Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 5

1) Let X, Y be Banach spaces and let $F : X \to Y$ be Newton differentiable near $x^* \in X$ with $F(x^*) = 0$ with Newton derivative $D_N F(x^*)$. Also assume to be given $M(x) \in \mathcal{L}(X, Y)$ that satisfy $\|M(x) - D_N F(x)\|_{\mathcal{L}(X,Y)} \le \theta$ and $\|M(x)^{-1}\|_{\mathcal{L}(Y,X)} \le C$ for all $x \in B_\delta(x^*)$ for some $\delta, \theta > 0$ and $0 < C < \theta^{-1}$. Then $x_{k+1} = x_k - M(x_k)^{-1}F(x_k)$ converges q-linearly to x^* for all x_0 sufficiently close to x^*.

2) For bounded and open domain $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ consider the function

$$F : L^2(\Omega) \to L^1(\Omega) \quad F(u) = \underset{v \in L^2(\Omega)}{\arg\min} \frac{1}{2}\|v - u\|^2_{L^2(\Omega)} + \|u\|_{L^1(\Omega)}$$

(i) Show that $F : L^2(\Omega) \to L^1(\Omega)$ is well defined, i.e. it is single valued with range in $L^1(\Omega)$.

(ii) Show that F is Newton-differentiable and compute the Newton derivative.
Hint: Display F as a suitable superposition operator and use one of the earlier exercises.

3) Let $\Omega \subset \mathbb{R}^n$ be bounded and set $X = L^\infty(\Omega)$. Consider the following operator

$$G : u \mapsto \mathcal{L}(X) \quad G(u)[h](x) = \begin{cases} h(x) & \text{if } u(x) \ge 0 \\ 0 & \text{if } u(x) < 0 \end{cases}$$

Show that $G : u \mapsto \mathcal{L}(X)$ is not the Newton derivative of the function

$$F : X \to X \quad F(u)(x) = \max(0, u(x))$$

4) (A globalized non-smooth method) Suppose that $F : X \to Y$ between Banach spaces is continuous and directionally differentiable on $S = \overline{B(x_0, r)}$, i.e.

$$F'(x, d) = \lim_{t \to 0^+} \frac{F(x + td) - F(x)}{t}$$

exists for every $x \in S$. Assume also the existence of bounded operators $G(\cdot) \in \mathcal{L}(X, Z)$ and constants $\beta, \gamma > 0$ such that

$$\|G(x)^{-1}\|_{\mathcal{L}(Z,Y)} \le \beta\,, \quad \|G(x)(y - x) - F'(x, y - x)\| \le \gamma\|y - x\|\,,$$
$$\|F(y) - F(x) - F'(x, y - x)\| \le \delta\|y - x\|$$

for all $x, y \in S$, where $\alpha = \beta(\gamma + \delta) < 1$, and $\beta\|F(x_0)\| \le r(1 - \alpha)$. Then the iterates defined by $x_{k+1} = x_k - G(x_k)^{-1}F(x_k)$ for $k = 0, 1, \ldots$ remain in S and converge to the unique solution x^* of $F(x) = 0$ in S. Moreover, we have the error estimate

$$\|x_k - x^*\| \le \frac{\alpha}{1 - \alpha}\|x_k - x_{k-1}\|$$