Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Semismooth Newton Method
Winter semester 2023/24

Exercise Sheet 1

1) (Isolated solutions) Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be continuously differentiable with $DF(x^*) \in \mathcal{L}(\mathbf{R}^n)$ invertible and $F(x^*) = 0$. Is it possible to construct a sequence $(x_n)_{n\in\mathbb{N}}$ of solutions with $F(x_n) = 0$ for every $n \in \mathbf{N}$, $x_n \neq x^*$ and $x_n \to x^*$ as $n \to \infty$?

2) (Newton algorithm) Consider for continuously differentiable $F : \mathbf{R}^n \to \mathbf{R}^n$ the problem of finding $x^* \in \mathbf{R}^n$ such that $F(x^*) = 0$. Assume F has a γ-Hölder continuous derivative, i.e.

$$\|DF(x) - DF(y)\|_{\mathcal{L}(\mathbf{R}^n)} \leq L\|x - y\|^\gamma \quad \text{for every } x, y \in \mathbf{R}^n.$$

(i) Show that for $x, h \in \mathbf{R}^n$ the following bound holds true

$$\|F(x + h) - F(x) - DF(x)[h]\| \leq \frac{L\|h\|^{1+\gamma}}{2}$$

(ii) Let $x^* \in \mathbf{R}^n$ such that $F(x^*) = 0$ and $DF(x^*)$ invertible with $\|DF(x^*)\|^{-1} \leq \beta$. Show that for fixed $c \in (0, 1)$ and $0 < \eta \leq \left(\frac{c}{L\beta}\right)^{\frac{1}{\gamma}}$ the following holds: $DF(x)$ is invertible and

$$\|DF(x)^{-1}\|_{\mathcal{L}(\mathbf{R}^n)} \leq \frac{\beta}{1 - c} \quad \text{for every } x \in B_\eta(x^*)$$

(iii) Assume $F(x^*) = 0$ and $DF(x^*)$ invertible with $\|DF(x^*)^{-1}\| \leq \beta$. Let again $c \in (0, 1)$ fixed. Show that there is $r > 0$ such that the Newton method when initialized from $x_0 \in B_r(x^*)$ is well defined and the following bound holds

$$\|x_{k+1} - x^*\| \leq \frac{\beta L}{2(1 - c)}\|x_k - x^*\|^{1+\gamma}$$

3) (Example) Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sqrt{x^2 + 1}$.

(i) Write down the Newton algorithm to find first-order stationary points of f, i.e. points x where $f'(x) = 0$. Are the iterates well defined?

(ii) Which convergence rates can we expect? Hint: Distinguish the cases $|x_0| < 1$, $|x_0| = 1$ and $|x_0| > 1$.

4) (Newton for optimization) Let $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|^p$ for $p > 2$. Consider the Newton method for finding first-order stationary points of f. Initialize with $x_0 > 0$. Show that the method converges to the unique minimum of f with a q-linear rate, but not with a q-superlinear rate. Does this contradict the statement from exercise 2)?