Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 4

1) (Superposition operators) Consider a bounded, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f' : \mathbb{R} \to \mathbb{R}$ is also bounded. Define the operator

$$\Phi : L^p(\Omega) \to L^p(\Omega) \quad \Phi(u)(x) = f(u(x))$$

for a bounded domain $\Omega \subset \mathbb{R}^d$.

(i) Show for $p = \infty$, that Φ is Frechét differentiable and find the derivative.
(ii) For $p = 2$ show that Φ is Gateaux-differentiable and compute its Gateaux derivative.

2) (Problems in infinite dimensions) Consider the same function $\Phi : L^2(\Omega) \to L^2(\Omega)$ from exercise 1. Assume that Φ is Frechét-differentiable. Show that there are $a, b \in \mathbb{R}$ such that $f(x) = ax + b$, i.e. f is affine linear.
Hint: Assume Φ is differentiable and consider the following difference quotient

$$\|\Phi(u+h) - \Phi(u) - D\Phi(u)[h]\| = o(\|h\|)$$

for $u = 0$ and $h = \lambda \mathbf{1}_{B(x^*,\delta)}$ for given $x^* \in \Omega$ and $\delta \to 0$.

3) Define for bounded $\Omega \subset \mathbb{R}^d$ and bounded convex subset $U_{ad} \subset L^2(\Omega)$ the projection operator

$$\operatorname{proj}_{U_{ad}} : L^2(\Omega) \to L^2(\Omega) \quad \operatorname{proj}_{U_{ad}}(v) = \arg\min_{u \in U_{ad}} \frac{1}{2}\|u - v\|^2_{L^2(\Omega)}$$

(i) Show that the minimization problem has a solution that is unique.
(ii) Assume now that $U_{ad} = \{u \in L^2(\Omega) : \alpha(x) \leq u(x) \leq \beta(x) \text{ for a.e } x \in \Omega\}$ for functions $\alpha, \beta \in L^\infty(\Omega)$. Show that

$$\operatorname{proj}_{U_{ad}}(u)(x) = \begin{cases} \beta(x) & \text{if } u(x) \geq \beta(x) \\ u(x) & \text{if } u(x) \in (\alpha(x), \beta(x)) \\ \alpha(x) & \text{if } u(x) \leq \alpha(x) \end{cases}$$

for a.e. $x \in \Omega$.

4) Let X, Y be Banach spaces and let $F : X \to Y$ be Newton differentiable near $x^* \in X$ with $F(x^*) = 0$ with Newton derivative $D_N F(x^*)$. Assume further that there exist $\delta > 0$ and $C > 0$ with $\|D_N F(x)^{-1}\|_{\mathcal{L}(X,Y)} \leq C$ for all $x \in B(x^*, \delta)$. Then the semismooth Newton method converges superlinearly x^* for all x_0 sufficiently close to x^*.

(5) Let the superposition operator for the max-function be defined as

$$\Phi : L^{\infty}(0,1) \to L^{\infty}(0,1) \quad \Phi(u)(x) = \max(u(x), 0)$$

Show that

$$D_N F(u)[h] = \begin{cases} h(x) & \text{if } u(x) \geq 0. \\ \delta h(x) & \text{if } u(x) = 0. \\ 0 & \text{if } u(x) < 0. \end{cases}$$

is not the Newton derivative for Φ.
Hint: Consider $u(x) = x$ and directions $h_n(x) = (nx - 1)\mathbf{1}_{[0,1/n]}(x)$.