Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 10

1) Let W, U, Y be reflexive Banach spaces, $J : Y \times U \to \mathbb{R}$ and $e : Y \times U \to W$ be continuously differentiable. Moreover let $D_1 e(y,u) : Y \to W$ be continuously invertible and let $e(y,u) = 0$ have a single-valued and bounded solution operator $u \mapsto y(u)$ with $e(y(u),u) = 0$. In addition assume that the mapping $e : Y \times U \to W$ is continuous under weak convergence. Consider the following general control problem

$$\min_{(y,u)\in Y\times U} J(y,u) \quad \text{s.t. } e(y,u) = 0 \text{ and } u \in U_{ad}$$

where $U_{ad} \subset U$ is nonempty, bounded and convex in U.

(i) Show that the problem above has a solution.

(ii) Show that for any locally optimal solution $(y,u) \in Y \times U_{ad}$ there is a $p \in W^*$ such that

$$\begin{aligned} e(y,u) &= 0, \\ D_1 e(y,u)^* p &= -D_1 J(y,u), \\ \langle D_2 J(y,u) + D_2 e(y,u)^* p, v - u\rangle_{U^*,U} &\geq 0 \quad \text{for all } v \in U_{ad}. \end{aligned}$$

(iii) Apply the abstract framework presented above to $n = 2$, with $\alpha > 0$

$$\begin{aligned} J(y,u) &= \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ e(y,u) &= -\Delta y + y^3 - u \quad \text{in } H^{-1}(\Omega) \\ U_{ad} &= \{u \in L^2(\Omega) : a \leq u(x) \leq b \text{ for a.e. } x \in \Omega.\} \end{aligned}$$

where $y_d \in L^2(\Omega)$ and $a, b \in L^\infty(\Omega)$. Hint: For the weak-to-weak continuity consider compact embedding of $H^1_0(\Omega) \underset{c}{\hookrightarrow} L^5(\Omega)$ and analyse the convergence of y^3 in $L^{5/3}(\Omega)$. For the differentiability of the superposition operator use $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$.

2) Let $\Omega \subset \mathbb{R}^n, n = 2, 3$ be convex, bounded and Lipschitz. Moreover consider $f : \mathbb{R} \to \mathbb{R}$ of class C^3, with $f' \geq 0$, $f(0) = 0$ and assume the existence of constants $c_1, c_2 > 0$ with

$$|f'''(u)| \leq c_1 + c_2|u|^{\frac{p-6}{2}} \quad \text{for all } u \in \mathbb{R},$$

where we fix $p \in [6, +\infty]$ for $n = 2$ and $p = 6$ for $n = 3$. As in the lecture consider the semi-linear differential operator formally given by

$$E : H^1_0 \to H^{-1}(\Omega) \quad E(u) = -\Delta u + f(u)$$

Please show the following:

(i) E is twice continuously Frechet-differentiable. Compute the 1st and 2nd derivatives.

(ii) The operator $E : H^1_0(\Omega) \to H^{-1}(\Omega)$ is ν-strongly monotone and uniquely invertible.

(iv) The inverse $E : H^{-1}(\Omega) \to H^1_0(\Omega)$ is Lipschitz continuous with modulus ν^{-1}.

(iii) For every $u \in L^2(\Omega)$ the linear operator $E'(u) \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is continuously invertible with

$$\|E'(u)^{-1}\|_{\mathcal{L}(H^{-1}(\Omega),H_0^1(\Omega))} \leq \nu^{-1}$$

3) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We consider a division of Ω into subdomains such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \quad T_1, T_2 \in \mathcal{T}_h \text{ with } T_1 \neq T_2 \Rightarrow T_1 \cap T_2 \subset \partial T_1 \cap \partial T_2$$

Here $h = \max_{T \in \mathcal{T}_h} \operatorname{diam}(T) > 0$. Define the subspace $Y_h := \operatorname{span}(\{\mathbf{1}_T : \Omega \to \mathbb{R} : T \in \mathcal{T}_h\})$ and prove the following assertions:

(i) The projection $\Pi_h := \operatorname{proj}_{Y_h} : L^2(\Omega) \to Y_h$ is given by

$$\Pi_h(u)|_T = \frac{1}{|T|} \int_T u(x) \mathrm{d}x.$$

(ii) For every $q \in [0, +\infty]$ and every $v \in L^q(\Omega)$ we have $\|\Pi_h v\|_{L^q(\Omega)} \leq \|v\|_{L^q(\Omega)}$.