Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 3

1) (Generalized directional derivative) Let X be a Banach space and $F : X \to \mathbf{R}$ be locally Lipschitz continuous and denote the local Lipschitz constant on neighborhood $U_x \subset \mathbf{R}^n$ of x by $L(U_x) \geq 0$. Moreover consider the generalized directional derivative

$$F^\circ(x,h) = \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{F(y+th) - F(y)}{t}.$$

Prove the following statements of $h \mapsto F^\circ(x,h)$:

(i) Lipschitz continuity, i.e. $|F^\circ(x,h)| \leq L(U_x)\|h\|$ for every $h \in \mathbf{R}^n$.
(ii) Subadditivity, i.e. $F^\circ(x, g+h) \leq F^\circ(x,g) + F^\circ(x,h)$ for all $g, h \in \mathbf{R}^n$.
(iii) Positive homogeneity, i.e. $F^\circ(x, \alpha h) = (\alpha F)^\circ(x,h)$ for every $\alpha > 0, h \in \mathbf{R}^n$.
(iv) The function $(u,h) \mapsto F^\circ(x,h)$ is upper-semicontinuous.

2) For $F, G : \mathbf{R}^n \to \mathbf{R}$ locally Lipschitz, define the (Clarke-) subdifferential as the set

$$\partial F(x) := \{v \in \mathbb{R}^n | \langle v, h\rangle \leq f^\circ(x,h) \text{ for every } h \in \mathbb{R}^n\}$$

(i) Show that $\partial F(x)$ is nonempty, closed and bounded.
(ii) Show the sumrule: $\partial(F+G)(x) \subset \partial F(x) + \partial G(x)$. Give an example that shows that in general the inclusion is strict.
(iii) If $F : \mathbf{R}^n \to \mathbf{R}$ is continuously differentiable at x, we have $\partial F(x) = \{F'(x)\}$.
(iv) If F is convex, then F is Clarke-regular, i.e.

$$F^\circ(x,h) = F'(x,h) := \lim_{t \to 0^+} \frac{f(x+th) - f(x)}{t}$$

Hint: To show that this limit exists, you can prove that $t \mapsto (f(x+th) - f(t))/t$ satisfies some monotonicity properties.

3) (Difficulties in infinite dimensions) Consider the following operators

$$\Phi_1 : L^2([0,1]) \to L^2([0,1]) \quad \Phi_1[u](x) = \sin(u(x))$$
$$\Phi_2 : L^p([-1,1]) \to L^p([-1,1]) \quad \Phi_2[u](x) = \max(u(x), 0)$$

Show that both functions are well defined. However Φ_1 is not Frechet differentiable and the following operators cannot serve as a Newton derivative for Φ_2 at $u^* = 0$ for any $p \in [1, +\infty)$

$$D_N F(u)[h] = \begin{cases} h(x) & \text{if } u(x) \geq 0. \\ \delta h(x) & \text{if } u(x) = 0. \\ 0 & \text{if } u(x) < 0. \end{cases}$$

Here $\delta \in [0,1]$ is an arbitrary number.

4) Let X, Y, Z Banach spaces and let $F : X \to Y$ and $G : Y \to Z$ such that F is Newton-differentiable at x with Newton derivative $D_N F(x)$ and G is Newton differentiable at $F(x)$ with derivative $D_N G(F(x))$. Assume that $D_N F$ and $D_N G$ are uniformly bounded in neighborhoods of x and $F(x)$. Show the following: $G \circ F$ is Newton differentiable at x with derivative

$$D_N(F \circ G)(x) = D_N G(F(x)) \circ D_N F(x).$$