Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 11 (sample questions for the exam)

1) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function and let $x \in \mathbb{R}^n$.

(i) Define the $B-$Subdifferential $\partial_B F(x)$ and the set of Clarke generalized Jacobians $\partial F(x)$ and show that the set $\partial F(x)$ is compact.

(ii) Define the notion of semismoothness of F at $x \in \mathbb{R}^n$ and define the directional derivative $F'(x, d)$ for a direction $d \in \mathbb{R}^n$.

(iii) Show the following statement: IF $F : \mathbb{R}^n \to \mathbb{R}^n$ is directionally differentiable at x and

$$\sup_{V \in \partial F(x+h)} \|Vh - F'(x,h)\| = o(\|h\|) \quad \text{as } h \to 0.$$

then F is semismooth at x.

2) Let X, Y be Banach spaces and let $F : X \to Y$.

(i) State the definition of Newton differentiability of F at a point $x \in X$ in infinite dimensions and write down the semismooth Newton method to solve $F(x) = 0$.

(ii) Assume that $x^* \in X$ with $F(x^*) = 0$ with Newton derivative $D_N F(x^*)$. Assume further that there exist $\delta > 0$ and $C > 0$ with $\|D_N F(x)^{-1}\|_{\mathcal{L}(Y,X)} \leq C$ for all $x \in B_\delta(x^*)$. Then the semismooth method method converges superlinearly to x^* for all initializations $x_0 \in X$ sufficiently close to x^*.

(iii) Write down a Newton derivative of the operator

$$\Phi : L^p(\Omega) \to L^q(\Omega), \quad \Phi(u)(x) = \max(u(x), 0)$$

for $\Omega \subset \mathbb{R}^n$ a bounded domain. For which p, q does the Newton differentiability hold?

(iv) Use (iii) to prove the Newton differentiability of the function $\text{proj}_{U_{ad}} : L^2(\Omega) \to L^q(\Omega)$ where $U_{ad} := \{u \in L^p(\Omega) : a \leq u(x) \leq b \text{ for a.e. } x \in \Omega\}$ for some scalars $a \leq b$ and appropriate choice of q.

3) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the following bilevel optimization problem for $y_d \in L^2(\Omega)$ and $\alpha, \beta > 0$

$$\min_{(u,y) \in U_{ad} \times H_0^1(\Omega)} \frac{1}{2}\|y - y^d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \quad \text{s.t. } y \in \arg\min_{v \in H_0^1(\Omega)} \frac{1}{2}\|v - u\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\nabla v\|^2_{L^2(\Omega)}.$$

Here $U_{ad} = \{u \in L^2(\Omega) : u \leq \psi\}$ for some $\psi \in L^\infty(\Omega)$.

(i) Show that the problem has a unique solution $(y, u) \in U_{ad} \times H_0^1(\Omega)$.

(ii) Provide the first order optimality system for the problem above using the adjoint approach.

(iii) Write the optimality system in the form $F(x) = 0$, where $F : X \to Y$ is a suitable mapping between function spaces. Specify the function spaces such that the operator $F : X \to Y$ is Newton differentiable.

(iv) Write down the semismooth Newton-method in infinite dimensions for solving the system $F(x) = 0$.

4) Consider for a bounded Lipschitz-domain $\Omega \subset \mathbb{R}^n$, $f \in L^2(\Omega)$, a continuous linear operator $A : L^2(\Omega) \to L^2(\Omega)$ and a scalar parameter $\alpha > 0$ the following optimization problem

$$(P_1) \qquad \min_{u \in U_{ad}} J(u) := \frac{1}{2}\|Au - f\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{H^1(\Omega)} \quad \text{over } H^1_0(\Omega),$$

where $U_{ad} \subset L^2(\Omega)$ is given by $U_{ad} := \{u \in L^2(\Omega) : a \leq u(x) \leq b \text{ for a.e. } x \in \Omega\}$ for some scalars $a \leq b$.

(i) Show that the problem has a unique solution $u \in H^1_0(\Omega)$.

(ii) Define for a general function $F : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ the Moreau-Yosida regularization $F_\gamma : X \to \mathbb{R}$.

(iii) Consider the (unconstrained) regularized problem

$$(P_2) \qquad \min_{u \in H^1_0(\Omega)} J_\gamma(u) := J(u) + (\mathcal{I}_{U_{ad}})_\gamma(u)$$

and show that for $\mathcal{I}_{U_{ad}} : L^2(\Omega) \to L^2(\Omega)$ the Moreau-Yosida regularization is given by

$$(\mathcal{I}_{U_{ad}})_\gamma(u) = \frac{1}{2\gamma}\left(\|\min(0, u - a)\|^2_{L^2(\Omega)} + \|\max(u - b, 0)\|^2_{L^2(\Omega)}\right)$$

(iv) Show that any sequence of solutions $(u_\gamma)_{\gamma > 0}$ of (P_2) converges weakly in $H^1_0(\Omega)$ to the solution of (P_1) for $\gamma \downarrow 0$.

(v) Show that the first order optimality system for (P_2) can be written as $F(u) = 0$ where

$$F : H^1_0(\Omega) \to H^{-1}(\Omega) \quad F(u) := \alpha\Delta u + \alpha u - A^*(Au - f) + \frac{1}{2\gamma}P(a, b, u),$$
$$P(a, b, u) := \min(0, u - a) + \max(u - b, 0)$$

and discuss Newton differentiablity of F.