Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 9

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1) Let X be a Hilbert space and let $F : X \to \overline{\mathbb{R}}$ be convex, proper and lower semi-continuous. Let again the Moreau-Yosida regularization $F_\gamma : X \to \mathbb{R}$ and the prox-operator $P_\gamma : X \to X$ be defined as on the last sheets. Show the following properties:

(i) For every real sequence $\gamma_n \downarrow 0$:

$$\liminf_{n\to\infty} F_n(u_n) \geq F(u) \quad \text{whenever } u_n \to u \text{ in } X. \tag{0.1}$$

$$\limsup_{n\to\infty} F_n(v_n) \leq F(u) \quad \text{for at least one sequence } (v_n)_n \text{ with } v_n \to u \text{ in } X. \tag{0.2}$$

Here we used the abbreviation $F_n := F_{\gamma_n}$.

(ii) Construct a sequence of functions $f_n : X \to \overline{\mathbb{R}}$ that satisfy (0.1) and (0.2) but does not satisfy $f_n(x) \to f(x)$ for every $x \in X$.

(iii) Consider a continuous function that is bounded from below $J : L^2(\Omega) \to \mathbb{R}$ for some bounded domain $\Omega \subset \mathbb{R}^n$ and the problems

$$\min_{u\in X} F_n(u) := J(u) + \frac{1}{2\gamma_n}\|\max(0, u-1)\|_{L^2}^2 \tag{0.3}$$

$$\min_{u\in K} J(u) \tag{0.4}$$

where $K = \{u \in L^2(\Omega) : u(x) \leq 1 \text{ for a.e. } x \in \Omega\}$. Prove that every bounded sequence $(u_n)_n$ of minimizers of F_n in (0.3) converges (up to a subsequence) to a solution of (0.4).

2) For and $\Omega \subset \mathbb{R}^n$ open, bounded with Lipschitz-boundary consider the following optimal control problem. Here $\alpha, \beta > 0$ and $y_d \in L^2(\Omega)$.

$$(P_1) \qquad \min_{(y,u)\in H_0^1(\Omega)\times L^2(\Omega)} J(y,u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \frac{\beta}{2}\|u\|_{L^1(\Omega)}$$
$$\text{subject to} \qquad -\Delta y = u \text{ on } \Omega, \qquad y = 0 \text{ on } \partial\Omega$$

(i) Show that (P_1) has a unique solution in $H_0^1(\Omega) \times L^2(\Omega)$.

(ii) Show that if $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^2(\Omega)$ solves (P_1) there is a $\bar{\mu} \in L^2(\Omega)$ and $p \in H_0^1(\Omega)$ such that

$$(\text{Opt}_1) \qquad \begin{aligned} -\Delta\bar{y} &= \bar{u} && \text{in } H_0^1(\Omega) \\ -\Delta\bar{p} &= (\bar{y} - y_d) && \text{in } H_0^1(\Omega) \\ \alpha\bar{u} - \bar{p} + \bar{\mu} &= 0 && \text{in } \Omega \\ \bar{\mu} &= \beta && \text{on } \{x \in \Omega : \bar{u} > 0\}. \\ |\bar{\mu}| &\leq \beta && \text{on } \{x \in \Omega : \bar{u} = 0\}. \\ \bar{\mu} &= -\beta && \text{on } \{x \in \Omega : \bar{u} < 0\}. \end{aligned}$$

(iii) How could a Semismooth Newton method be applied here?

3) (Still from the last sheet) For $n \leq 3$ and $\Omega \subset \mathbb{R}^n$ open, bounded with C^2 boundary consider the following state constrained optimal control problem. Here $b \in H^2(\Omega), b > 0, \alpha > 0$ and $y_d \in L^2(\Omega)$.

$$(P_2) \qquad \min_{(y,u)\in H_0^1(\Omega)\times L^2(\Omega)} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

$$\text{subject to} \quad -\Delta y = u \text{ on } \Omega, \quad y = 0 \text{ on } \partial\Omega, \quad y \leq b \text{ a.e. on } \Omega,$$

and the associated first order optimality system: $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^2(\Omega)$ solves (P_2) iff there is a $\bar{\mu} \in \mathcal{M}(\Omega) := C_0(\Omega)'$ and $p \in L^2(\Omega)$ such that

$$(\text{Opt}) \qquad \begin{aligned} -\Delta\bar{y} &= \bar{u} && \text{in } H_0^1(\Omega) \\ \langle \bar{p}, -\Delta v\rangle_{L^2(\Omega)} + \langle \bar{\mu}, v\rangle_{\mathcal{M}(\Omega)} &= -\langle \bar{y} - y_d, v\rangle_{L^2(\Omega)} && \text{for every } v \in H^2(\Omega)\cap H_0^1(\Omega) \\ \alpha\bar{u} - \bar{p} &= 0 && \text{in } \Omega \\ \bar{y} \leq b \quad \langle \bar{\mu}, v - \bar{y}\rangle_{\mathcal{M}(\Omega)} &\leq 0 && \text{for all } v \in C(\overline{\Omega}), v \leq b \end{aligned}$$

Consider also the regularized problem with $\gamma > 0$:

$$(\text{P}_\gamma) \qquad \min_{(y,u)\in H_0^1(\Omega)\times L^2(\Omega)} J_\gamma(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2\gamma}\|\max(0, \hat{\mu} + \gamma(y - b))\|^2_{L^2(\Omega)}$$

$$\text{subject to} \quad -\Delta y = u \text{ on } \Omega, \quad y = 0 \text{ on } \partial\Omega.$$

Here $\hat{\mu} \in L^2(\Omega), \hat{\mu} \geq 0$ is a fixed, optional shift-parameter.

(i) Show that (P_γ) has a unique solution $(\bar{y}_\gamma, \bar{u}_\gamma) \in \left[H_0^1(\Omega) \cap H^2(\Omega)\right] \times L^2(\Omega)$, which satisfies the first order system

$$(\text{Opt}_\gamma) \qquad \begin{aligned} -\Delta\bar{y}_\gamma &= \bar{u}_\gamma && \text{in } H_0^1(\Omega) \\ -\Delta\bar{p}_\gamma &= (\bar{y}_\gamma - y_d) - \max(0, \hat{\mu} + \gamma(\bar{y}_\gamma - b)) && \text{in } H_0^1(\Omega) \\ \alpha\bar{u}_\gamma - \bar{p}_\gamma &= 0 && \text{in } \Omega \end{aligned}$$

(ii) Define $\bar{\mu}_\gamma = \max(0, \hat{\mu} + \gamma(\bar{y}_\gamma - b))$ and show that given $\gamma_0 \geq 0$ the path $(\bar{y}_\gamma, \bar{u}_\gamma, \bar{p}_\gamma, \bar{\mu}_\gamma)_{\gamma\geq\gamma_0}$ is bounded in the space $Y \times L^2(\Omega) \times L^2(\Omega) \times Y'$. Here the space Y is defined as

$$Y := H_0^1(\Omega) \cap H^2(\Omega).$$

Moreover $(\bar{y}_\gamma, \bar{u}_\gamma, \bar{p}_\gamma, \bar{\mu}_\gamma) \rightharpoonup (\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ in $Y \times L^2(\Omega) \times L^2(\Omega) \times Y'$ as $\gamma \to \infty$ where $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ solves (P_2).

(iii) Bonus: Show that for any $\gamma_0 > 0$ the mapping

$$[\gamma_0, +\infty) \to Y \times L^2(\Omega) \times L^2(\Omega) \times Y' \quad \gamma \mapsto (\bar{y}_\gamma, \bar{u}_\gamma, \bar{p}_\gamma, \bar{\mu}_\gamma)$$

is Lipschitz continuous.