Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 6

1) Let X be a Hilbert space and let $F : X \to \overline{\mathbb{R}}$ be convex, proper and lower semi-continuous. Consider the function

$$F_\gamma(u) = \inf_{v\in X} J_\gamma(u,v) := F(v) + \frac{1}{2\gamma}\|u-v\|_X^2$$

Show the following properties:

(i) The optimization problem above has exactly one solution.

$$P_\gamma(u) = \arg\min_{v\in X} J_\gamma(u,v).$$

Show that $P_\gamma : X \to X$ is Lipschitz continuous.

(ii) The function $F_\gamma : X \to \overline{\mathbb{R}}$ is convex and finite everywhere.

(iii) The function $F_\gamma : X \to \mathbb{R}$ is Lipschitz continuously differentiable. Show that the gradient is given by

$$\nabla F_\gamma(u) = \frac{u - P_\gamma(u)}{\gamma}$$

(iv) The following equality holds

$$\inf_{u\in X} F(u) = \inf_{u\in X} F_\gamma(u).$$

Moreover if u^* minimizes F on X if and only if u^* minimizes F_γ on X.

2) Consider the same setting as in exercise 1. Compute P_γ and F_γ in the following situations:

(i) Let $C \subset H$ be a closed non-empty and convex subset of a Hilbert space H. Consider then $F : H \to \overline{\mathbb{R}}$ defined by $F(u) = \mathbf{1}_C(u)$.

(ii) Let H be a Hilbert space. Consider then $F : H \to \mathbb{R}$ given by $F(u) = \|u\|_H^2$.

(iii) Consider for bounded domain $\Omega \subset \mathbb{R}^d$ the set $C = \{u \in L^2(\Omega) : \|u\|_{L^\infty} \le 1\}$ and the function $F : L^2(\Omega) \to \overline{\mathbb{R}}$ defined by

$$F(u) = \mathbf{1}_C(u)$$

3) (Exact Penalty) Let $F : H \to \mathbb{R}$ be L-Lipschitz continuous and defined on a Hilbert space H and $C \subset H$ closed, convex and non-empty. Consider the minimization problem

$$\min_{u\in H} F(u) \quad \text{subject to } u \in C \tag{P}$$

Show the following

(i) If $x^* \in C$ solves (P), then, for any $K \ge L$, the function $u \mapsto F(u) + K\text{dist}(u, C)$ attains its (unconstrained) minimum over H at $u = u^*$.

(ii) Suppose that, for some $K > L$, the function $u \mapsto F(u) + K\text{dist}(u, C)$ attains its minimum over H at $u = u^*$. Then u^* belongs to C and solves (P).

(iii) Assume now that F is differentiable and consider a general function $P : H \to \mathbb{R}$ satisfying $P(u) \ge 0$ and $P(u) = 0 \Leftrightarrow u \in C$ for every $u \in H$. Define the problem

$$\min_{u\in H} G_\alpha(u) := F(u) + \alpha P(u) \quad \alpha > 0$$

Show the following statement: If u^* is a local minimum of F over C and $\nabla F(u^*) \neq 0$. Assume further that there is an $\overline{\alpha} > 0$ such that for all $\alpha > \overline{\alpha}$ the element u^* is also a local minimizer of G_α. Then P cannot be differentiable.