generalize branched Rough Paths.

All of these objects consist of a mixture of algebraic and analytic properties.

A crucial tool in Regularity Structures is the renormalization step.

This step relies on knowledge of the group of automorphisms of the space of models.

In this setting, an answer has been given by Bruned, Hairer and Zambotti (2016) for stationary models.

Now we will discuss the same problem for branched Rough Paths.

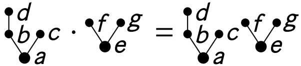
Some work on this has already been carried by Bruned, Chevyrev, Friz and Preiß (2017).

Branched rough paths

Let $(\mathscr{H}, \cdot, \Delta)$ be the Butcher–Connes–Kreimer Hopf algebra.

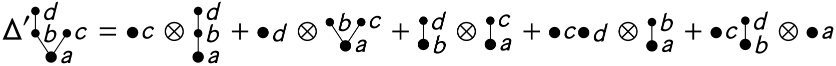
As an algebra, $\mathscr{H}$ is the commutative polynomial algebra over the set $\mathscr{T}$ of non-planar trees decorated by some alphabet A.

The product is simply the disjoint union of forests, e.g.



The empty forest 1 acts as the unit.

The coproduct Δ is described in terms of admissible cuts. For example



The Hopf algebra $\mathscr{H}$ is graded by the number of vertices in a forest. It is also connected.

Let G be the characters on $\mathscr{H}$.

Definition (Gubinelli (2010))

A branched Rough Path is a map $X : [0,1]^2 \to G$ *such that* $X_{tt} = \varepsilon$ *and*

$$X_{su} \star X_{ut} = X_{st}, \quad |\langle X_{st}, \tau \rangle| \lesssim |t-s|^{\gamma|\tau|}.$$

Example: let $(B_t)_{t\geq 0}$ be a Brownian motion, set $\langle X_{st}, \bullet \rangle := B_t - B_s$ and

$$\langle X_{st}, [\tau_1 \cdots \tau_k] \rangle = \int_s^t \langle X_{su}, \tau_1 \rangle \cdots \langle X_{su}, \tau_k \rangle \, \mathrm{d}B_u.$$

Let $\mathscr{C}_k$ be the continuous functions in k variables vanishing when consecutive variables coincide.

Gubinelli (2003) defines an exact cochain complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{C}_1 \xrightarrow{\delta_1} \mathscr{C}_2 \xrightarrow{\delta_2} \mathscr{C}_3 \xrightarrow{\delta_3} \cdots$$

that is $\delta_{k+1} \circ \delta_k = 0$ and $\operatorname{im} \delta_k = \ker \delta_{k+1}$.

Remark

If $F \in \ker \delta_2$ then there exists $f \in \mathscr{C}_1$ such that $F_{st} = f_t - f_s$.
If $C \in \ker \delta_3$ then there exists $F \in \mathscr{C}_2$ such that $C_{sut} = F_{st} - F_{su} - F_{ut}$.

In general, none of these operators are injective: if $F = G + \delta_{k-1} H$ then $\delta_k F = \delta_k G$.

Can do more if we restrict to smaller spaces: let $\mathscr{C}_2^\mu$ be the $F \in \mathscr{C}_2$ such that

$$\|F\|_\mu := \sup_{s<t} \frac{|F_{st}|}{|t-s|^\mu} < \infty.$$

Similarly, $\mathscr{C}_3^\mu$ are the $C \in \mathscr{C}_3$ such that $\|C\|_\mu < \infty$ for some suitable norm.

Theorem (Gubinelli (2004))

There is a unique linear map $\Lambda : \mathscr{C}_3^{1+} \cap \ker \delta_3 \to \mathscr{C}_2^{1+}$ *such that* $\delta_2 \Lambda = \mathrm{id}$. *In each of* $\mathscr{C}_3^\mu$ *for* $\mu > 1$ *it satisfies*

$$\|\Lambda C\|_\mu \le \frac{1}{2^\mu - 2} \|C\|_\mu.$$

Chen's rule reads

$$\langle X_{st}, \tau \rangle = \langle X_{su}, \tau \rangle + \langle X_{ut}, \tau \rangle + \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle.$$

or

$$\delta_2 F^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle$$

where $F^\tau_{st} := \langle X_{st}, \tau \rangle$.

The norm on $\mathscr{C}_3$ is such that the bound for X implies $\delta_2 F^\tau \in \mathscr{C}_3^{\gamma|\tau|}$.

The integer $N := \lfloor \gamma^{-1} \rfloor$ is special. Let G_N denote the multiplicative maps on the subcoalgebra

$$\mathscr{H}_N := \bigoplus_{n=0}^{N} \mathscr{H}_{(n)}.$$

Theorem (Gubinelli (2010))

Suppose $X : [0,1]^2 \to G_N$ *satisfies* $|\langle X_{st}, \tau\rangle| \lesssim |t-s|^{\gamma|\tau|}$. *Then there exists a unique map* $\hat{X} : [0,1]^2 \to G$ *on* $\mathscr{H}$ *such that* $\hat{X}\big|_{\mathscr{H}_N} = X$.

Proof.

Suppose $|\tau| = N + 1$ is a tree and set $C^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta'\tau\rangle$.
First one shows that $C^\tau \in \ker \delta_3$ by using the coassociativity of Δ'.
The bound above implies that $C^\tau \in \mathscr{C}_3^{\gamma|\tau|}$.
Therefore C^τ lies in the domain of Λ and we can set

$$\langle X_{st}, \tau\rangle := (\Lambda C^\tau)_{st}.$$

Continue inductively. □

Results

The previous argument works only because $\gamma|\tau| > 1$ i.e. $|\tau| > N$.

If $\gamma|\tau| \leq 1$, for any $g^\tau \in C^{\gamma|\tau|}$ (Hölder space) the function

$$G^\tau_{st} := F^\tau_{st} + \delta_1 g^\tau_{st}$$

also satisfies $\delta_2 G^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle$.

Let X and X' be two BRPs coinciding on $\mathscr{H}_{(1)}$.

Fix τ with $|\tau| = 2$ and let $F^\tau_{st} := \langle X_{st}, \tau \rangle$, $G^\tau_{st} := \langle X'_{st}, \tau \rangle$.

Then $\delta_2 F^\tau = \delta_2 G^\tau$ so there is $g^\tau \in \mathscr{C}_1$ such that

$$F^\tau_{st} = G^\tau_{st} + \delta_1 g^\tau_{st}.$$

Moreover $g^\tau \in C^{2\gamma}$.

This suggests that there might be an action of

$$\mathscr{D}^\gamma := \{(g^\tau)_{|\tau|\leq N} : g^\tau \in C^{\gamma|\tau|}, g_0^\tau = 0\}$$

on the space $\mathbf{BRP}^\gamma$ of branched Rough Paths.

Theorem (T.-Zambotti (2018))

Let $\gamma \in (0,1)$ such that $\gamma^{-1} \notin \mathbb{N}$. There is a regular action of $\mathscr{D}^\gamma$ on $\mathbf{BRP}^\gamma$.

This means we have a mapping

$$\mathscr{D}^\gamma \times \mathbf{BRP}^\gamma \ni (g, X) \to gX \in \mathbf{BRP}^\gamma$$

such that

- $g'(gX) = (g' + g)X$ for all $g, g' \in \mathscr{D}^\gamma$ and,
- for every pair $X, X' \in \mathbf{BRP}^\gamma$ there exists a *unique* $g \in \mathscr{D}^\gamma$ such that $X' = gX$.

$\mathbf{BRP}^\gamma$ is a *principal homogeneous space* for $\mathscr{D}^\gamma$.

Very rough sketch of proof

If $\gamma > \frac{1}{2}$ the result is easy: just set

$$\langle gX_{st}, \bullet_i \rangle = \langle X_{st}, \bullet_i \rangle + \delta g^{\bullet i}_{st}$$

and $\langle gX, \tau \rangle$ for $|\tau| \geq 2$ is given by the Sewing Lemma.

If $\frac{1}{3} < \gamma < \frac{1}{2}$ the action is the same in degree 1. In degree 2 we must have

$$\delta_2 \langle gX, \substack{\bullet j \\ | \\ \bullet i} \rangle_{sut} = (\delta x^j_{su} + \delta g^{\bullet j}_{su})(\delta x^i_{ut} + \delta g^{\bullet i}_{ut}).$$

The canonical choice (Young integral)

$$\int_s^t (\delta x^j_{su} + \delta g^{\bullet j}_{su}) \, \mathrm{d}(x^i_u + g^{\bullet i}_u)$$

is not well defined since $2\gamma < 1$.

In higher degrees the expressions are more complicated.

We handle this by constructing an *anisotropic* geometric Rough Path $\bar{X}$ such that

$$\langle X_{st}, \tau \rangle = \langle \bar{X}_{st}, \psi(\tau) \rangle$$

where $\psi : (\mathscr{H}, \cdot, \Delta) \to (T(\mathscr{T}_n), ⧢, \bar{\Delta})$ is the Hairer–Kelly map.

Anisotropic means that letters (trees) are allowed to have different weights.

In addition to the standard grading by the number of letters we have a weight function, e.g.

$$\omega\left(\bullet_a \otimes \begin{smallmatrix}\bullet_c\\ |\\ \bullet_b\end{smallmatrix}\right) = 3\gamma.$$

More concretely, $\bar{X}$ is a character over the shuffle algebra on the alphabet $\mathscr{T}_N$.

Single trees become letters in $T(\mathscr{T}_N)$, hence they are in degree one!

Set

$$\langle g\bar{X}, \tau\rangle := \langle \bar{X}, \tau\rangle + \delta g^{\tau}.$$

Then define

$$\langle gX, \tau\rangle = \langle g\bar{X}, \psi(\tau)\rangle.$$

1. Lifting of Chen's rule to the Lie algebra $\mathfrak{g}$. If $X_{st} = \exp_\star(\alpha_{st})$ then

$$\alpha_{st} = \mathrm{BCH}(\alpha_{su}, \alpha_{ut}) = \alpha_{su} + \alpha_{ut} + \mathrm{BCH}'(\alpha_{su}, \alpha_{ut}).$$

2. We use an explicit BCH formula due to Reutenauer.
3. We use the Lyons–Victoir (2007) method but in a constructive way, without invoking the axiom of choice.
4. However, the action is not unique nor canonical. The construction depends on a finite number of arbitrary choices.
5. We are able to construct γ-regular $\mathscr{H}$-rough paths over any $x \in C^\gamma(\mathbb{R}^d)$.

The geometric case

Now $(H, ⧢, \bar{\Delta})$ is the shuffle algebra over A. The product is defined recursively as

$$ua ⧢ vb = (u ⧢ vb)a + (ua ⧢ v)b,$$

e.g.

$$ab ⧢ cd = acdb + cadb + cdab + abcd + acbd + cabd$$

The coproduct is deconcatenation

$$\bar{\Delta}'(a_1 \cdots a_n) = \sum_{j=1}^{n-1} a_1 \cdots a_j \otimes a_{j+1} \cdots a_n.$$

Denote by $\bar{G}_N$ the multiplicative functionals on H_N.

Definition

A geometric rough path is a map $\bar{X} : [0,1]^2 \to \bar{G}_N$ *such that* $\bar{X}_{tt} = \bar{\varepsilon}$, *and*

$$\bar{X}_{su} \bar{\star} \bar{X}_{ut} = \bar{X}_{st}, \quad |\langle \bar{X}_{st}, w \rangle| \lesssim |t-s|^{\gamma|w|}.$$

Chen's rule again becomes $\delta_2 F^w_{sut} = \langle \bar{X}_{su} \otimes \bar{X}_{ut}, \Delta' w \rangle$ and so $\delta_2 F^w \in \mathscr{C}_3^{\gamma|w|}$ for all words.

Problem: H is not the polynomial algebra over all words.

Defining $\bar{X}$ over all words might give too much information: if we have $\langle \bar{X}_{st}, ab \rangle$ then

$$\langle \bar{X}_{st}, ba \rangle = \langle \bar{X}_{st}, a \rangle \langle \bar{X}_{st}, b \rangle - \langle \bar{X}_{st}, ab \rangle.$$

Next goals

1. Understand the algebraic picture. The action gX is not very easy to compute.
2. Relation with modification of products as explored in Ebrahimi-Fard, Patras, T. and Zambotti (2017).
3. Actions of an appropriate $\mathscr{D}^\gamma$ for the geometric case.
4. Clarify what the action means for controlled paths and RDEs.

Merci !