Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 8

1) Let X be a Hilbert space and let $F : X \to \overline{\mathbb{R}}$ be convex, proper and lower semi-continuous. Consider again the function

$$F_\gamma(u) = \inf_{v \in X} J_\gamma(u, v) = F(v) + \frac{1}{2\gamma}\|u - v\|_X^2$$

from sheet 6, together with its unique minimizer $P_\gamma : X \to X$ defined by

$$P_\gamma(u) = \operatorname*{arg\,min}_{v \in X} J(u, v).$$

Show the following properties:

(i) $F_\gamma(u) \uparrow F(u)$ as $\gamma \downarrow 0$.

(ii) $P_\gamma(u) \to u$ as $\gamma \to 0$ for all $u \in \overline{\operatorname{dom}(F)}$ and $P_\gamma(u) \to \operatorname{Proj}_{\overline{\operatorname{dom}(F)}}(u)$ as $\gamma \to 0$ for $u \in X$.

(iii) For $u \in \operatorname{dom}(F)$ we have $\nabla P_\gamma(u) \to p^* \in X$ as $\gamma \to 0$ where

$$p^* \in \operatorname*{arg\,min}_{p \in \partial F(u)} \|p\|_X^2$$

2) For $n \leq 3$ and $\Omega \subset \mathbb{R}^n$ open, bounded with C^2 boundary consider the following state constrained optimal control problem. Here $b \in H^2(\Omega), b > 0, \alpha > 0$ and $y_d \in L^2(\Omega)$.

$$\text{(P)} \qquad \begin{aligned} &\min_{(y,u) \in H_0^1(\Omega) \times L^2(\Omega)} && J(y, u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 \\ &\text{subject to} && -\Delta y = u \text{ on } \Omega, \\ &&& y = 0 \text{ on } \partial\Omega, \\ &&& y \leq b \text{ a.e. on } \Omega, \end{aligned}$$

(i) Show that (P) has a unique solution in $\left[H_0^1(\Omega) \cap H^2(\Omega)\right] \times L^2(\Omega)$.

(ii) Show that if $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^2(\Omega)$ solves (P) there is a $\bar{\mu} \in \mathcal{M}(\Omega) := C_0(\Omega)'$ and $p \in L^2(\Omega)$ such that

$$\text{(Opt)} \qquad \begin{aligned} -\Delta \bar{y} &= \bar{u} && \text{in } H_0^1(\Omega) \\ \langle \bar{p}, -\Delta v \rangle_{L^2(\Omega)} + \langle \bar{\mu}, v \rangle_{\mathcal{M}(\Omega)} &= -\langle \bar{y} - y_d, v \rangle_{L^2(\Omega)} && \text{for every } v \in H^2(\Omega) \cap H_0^1(\Omega) \\ \alpha \bar{u} - \bar{p} &= 0 && \text{in } \Omega \\ \bar{y} \leq b \quad \langle \bar{\mu}, v - \bar{y} \rangle_{\mathcal{M}(\Omega)} &\leq 0 && \text{for all } v \in C(\overline{\Omega}), v \leq b \end{aligned}$$

3) Consider for the same assumptions as in exercise 2) the regularized problem with $\gamma > 0$:

$$(\mathrm{P}_\gamma) \qquad \begin{aligned} &\min_{(y,u) \in H_0^1(\Omega) \times L^2(\Omega)} && J_\gamma(y, u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \frac{1}{2\gamma}\|\max(0, \hat{\mu} + \gamma(y - b))\|_{L^2(\Omega)}^2 \\ &\text{subject to} && -\Delta y = u \text{ on } \Omega, \\ &&& y = 0 \text{ on } \partial\Omega. \end{aligned}$$

Here $\hat{\mu} \in L^2(\Omega), \hat{\mu} \geq 0$ is a fixed, optional shift-parameter.

(i) Show that (P_γ) has a unique solution $(\bar{y}_\gamma, \bar{u}_\gamma) \in \left[H_0^1(\Omega) \cap H^2(\Omega)\right] \times L^2(\Omega)$, which satisfies the first order system

$$
\begin{aligned}
-\Delta \bar{y}_\gamma &= \bar{u}_\gamma && \text{in } H_0^1(\Omega) \\
-\Delta \bar{p}_\gamma &= (\bar{y}_\gamma - y_d) - \max(0, \hat{\mu} + \gamma(\bar{y}_\gamma - b)) && \text{in } H_0^1(\Omega) \\
\alpha \bar{u}_\gamma - \bar{p}_\gamma &= 0 && \text{in } \Omega
\end{aligned}
\tag{Opt_γ}
$$

(ii) Define $\bar{\mu}_\gamma = \max(0, \hat{\mu}+\gamma(\bar{y}_\gamma - b))$ and show that given $\gamma_0 \geq 0$ the path $(\bar{y}_\gamma, \bar{u}_\gamma, \bar{p}_\gamma, \bar{\mu}_\gamma)_{\gamma \geq \gamma_0}$ is bounded in the space $Y \times L^2(\Omega) \times L^2(\Omega) \times Y'$. Here the space Y is defined as

$$Y := H_0^1(\Omega) \cap H^2(\Omega).$$

Moreover $(\bar{y}_\gamma, \bar{u}_\gamma, \bar{p}_\gamma, \bar{\mu}_\gamma) \rightharpoonup (\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ in $Y \times L^2(\Omega) \times L^2(\Omega) \times Y'$ as $\gamma \to \infty$ where $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ solves (P).

(iii) Bonus: Show that for any $\gamma_0 > 0$ the mapping

$$[\gamma_0, +\infty) \to Y \times L^2(\Omega) \times L^2(\Omega) \times Y' \quad \gamma \mapsto (\bar{y}_\gamma, \bar{u}_\gamma, \bar{p}_\gamma, \bar{\mu}_\gamma)$$

is Lipschitz continuous.