Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 2

1) (Globalized Newton for equations) Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be continuously differentiable with Lipschitz continuous differential $x \mapsto DF(x)$. Consider the globalized Newton method from the lecture notes, see equation (11). Assume that the sequence $(x_k)_{k\in\mathbb{N}}$ of Newton iterates is bounded and that $DF(x_k)^{-1}$ exists and satisfies $\|DF(x_k)^{-1}\| \le M$ for every $k \in \mathbb{N}$. Show the following statements:

(i) For every $k \in \mathbb{N}, \omega, \nu \in (0,1)$ there is a smallest index $l = l(k,\nu) \in \mathbb{N}$ such that

$$\|F(x_k + \omega^l d_k)\| \le (1 - \nu\omega^l)\|F(x_k)\|$$

Moreover, the mapping $k \mapsto l(k)$ is uniformly bounded in k.

(ii) The sequence $(x_k)_{k\in\mathbb{N}}$ of Newton iterates converges to some x^* with $F(x^*) = 0$.

(iii) There is an index k_0 such that for every $k \ge k_0$ we have $l(k) = 0$, i.e. the full Newton step is accepted. Determine also the convergence rate.

2) (Characterization of q-superlinear convergence) Consider continuously differentiable function $F : \mathbf{R}^n \to \mathbf{R}^n$, an x^* with $DF(x^*)$ invertible and a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \to x^*$. Show the equivalence of the following assertions:

(i) $(x_k)_k$ converges superlinearly to x^* and $F(x^*) = 0$.

(ii) $\|F(x_k) + DF(x^*)[x_{k+1} - x_k]\| = o(\|x_{k+1} - x_k\|)$.

(iii) $\|F(x_k) + DF(x_k)[x_{k+1} - x_k]\| = o(\|x_{k+1} - x_k\|)$.

3) For continuously differentiable $F : \mathbf{R}^n \to \mathbf{R}^n$ consider the Newton-method to find x^* with $F(x^*) = 0$. Instead of solving $DF(x_k)[x_{k+1} - x_k] = -F(x_k)$ in every iteration, we only assume

$$\|DF(x_k)[x_{k+1} - x_k] + F(x_k)\| \le \eta_k\|F(x_k)\|$$

for some sequence $\eta_k \to 0$ with $\eta_k \in (0,1)$ in every iteration. Show the following:

(i) If $x_n \to x^*$ and $DF(x^*)$ is invertible, the convergence is even q-superlinear.

4) Give a rigorous proof of Theorem 2.2 in the lecture notes. For (c) only show the upper-semicontinuity of the Clarke generalized Jacobian $x \to \partial F(x)$. Moreover consider the following function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x_1,x_2) = \begin{cases} (x_1, x_2)^T & \text{if } x_1 \ge 0. \\ (x_1, -2x_1 - x_2)^T & \text{if } x_1 \le 0 \text{ and } x_2 \ge -x_1. \\ (-2x_1 - 2x_2, x_2)^T & \text{if } x_1 \le 0 \text{ and } 0 \le x_2 \le -x_1. \\ (-x_1 + 2x_2, x_2)^T & \text{if } x_1 \le 0 \text{ and } 0 \ge x_2 \ge x_1. \\ (x_1, 2x_1 - x_2)^T & \text{if } x_1 \le 0 \text{ and } x_2 \le x_1. \end{cases}$$

Compute $\partial f(x)$ and $\partial_C f(x)$ for $x = 0$. What do we learn from this example?

5) (Examples) Consider the following mappings $p_i : \mathbf{R}^n \to \mathbf{R}^n$

$$p_1(x) = \arg\min_{u \in \mathbf{R}^n} \frac{1}{2}\|u - x\|_2^2 + \|u\|_1$$

$$p_2(x) = \arg\min_{u \in C} \frac{1}{2}\|u - x\|_2^2$$

where $C = [0, 1]^n$ is the unit cube. Show that the functions are well-defined, i.e. the minimization problems have a unique solution. Moreover, compute the sets of Clarke generalized jacobians $\partial p_i(x)$.

Remark: For **4)** and **5)** you can assume that the mappings under consideration are sufficiently Lipschitz continuous.