Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Signatures in Shape Analysis

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- Practical Setup
- Technical Setup

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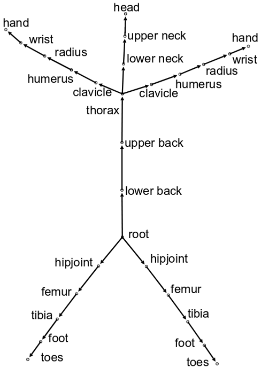
- Definition
- An example
- The case of SO(3)

3 Clustering

- Comparing signatures

The problem is to find some sort of similarity measure between shapes that is:

1. accurate enough, in that it distinguishes different types of motion (clustering), and
2. easy (and fast) to compute.



Our main application is to *computer motion capture*.

For each motion, we get a set of curves in $\mathrm{SO}(3)$, representing the rotation of joints relative to a fixed origin (root).

Given two motions, we want to compute some kind of distance between them.

We use data from the Carnegie Melon University MoCap Database

`http://mocap.cs.cmu.edu`.

Shapes are viewed as *unparametrized* curves taking values, in our case, on a finite-dimensional Lie group G whose Lie algebra is denoted by $\mathfrak{g}$.

We identify curves modulo reparametrization.

For technical reasons, we restrict to the space of *immersions*

$$\mathrm{Imm} := \{c \colon [0,1] \to G \mid c' \neq 0\}$$

The group D^+ of orientation-preserving diffeomorphisms of $[0,1]$ acts on Imm by composition

$$c.\varphi := c \circ \varphi.$$

We denote $\mathcal{S} := \mathrm{Imm}/D^+$.

Similarity between *shapes* is then measured by some distance

$$d_{\mathcal{S}}([c],[c']) := \inf_{\varphi} d_{\mathcal{P}}(c, c'.\varphi).$$

The (pseudo)distance $d_{\mathcal{P}}$ on *parametrized* curves must be reparametrization invariant.

The standard choice is obtained through a Riemannian metric on Imm. In the end one gets

$$d_{\mathcal{P}}(c, c') = \sqrt{\int_0^1 \|q(t) - q'(t)\|^2 \, \mathrm{d}t}$$

where

$$q(t) := \frac{(R_{c(t)}^{-1})_*(\dot{c}(t))}{\sqrt{|\dot{c}(t)|}}$$

is called the *Square root velocity transform* (SRVT) of the curve c.

Some observations about $d_{\mathcal{P}}$:

1. it is only a pseudometric.
2. it corresponds to the geodesic distance of a weak Riemannian metric on Imm, obtained as the pullback of the usual L^2 metric on curves in $\mathfrak{g}$ under the SRVT.
3. it is reparametrization invariant.

Hence, the similarity measure for shapes is

$$d_{\mathcal{S}}([c],[c']) = \inf_{\varphi \in D^+} \left(\int_0^1 \left\| q - (q'.\varphi)\sqrt{\dot{\varphi}} \right\|^2 \right)^{1/2}.$$

This optimization problem is often solved using *dynamic programming*.

Let G be a d-dimensional Lie group with Lie algebra $\mathfrak{g}$.

Definition

The Maurer–Cartan form of G is the $\mathfrak{g}$-valued 1-form

$$\omega_g(v) := (R_g^{-1})_*(v), \quad g \in G, v \in T_g G.$$

This means that ω is a bundle morphism $TG \longrightarrow (G \times \mathfrak{g})$, i.e. for each $g \in G$ we have a linear map $\omega_g : T_g G \longrightarrow \mathfrak{g}$.

In particular, if $X_1, \ldots, X_d$ is a basis for $\mathfrak{g}$ then we may write

$$\omega_g(v) = \omega_g^1(v)X_1 + \cdots + \omega_g^d(v)X_d.$$

Consider a curve $\alpha \in C^\infty([0,1], G)$.

Definition (Chen (1954))

The signature on G is the map $\alpha \mapsto S^G(\alpha)$ defined recursively by $\langle 1, S^G_{s,t}(\alpha)\rangle = 1$ and

$$\langle e_{i_1\cdots i_n}, S^G_{s,t}(\alpha)\rangle := \int_s^t \langle e_{i_1\cdots i_{n-1}}, S^G_{s,u}(\alpha)\rangle \omega^{i_n}_{\alpha(u)}(\dot\alpha(u))\,\mathrm{d}u$$

The Maurer–Cartan form can be computed explicitly in some situations, specially for matrix Lie groups where it takes the simple form

$$\omega_g = \mathrm{d}g\, g^{-1}.$$

An easy example is the Heisenberg group

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Then, we obtain

$$\omega_g = \begin{pmatrix} 0 & \mathrm{d}x & -y\,\mathrm{d}x + \mathrm{d}z \\ 0 & 0 & \mathrm{d}y \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$S^{H_3}_{s,t}(\alpha) = 1 + \int_s^t \dot{\alpha}^x(u)\,\mathrm{d}u\; e_1 + \int_s^t \dot{\alpha}^y(u)\,\mathrm{d}u\; e_2 + \int_s^t (\dot{\alpha}^z(u) - \alpha^y(u)\dot{\alpha}^x(u))\,\mathrm{d}u\; e_3 + \cdots$$

For $\mathrm{SO}(3)$ the computation is more difficult.

However, we can do something clever: we do “geodesic interpolation”.

Given $A, B \in \mathrm{SO}(3)$, let $\alpha : [0,1] \longrightarrow \mathrm{SO}(3)$ be given by

$$\alpha(t) := \exp(t \log(BA^\top))A,$$

so that $\alpha(0) = A$ and $\alpha(1) = B$.

Since $\mathrm{SO}(3)$ is a “nice” group, it has a unique bi-invariant Riemannian metric. For this metric, geodesics and one-parameter subgroups coincide, that is, geodesics correspond to flows of left-invariant vector fields.

For this choice,

$$\omega_{\alpha(t)}(\dot\alpha(t)) = \log(BA^\top).$$

We need a way of comparing signatures. There are several choices:

1. using the metric inherited from $T((\mathbb{R}^d))$, i.e.

$$d(g,h) := \|h - g\|_{T((\mathbb{R}^d))},$$

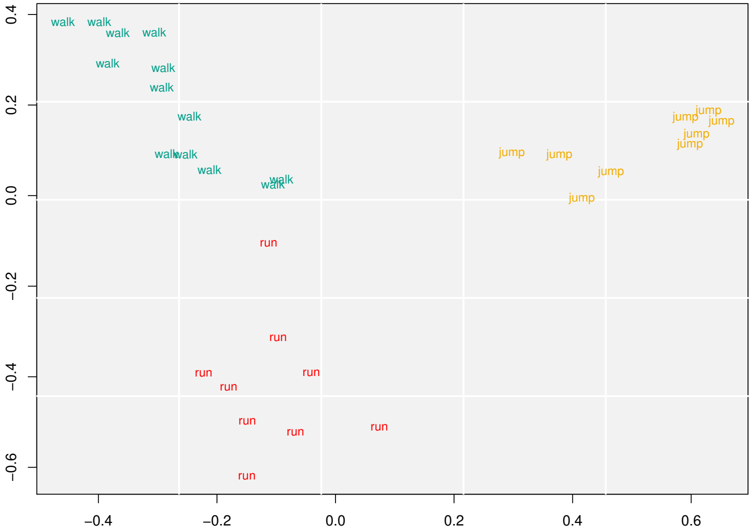
2. using the homogeneous norm on truncated characters, i.e.

$$\rho^n(g,h) := \|h^{-1} \otimes g\|_{G^{(n)}},$$

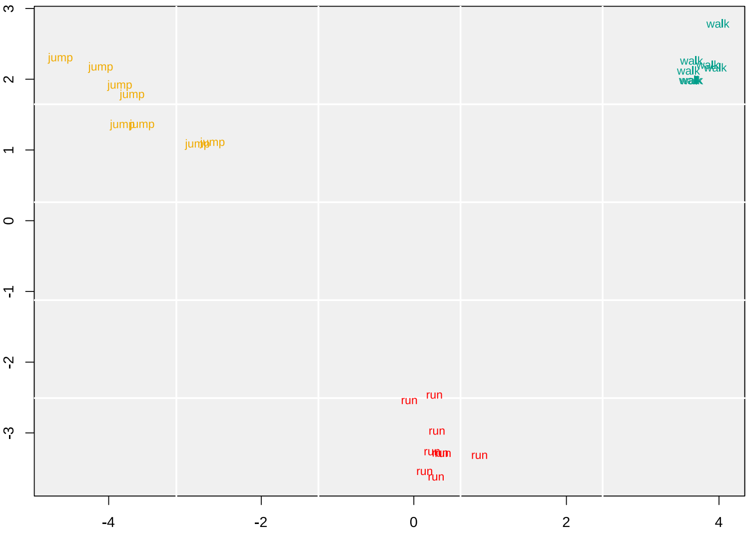
3. compare log-signatures.

We also compare with currently used methods based on the SRVT, i.e. dynamic programming.

Clustering



Clustering



jump
jump
jump
jump
jumpjump
jumpjump
walk
walk
walkwalk
walk
walk
run
run
run
runrun
run
run
run
3
2
1
0
-1
-2
-3
-4
-2
0
2
4

Let $a \in \mathbb{R}^2$ and consider $x : [0,1] \to \mathbb{R}^2$ given by $x(t) := at$.

Then:

$$S(x) = \exp_\otimes(a) = 1 + a + \tfrac{1}{2} a \otimes a + \tfrac{1}{6} a \otimes a \otimes a + \cdots$$

For small $\varepsilon > 0$ define $x_\varepsilon(t) := (a + \varepsilon v)t$.

Then:

$$S(x_\varepsilon) = \exp_\otimes(a + \varepsilon v)$$

By Baker–Campbell–Hausdorff:

$$g_\varepsilon := S(x)^{-1} \otimes S(x_\varepsilon) = \exp_\otimes(\varepsilon v + \mathrm{BCH}(-a, a + \varepsilon v)).$$

Truncating at $n = 2$,

$$g_\varepsilon = \exp_\otimes\left(\varepsilon v - \frac{1}{2}[a, a + \varepsilon v]\right) = 1 + \varepsilon v - \frac{\varepsilon}{2}[a, v] + \frac{1}{2}\varepsilon^2 v \otimes v,$$

hence

$$\rho(S(x_\varepsilon), S(x)) = \|g_\varepsilon\|_{G^{(2)}} = \max\left\{|\varepsilon|, \sqrt{\frac{|\varepsilon|}{2}}\left(4\varepsilon^2 + \tfrac{1}{2}(a_1 v_2 - v_1 a_2)^2\right)^{1/4}\right\} = O\left(\sqrt{\varepsilon}\right).$$

In general

$$\|g_\varepsilon\|_{G^{(n)}} = O(\varepsilon^{1/n}).$$

Now, let c_a and c_b correspond to “walking” and “jogging” animations.

We generate a geodesic interpolation $\bar{c}$ between the curves, i.e. $c(0,\cdot) = c_a$, $c(1,\cdot) = c_b$ and for $s \in (0,1)$ the animation $c(s,\cdot)$ is a mixture of both.

In practice, this is generated using the SRVT so in fact we are doing linear interpolation at the level of the Lie algebra.

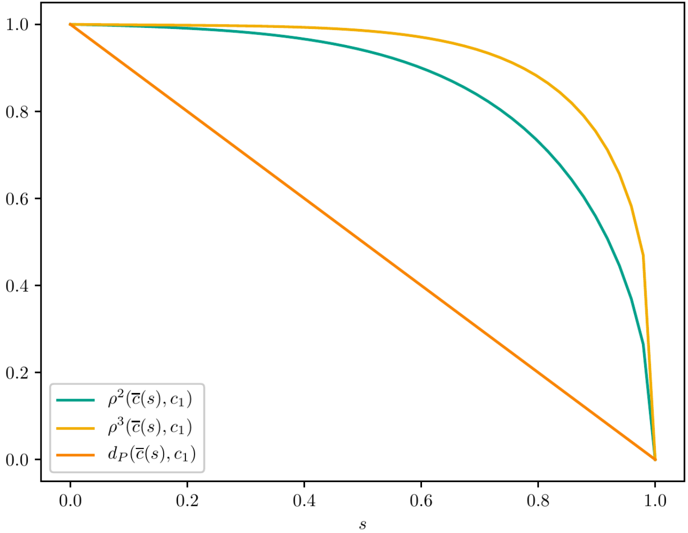
Signatures were computed using the `iisignature` Python package by J. Reizenstein and B. Graham.

We can then look at the behaviour of the different similarity measures when s varies.

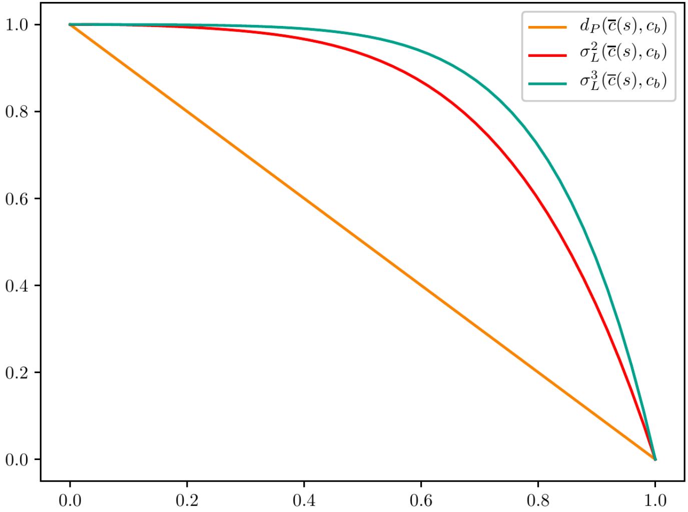
Remark

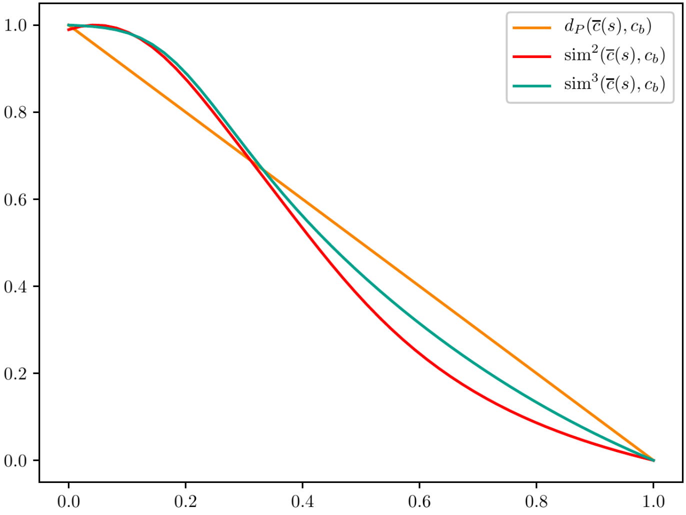
Since the distance $d_{\mathcal{S}}$ coincides with the geodesic distance, we will see a straight line for this metric.

Clustering: a concrete example (cont.)



Clustering: a concrete example (cont.)





$d_P(\overline{c}(s), c_b)$
$\mathrm{sim}^2(\overline{c}(s), c_b)$
$\mathrm{sim}^3(\overline{c}(s), c_b)$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Questions:

1. Pullback metric from signatures to curves.
2. How much does geometrical information help.
3. Better understanding of the various metrics.
4. Purely discrete approach.

Thanks!