Humboldt-Universität zu Berlin
Institut für Mathematik

Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 7

1) (Programming exercise) Consider for $\Omega = [0,1]^2$, given $y_d \in L^2(\Omega)$, $\psi \in L^\infty(\Omega)$ and $\alpha > 0$ the following problem:

$$\text{(P)} \qquad \min_{(y,u)\in H_0^1(\Omega)\times U_{ad}} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$
$$\text{subject to} \qquad -\Delta y = u \text{ on } \Omega,$$
$$y = 0 \text{ on } \partial\Omega,$$

where the PDE is given in the usual weak form and $U_{ad} = \{u \in L^2(\Omega) : u \leq \psi\}$.

(i) Show that (P) has a unique solution $(u, y) \in H_0^1(\Omega) \times U_{ad}$.

(ii) Show that $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times U_{ad}$ solves (P) if and only if there is a $p \in H_0^1(\Omega)$ and a $\lambda \in L^2(\Omega)$ such that the following system holds for some $c > 0$:

$$\text{(Opt)} \qquad \begin{aligned} -\Delta\bar{y} &= \bar{u} && \text{in } H_0^1(\Omega) \\ -\Delta p &= -(\bar{y} - y_d) && \text{in } H_0^1(\Omega) \\ \bar{u} &= \frac{1}{\alpha}(p - \lambda) \\ \lambda &= \max(0, \lambda + c(\bar{u} - \psi)). \end{aligned}$$

Here the last equation in (Opt) is to be understood pointwise almost everywhere.

(iii) Write down the (semi-smooth) Newton system to solve (Opt) for

$$(\bar{y}, \bar{u}, p, \lambda) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega).$$

Also consider the the primal dual active set strategy to solve (Opt). How are these two approaches related. Discuss Newton-differentiability and convergence properties.

(iv) Use a standard finite differences discretization of (Opt) and solve the resulting finite dimensional system using the semi-smooth Newton method in the following case

$$\begin{aligned} y_d(x_1, x_2) &= \sin(2\pi x_1)\sin(2\pi x_2)\exp(2x_1)/6 \\ \psi &\equiv 0 \\ \alpha &= 0.01 \\ c &= 0.1 \end{aligned}$$

(v) Visualize the convergence behaviour and experiment with different mesh sizes $h > 0$ and $c > 0$. Also discuss suitable stopping criteria.

Please use Python or Matlab for the implementation. We will discuss questions regarding the exercise in the class at 13.12.2023. The solutions will be presented in the first exercise class in the new year.