An Introduction to Mesh Generation Algorithms

Steve Owen

…a U.S. Department of Energy national security laboratory.

National Nuclear Security Administration

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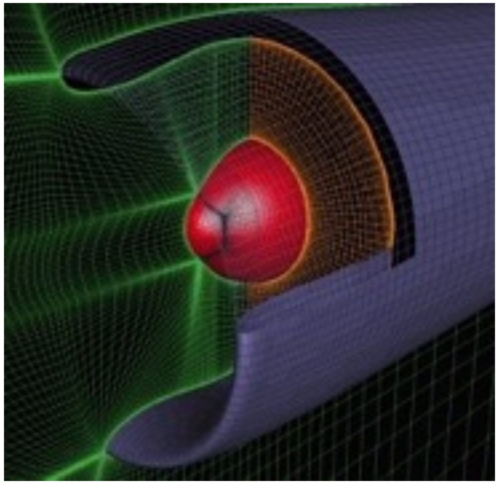
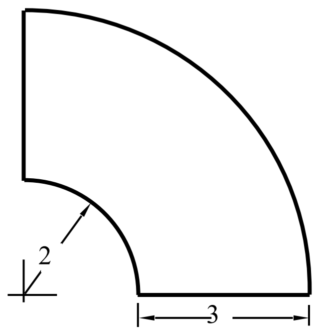
Sandia National Laboratories

Overview

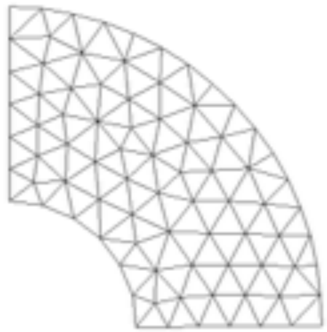
- **The Simulation Process**
- **Geometry Basics**
- **The Mesh Generation Process**
- **Meshing Algorithms**
  – Tri/Tet Methods
  – Quad/Hex Methods
  – Hybrid Methods
  – Surface Meshing
- **Algorithm Characteristics**



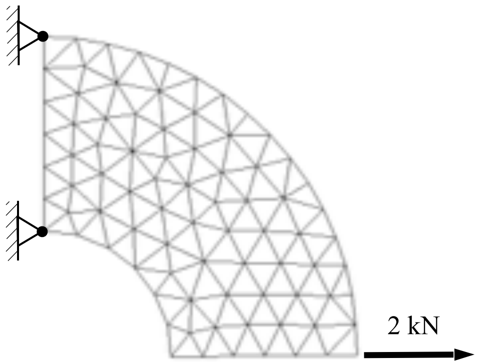
Simulation Process



1. Build CAD Model

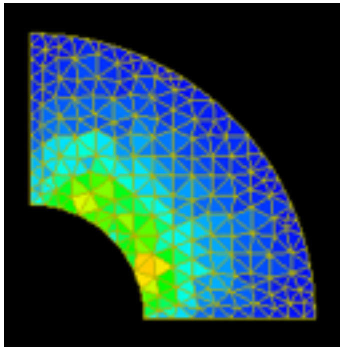


2. Mesh



3. Apply Loads and Boundary Conditions

4. Computational Analysis



5. Visualization

Adaptive Simulation Process

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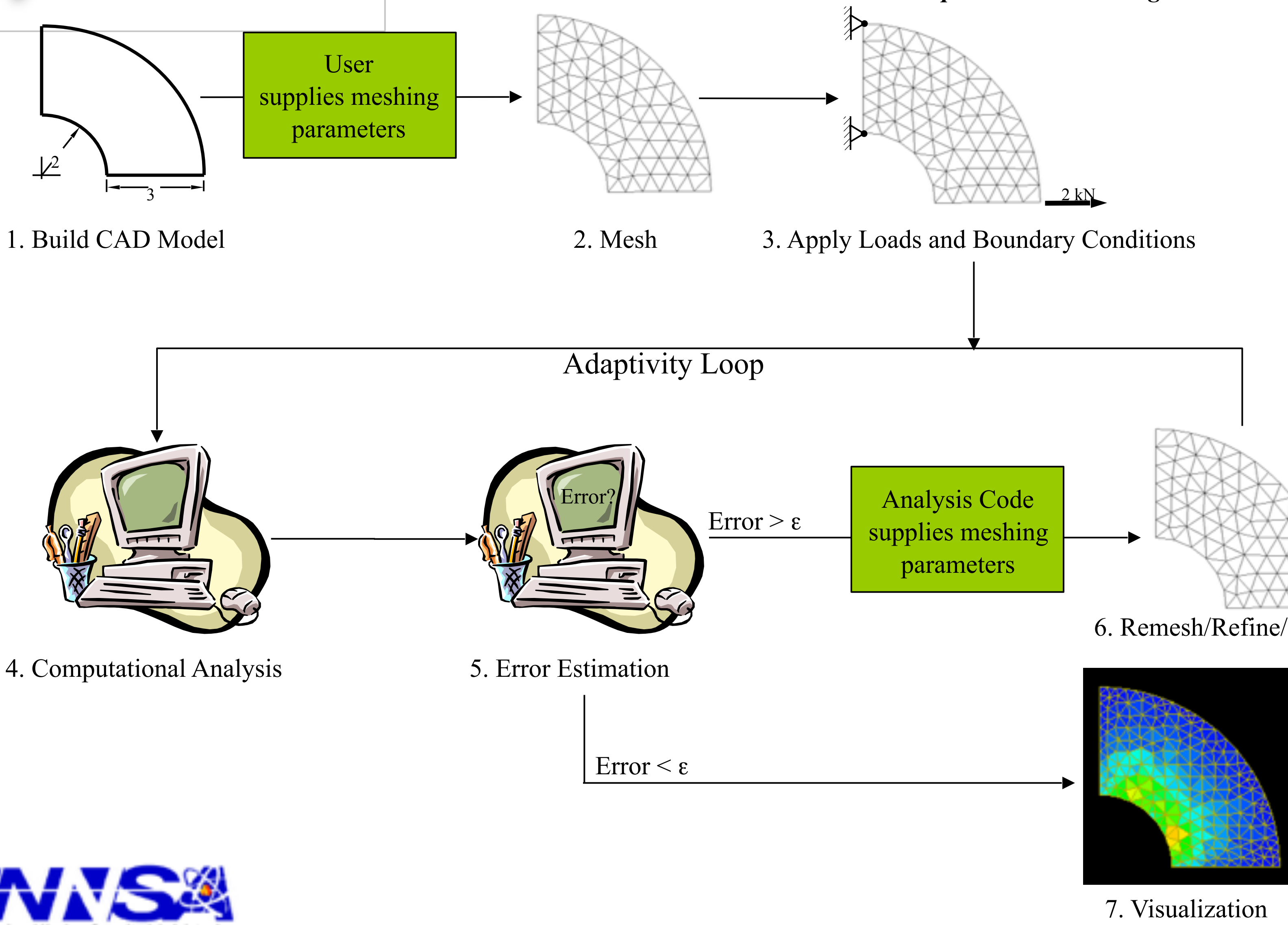
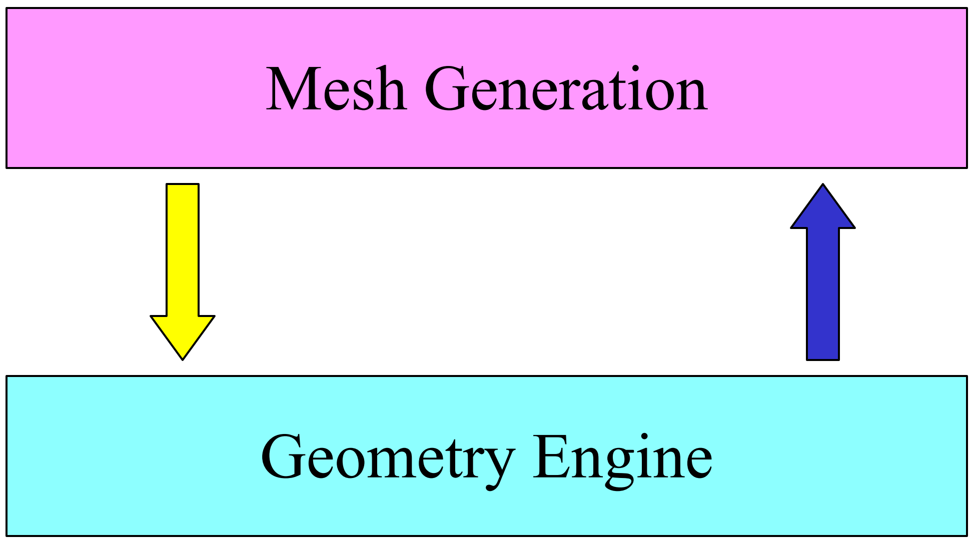
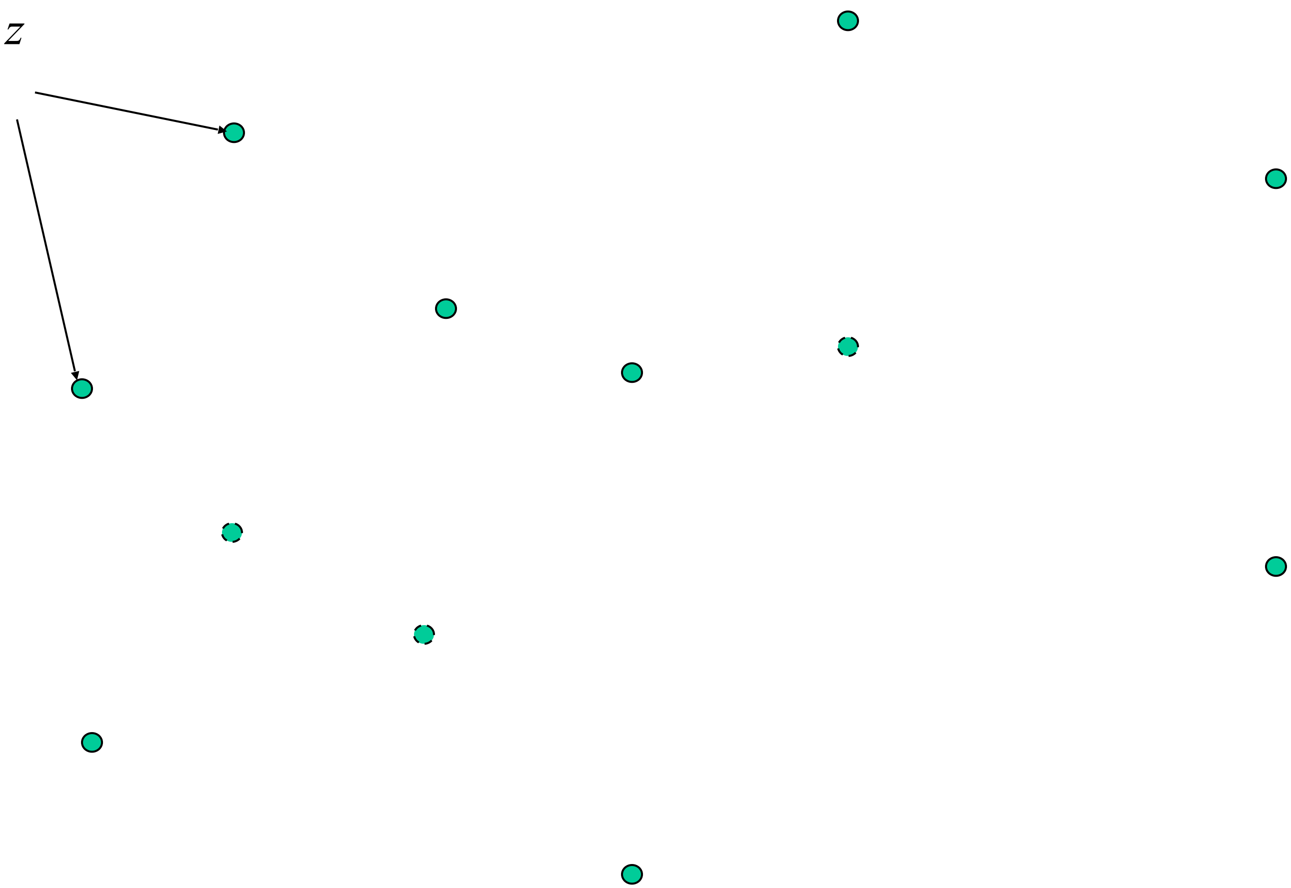


Geometry



Geometry

vertices: *x,y,z location*



Geometry

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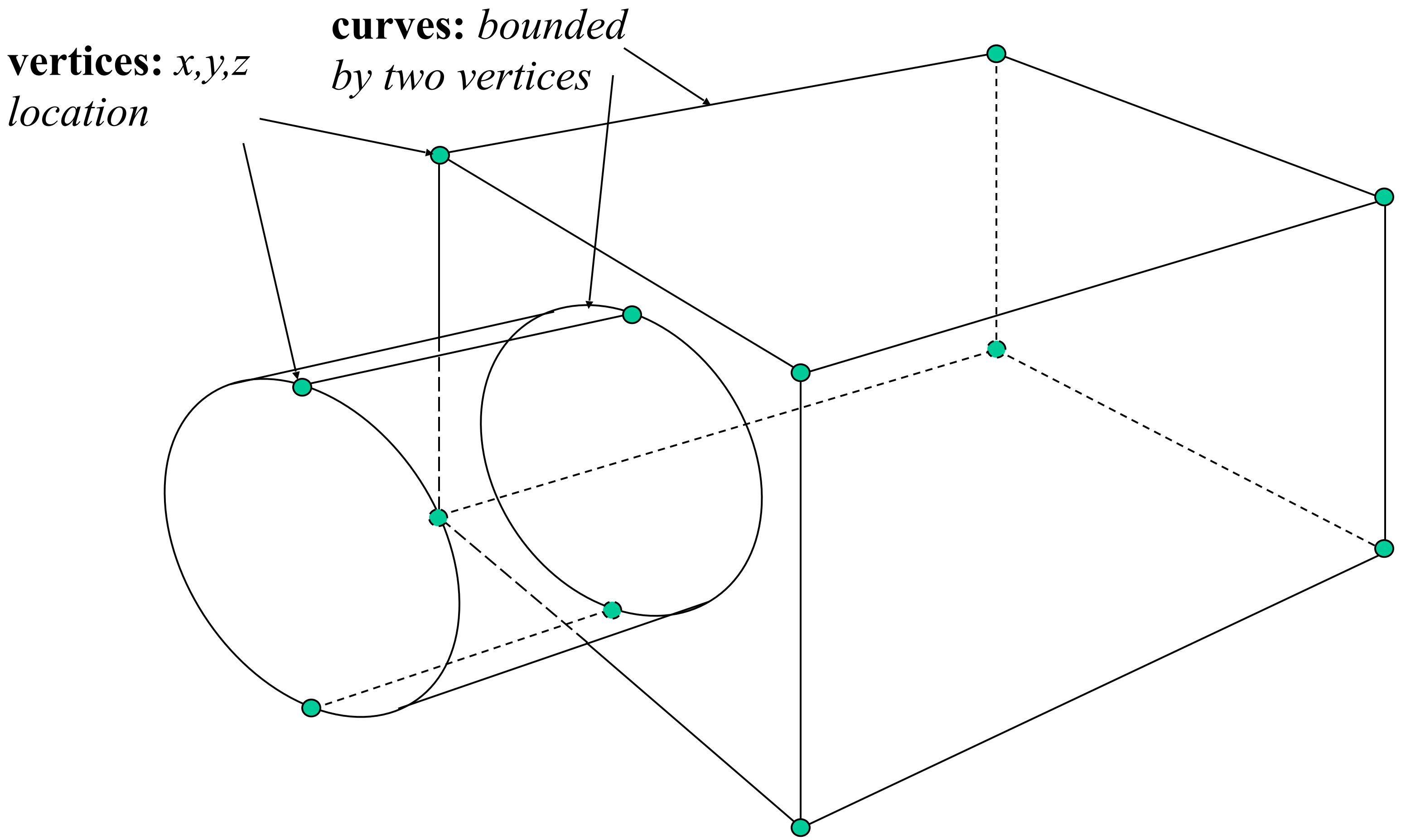

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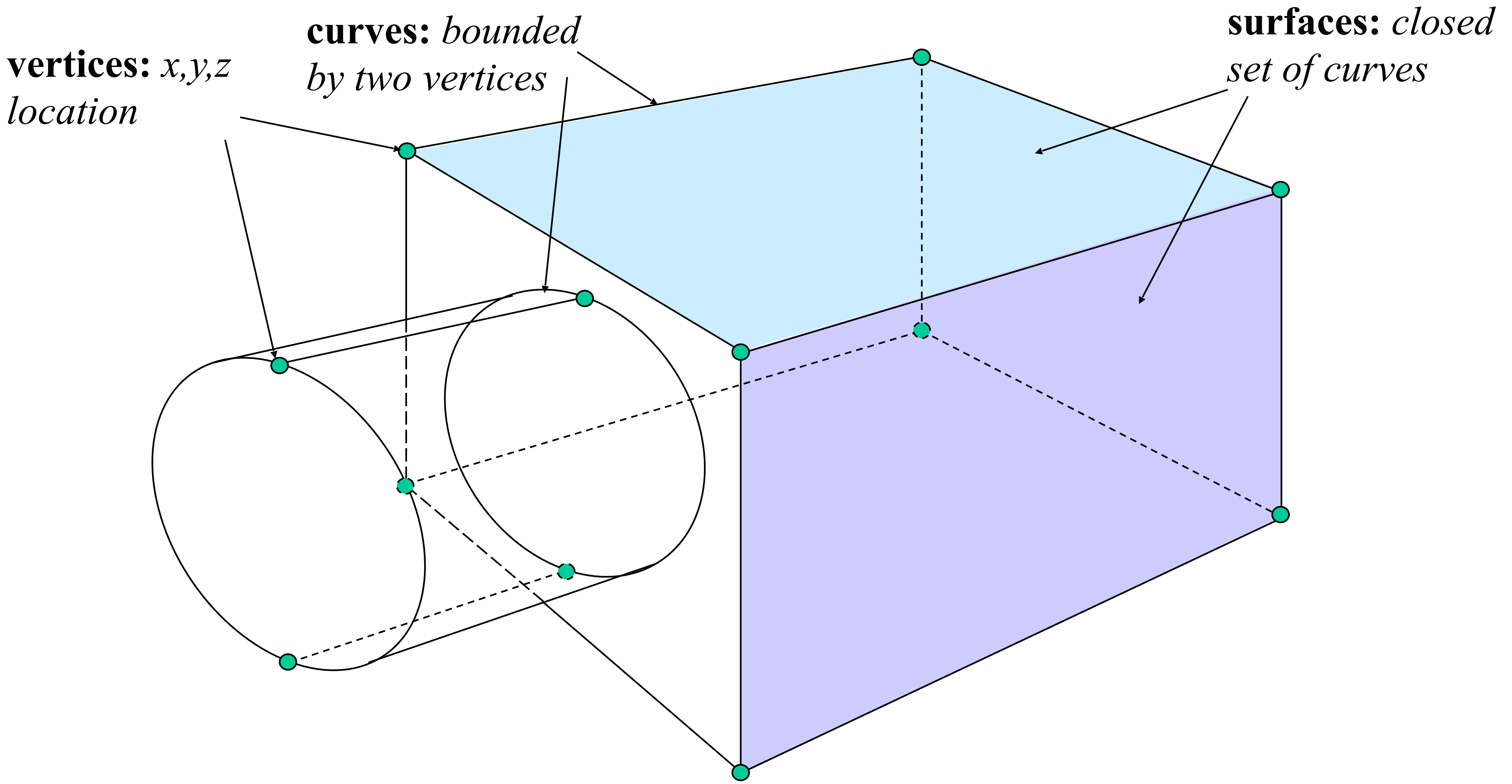
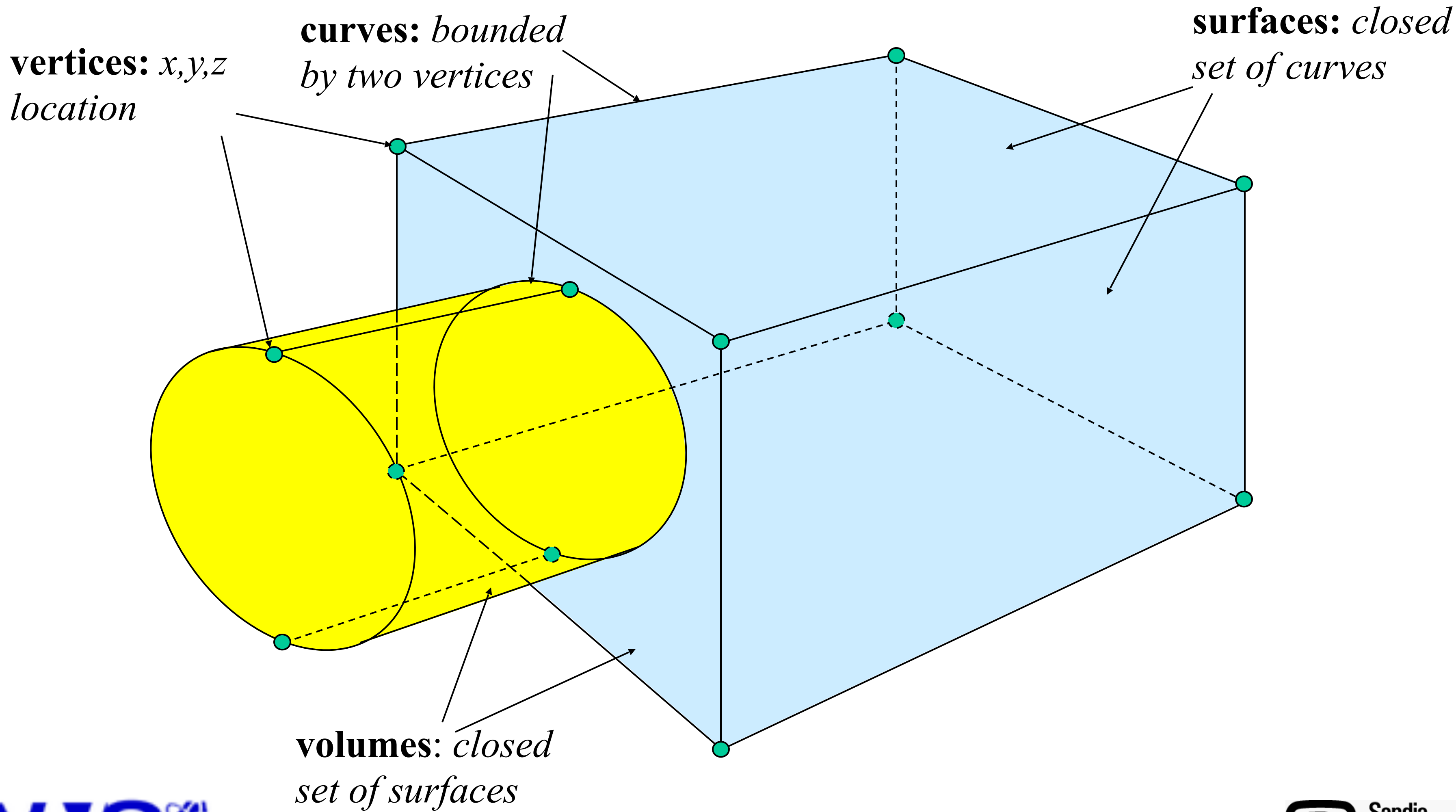
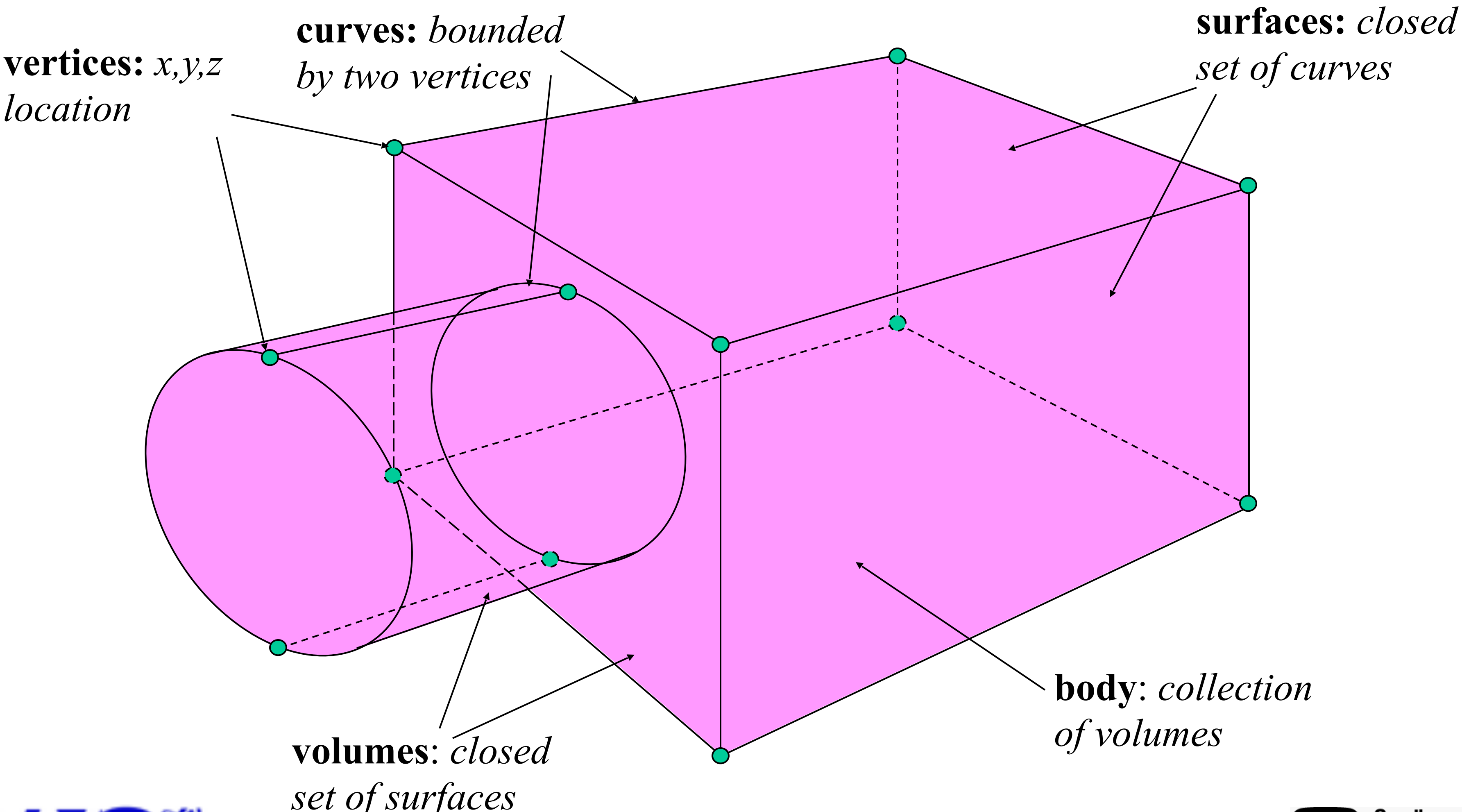

vertices: *x,y,z location*
curves: *bounded by two vertices*
surfaces: *closed set of curves*

Geometry


vertices: *x,y,z location*
curves: *bounded by two vertices*
surfaces: *closed set of curves*
volumes: *closed set of surfaces*

Geometry


vertices: *x,y,z location*
curves: *bounded by two vertices*
surfaces: *closed set of curves*
volumes: *closed set of surfaces*
body: *collection of volumes*

Geometry

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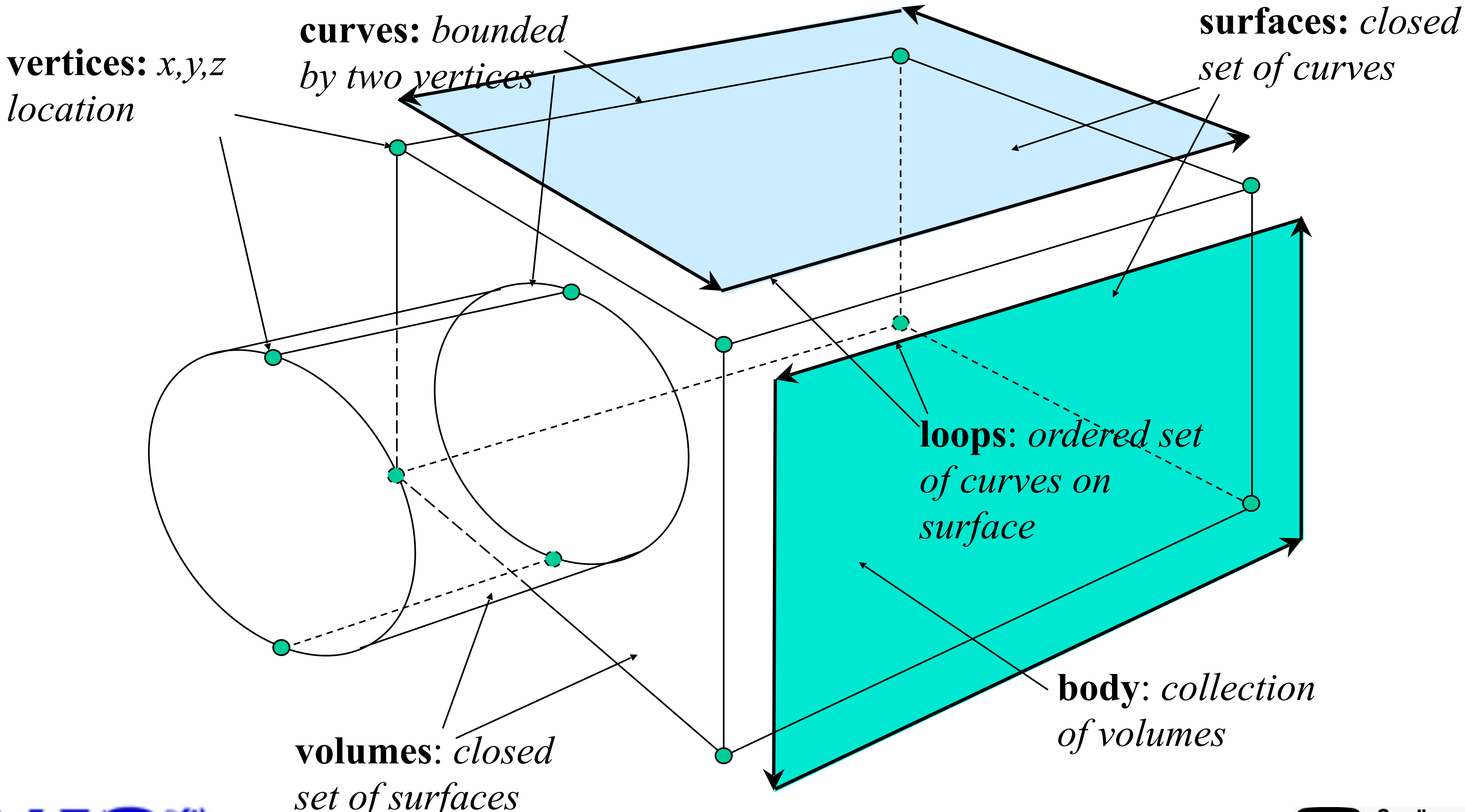

Geometry

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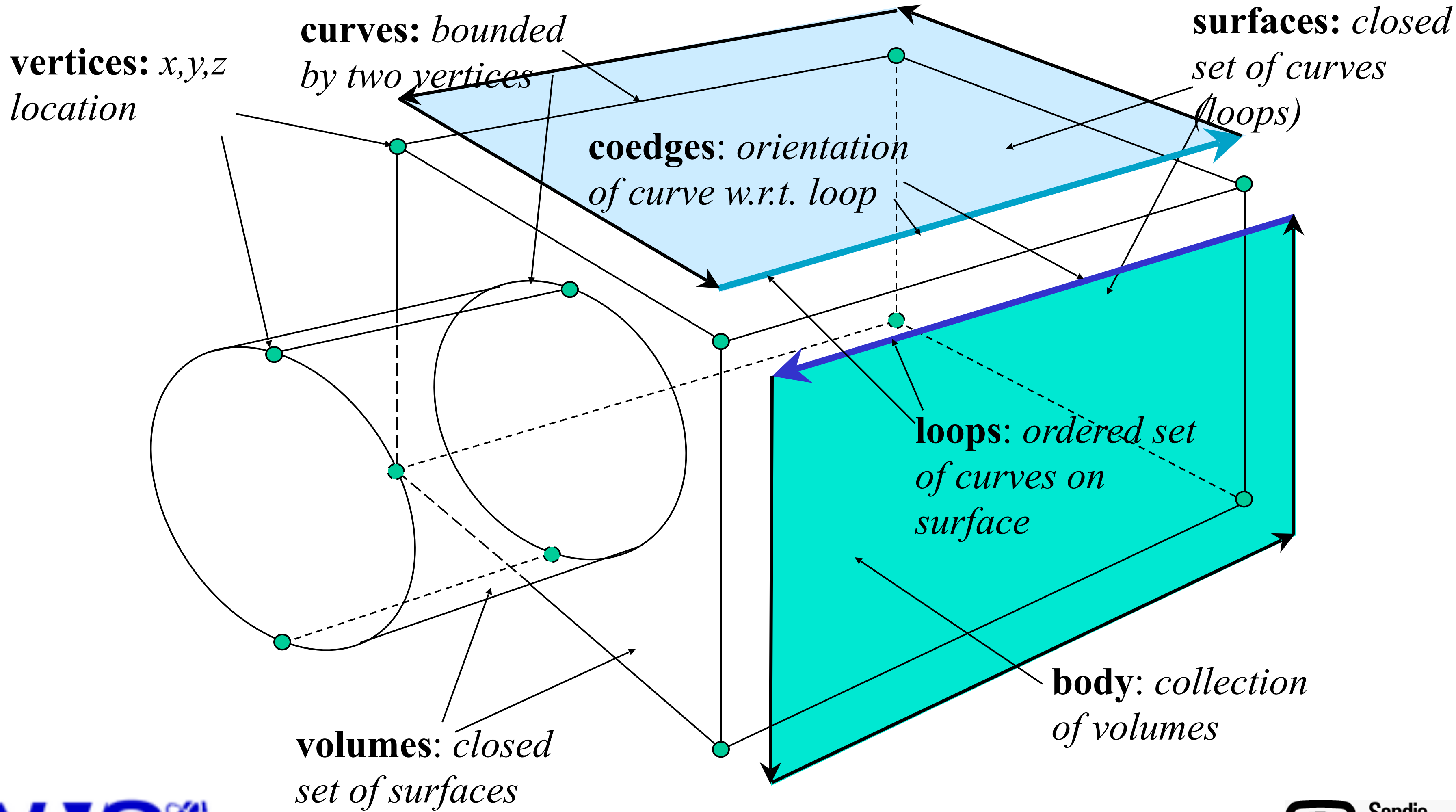

Geometry

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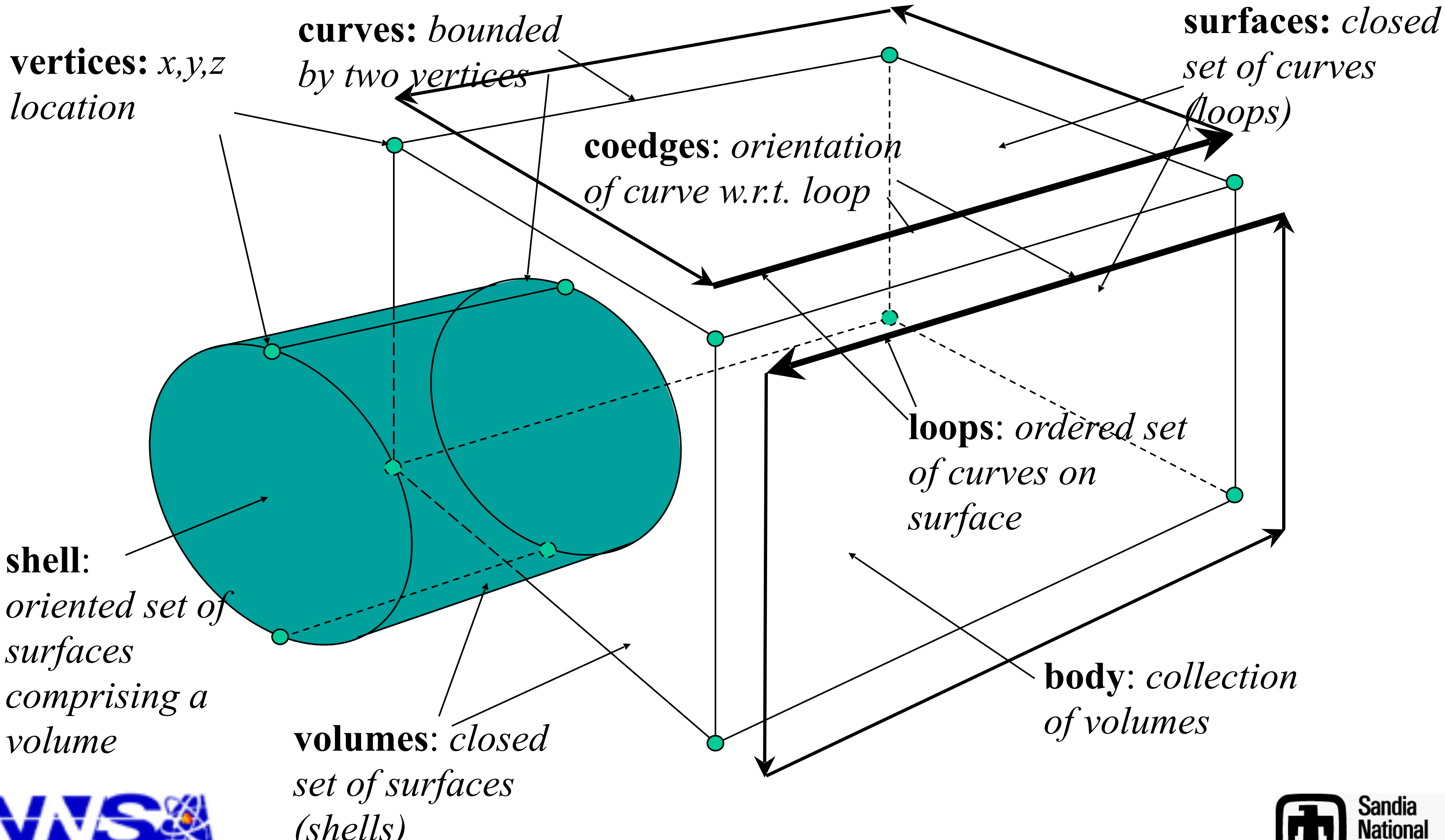

Geometry

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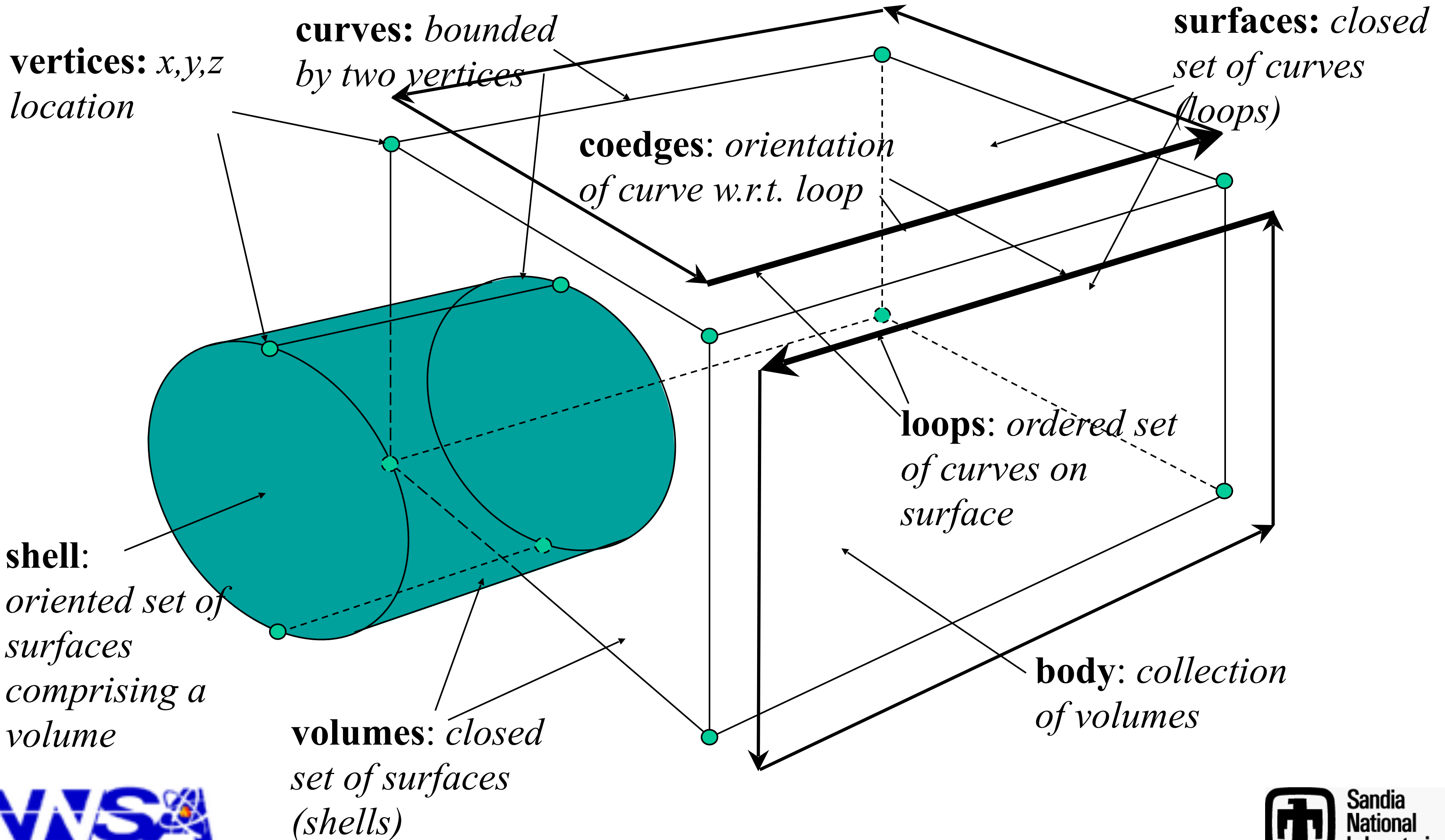

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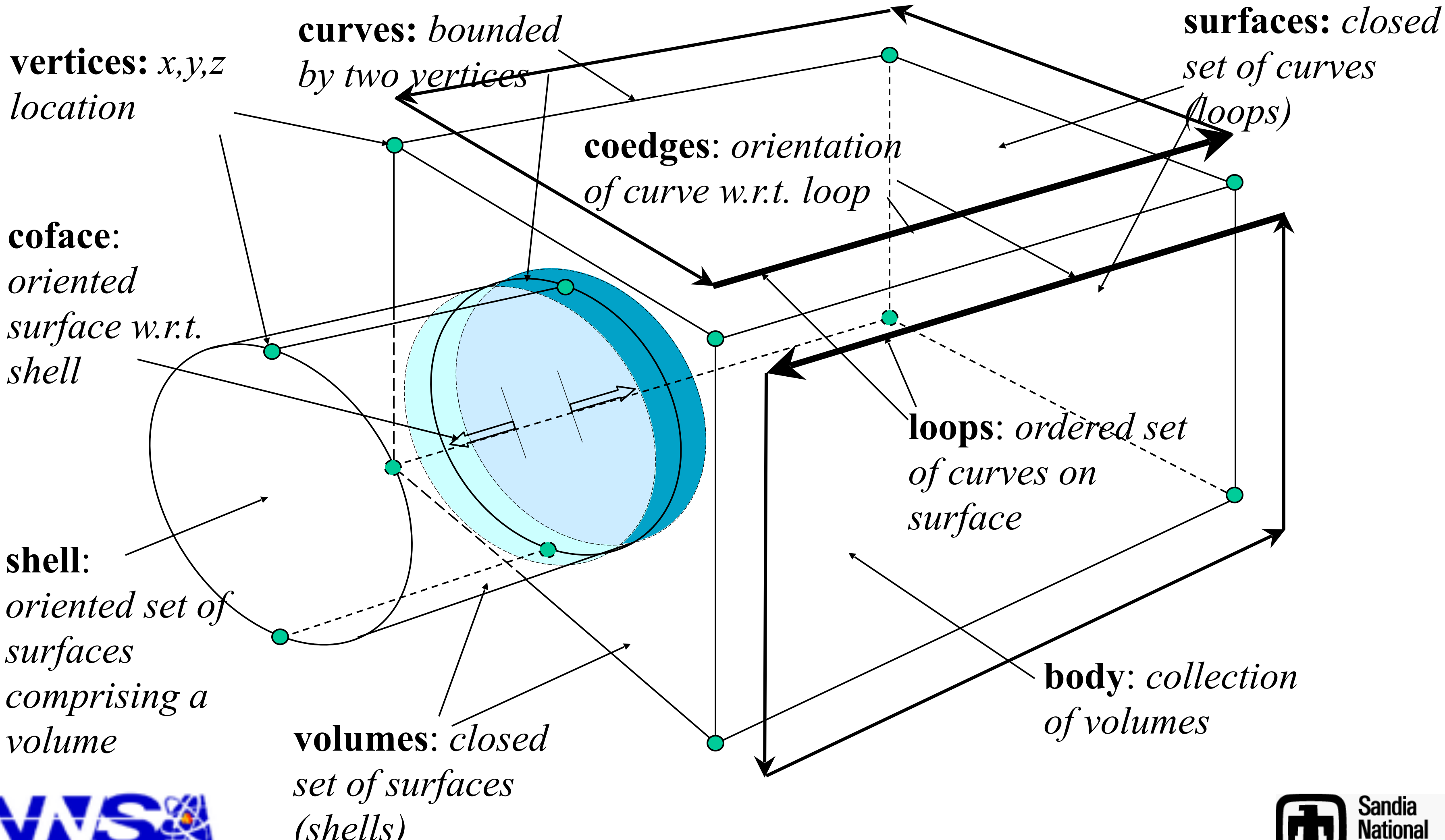
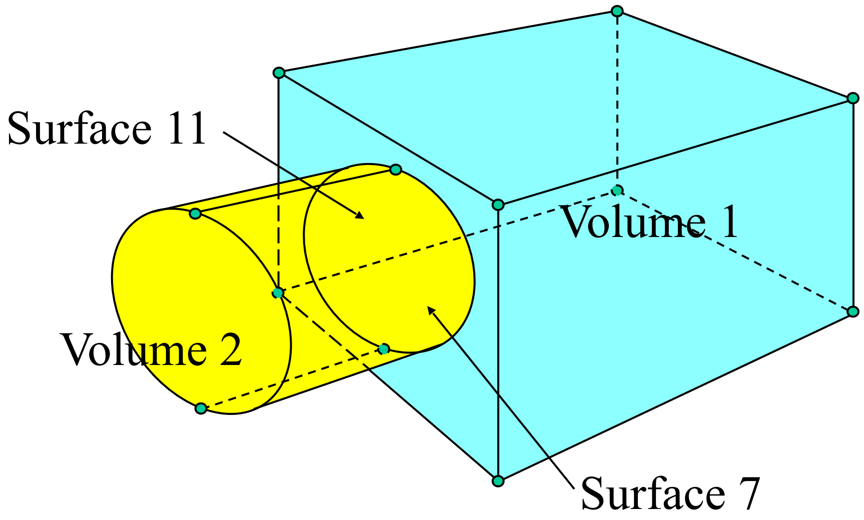
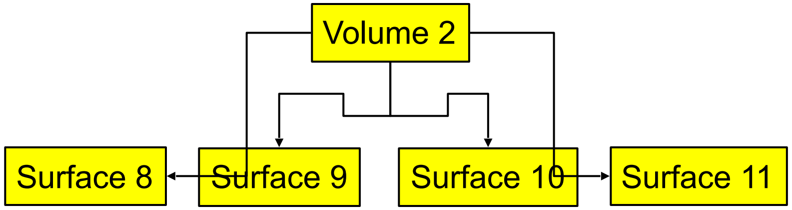


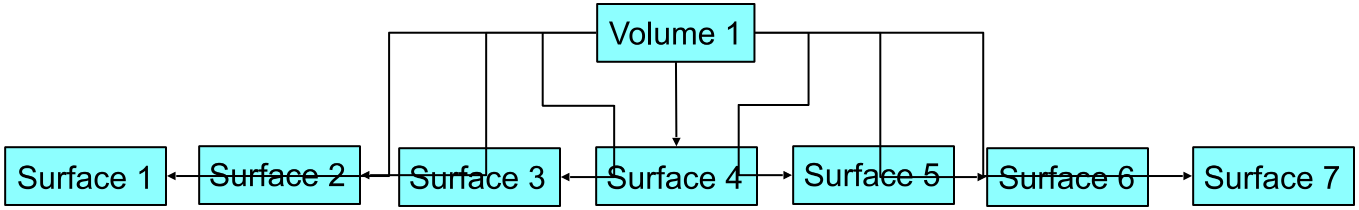
vertices: *x,y,z location*
curves: *bounded by two vertices*
coedges: *orientation of curve w.r.t. loop*
surfaces: *closed set of curves (loops)*
coface: *oriented surface w.r.t. shell*
loops: *ordered set of curves on surface*
shell: *oriented set of surfaces comprising a volume*
volumes: *closed set of surfaces (shells)*
body: *collection of volumes*

Geometry

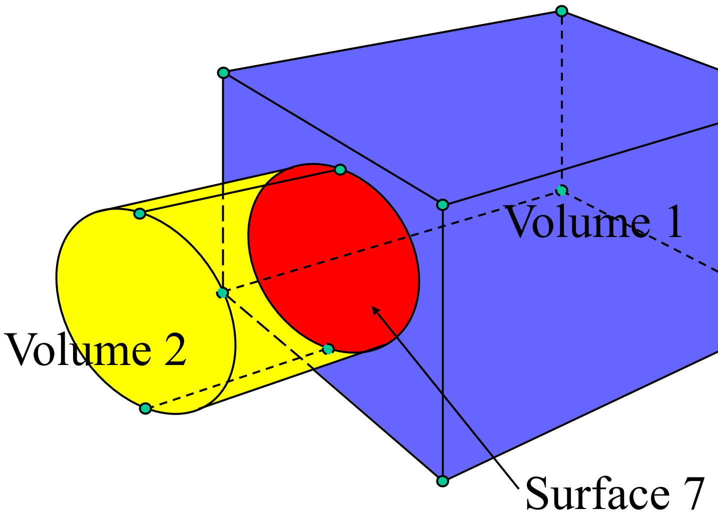


Manifold Geometry:
Each volume maintains its own set of unique surfaces

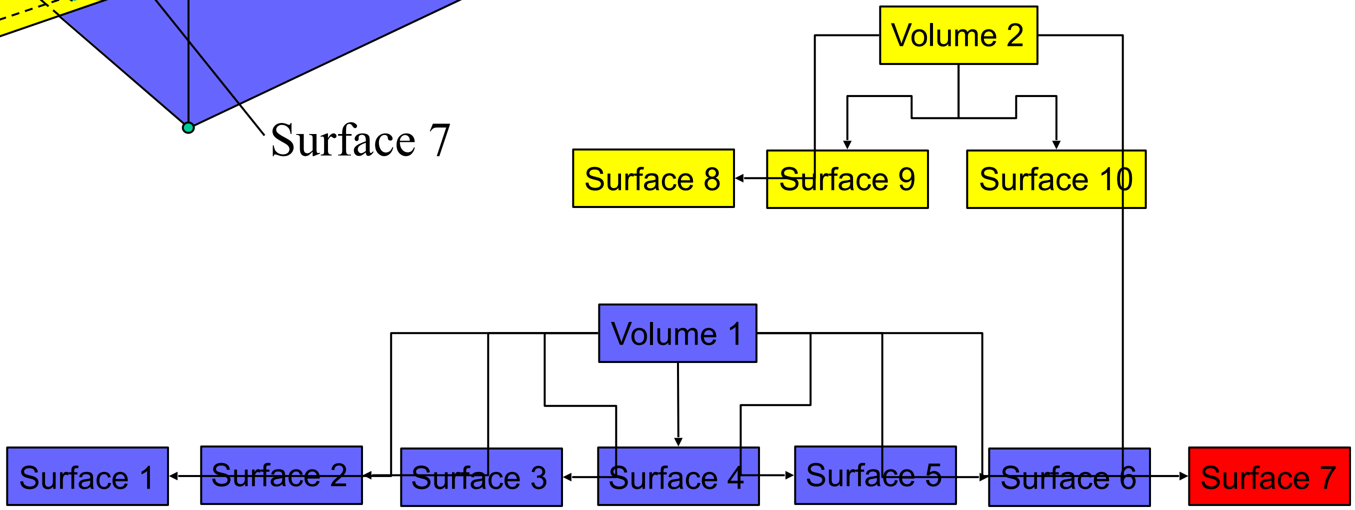




Geometry



Non-Manifold Geometry: *Volumes share matching surfaces*



Mesh Generation Process

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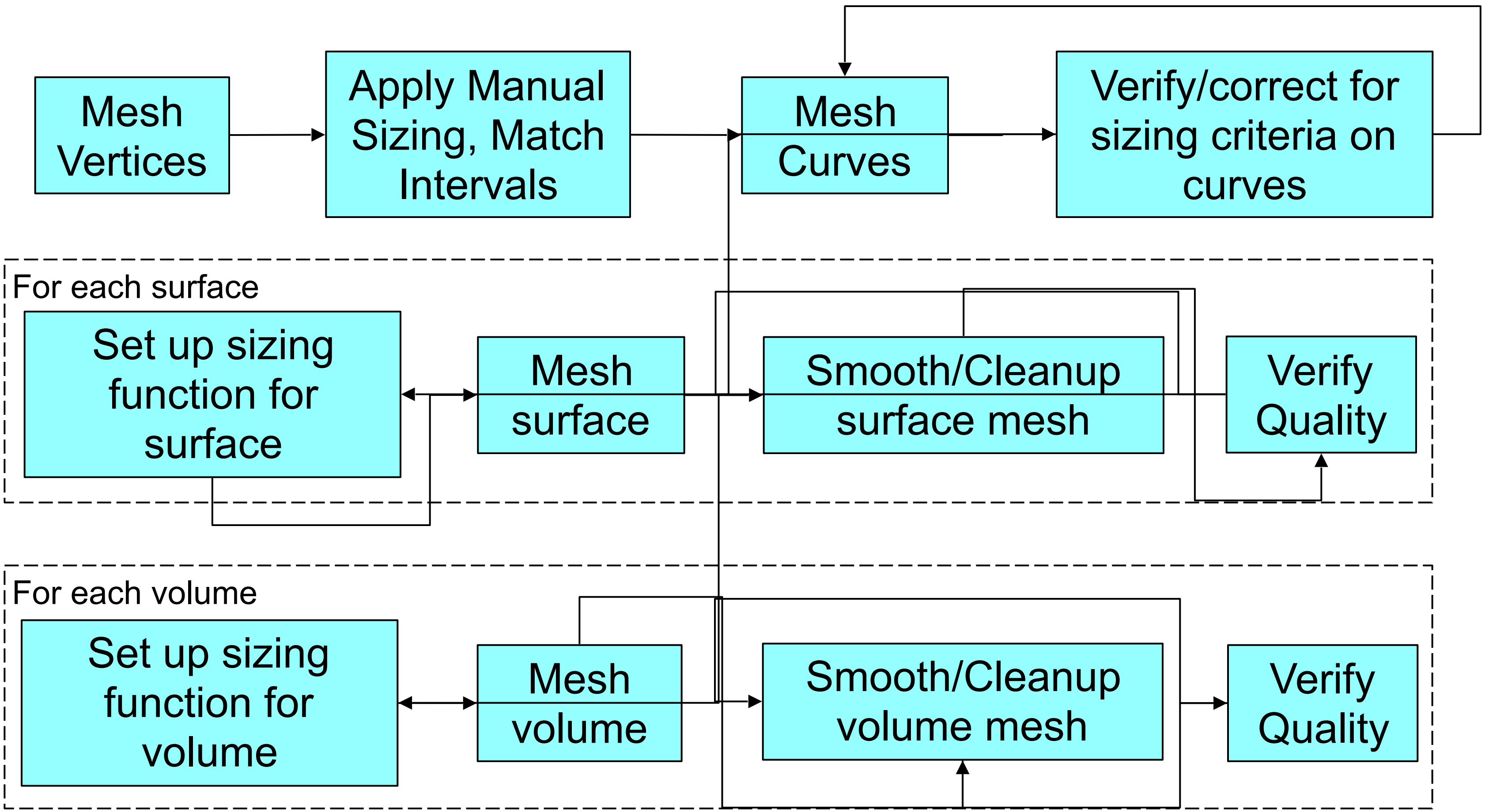

Meshing Algorithms

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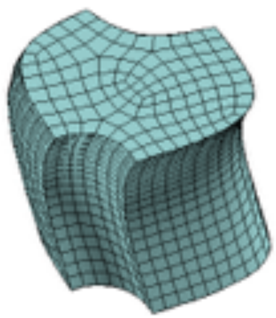

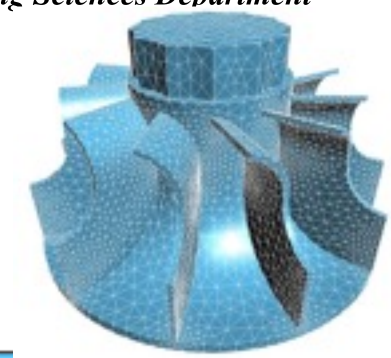

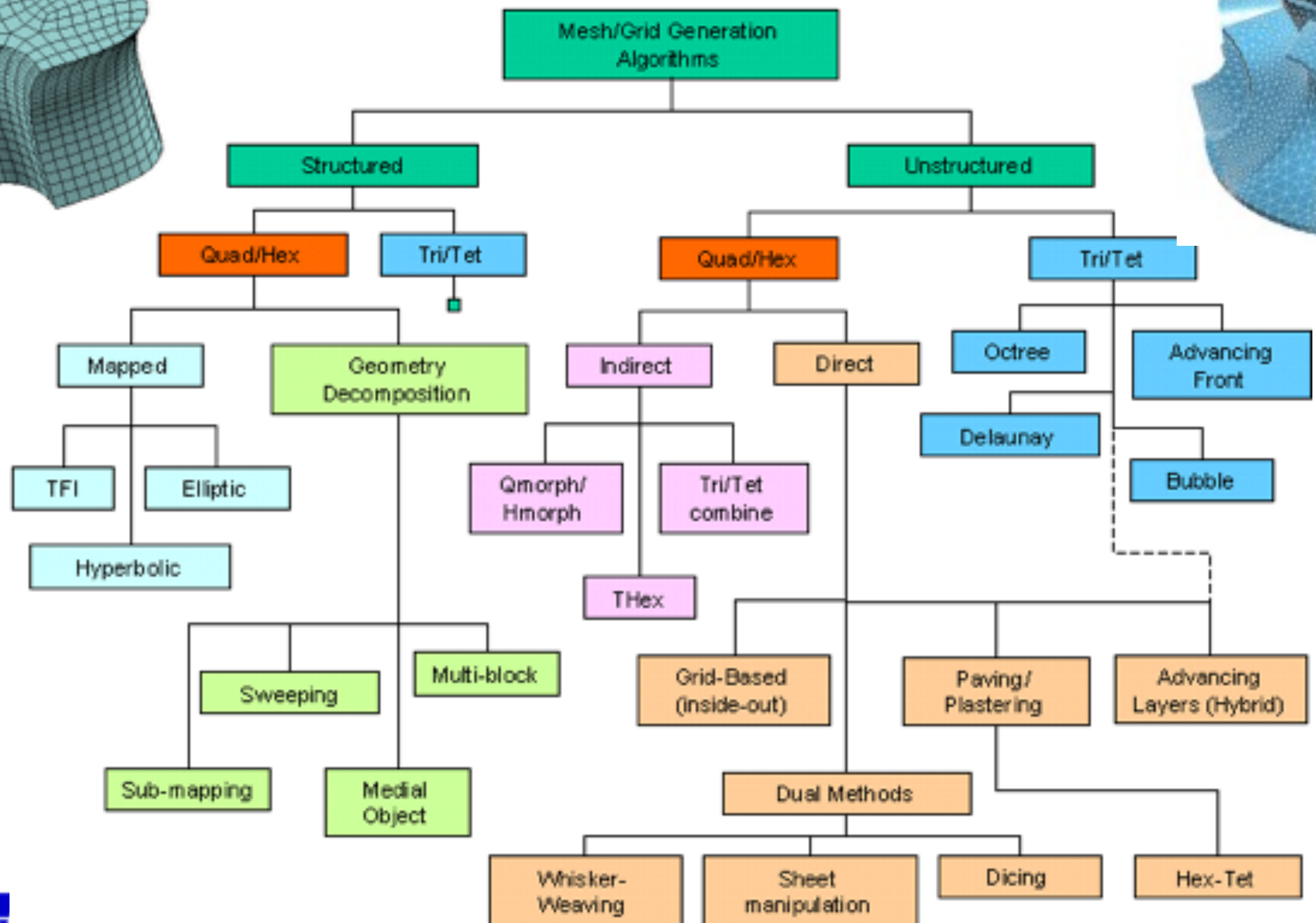

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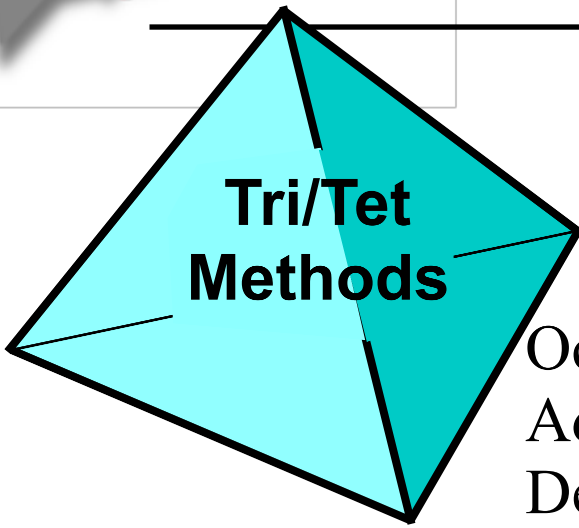
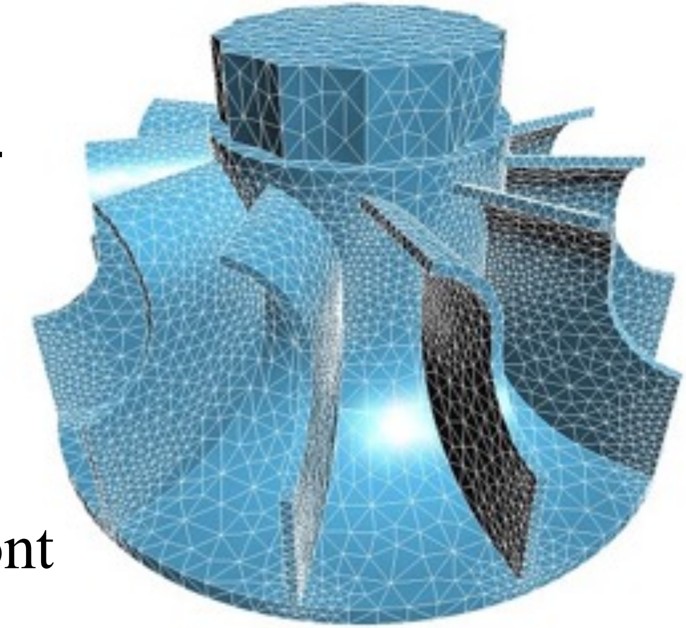
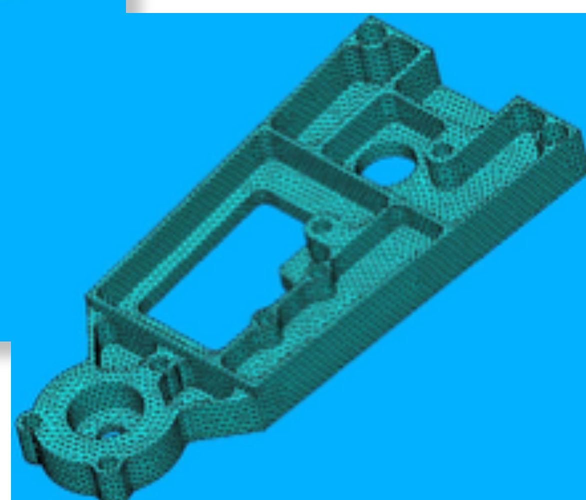


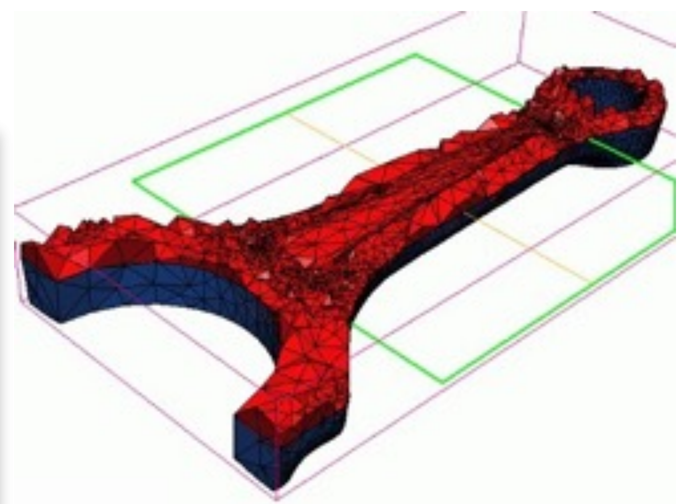
Octree
Advancing Front
Delaunay



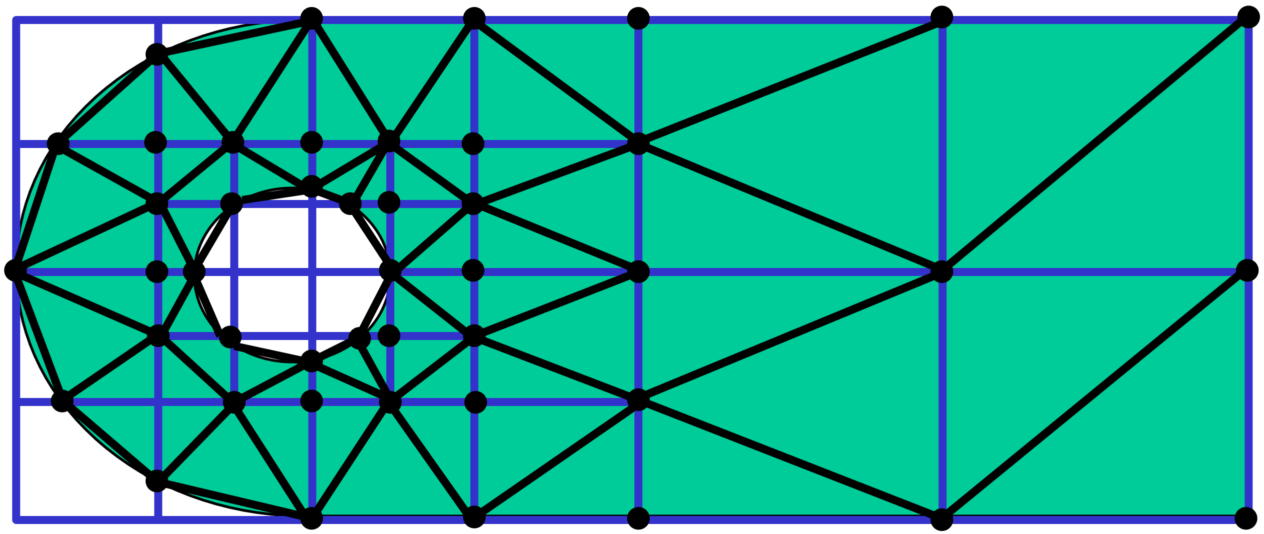
http://www.simulog.fr/mesh/gener2.htm



http://www.ansys.com



Octree/Quadtree



- Define intial bounding box (*root* of quadtree)
- Recursively break into 4 *leaves* per *root* to resolve geometry
- Find intersections of leaves with geometry boundary
- Mesh each *leaf* using corners, side nodes and intersections with geometry
- Delete Outside
- (Yerry and Shephard, 84), (Shepherd and Georges, 91)

Octree/Quadtree

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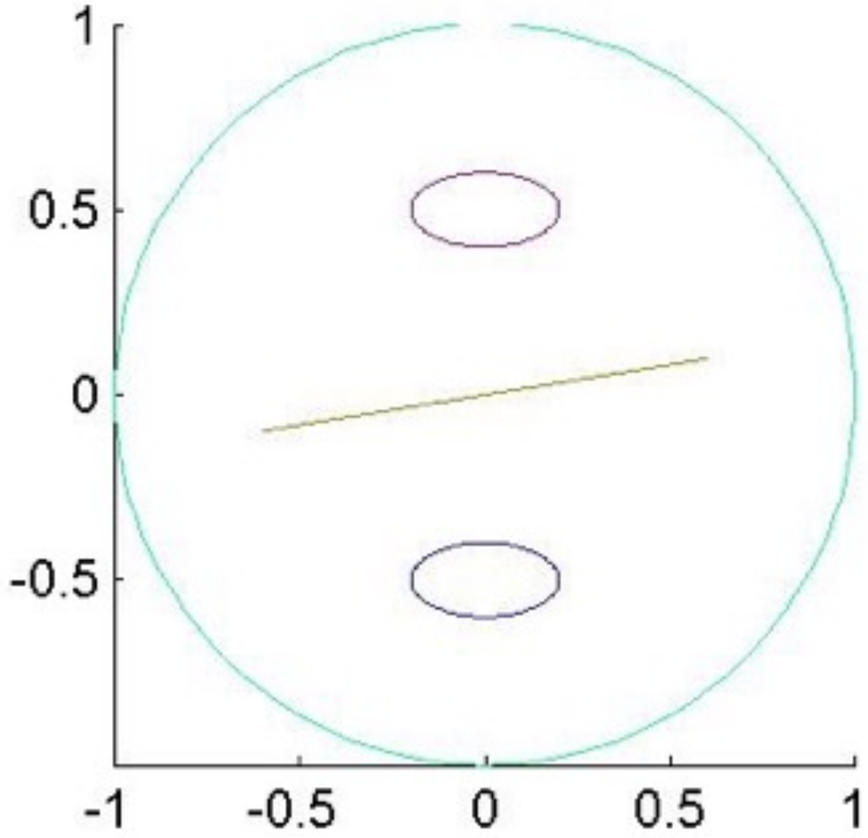
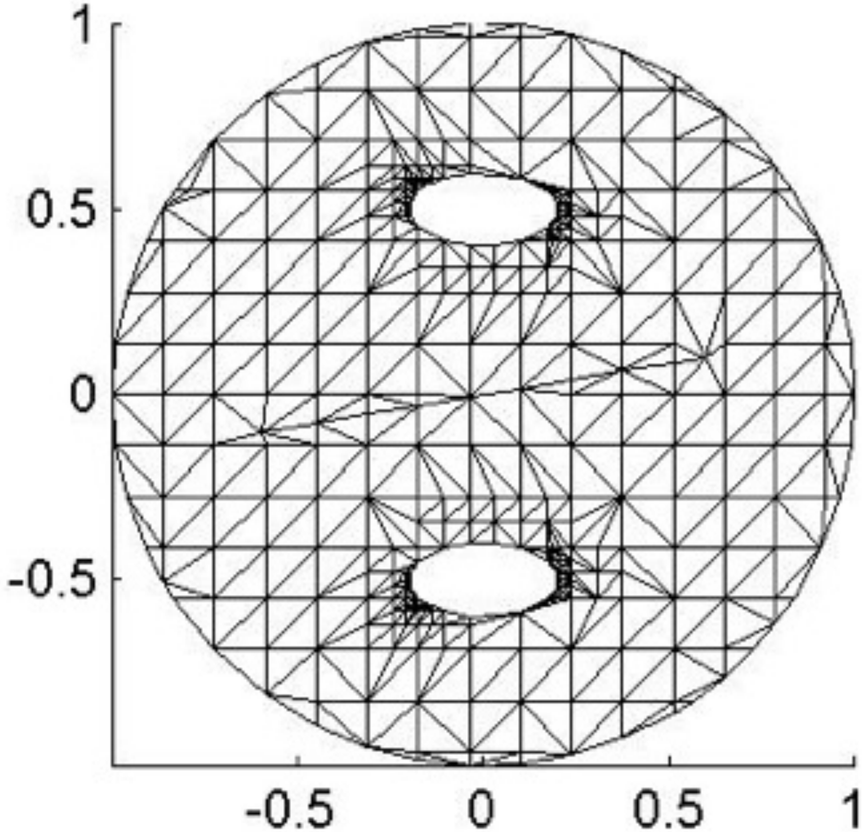


QMG,
Cornell University

Octree/Quadtree



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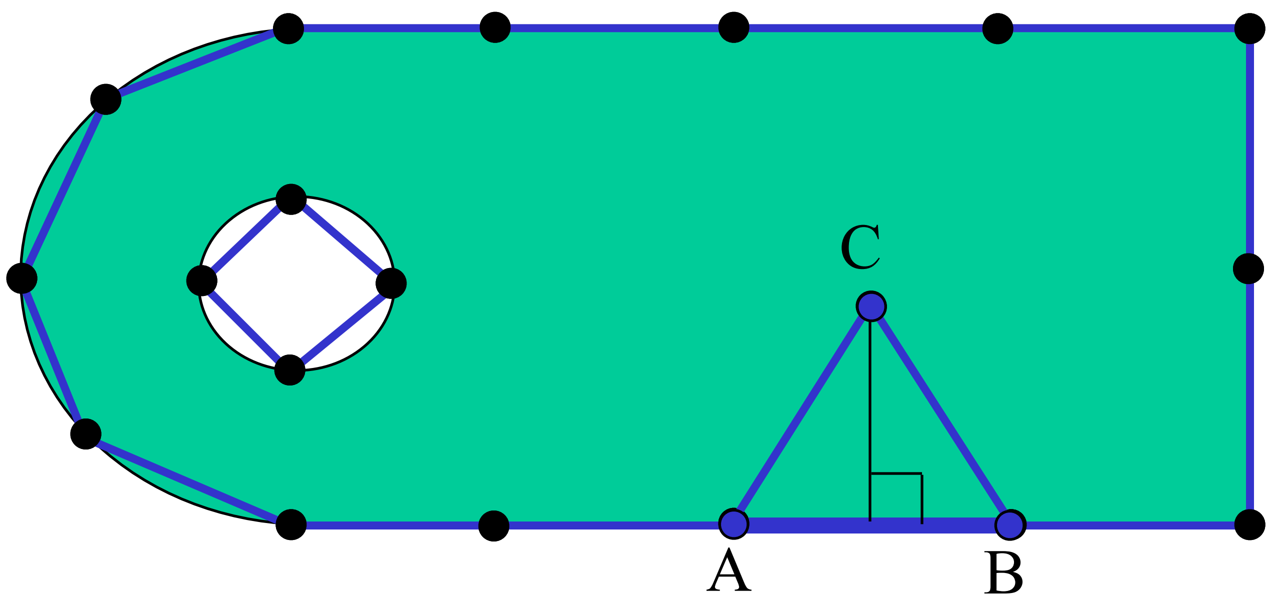
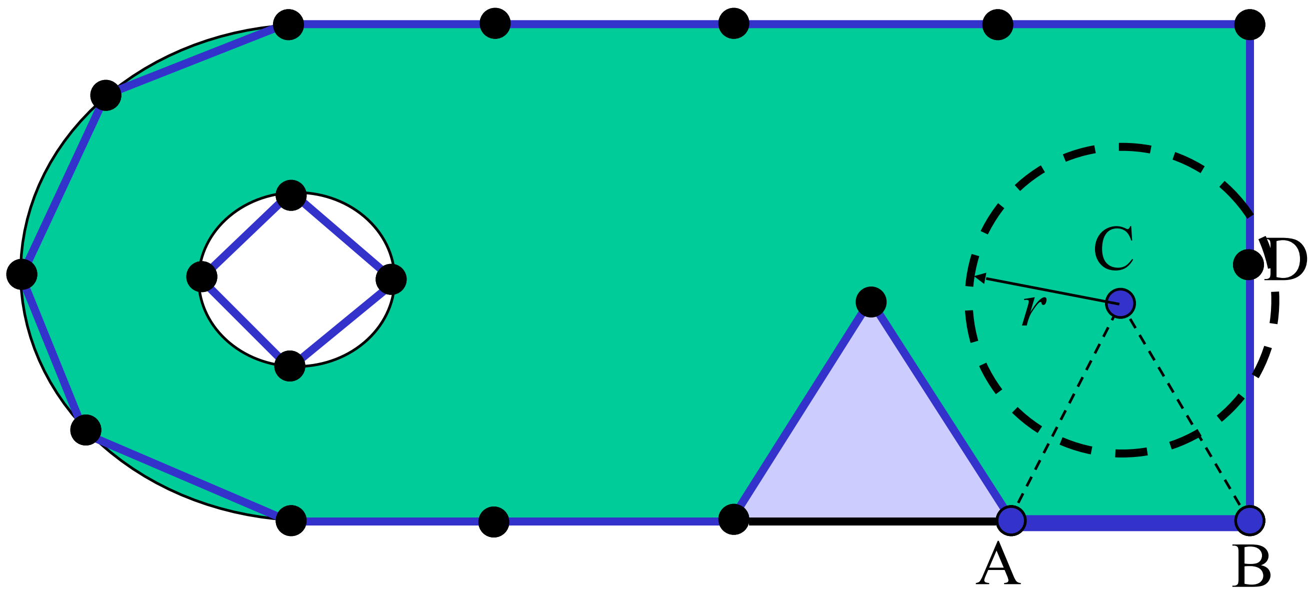
Advancing Front



- Begin with boundary mesh - define as initial *front*
- For each edge (face) on front, locate ideal node C based on front AB

Advancing Front



•Determine if any other nodes on current front are within search radius *r* of ideal location C (Choose D instead of C)

Advancing Front

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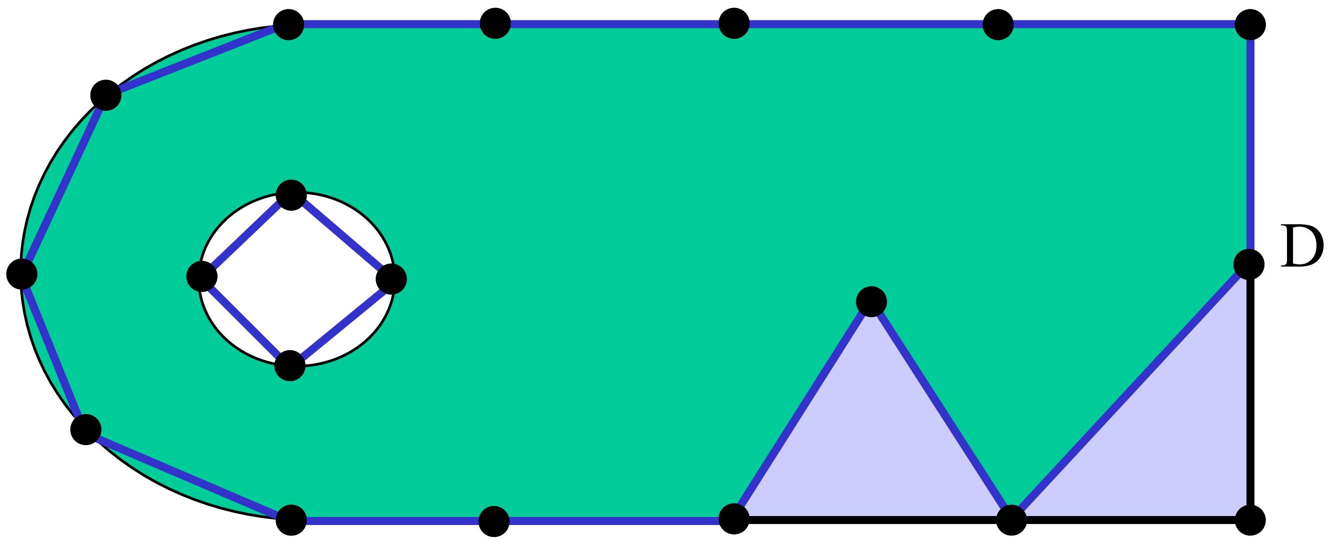
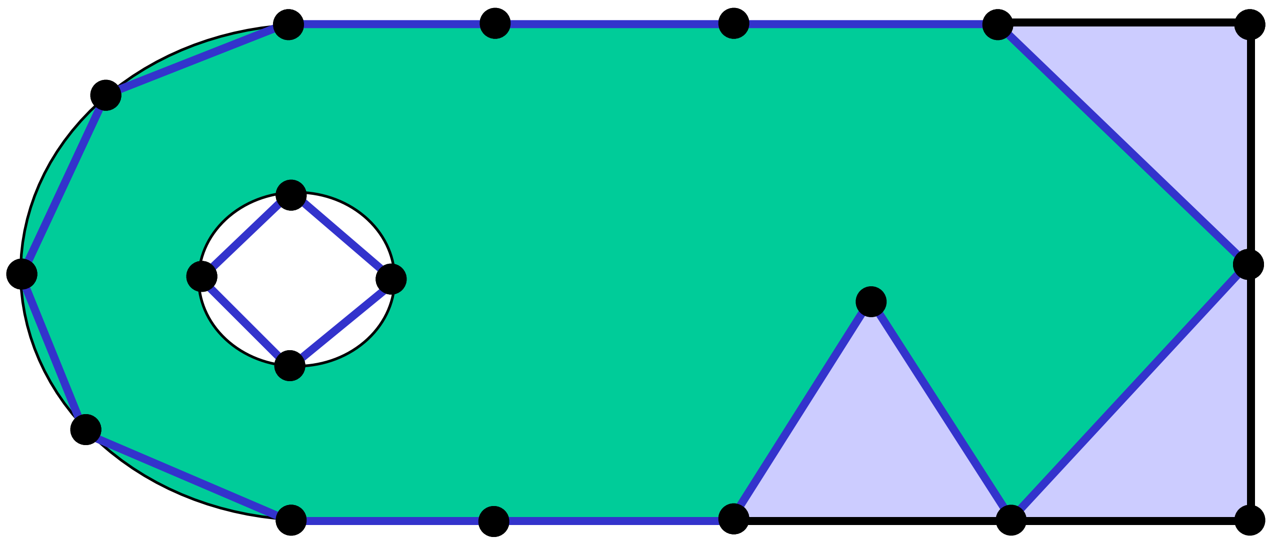


- Book-Keeping: New *front edges* added and deleted from *front* as triangles are formed
- Continue until no *front edges* remain on *front*

Advancing Front



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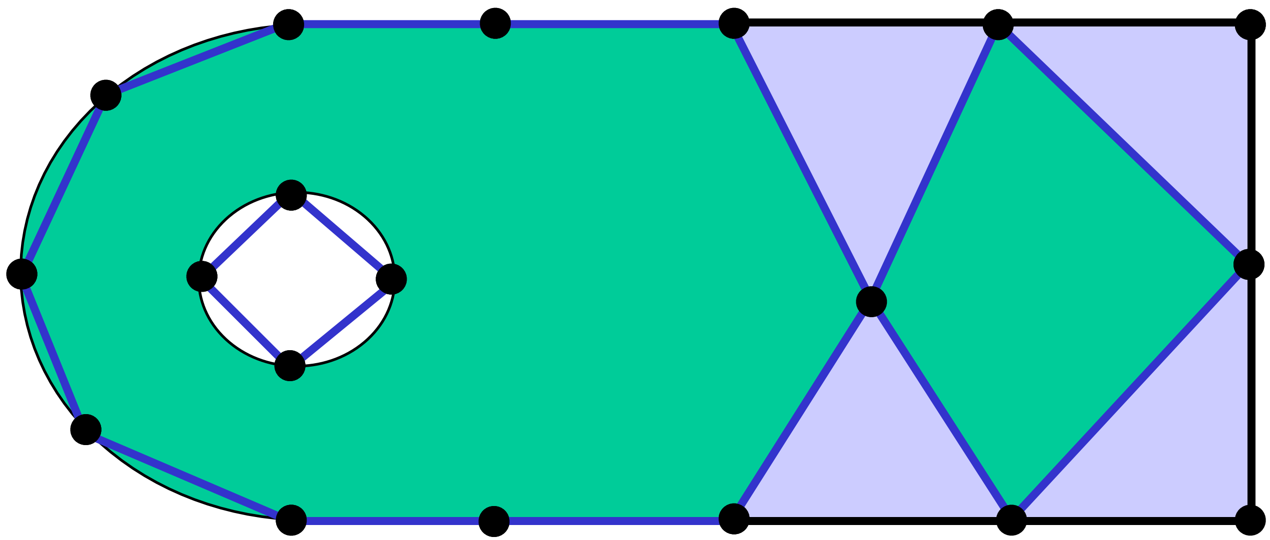

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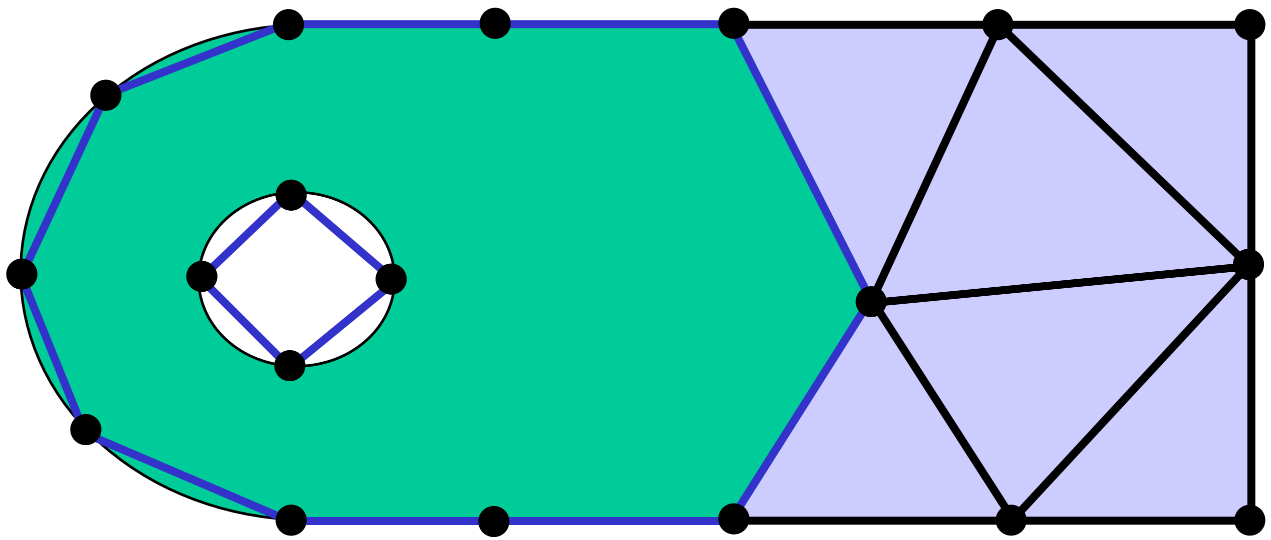

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Advancing Front

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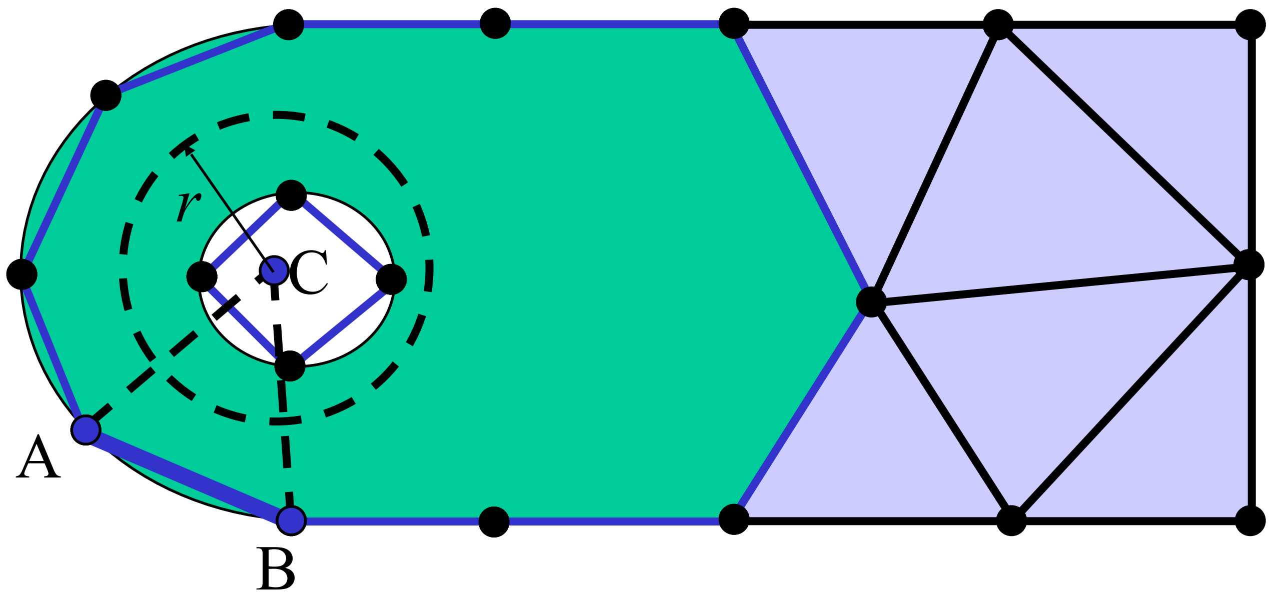
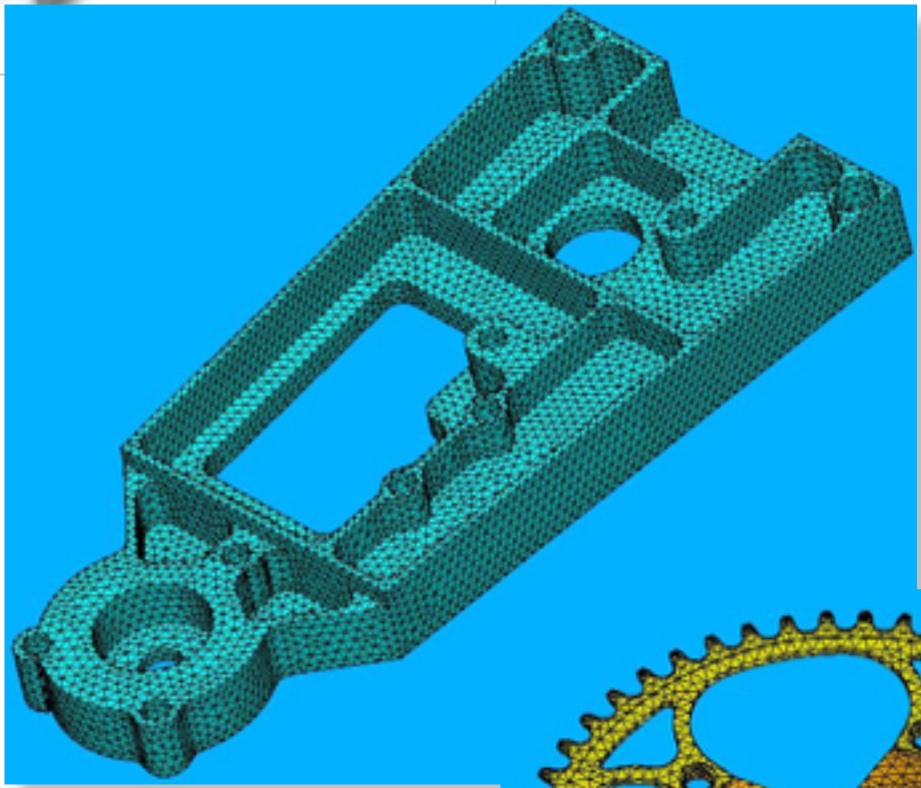
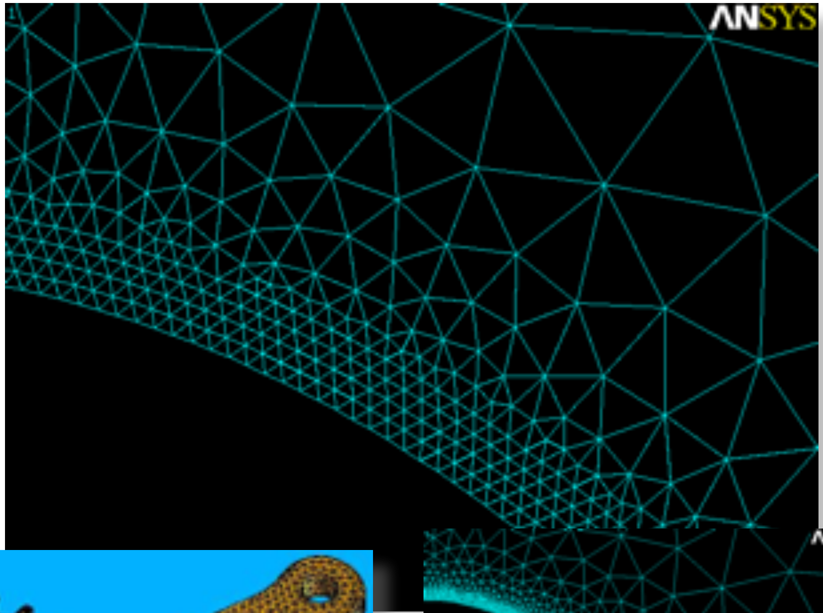
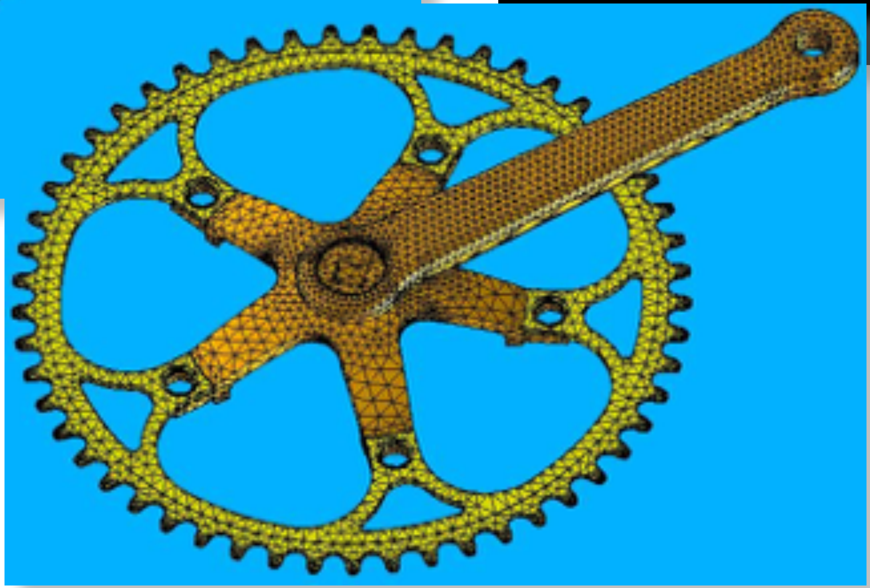


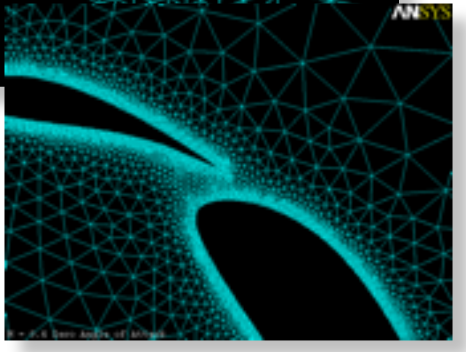
•Where multiple choices are available, use best quality (closest shape to equilateral)
•Reject any that would intersect existing front
•Reject any inverted triangles (|AB X AC| > 0)
•(Lohner,88;96)(Lo,91)

Advancing Front









Ansys, Inc.
www.ansys.com

Delaunay

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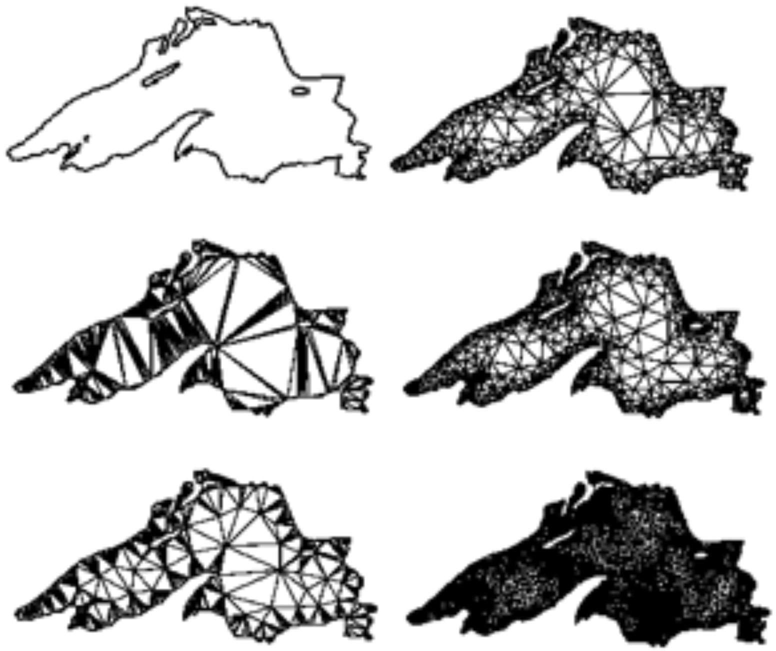
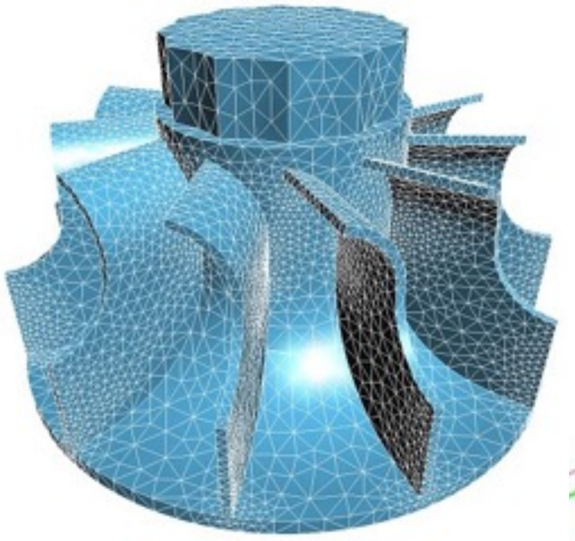
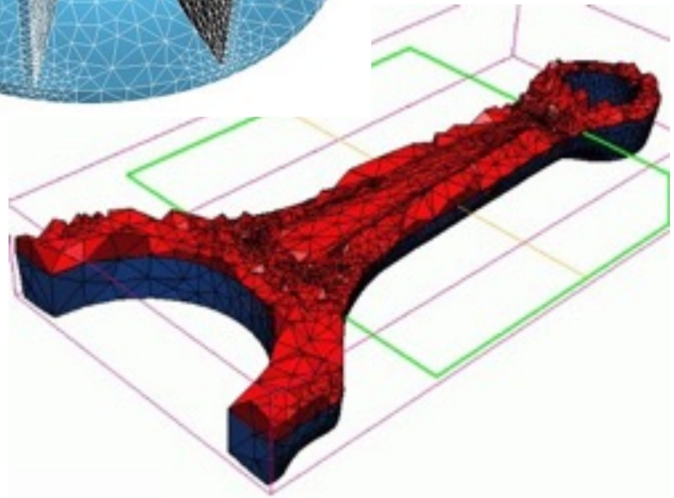


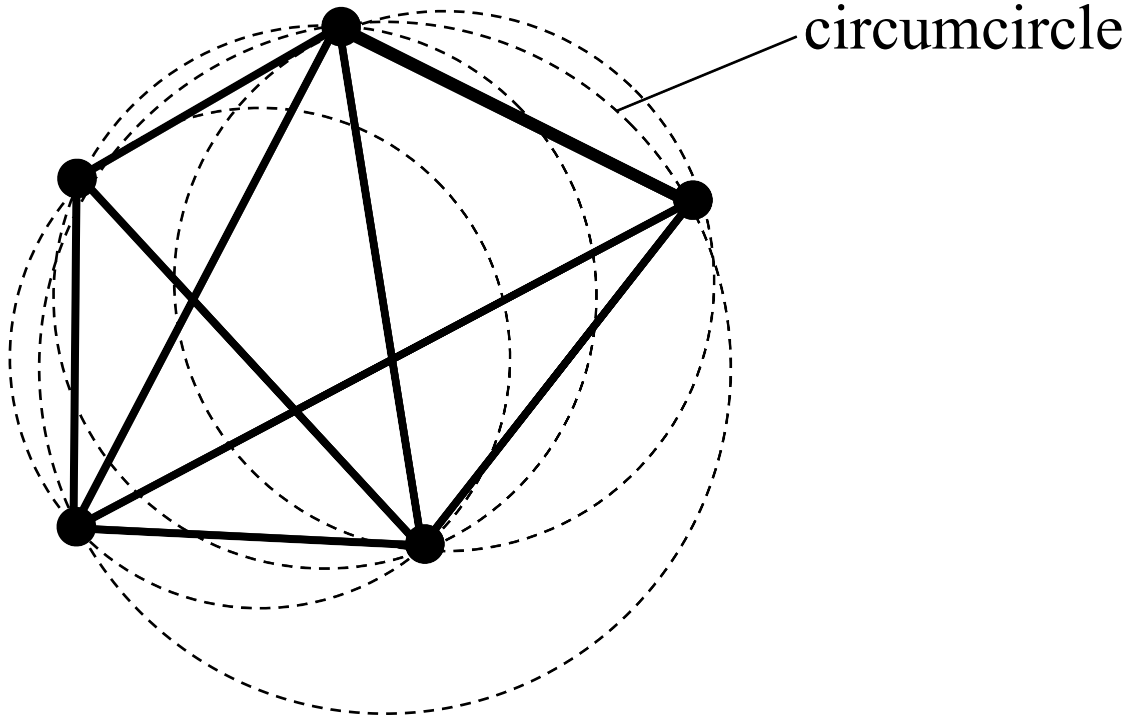
Triangle
Jonathon Shewchuk
http://www-2.cs.cmu.edu/~quake/triangle.html





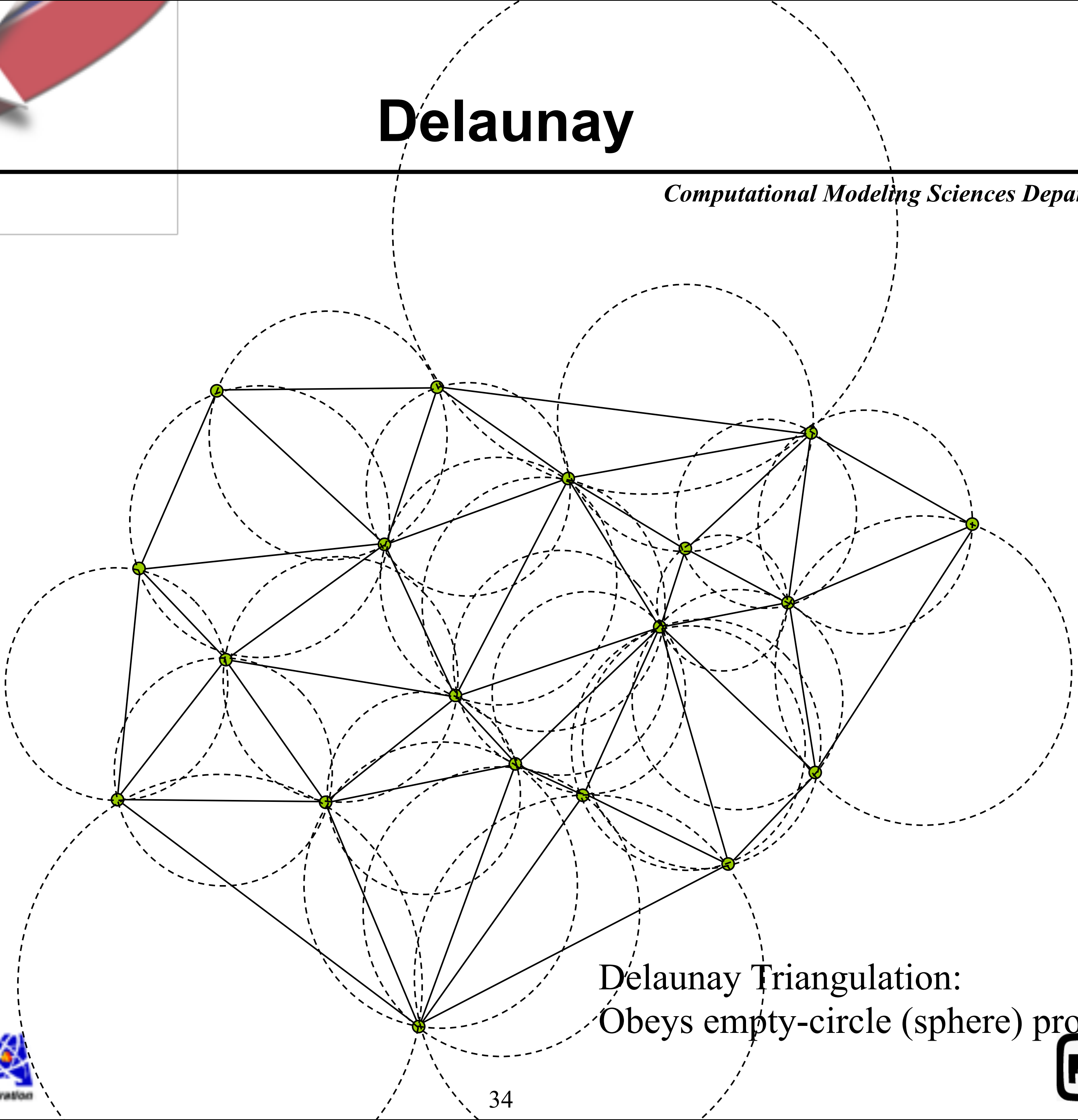
Tetmesh-GHS3D
INRIA, France
http://www.simulog.fr/tetmesh/

Delaunay



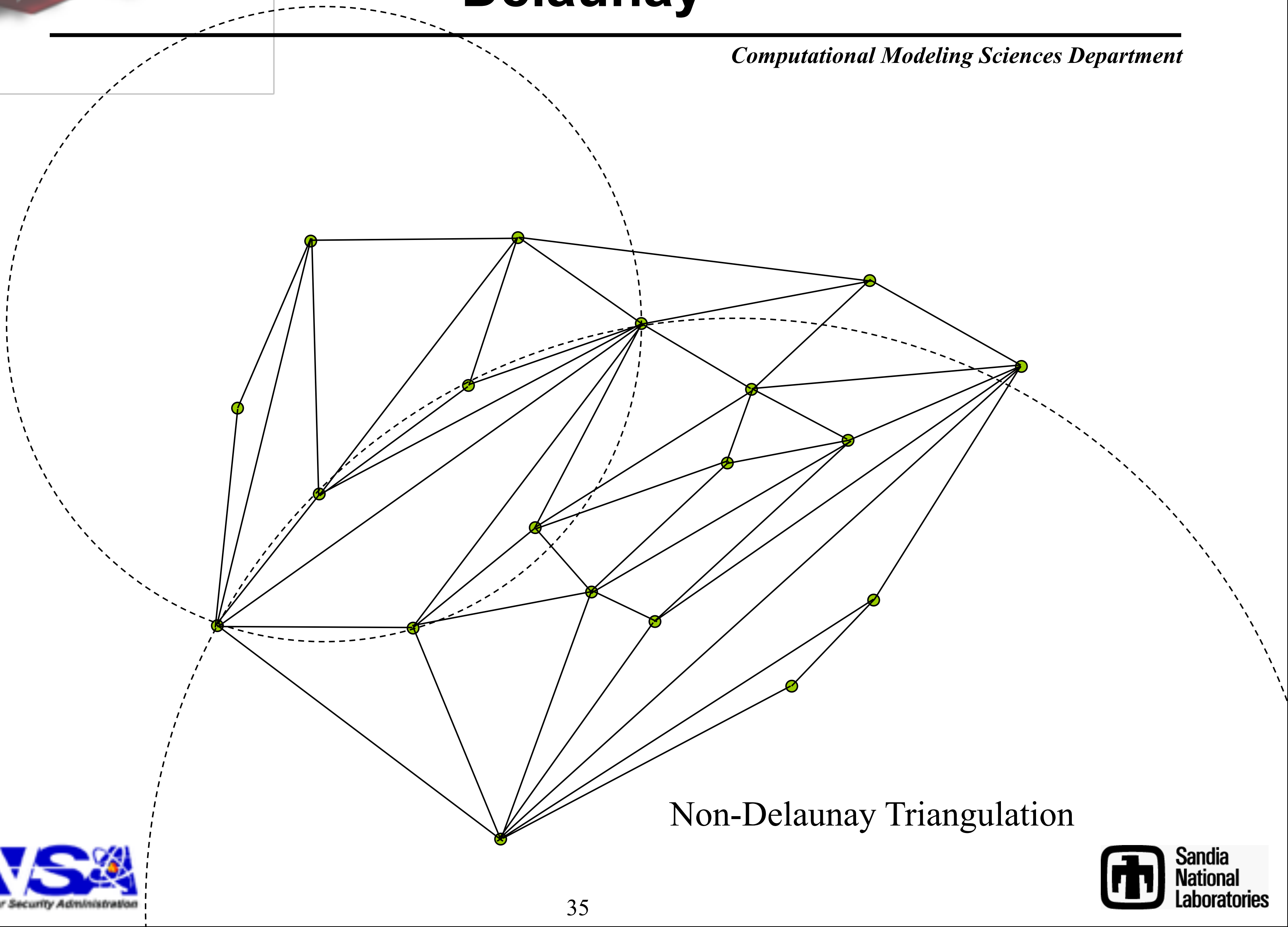
Empty Circle (Sphere) Property:
No other vertex is contained within the circumcircle (circumsphere) of any triangle (tetrahedron)

Delaunay



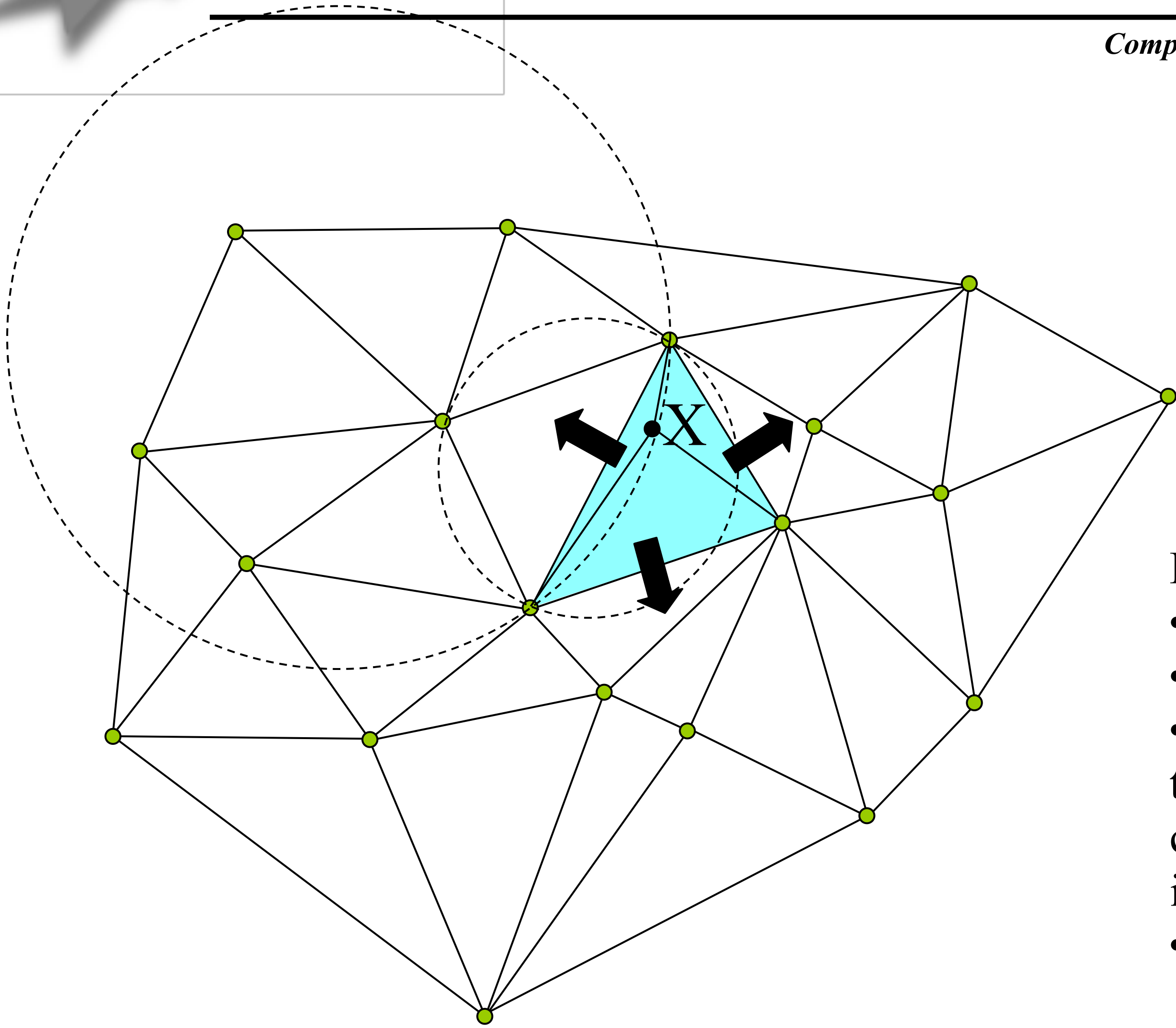
Delaunay Triangulation:
Obeys empty-circle (sphere) property

Delaunay



Non-Delaunay Triangulation

Delaunay



Given a Delaunay Triangulation of n nodes, How do I insert node n+1 ?

Lawson Algorithm

- Locate triangle containing X
- Subdivide triangle
- Recursively check adjoining triangles to ensure empty-circle property. Swap diagonal if needed
- (Lawson,77)

Delaunay

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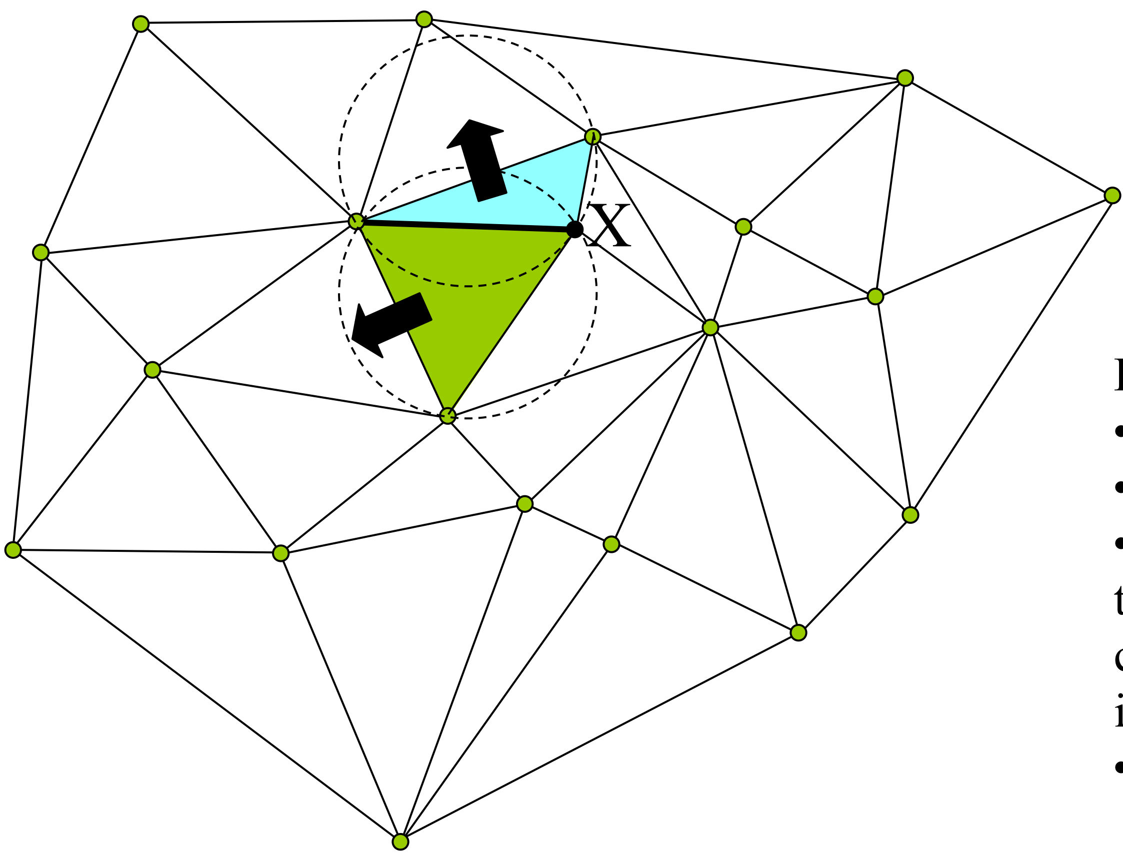

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Delaunay

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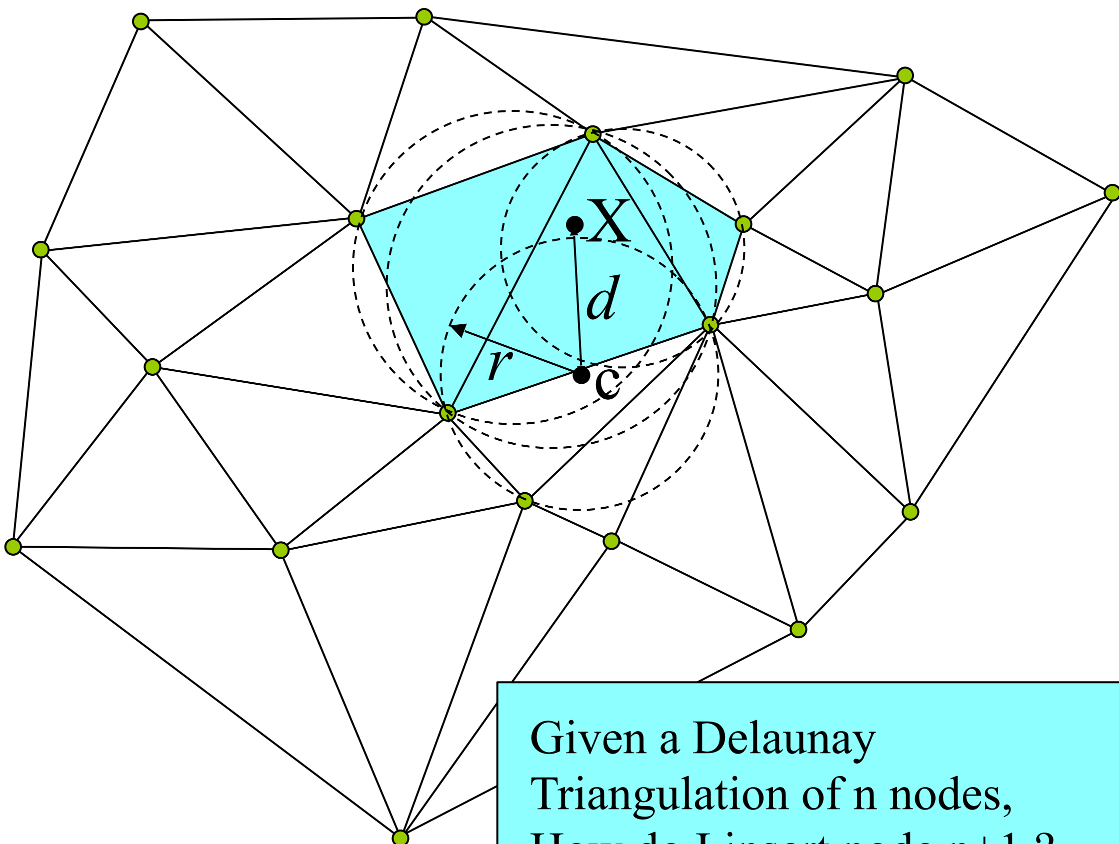
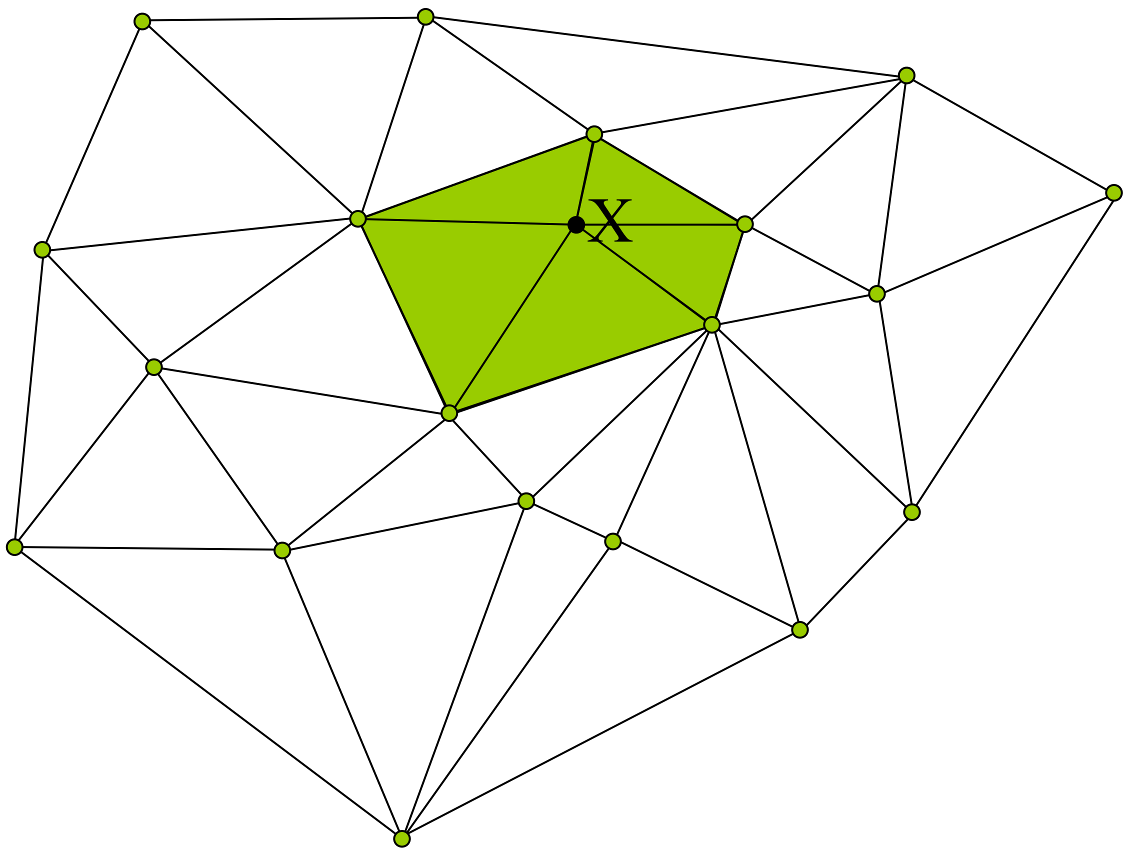


Bowyer-Watson Algorithm

- Locate triangle that contains the point
- Search for all triangles whose circumcircle contain the point ($d<r$)
- Delete the triangles (creating a void in the mesh)
- Form new triangles from the new point and the void boundary
- (Watson,81)

Given a Delaunay Triangulation of n nodes, How do I insert node n+1 ?

Delaunay



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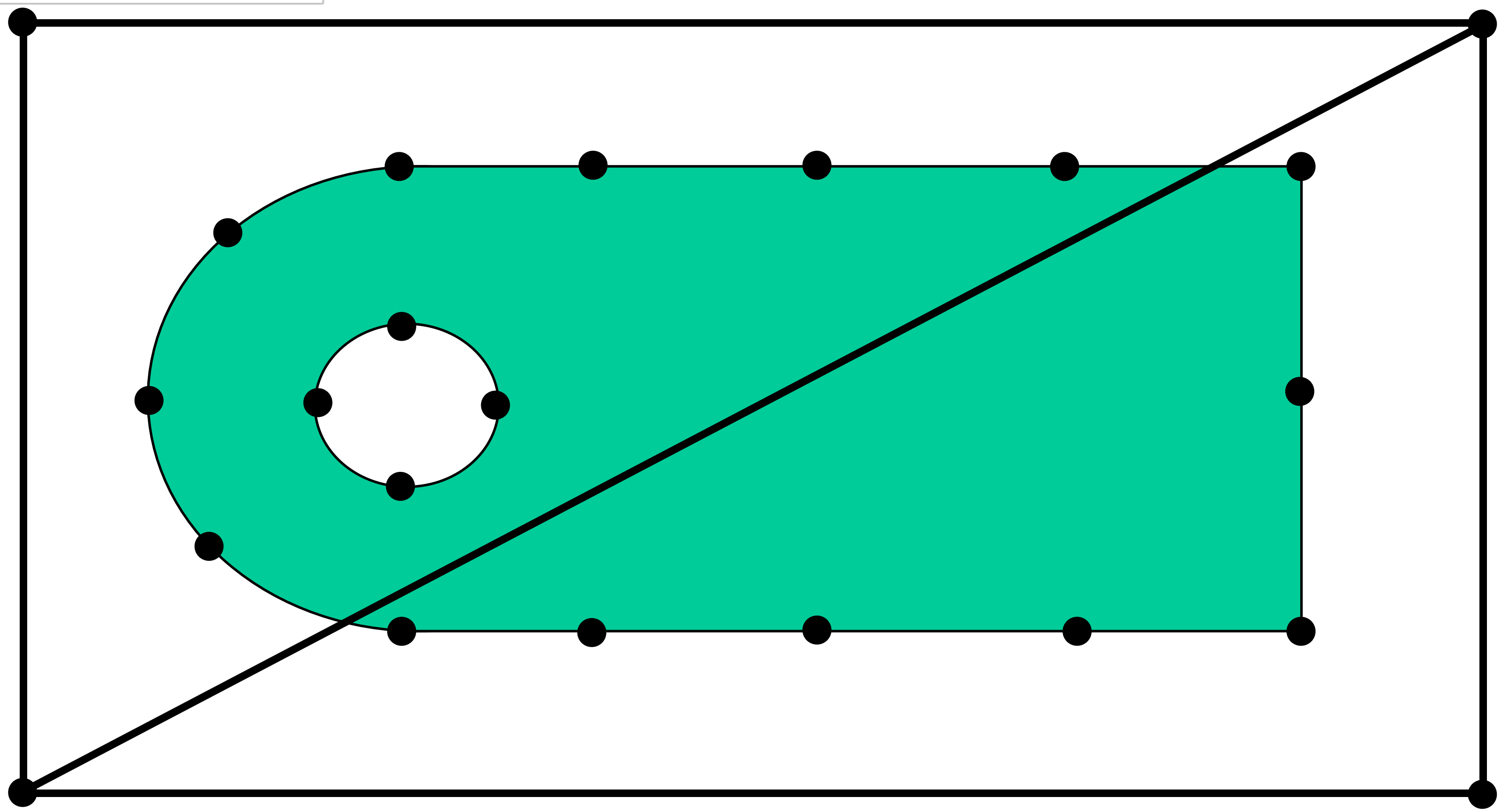
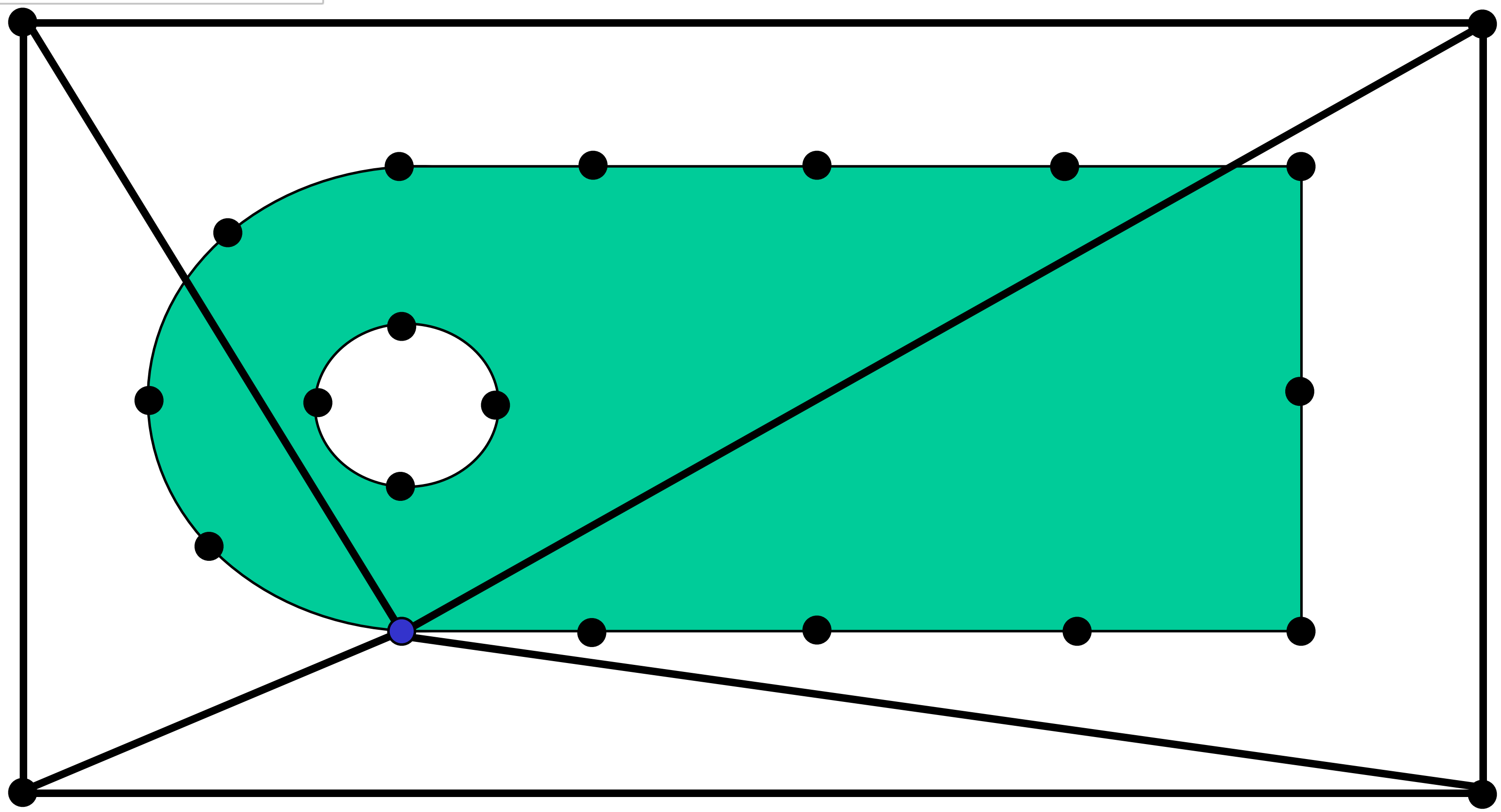
Given a Delaunay Triangulation of n nodes
How do I insert node n+

Delaunay



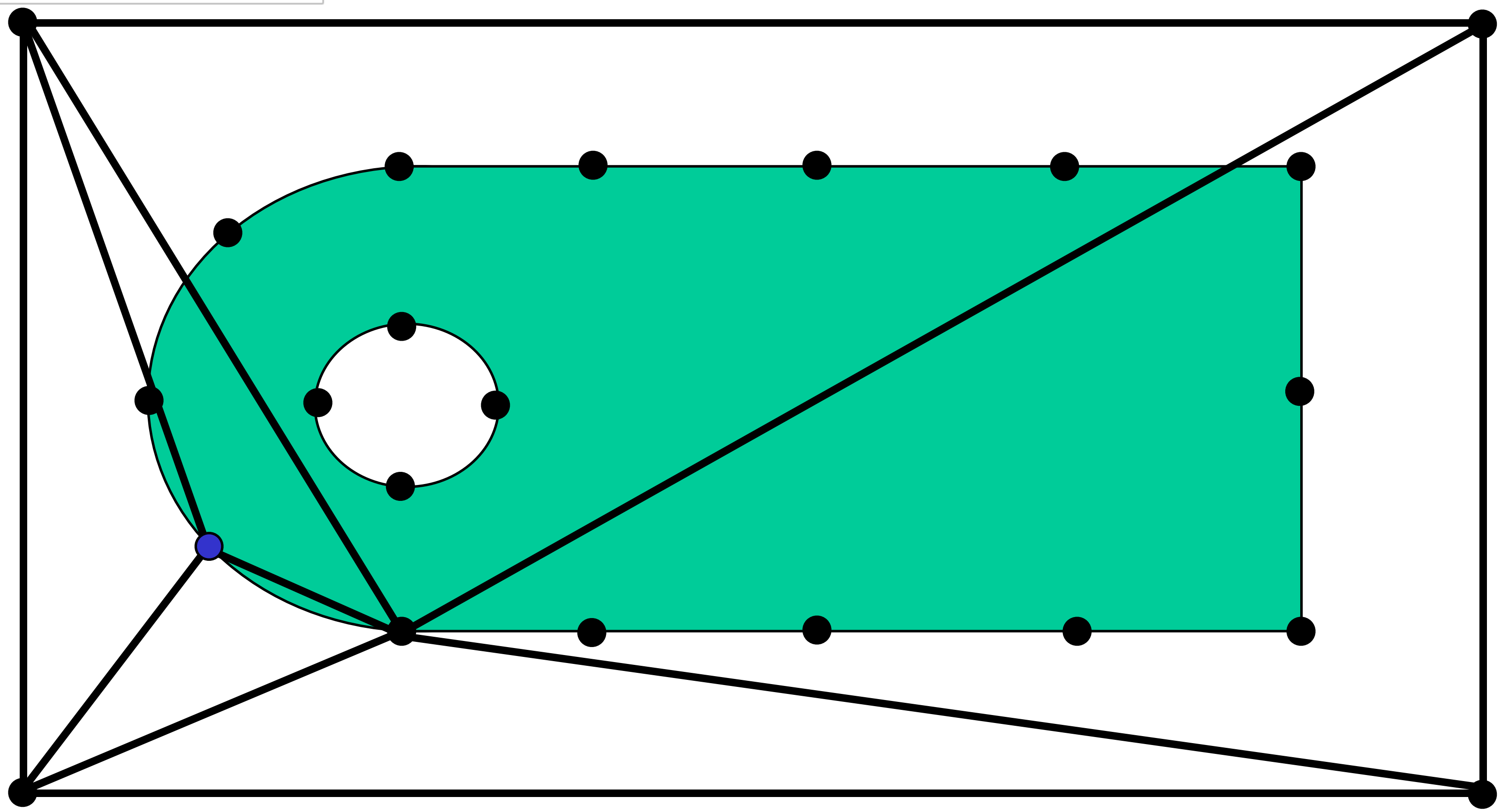
•Begin with Bounding Triangles (or Tetrahedra)

Delaunay



•Insert boundary nodes using Delaunay method (Lawson or Bowyer-Watson)

Delaunay



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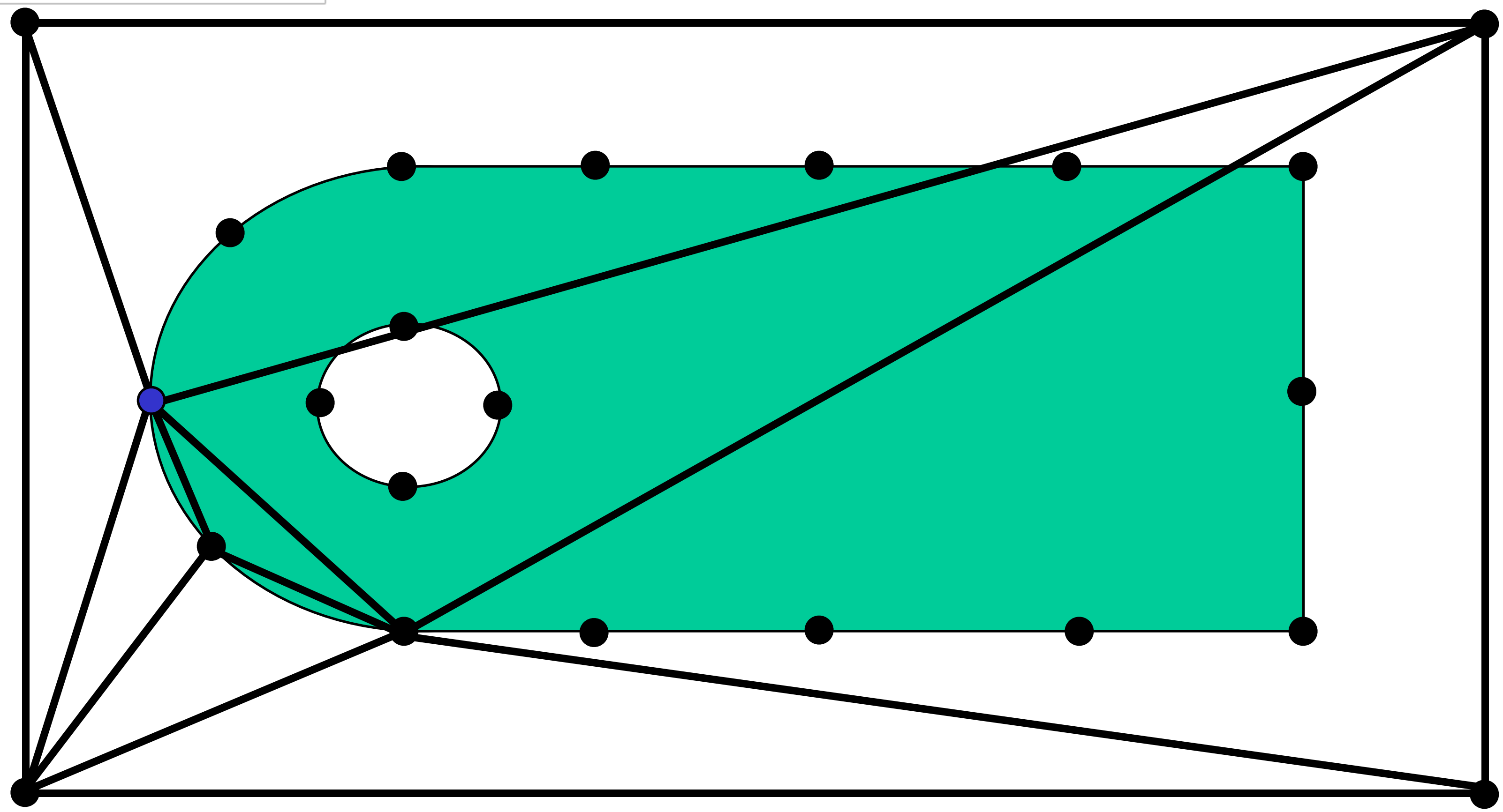

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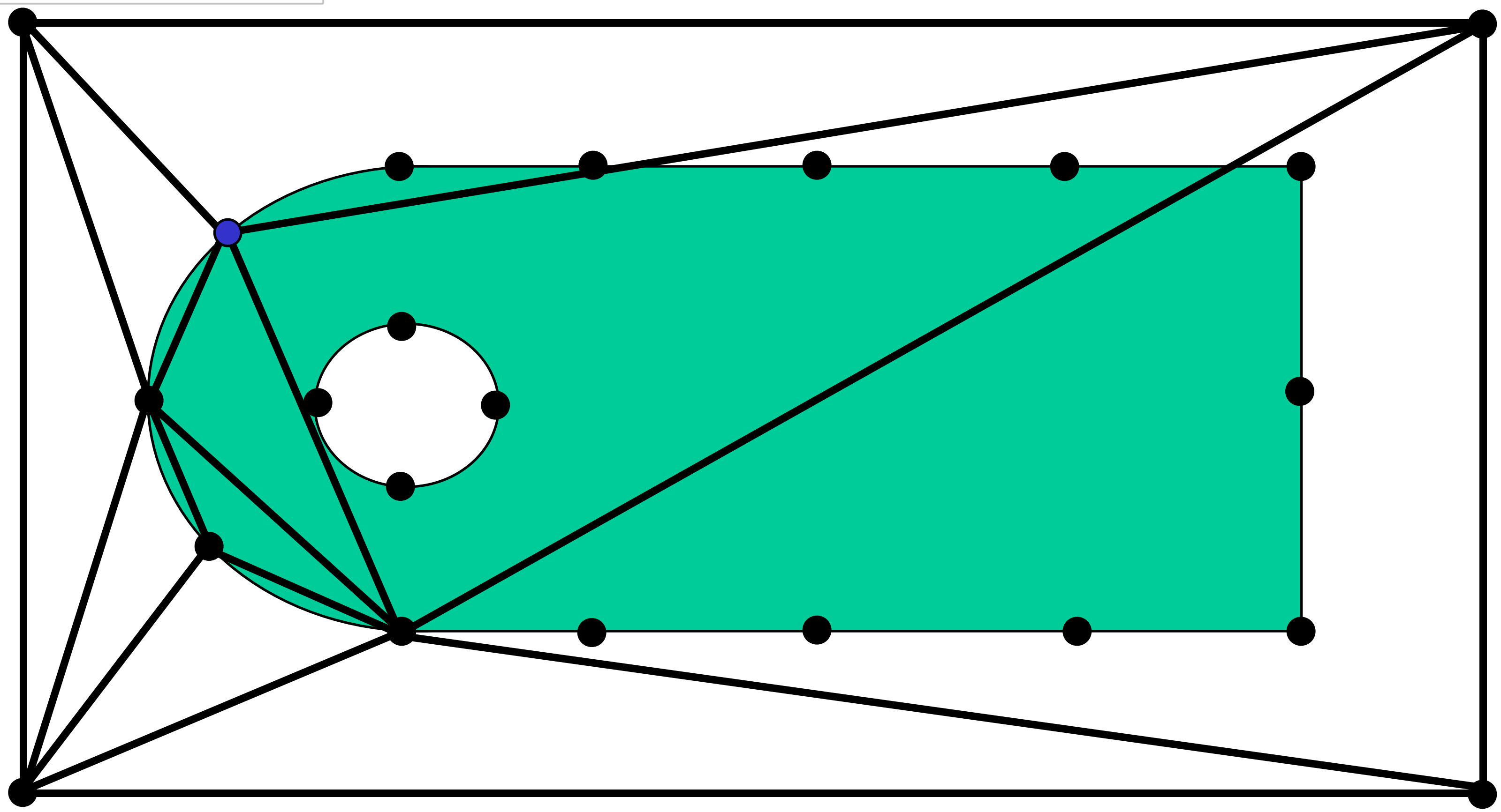

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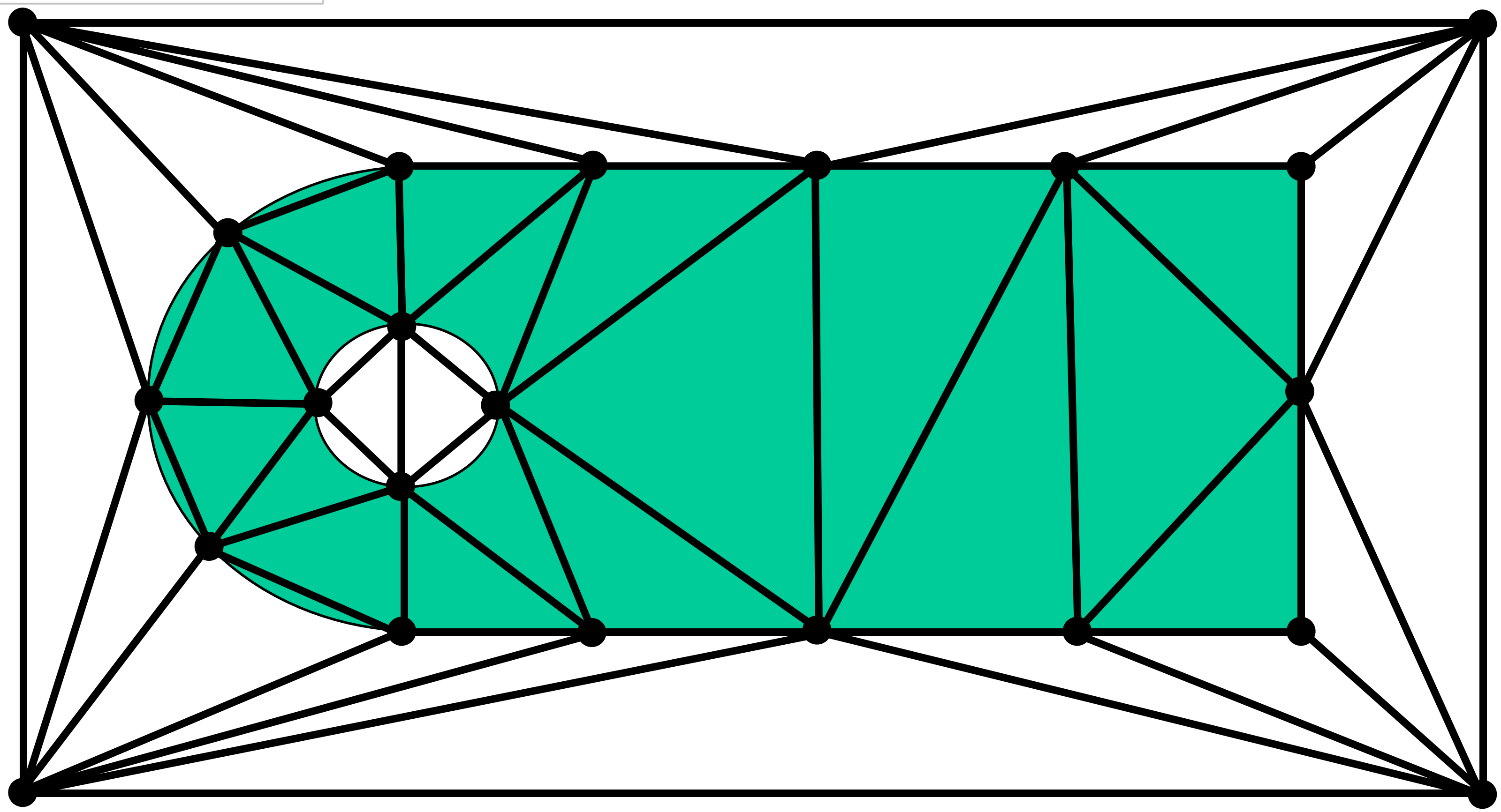

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Delaunay

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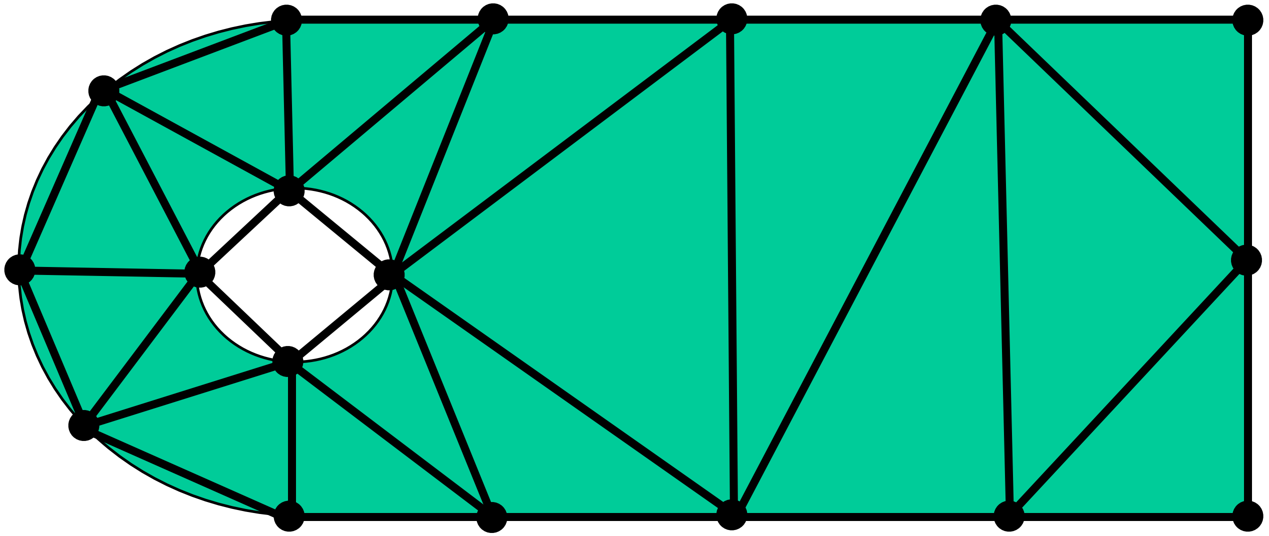
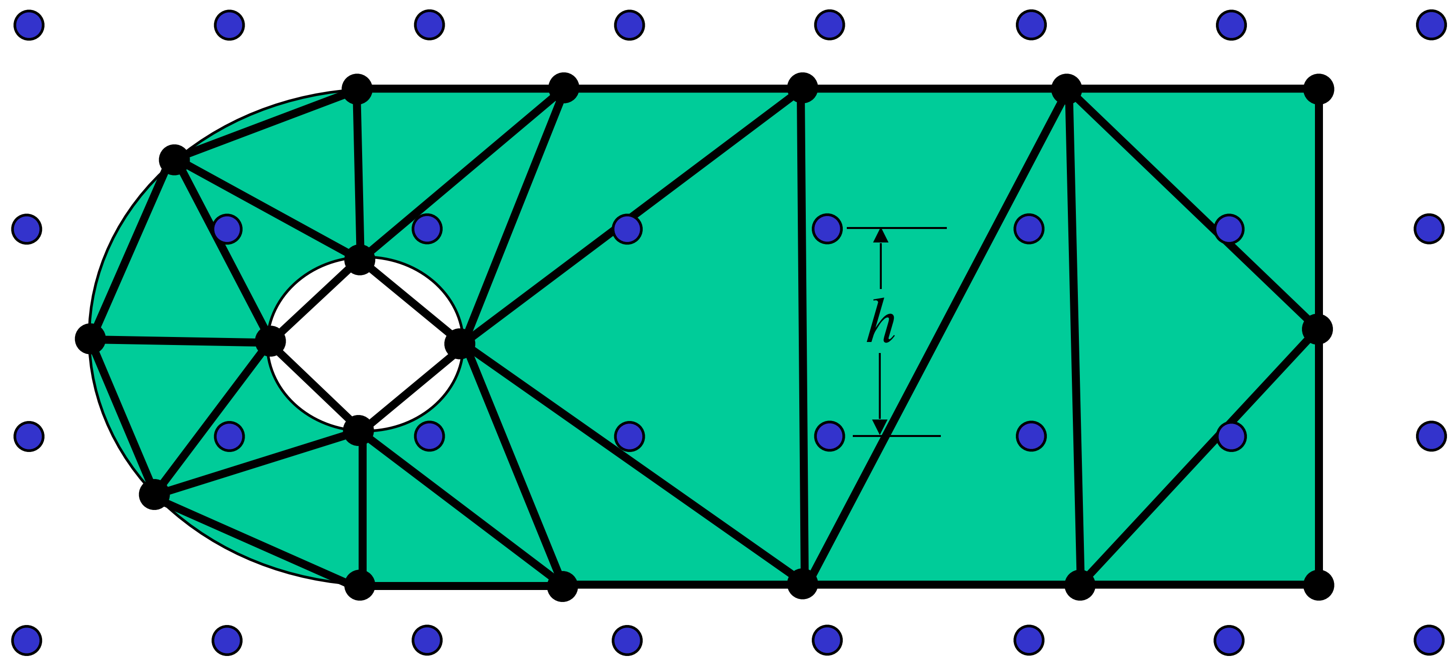


- Recover boundary
- Delete outside triangles
- Insert internal nodes

Delaunay



Grid Based

•Nodes introduced based on a regular lattice

•Lattice could be rectangular, triangular, quadtree, etc…

•Outside nodes ignored

Node Insertion

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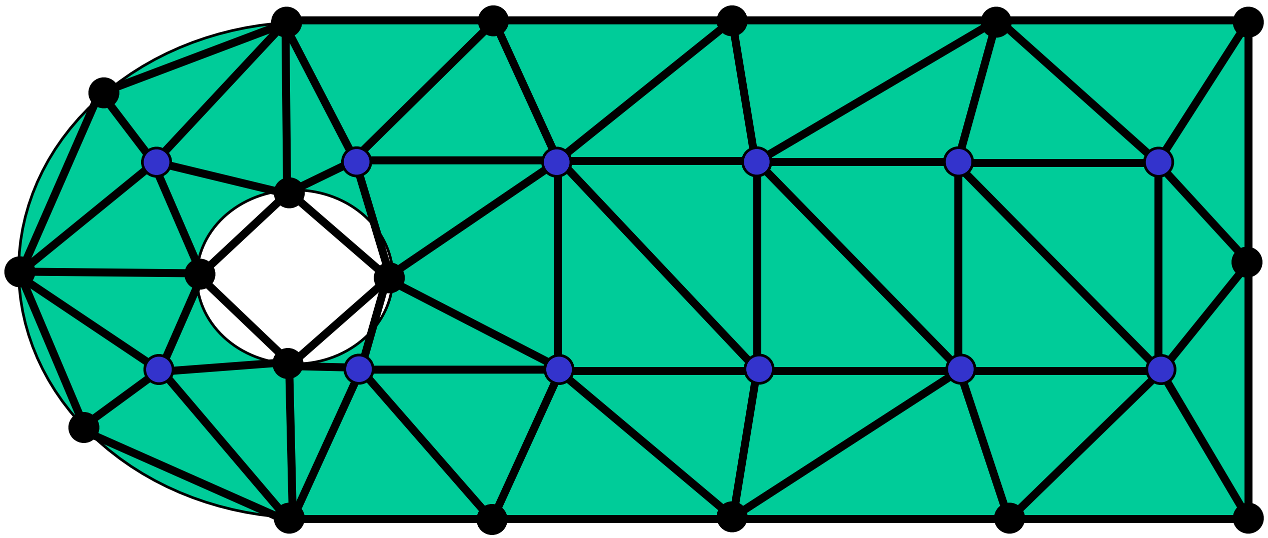

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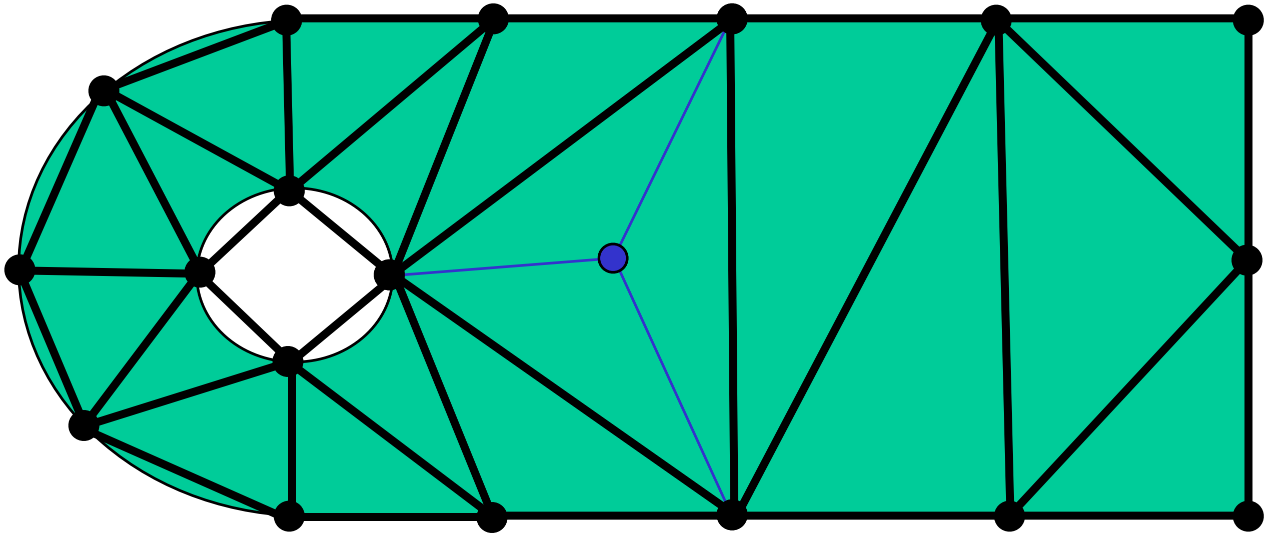
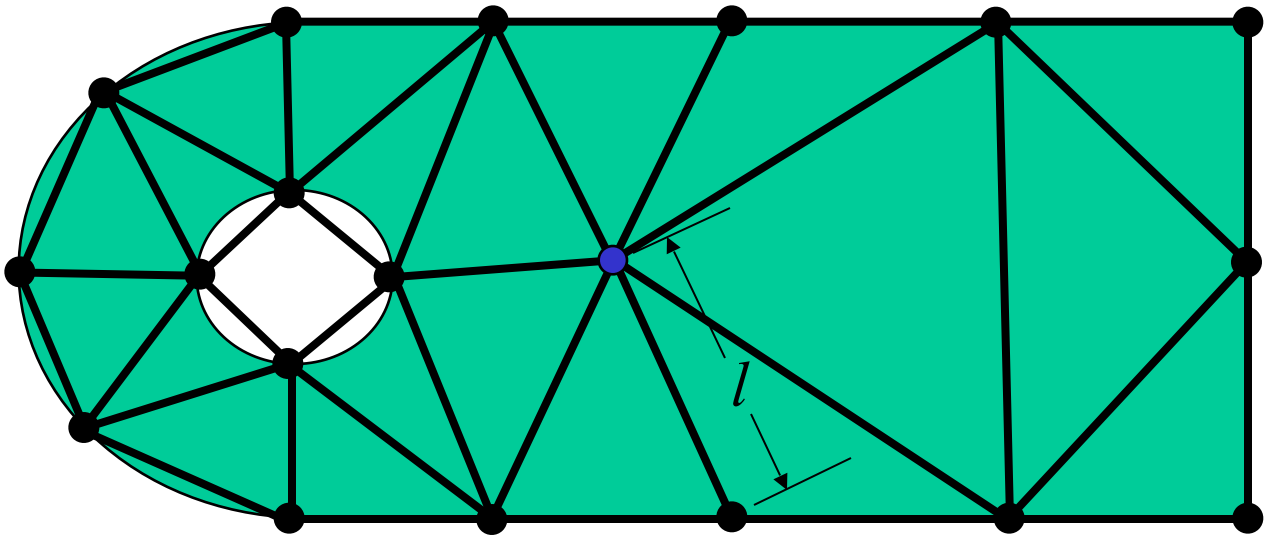


Centroid
- Nodes introduced at triangle centroids
- Continues until edge length, $l \approx h$

Node Insertion

Delaunay



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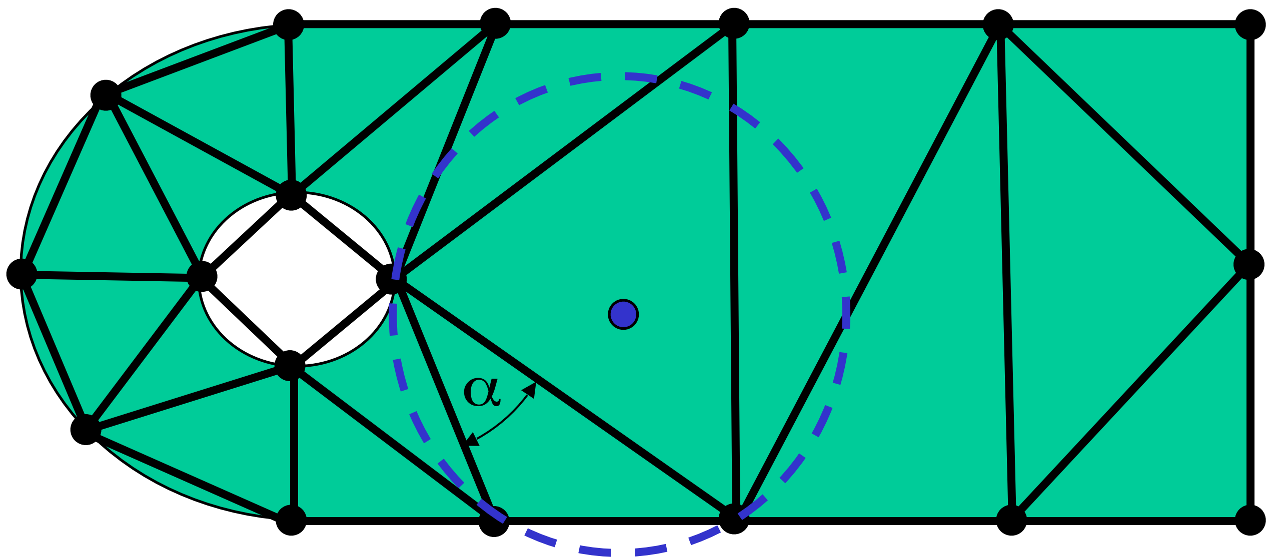


Circumcenter ("Guaranteed Quality")

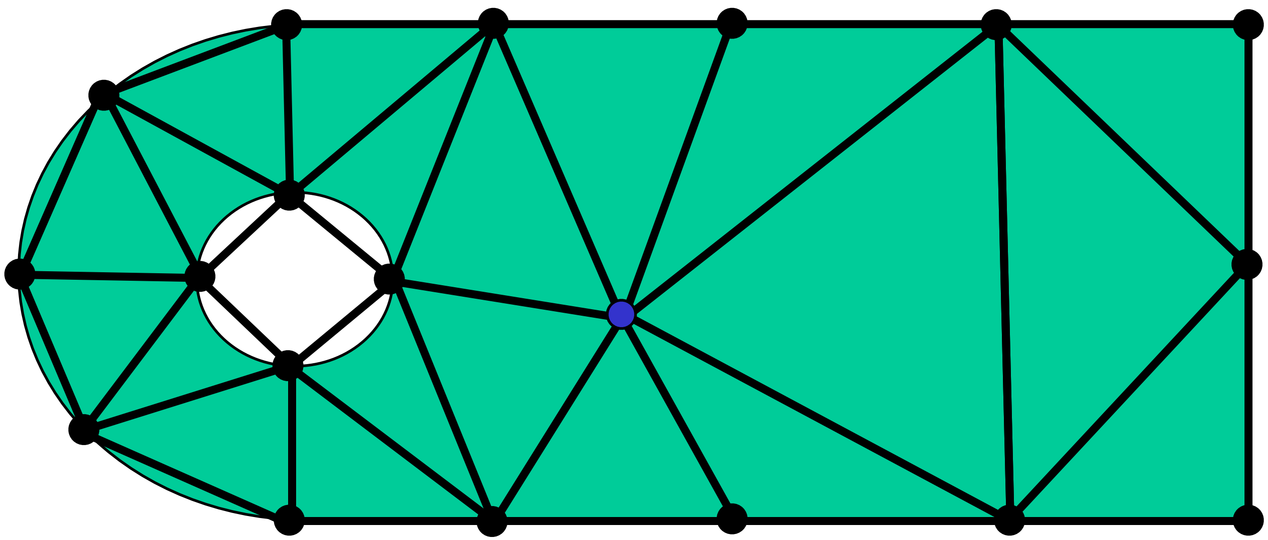
•Nodes introduced at triangle circumcenters

•Order of insertion based on minimum angle of any triangle

•Continues until minimum angle > predefined minimum $(\alpha \approx 30^{\circ})$

Node Insertion (Chew,Ruppert,Shewchuk)

Delaunay



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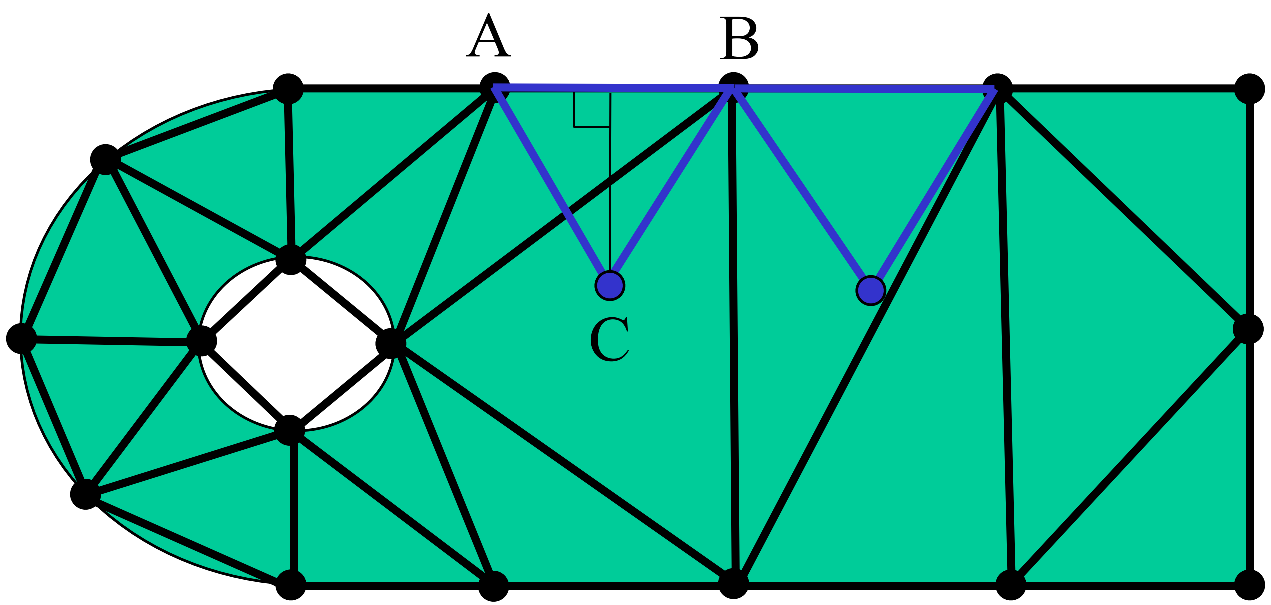


Advancing Front

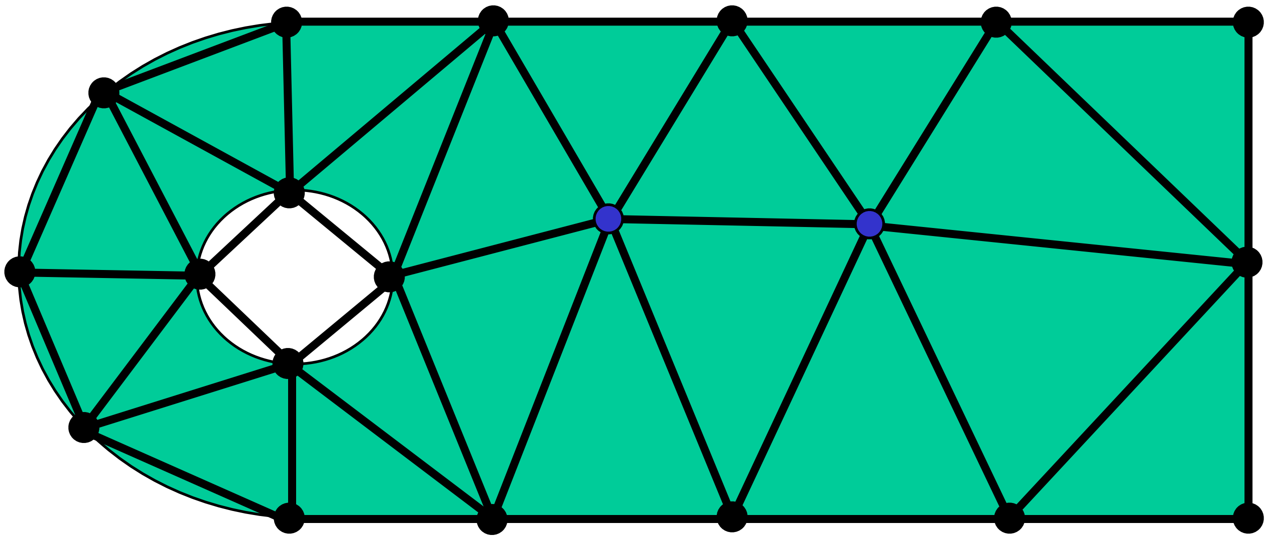
•"Front" structure maintained throughout

•Nodes introduced at ideal location from current front edge

(Marcum,95)

Node Insertion

Delaunay



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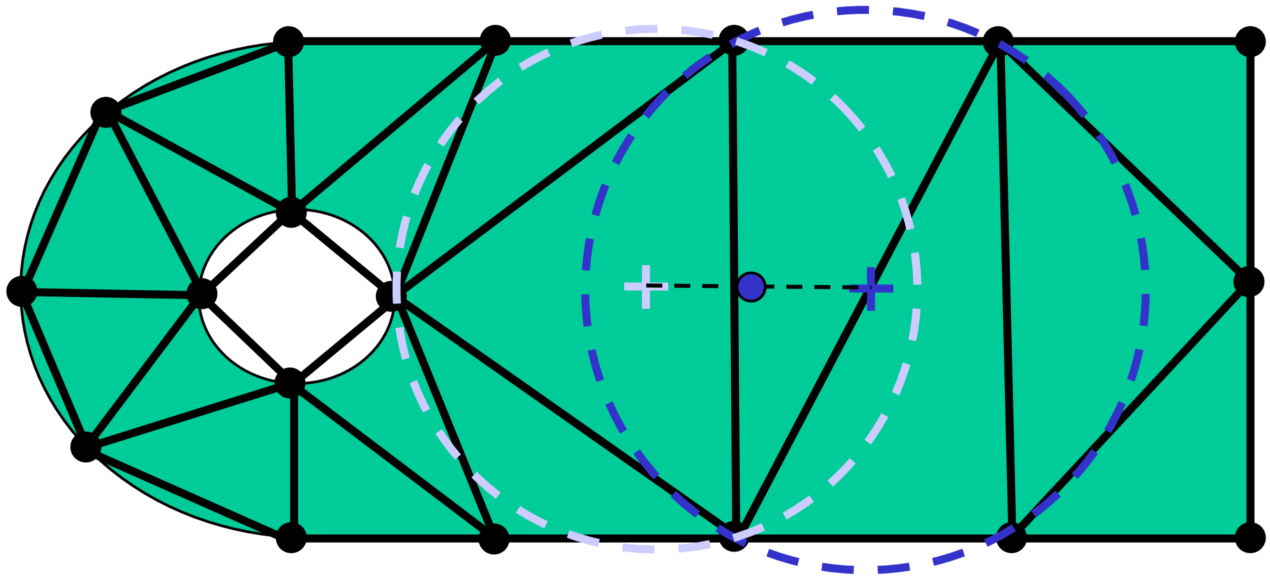
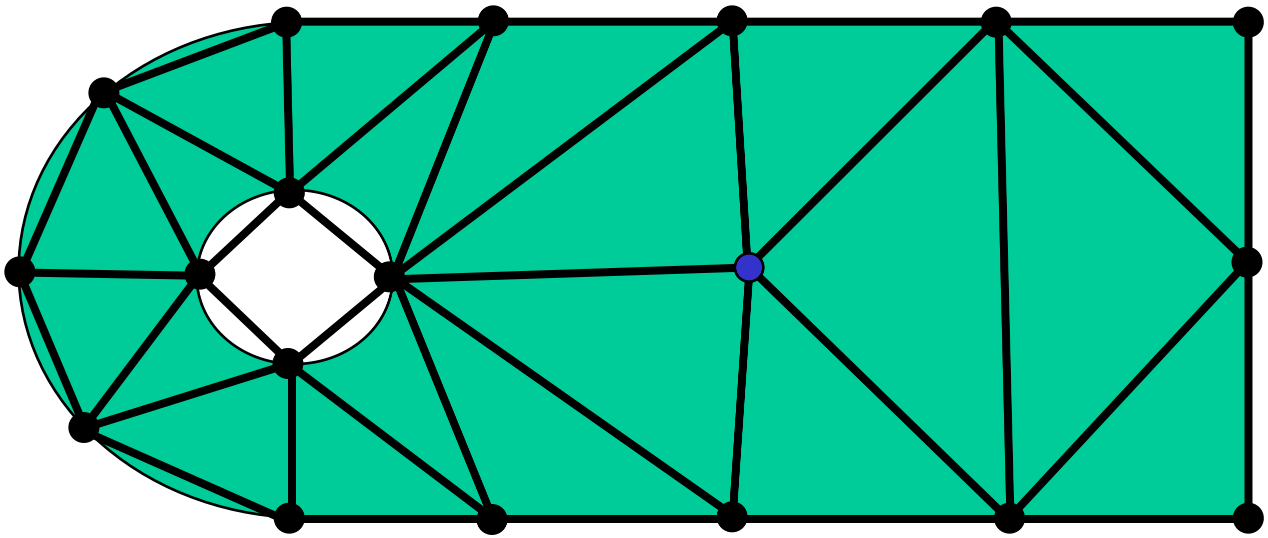


Voronoi-Segment

•Nodes introduced at midpoint of segment connecting the circumcircle centers of two adjacent triangles

(Rebay,93)

Node Insertion



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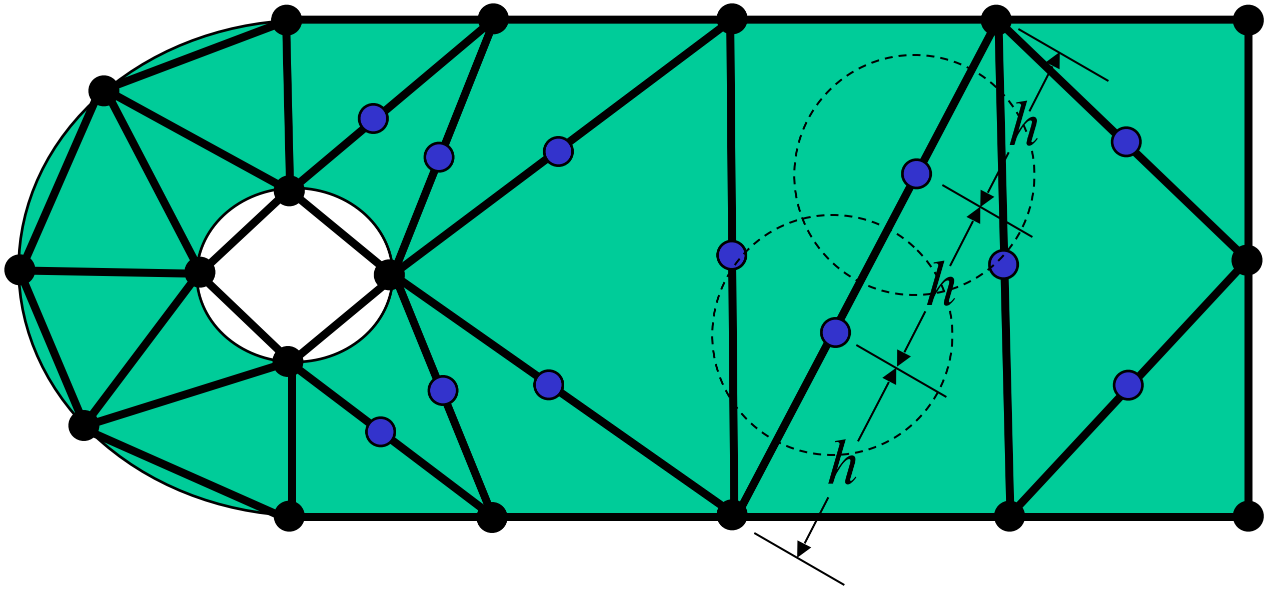


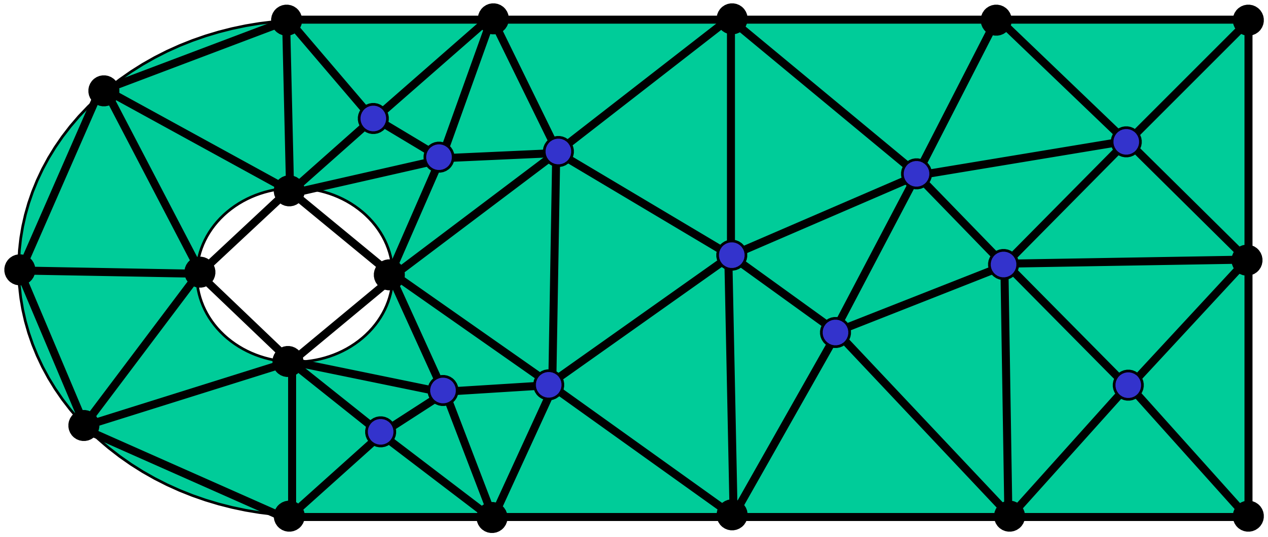
Edges

•Nodes introduced at along existing edges at *l=h*

•Check to ensure nodes on nearby edges are not too close

(George,91)

Node Insertion



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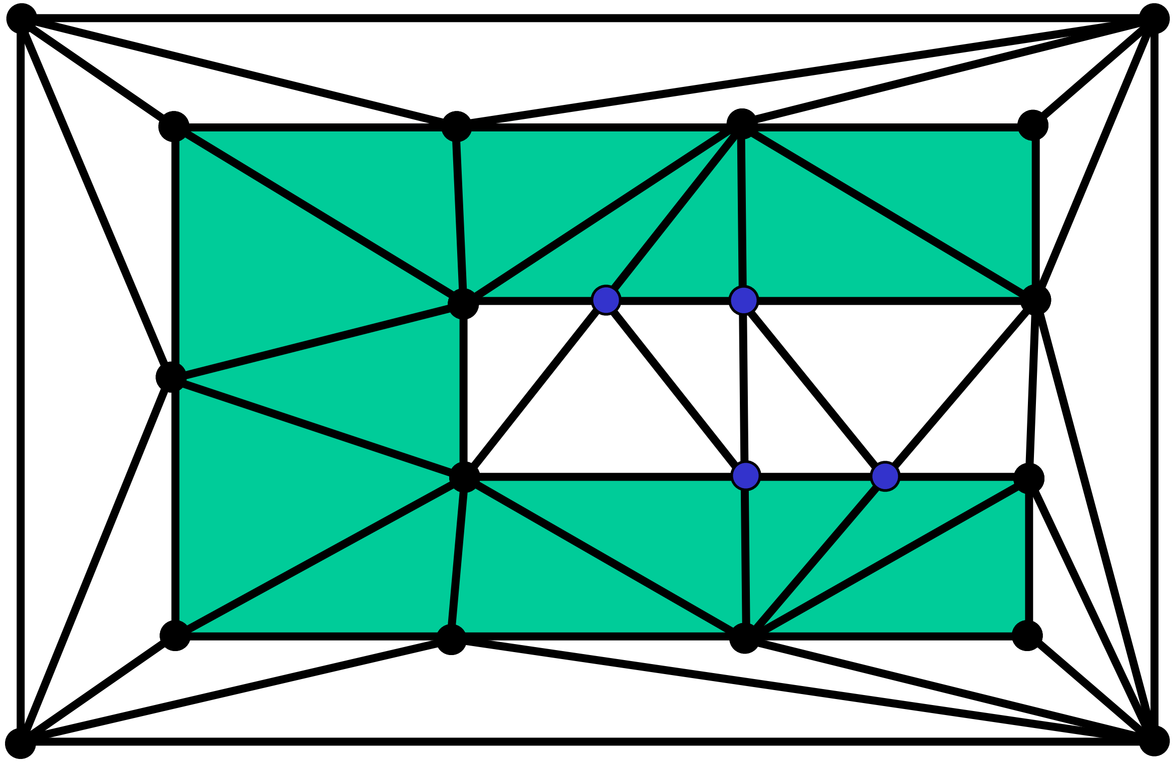
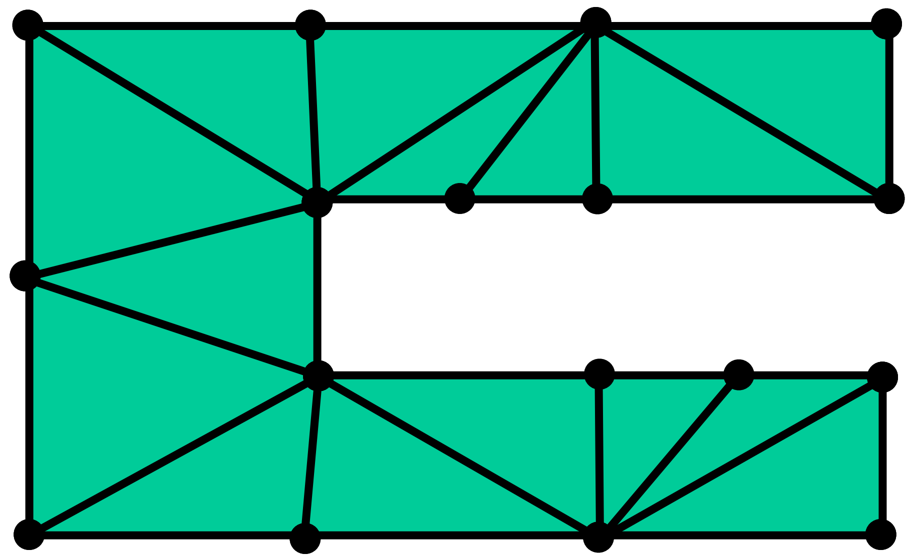


Boundary Intersection

•Nodes and edges introduced where Delaunay edges intersect boundary

Boundary Constrained

Delaunay



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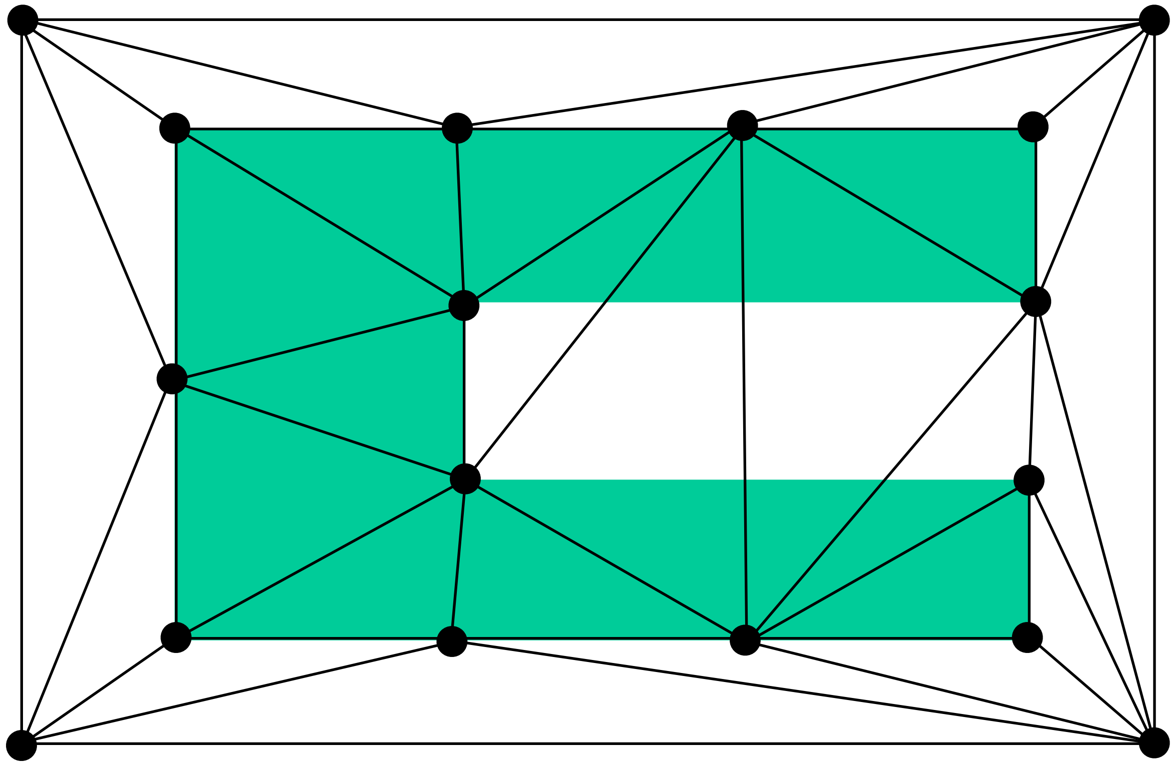


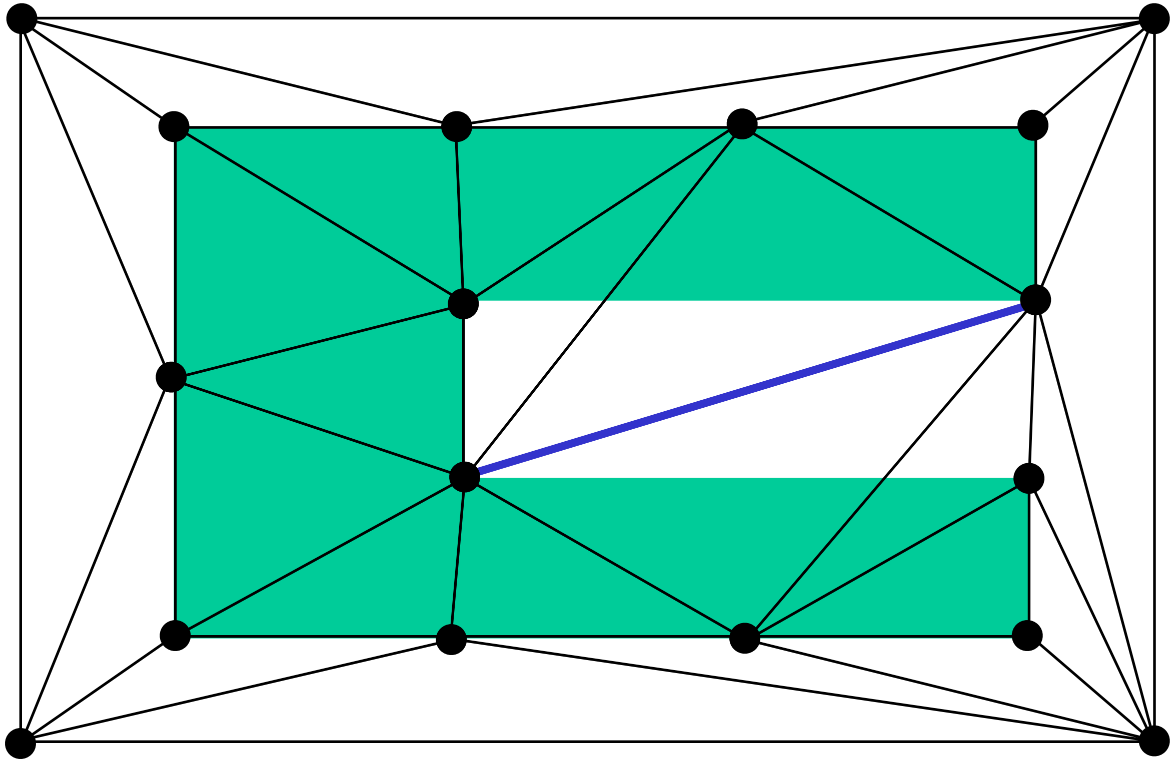
Local Swapping

•Edges swapped between adjacent pairs of triangles until boundary is maintained

Boundary Constrained

Delaunay

Computational Modeling Sciences Department



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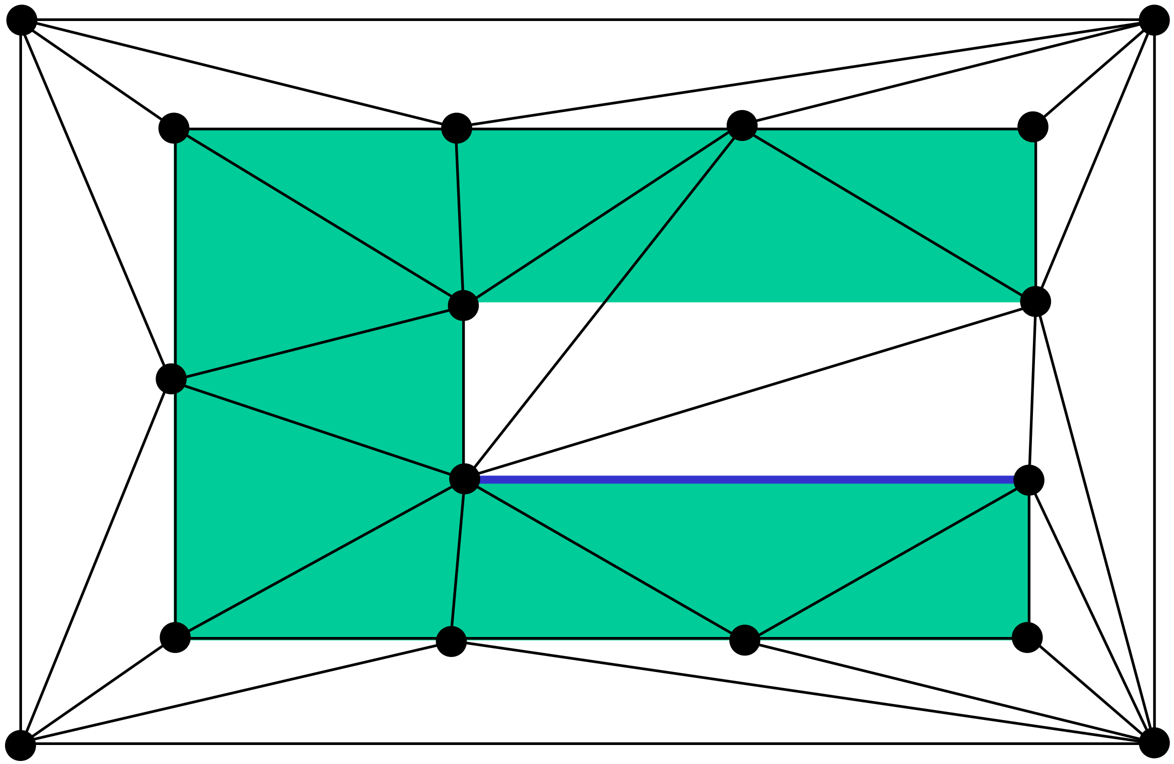

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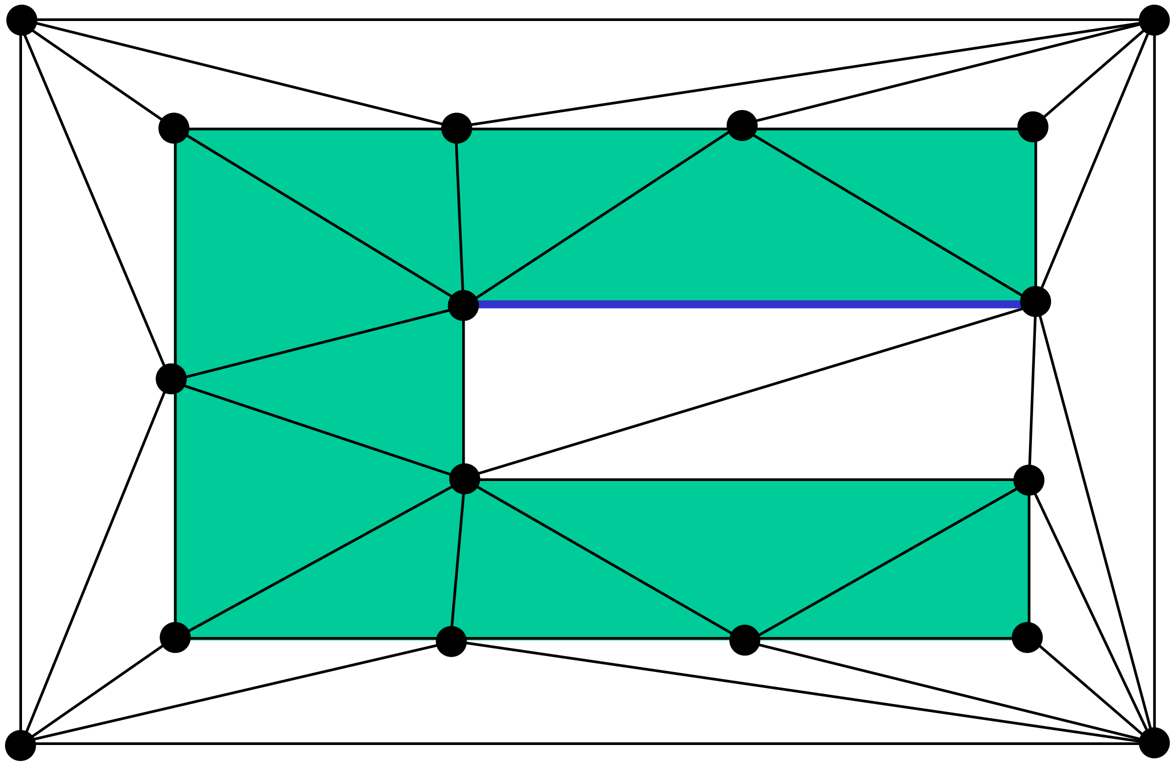

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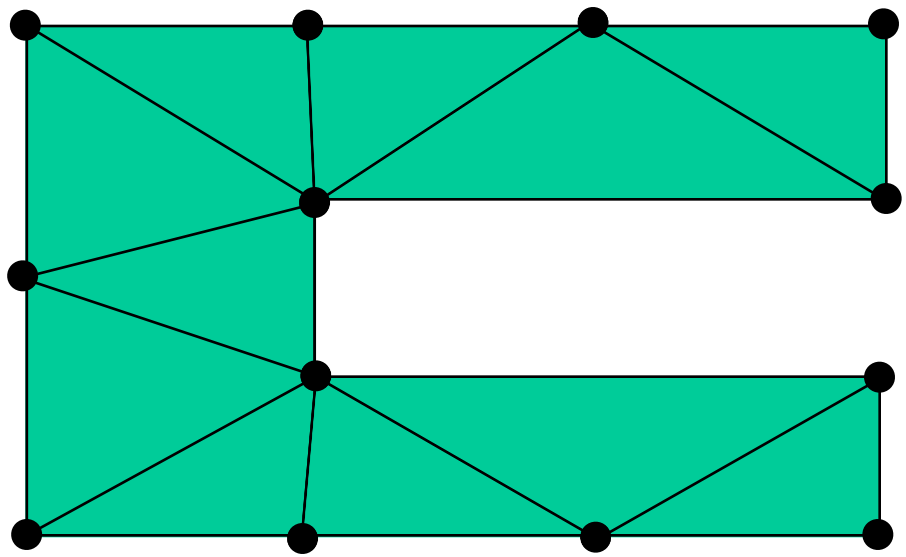

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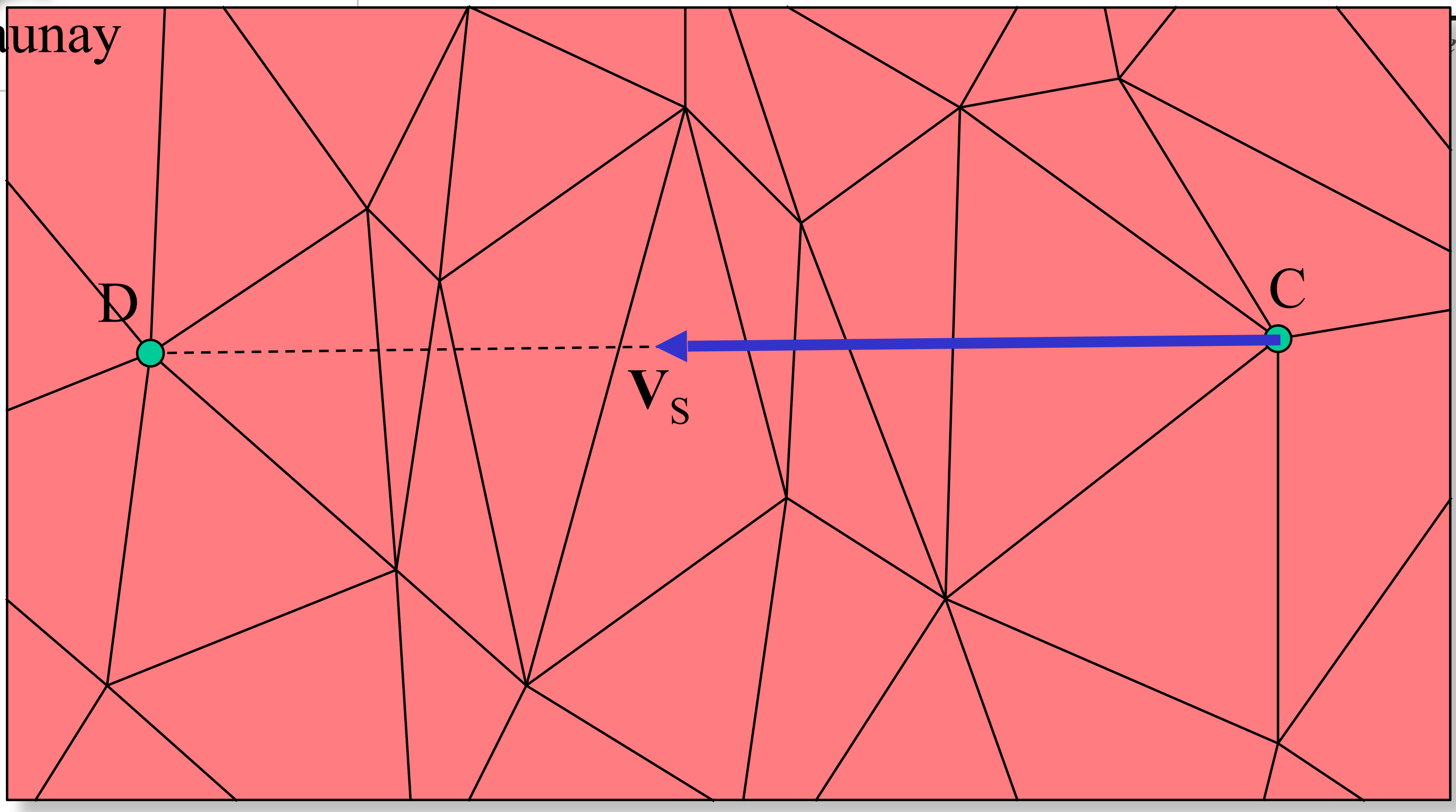
(George,91)(Owen,99)

Boundary Constrained

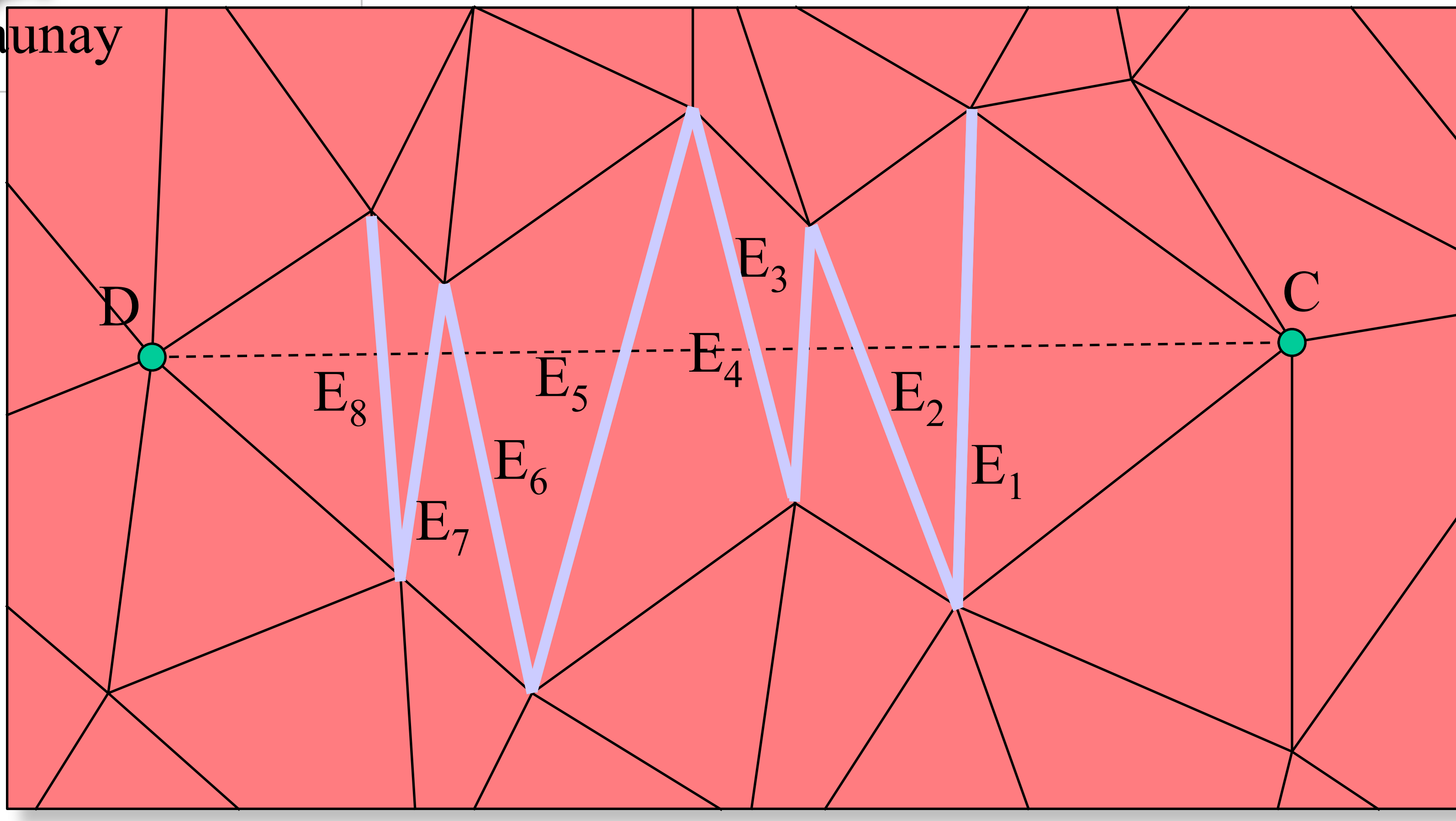


Local Swapping Example

•Recover edge CD at vector $\mathbf{V}_s$

Boundary Constrained

Delaunay

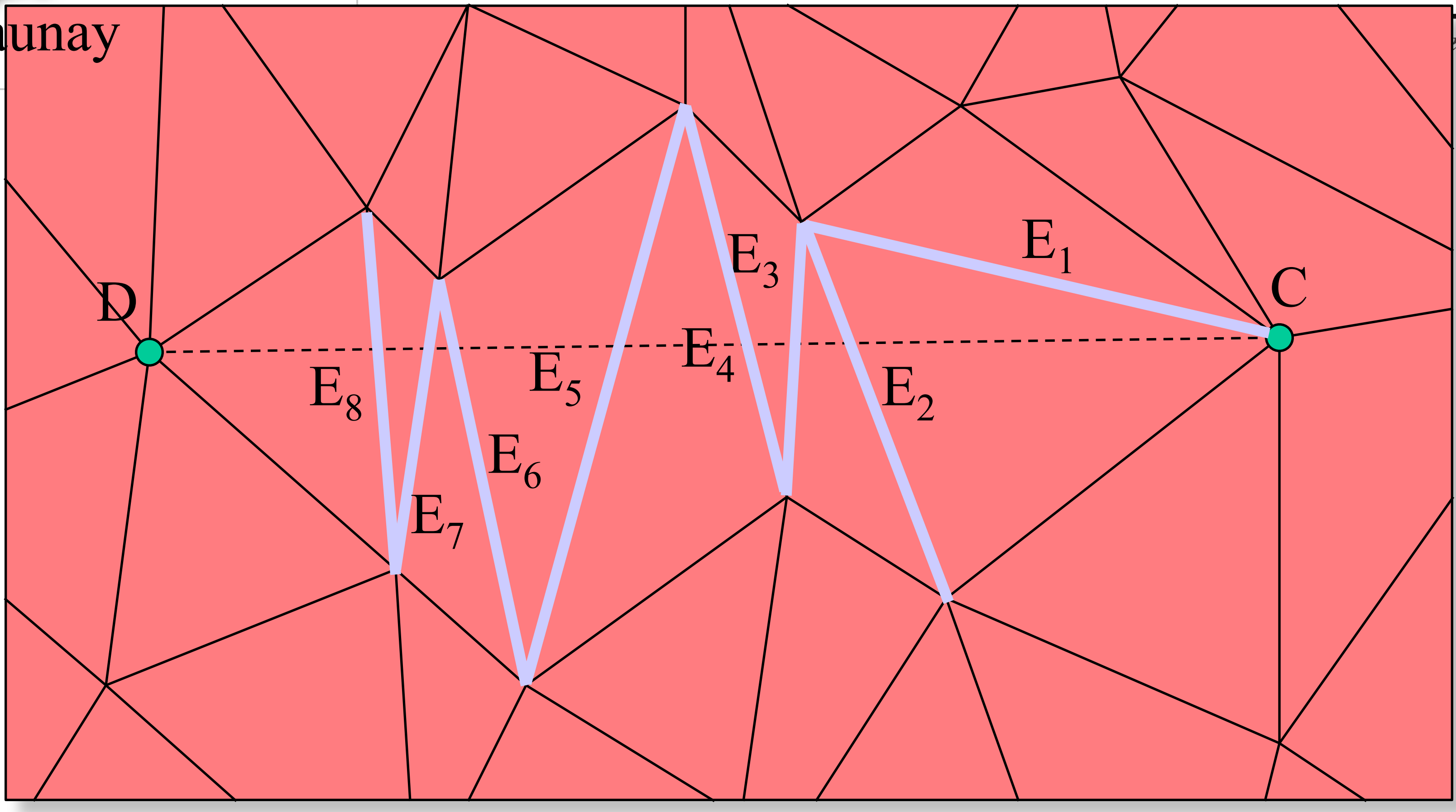


Local Swapping Example

•Make a list (queue) of all edges E_i, that intersect $\mathbf{V}_s$

Boundary Constrained

Delaunay

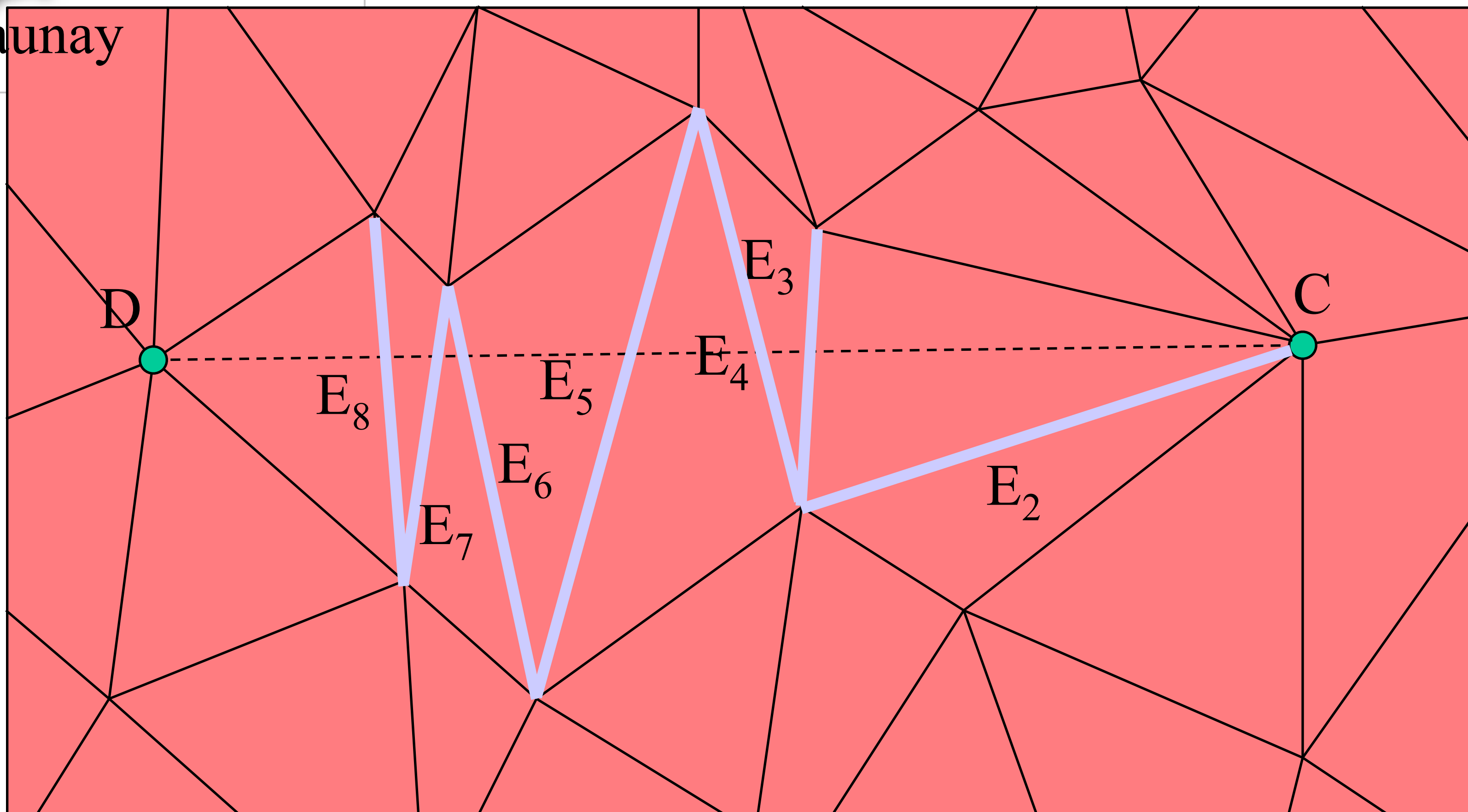


Local Swapping Example

•Swap the diagonal of adjacent triangle pairs for each edge in the list

Boundary Constrained

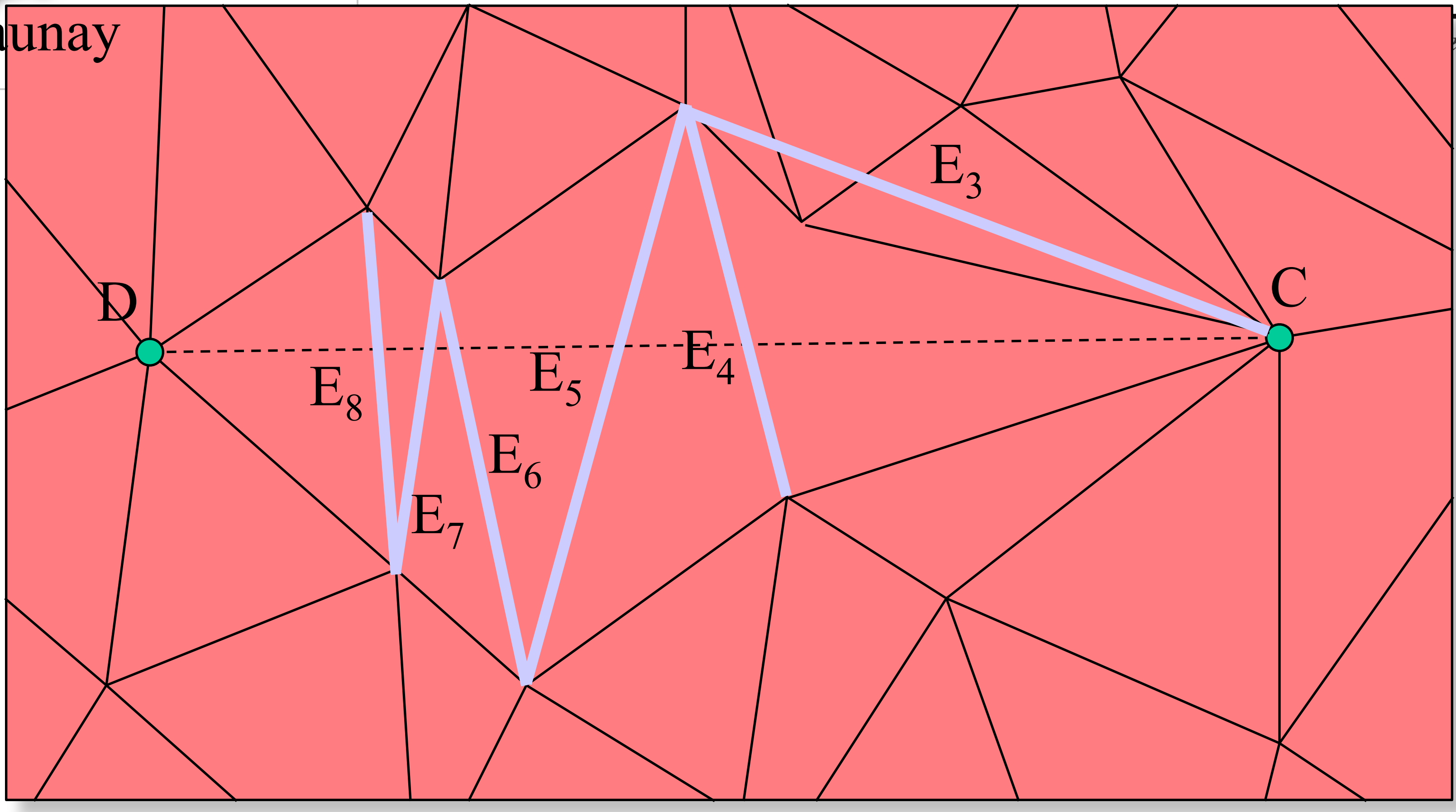
Delaunay



Local Swapping Example

•Check that resulting swaps do not cause overlapping triangles. I they do, then place edge at the back of the queue and try again later

Delaunay



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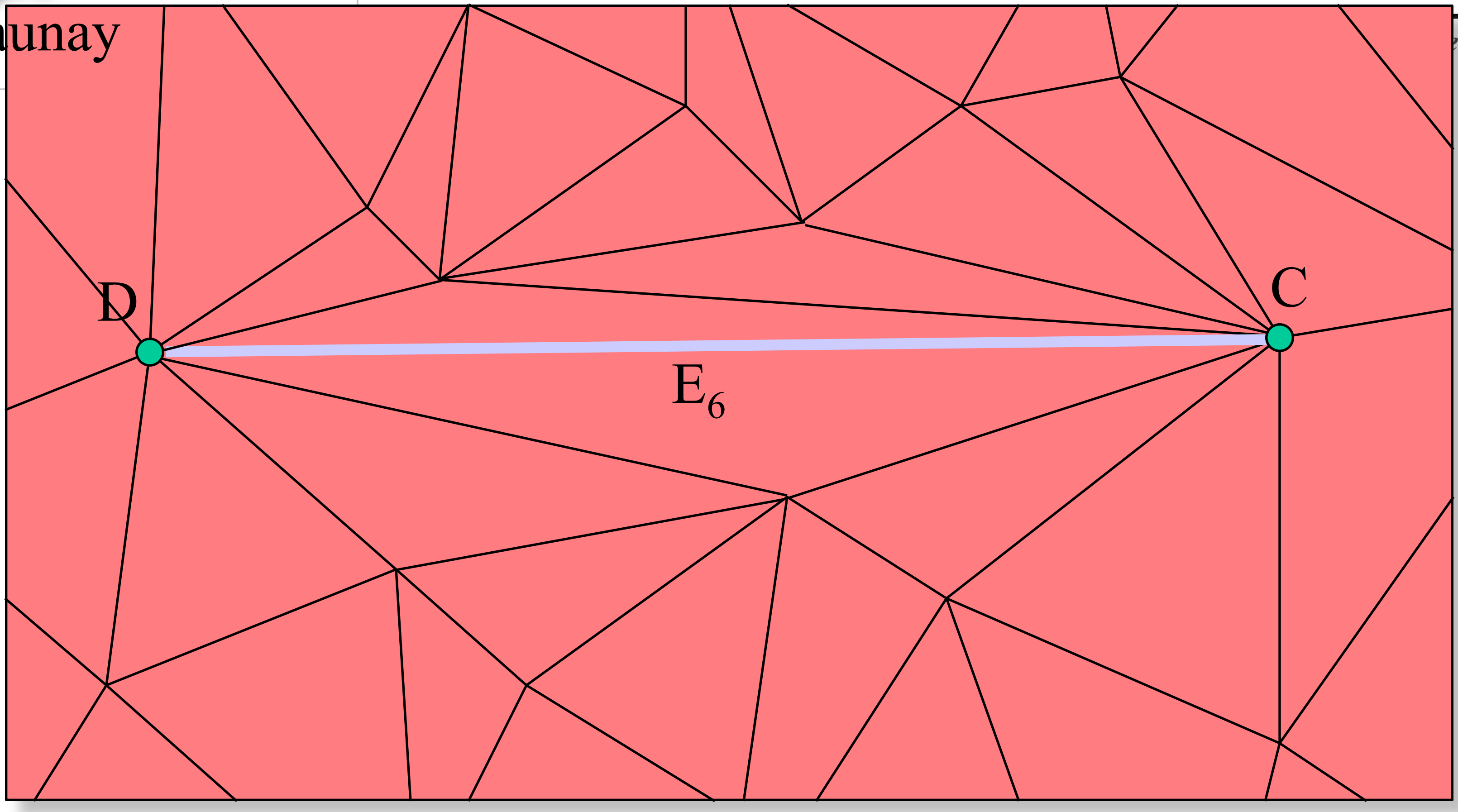


Local Swapping Example

•Final swap will recover the desired edge.

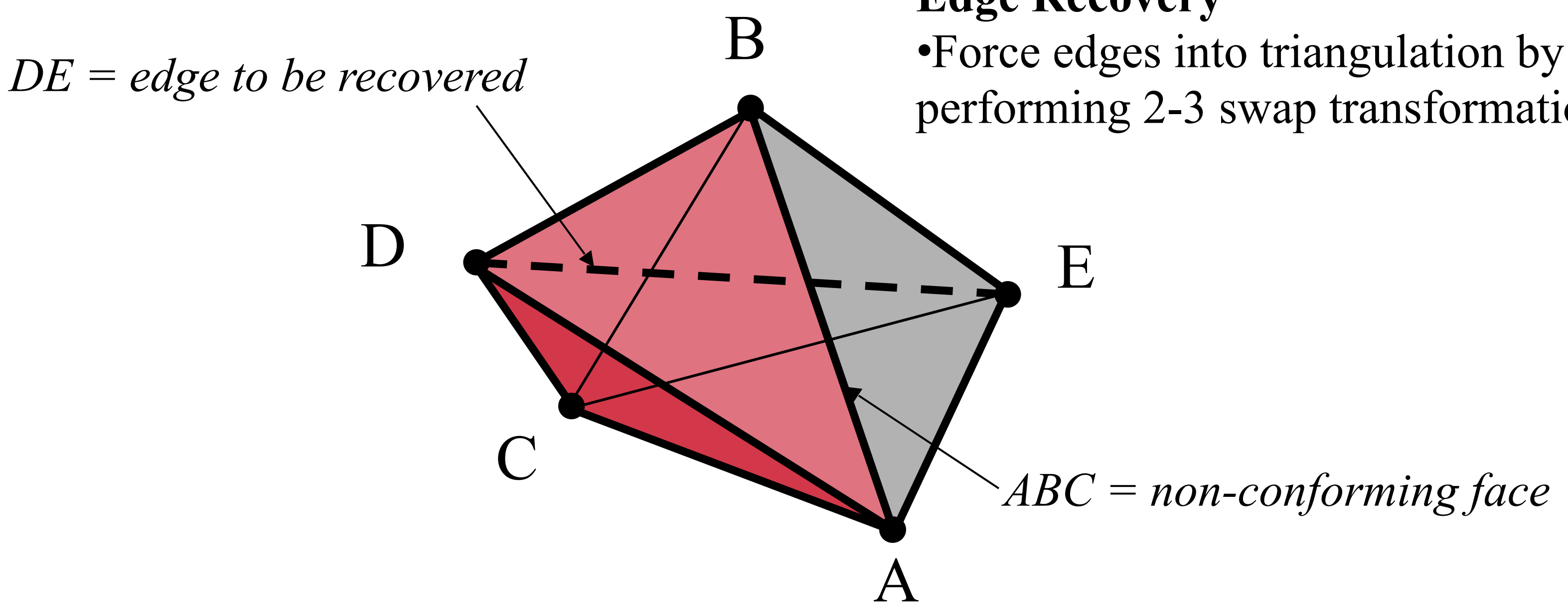
•Resulting triangle quality may be poor if multiple swaps were necessary

•Does not maintain Delaunay criterion!

Delaunay

Edge Recovery

•Force edges into triangulation by performing 2-3 swap transformation



3D Local Swapping

•Requires both boundary *edge* recovery and boundary *face* recovery

(George,91;99)(Owen,00)

Boundary Constrained

Delaunay

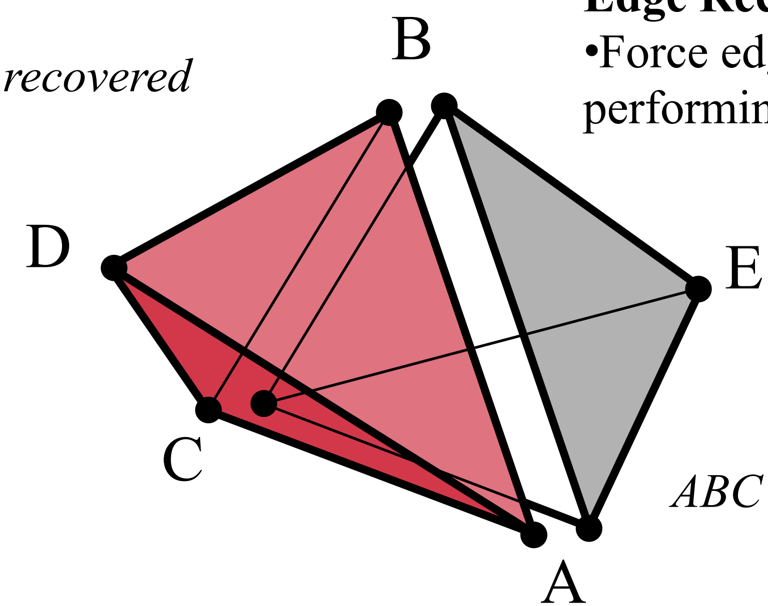
Computational Modeling Sciences Department

DE = edge to be recovered

Edge Recovery
- Force edges into triangulation by performing 2-3 swap transformation

2-3 Swap

ABCE
ACBD



ABC = non-conforming face

3D Local Swapping
- Requires both boundary *edge* recovery and boundary *face* recovery

(George,91;99)(Owen,00)

Boundary Constrained

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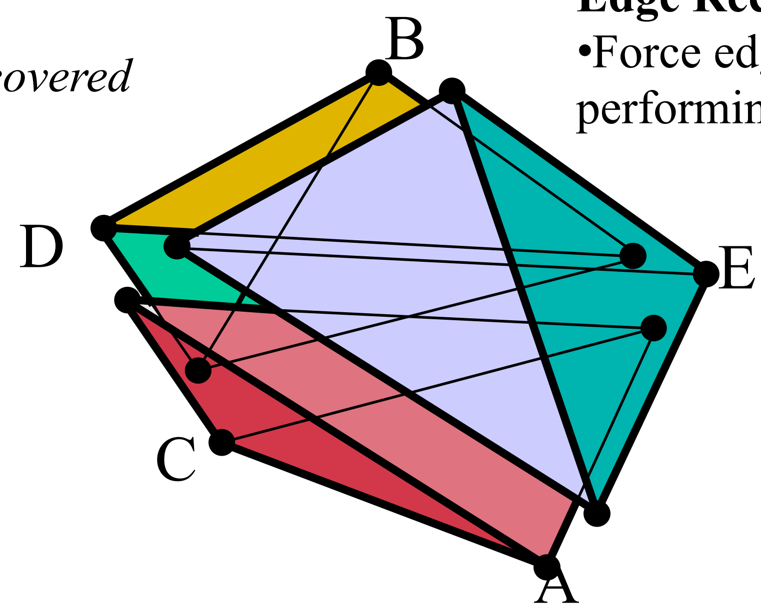

Edge Recovery

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2-3 Swap

ABCE *ACBD* ⇨ *BAED* *CBED* *ACED*

3D Local Swapping

•Requires both boundary *edge* recovery and boundary *face* recovery

(George,91;99)(Owen,00)

Boundary Constrained

Delaunay

Edge Recovery
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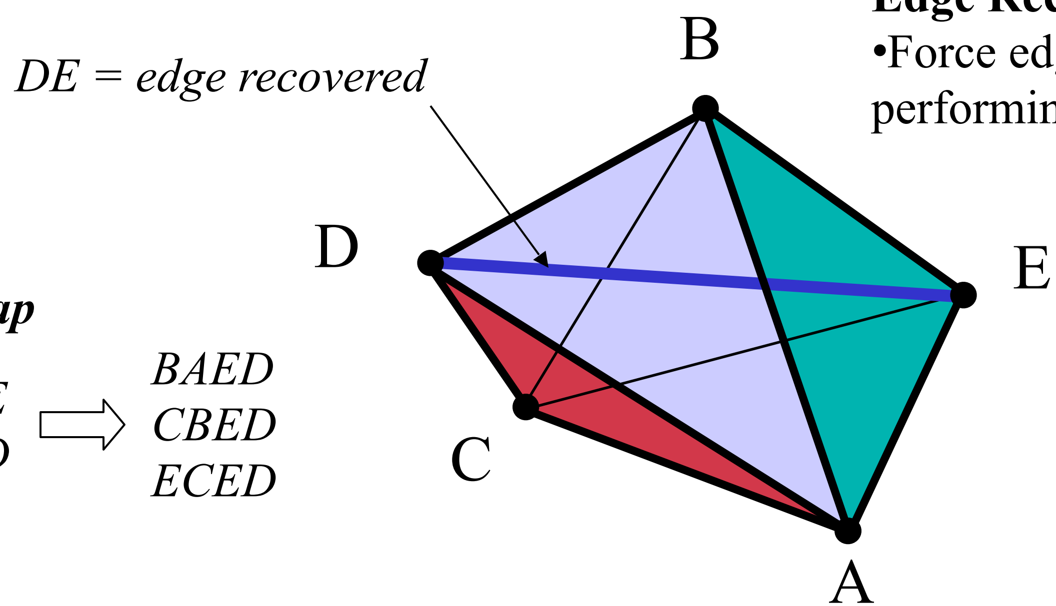

2-3 Swap

ABCE
ACBD ⇨ *BAED*
CBED
ECED

3D Local Swapping
•Requires both boundary *edge* recovery and boundary *face* recovery

(George,91;99)(Owen,00)

Boundary Constrained

Delaunay

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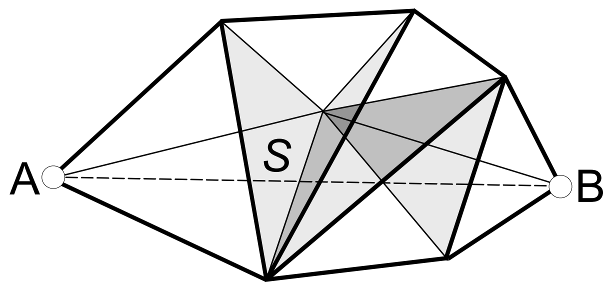
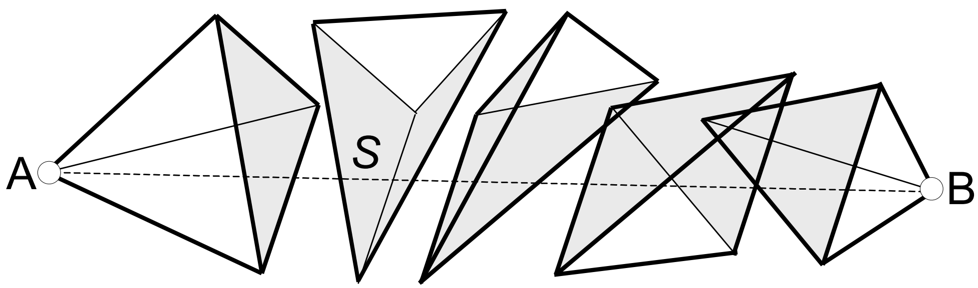


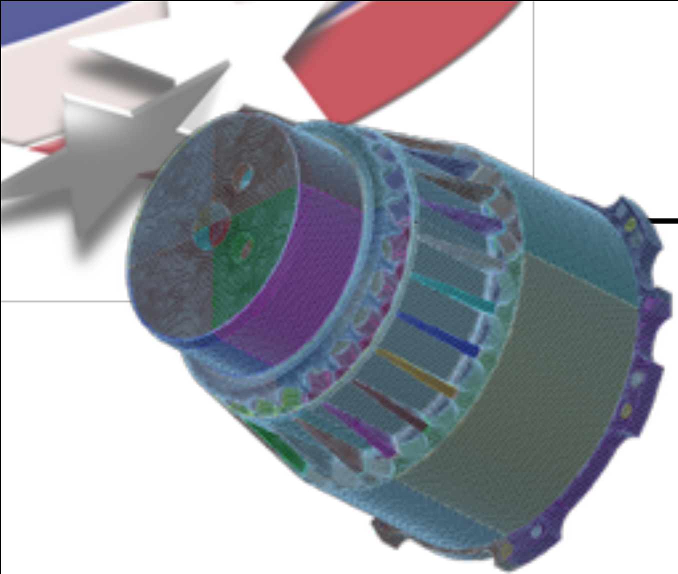
Edge AB to be recovered



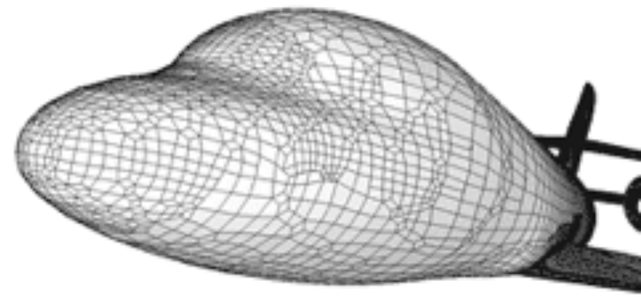
Exploded view of tets intersected by AB

3D Edge Recovery

- Form queue of faces through which edge AB will pass
- Perform 2-3 swap transformations on all faces in the list
- If overlapping tets result, place back on queue and try again later
- If still cannot recover edge, then insert "steiner" point

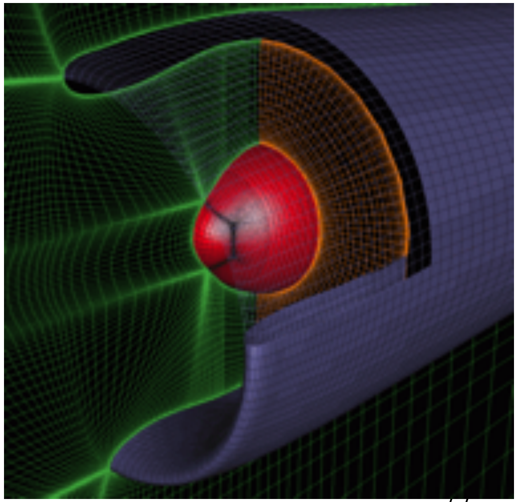


Quad/Hex Methods



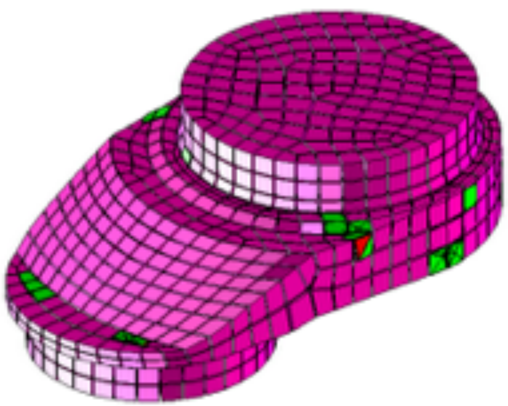
Structured

•Requires geometry to conform to specific characteristics
•Regular patterns of quads/hexes formed based on characteristics of geometry
•Internal nodes always attached to same number of elements



Unstructured

•No specific requirements for geometry
•quads/hexes placed to conform to geometry.
•No connectivity requirement (although optimization of connectivity is beneficial)



NNSA National Nuclear Security Administration

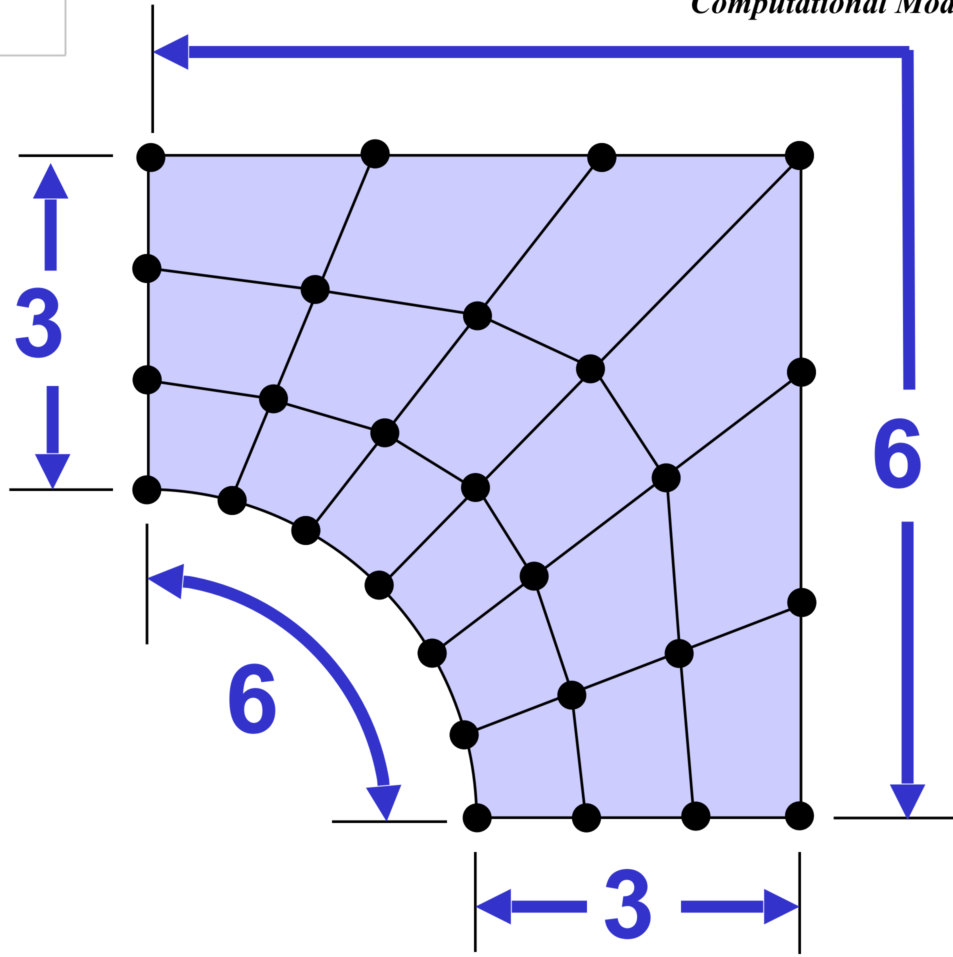
Sandia National Laboratories

Structured

Algorithm

- Trans-finite Interpolation (TFI)
- maps a regular lattice of quads onto polygon

(Thompson,88;99)
(Cook,82)

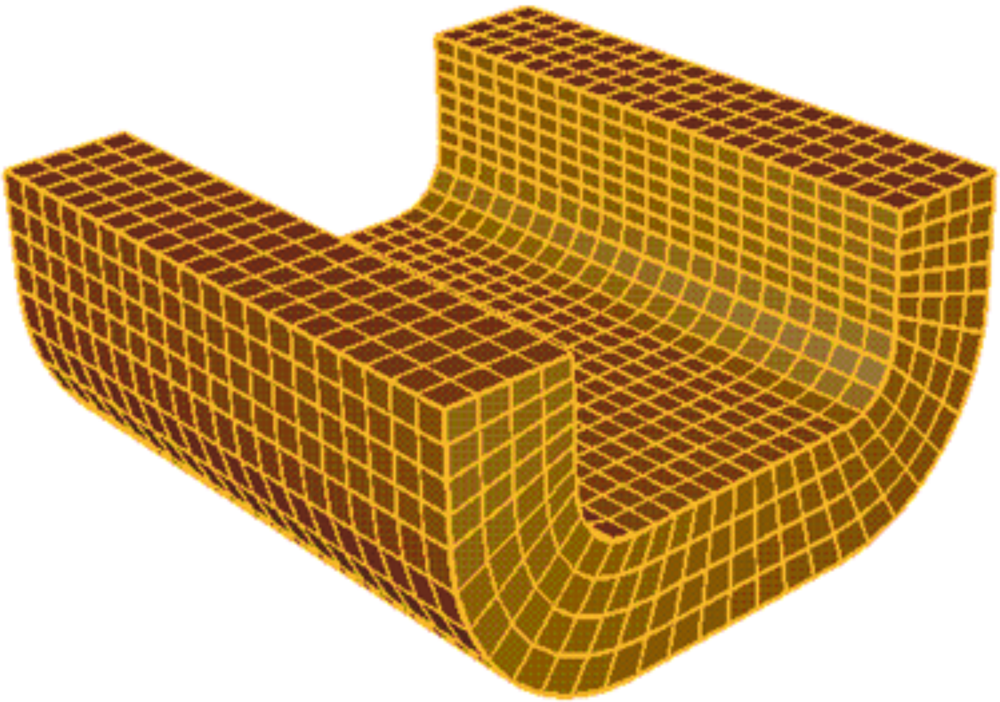


Geometry Requirements

- 4 topological sides
- opposite sides must have similar intervals

Mapped Meshing

Structured



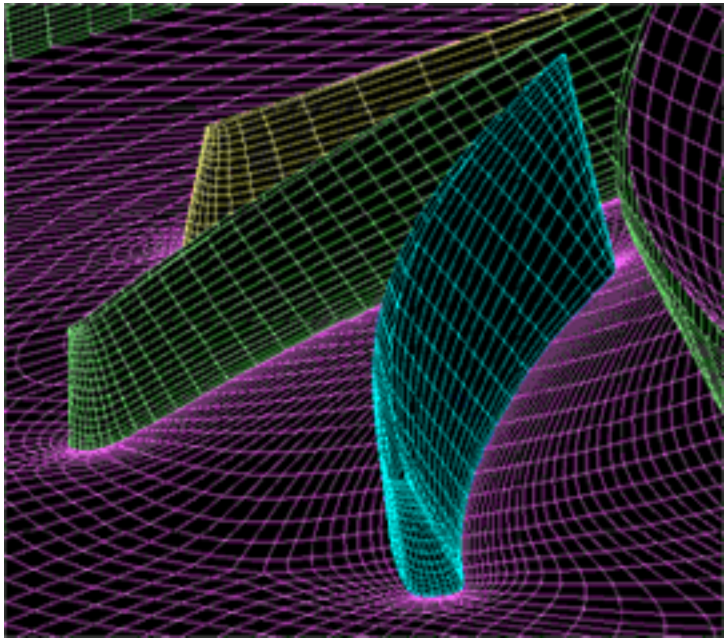
Geometry Requirements

- 6 topological surfaces
- opposite surfaces must have similar mapped meshes

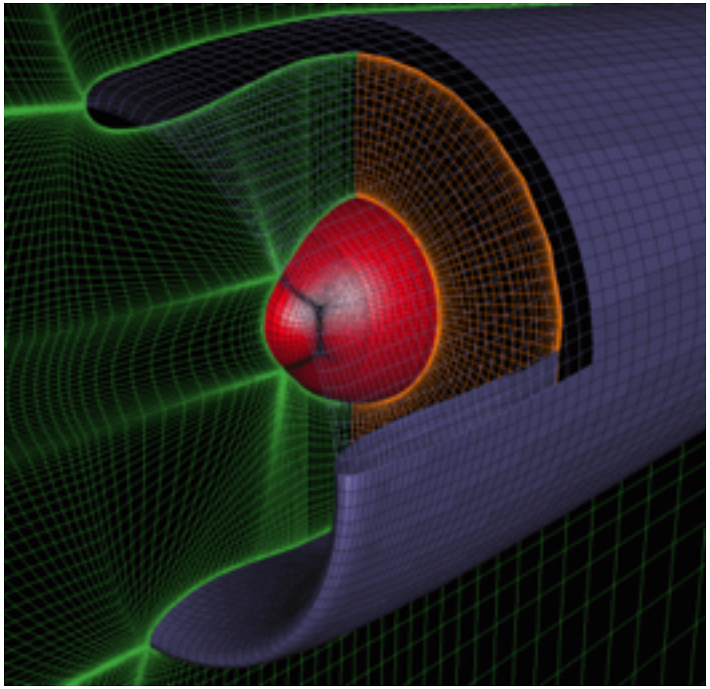
3D Mapped Meshing

Structured

Block-Structured



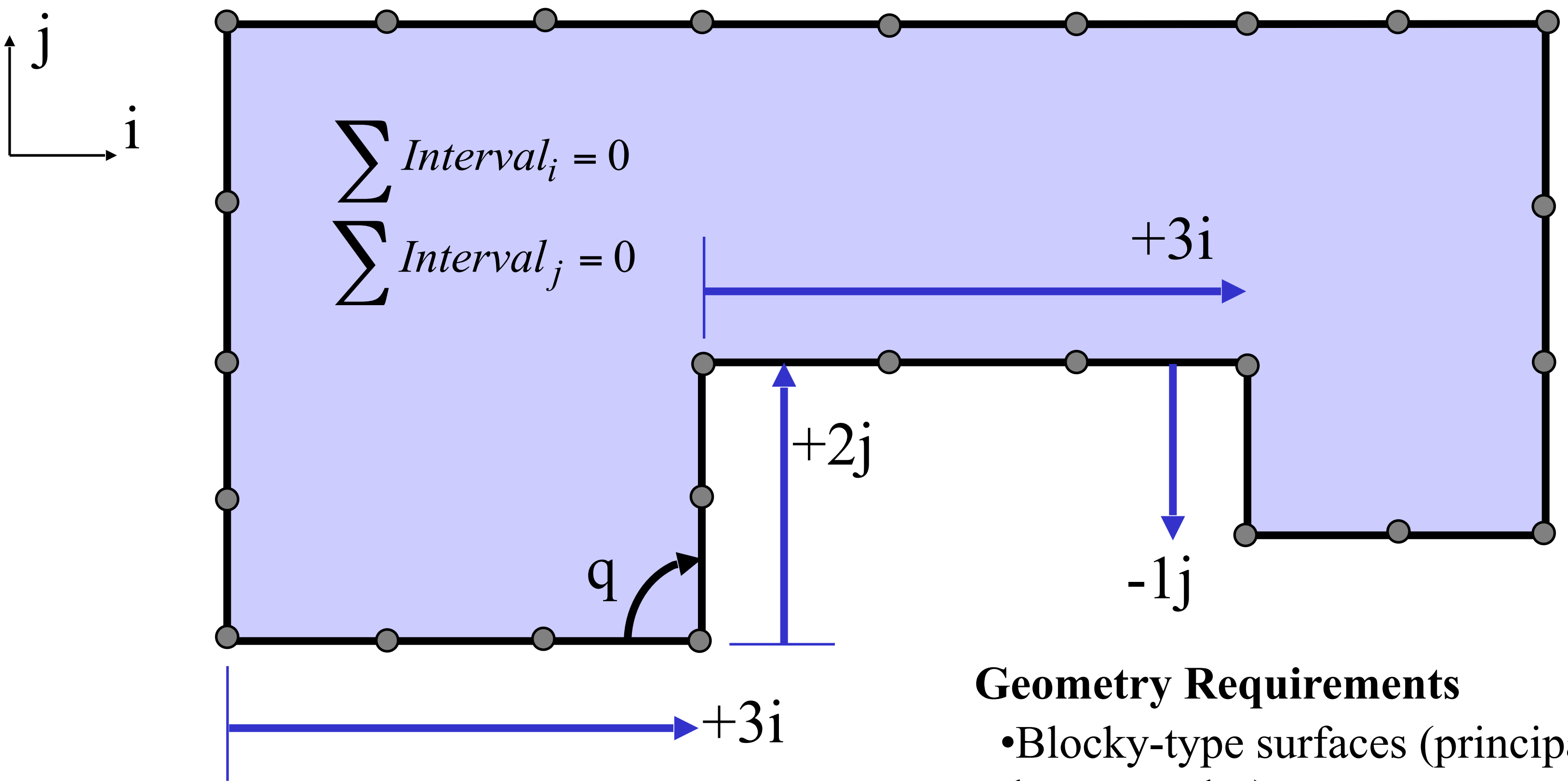
http://www.gridpro.com/gridgallery/tmachinery.html



http://www.pointwise.com/case/747.htm

Mapped Meshing

Structured



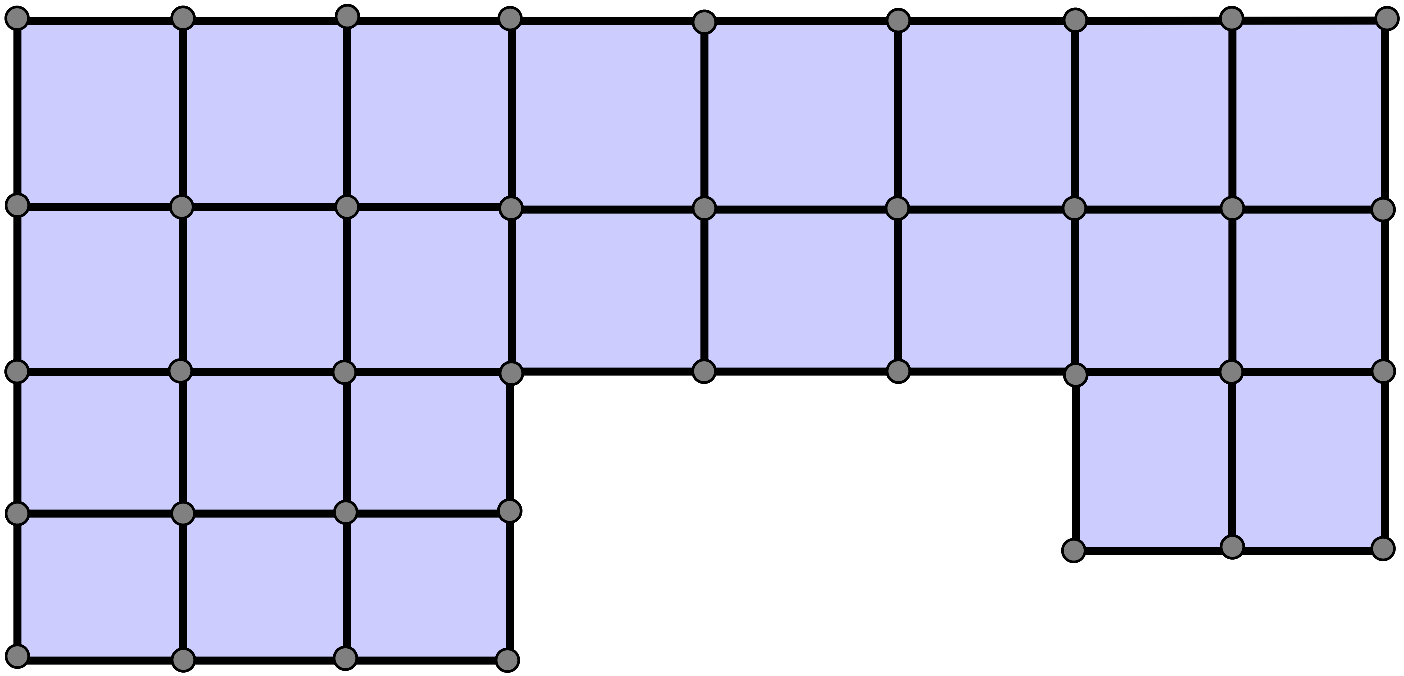
Geometry Requirements

•Blocky-type surfaces (principally 90 degree angles)

Sub-Mapping

(White,95)

Structured

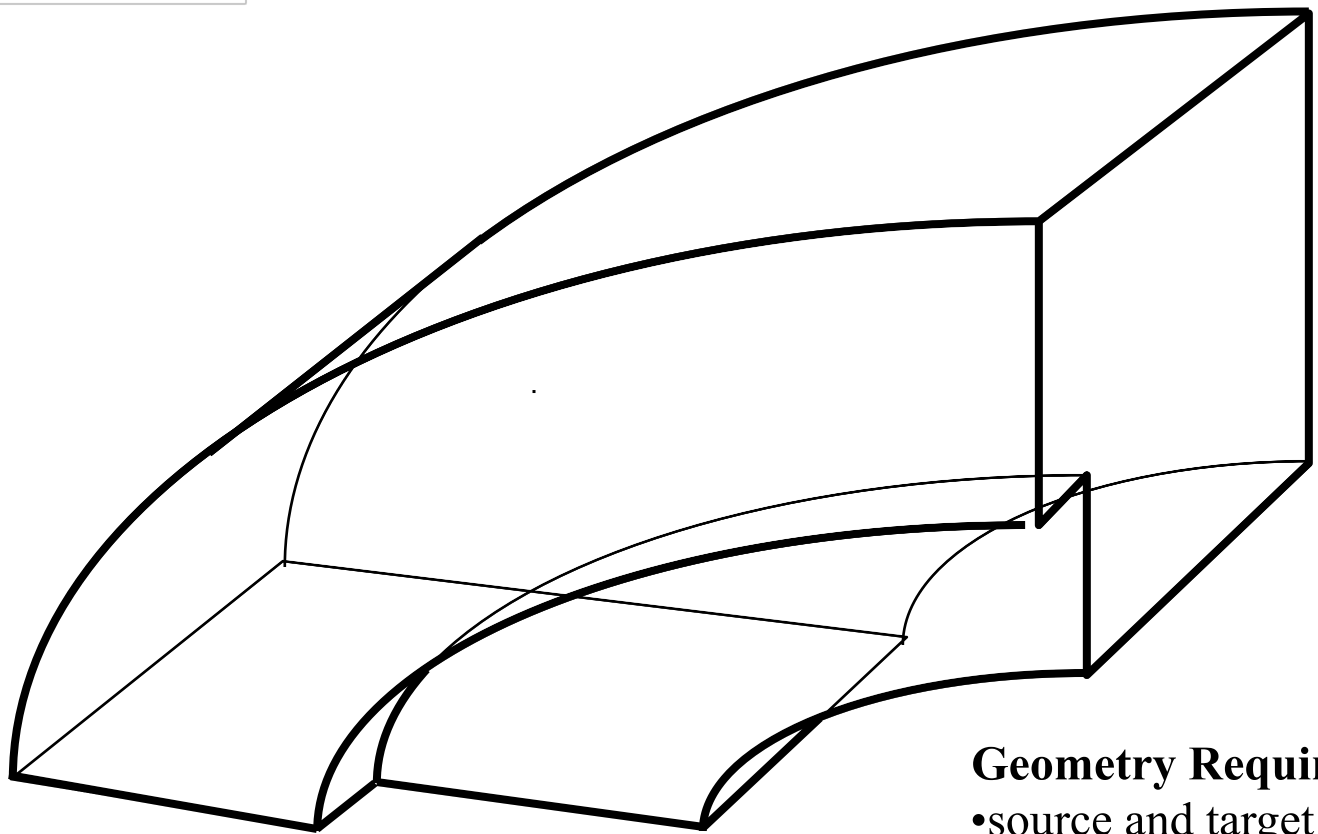


•Automatically decomposes surface into mappable regions based on assigned intervals

Sub-Mapping

(White,95)

Structured



Sweeping

Geometry Requirements
•source and target surfaces topologicaly similar
•linking surfaces *mapable* or *submapable*

Structured

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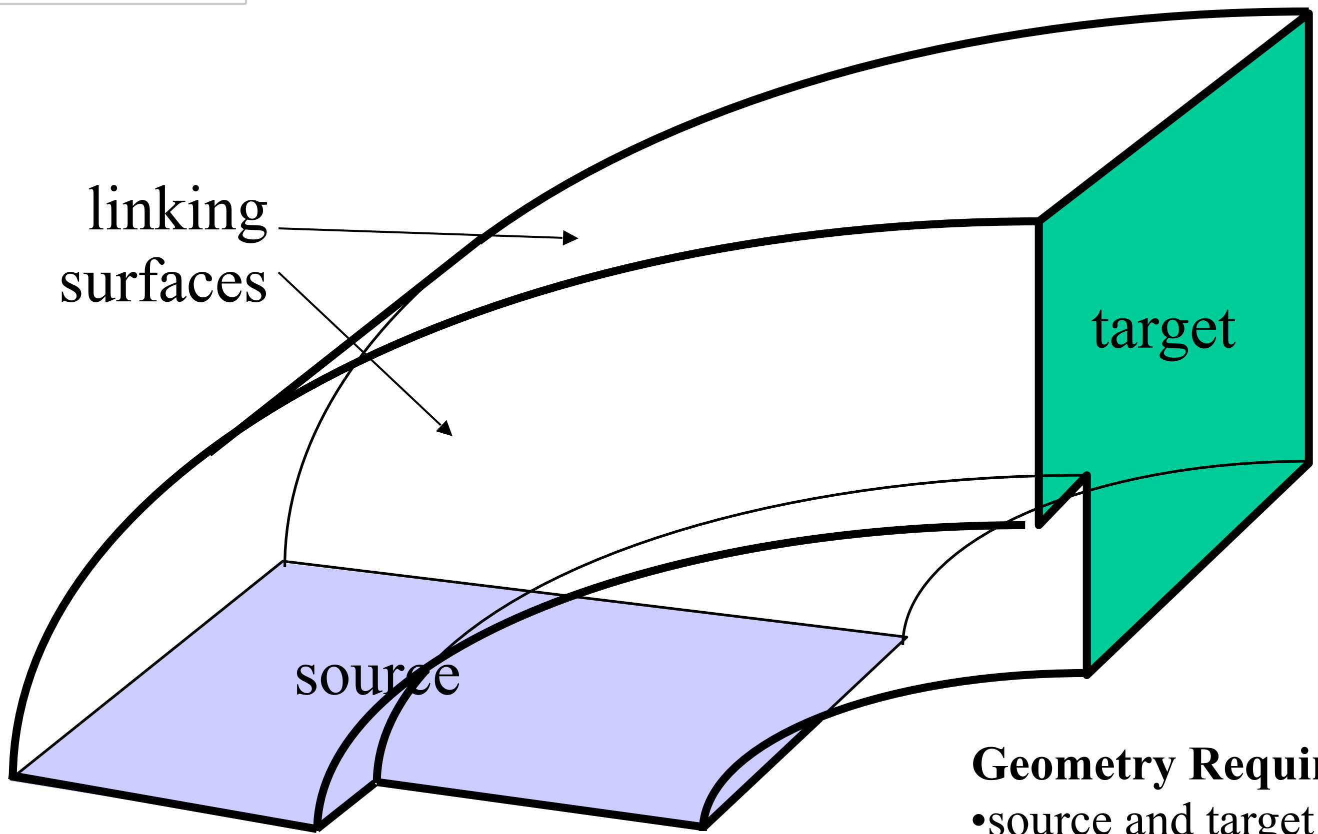

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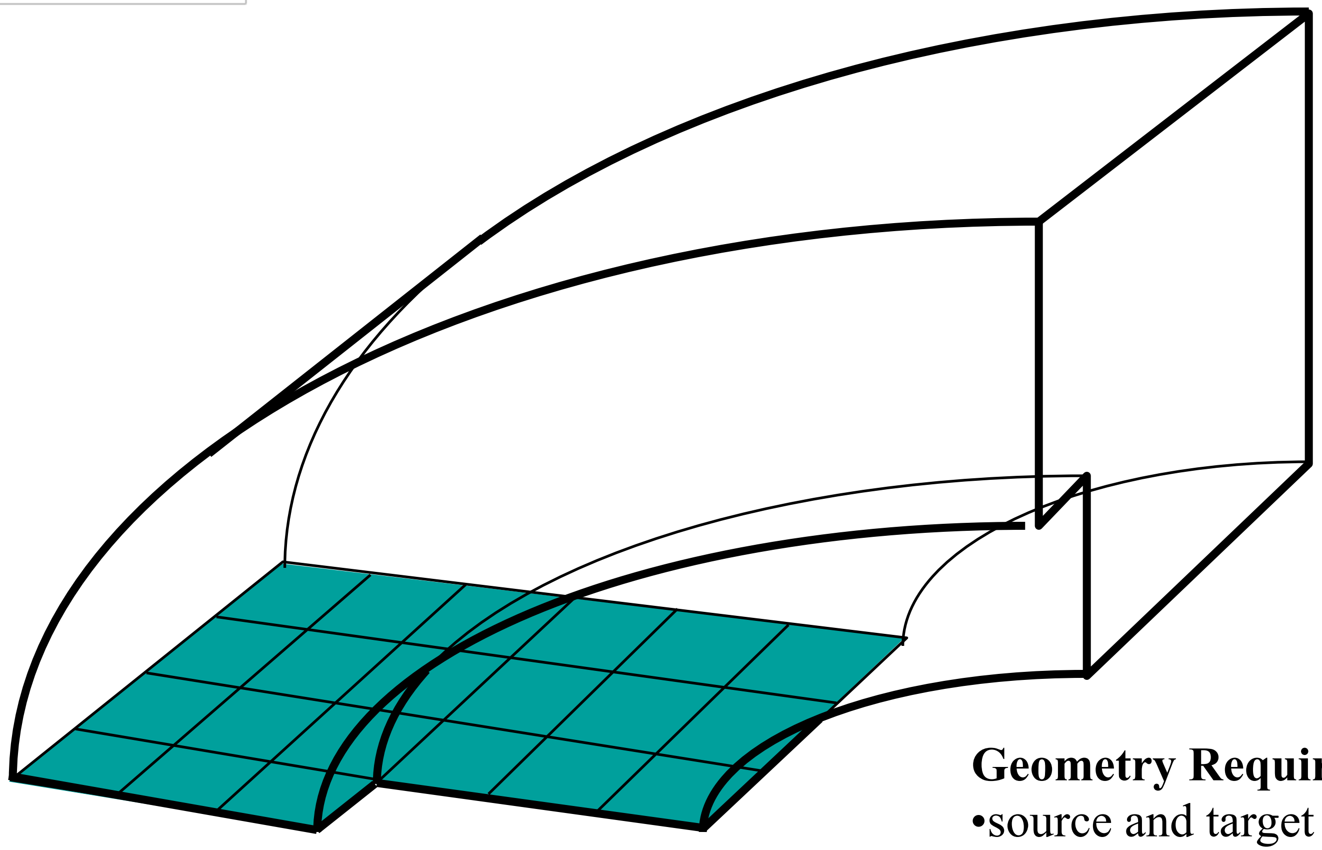

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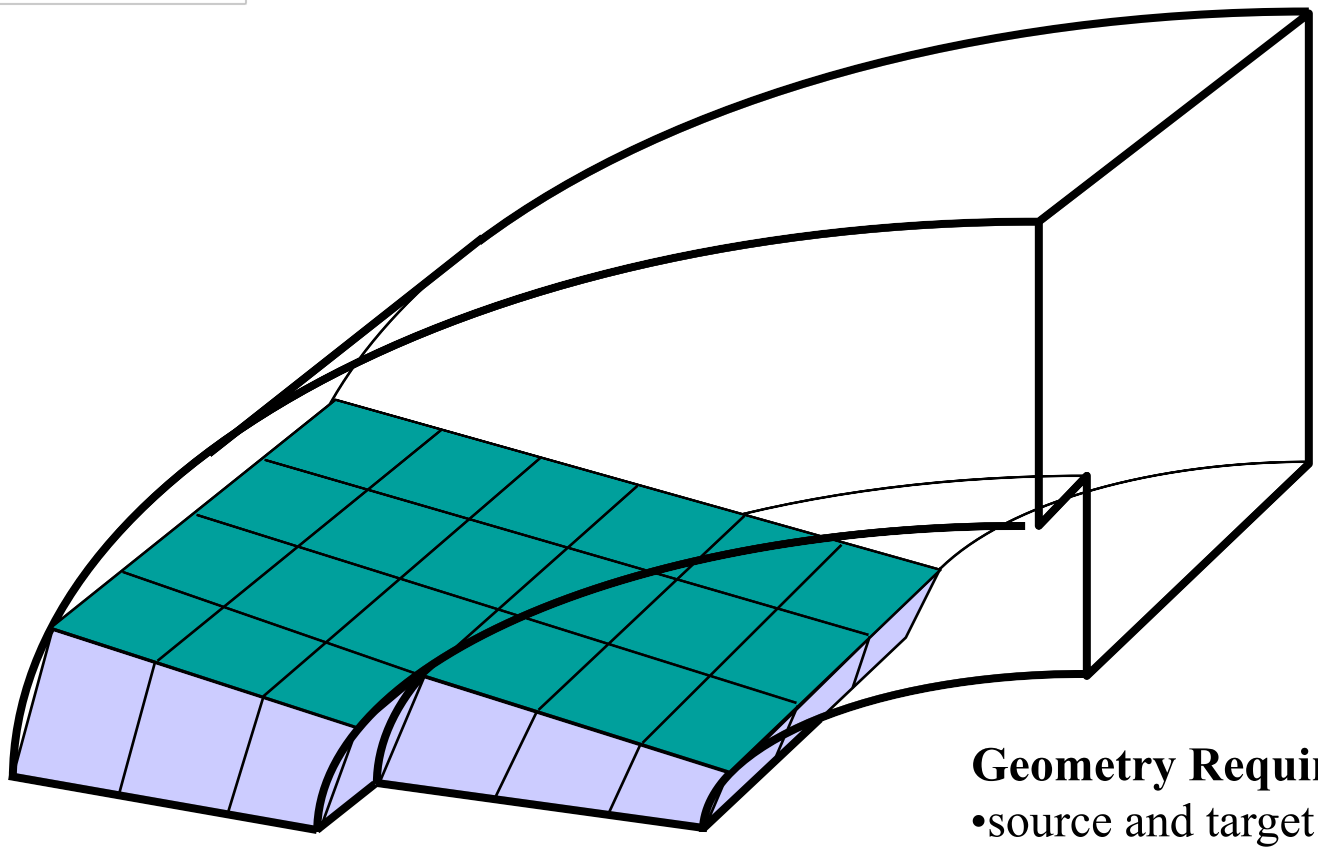

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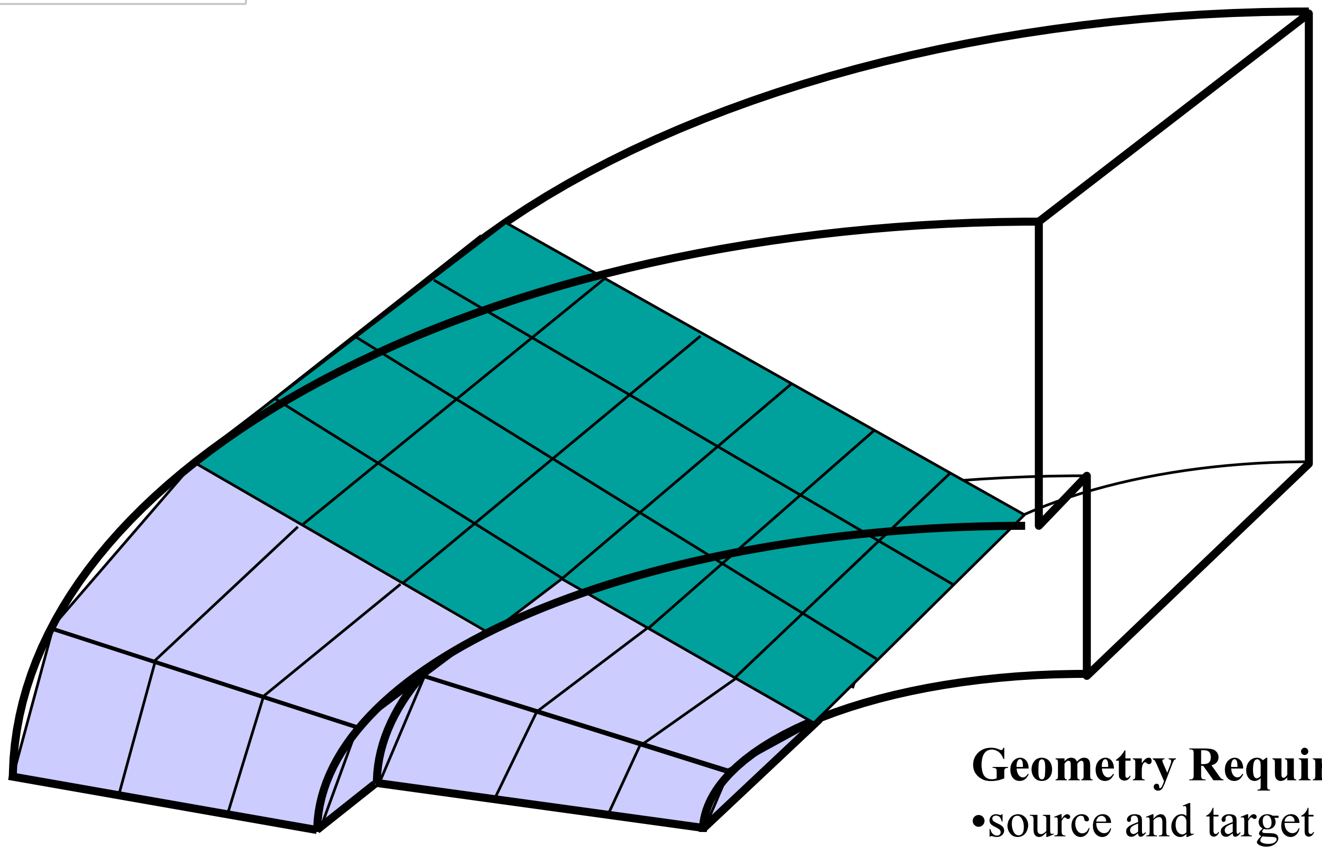
Computational Modeling Sciences Department



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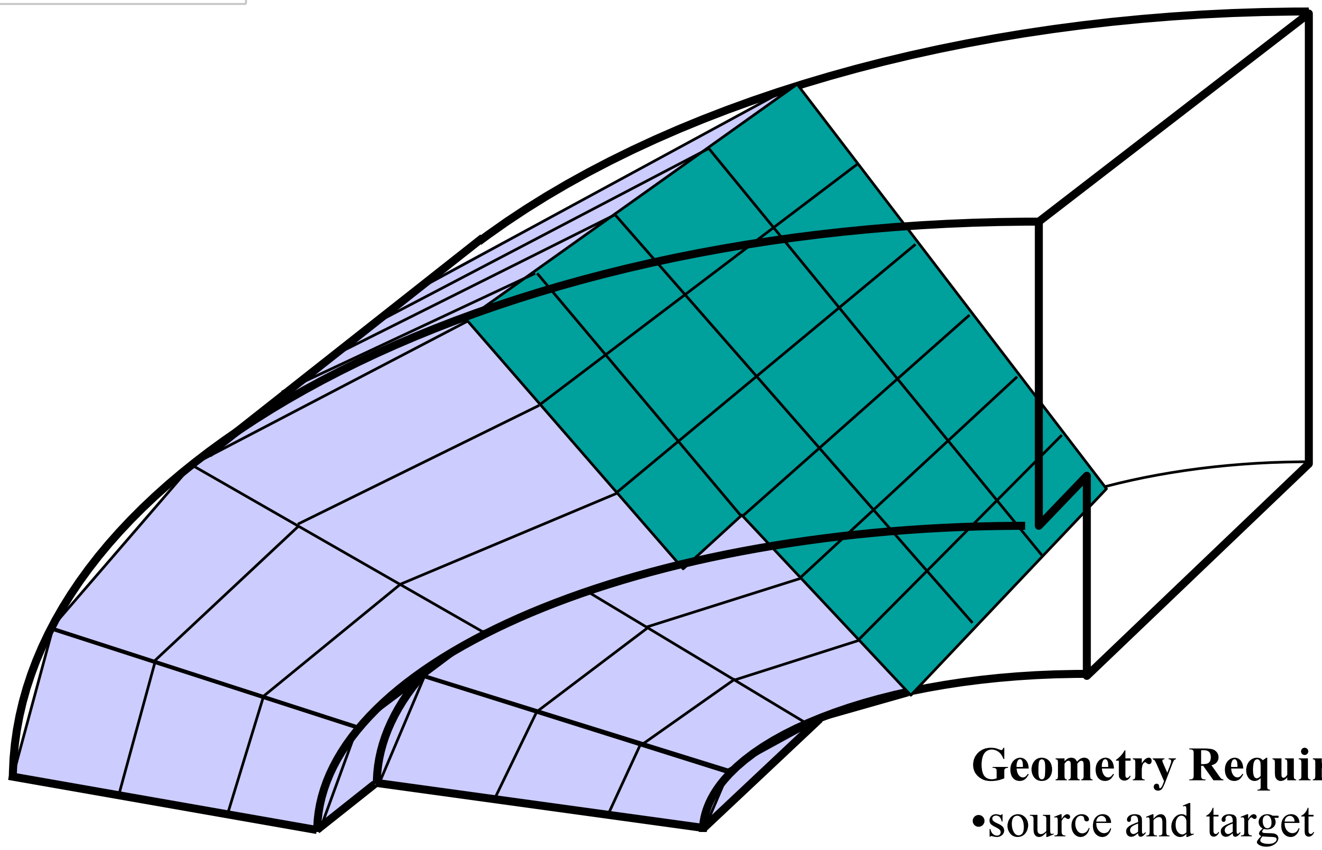

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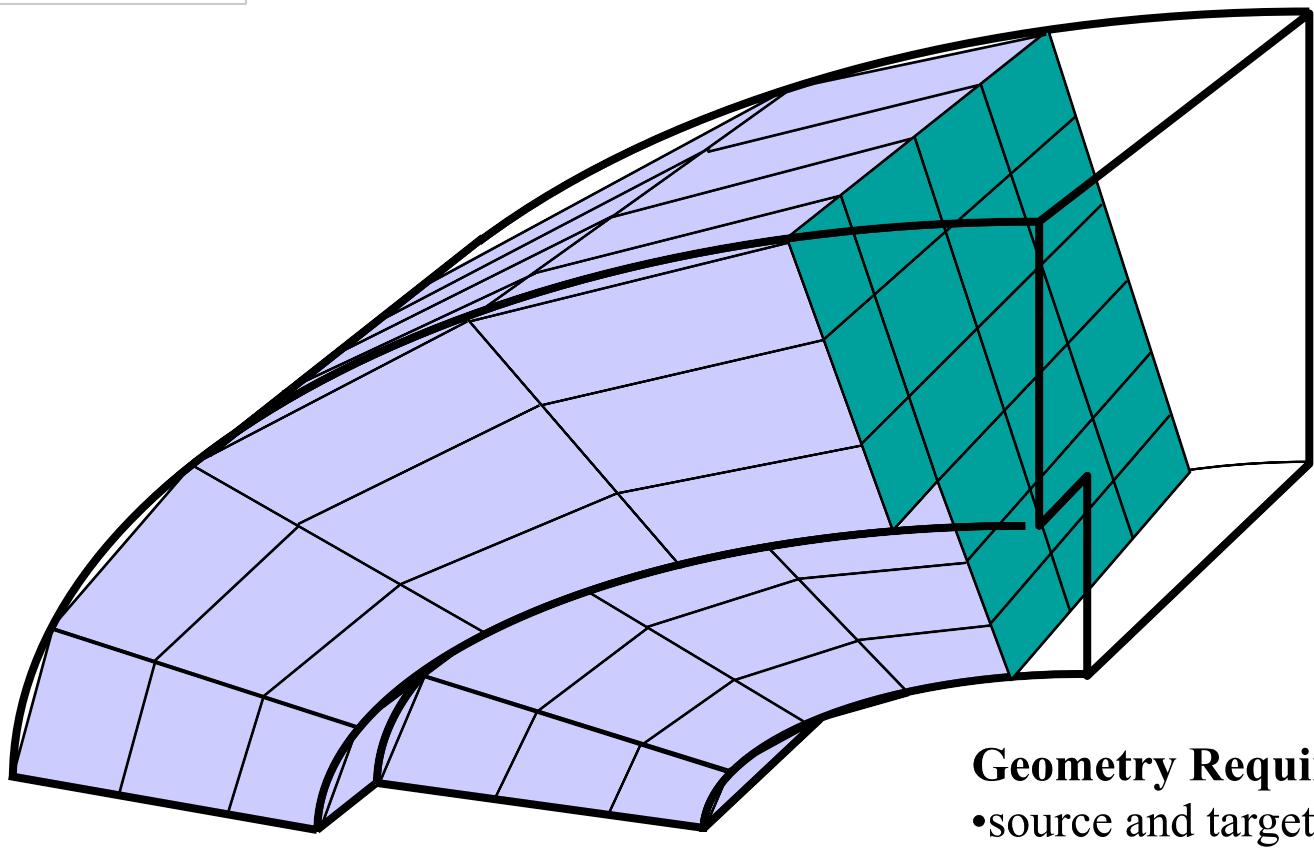

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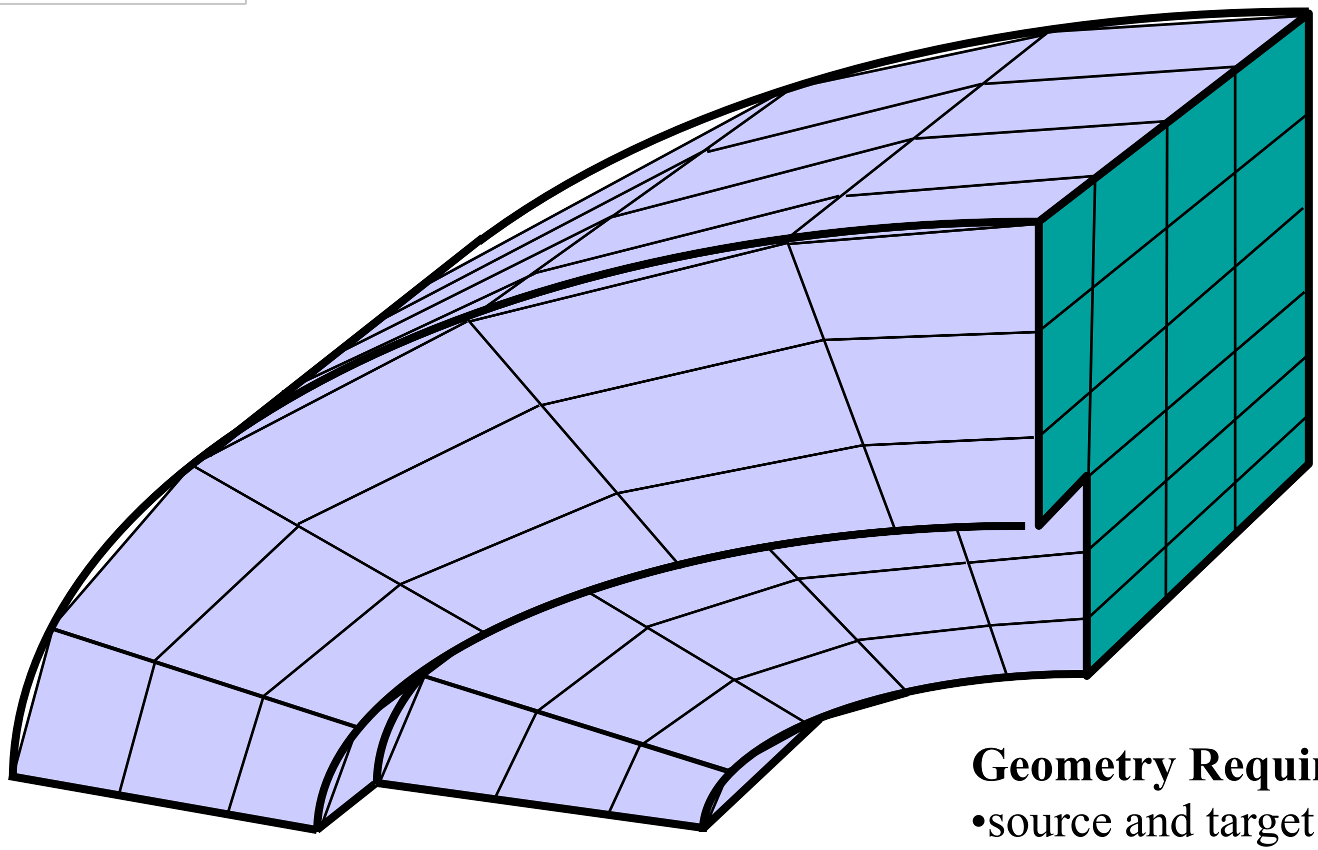

Sweeping

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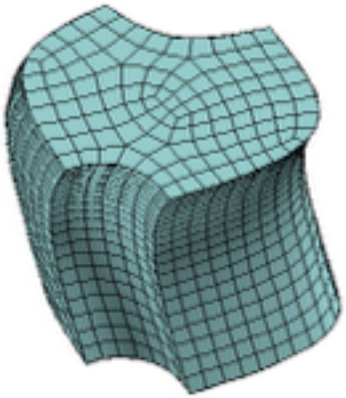
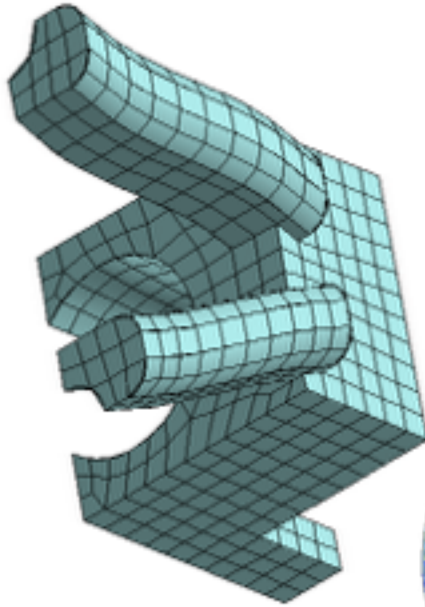
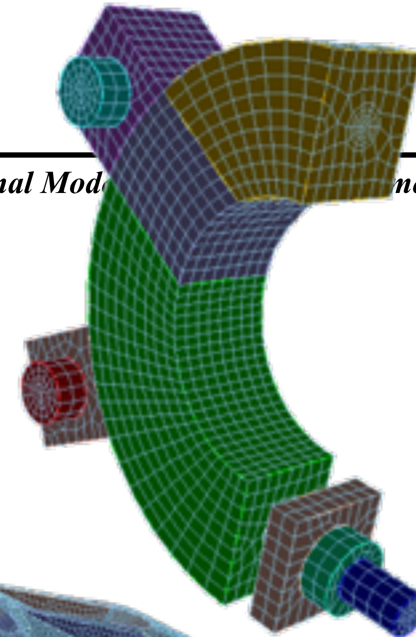
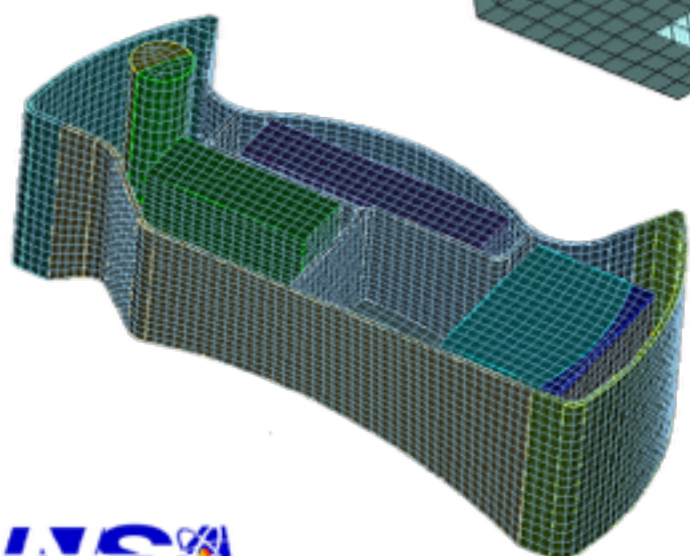
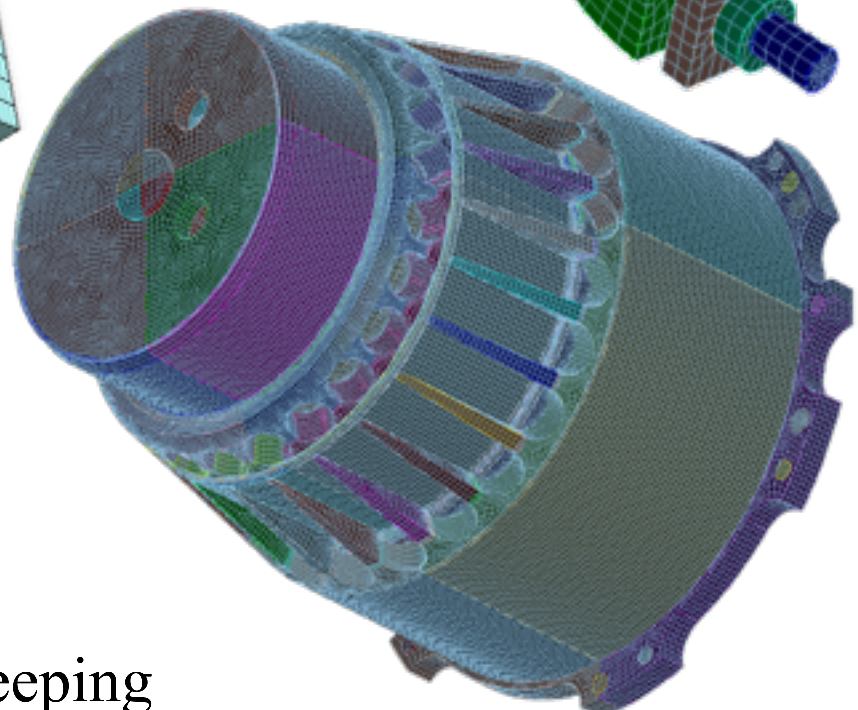


Gambit, Fluent Inc.





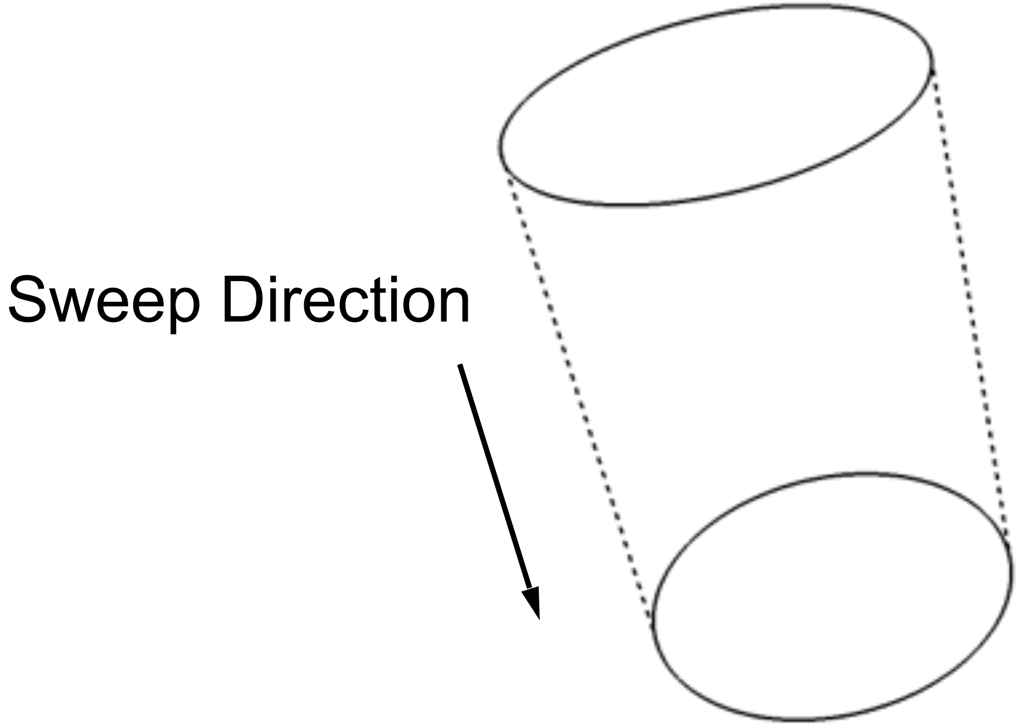




Sweeping

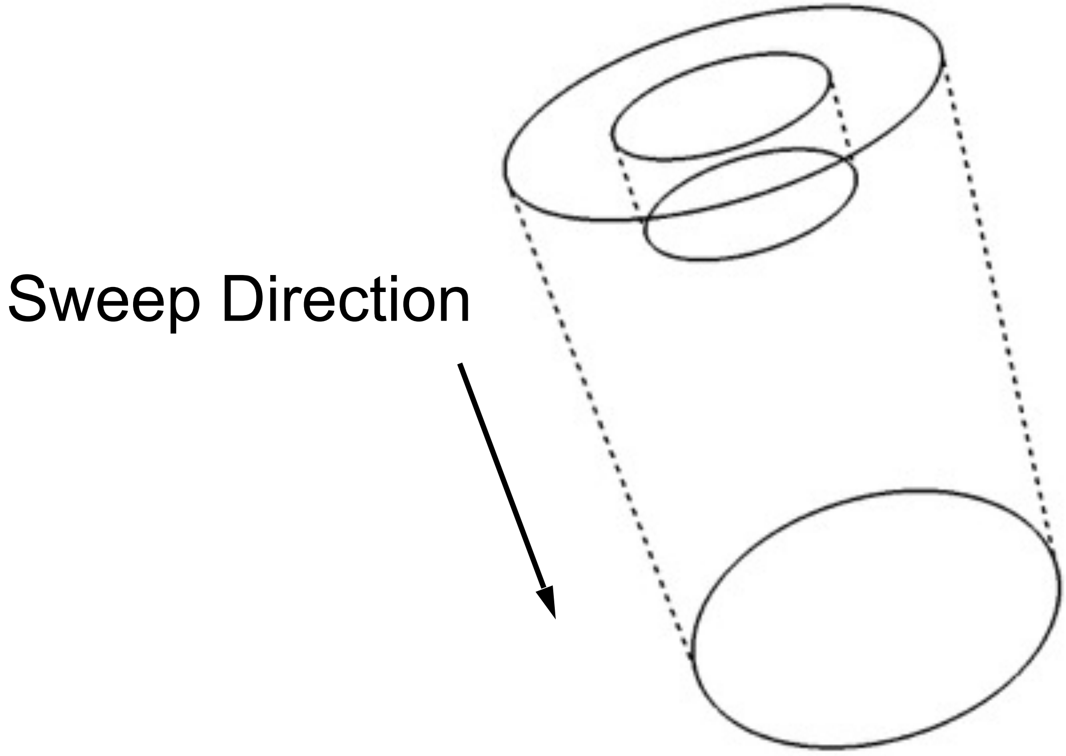
Cubit, Sandia National Labs

Sweeping



1-to-1 sweepable

Sweeping



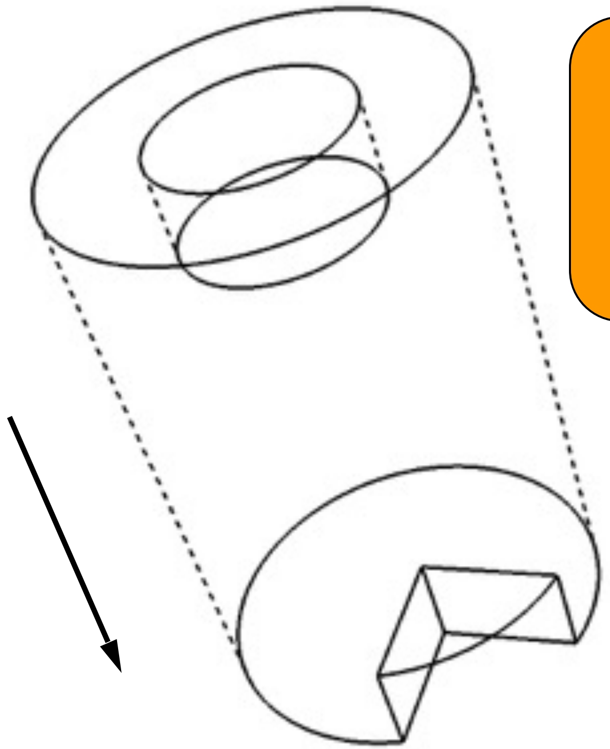
n-to-1 sweepable

Sweeping

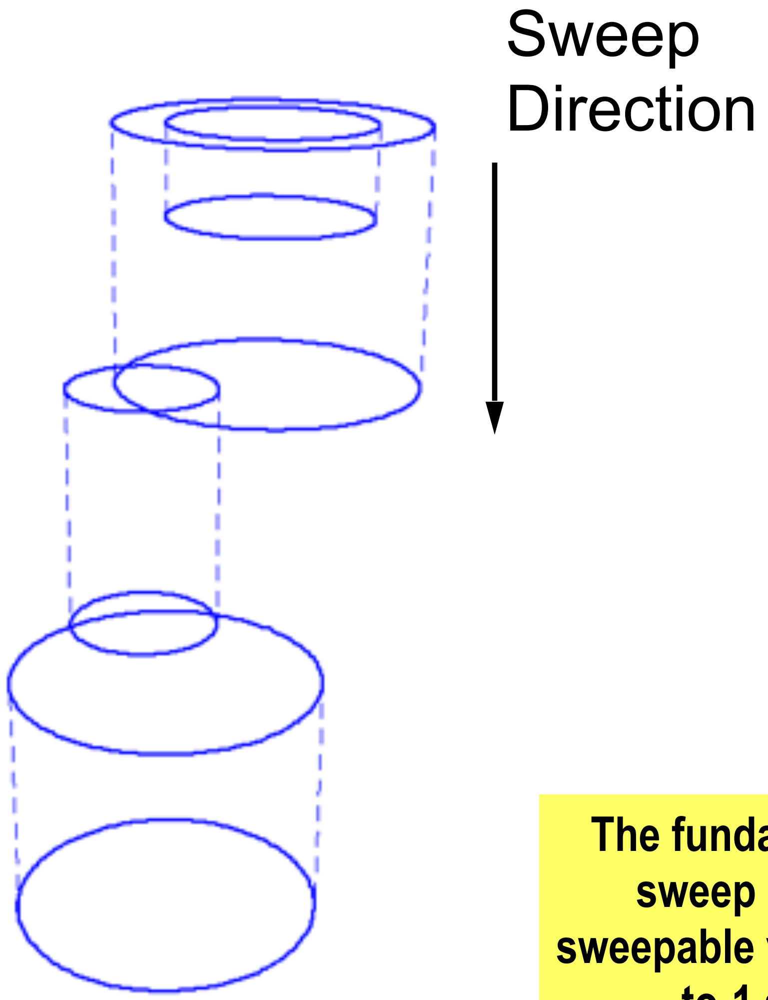
n-to-m sweepable

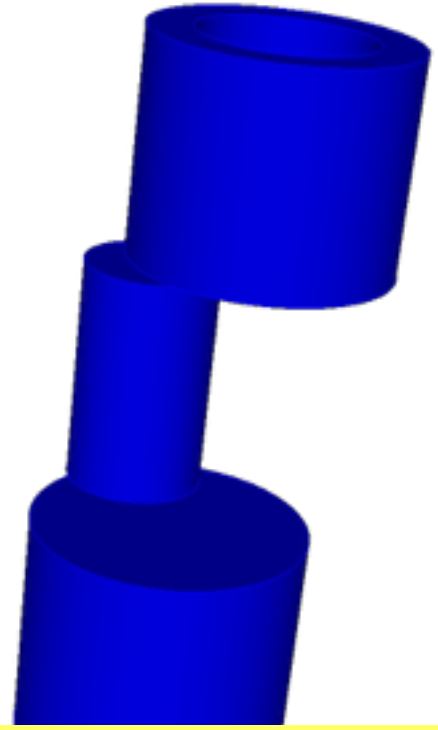
Multi-Sweep

Sweep Direction



Sweeping



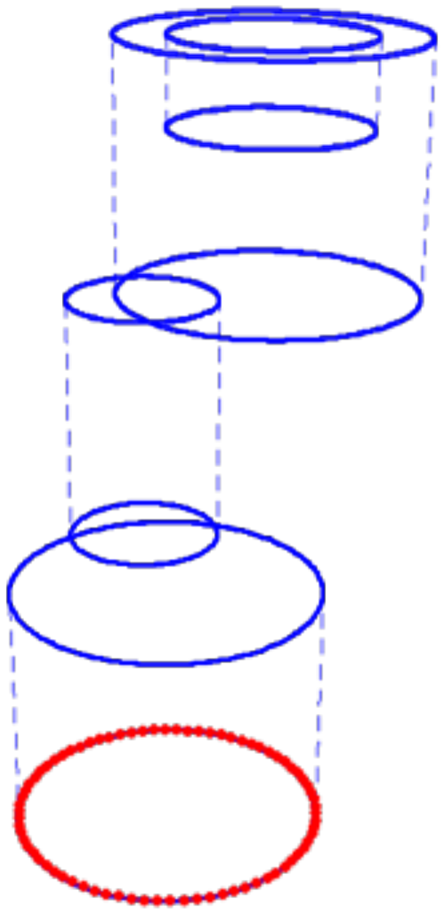


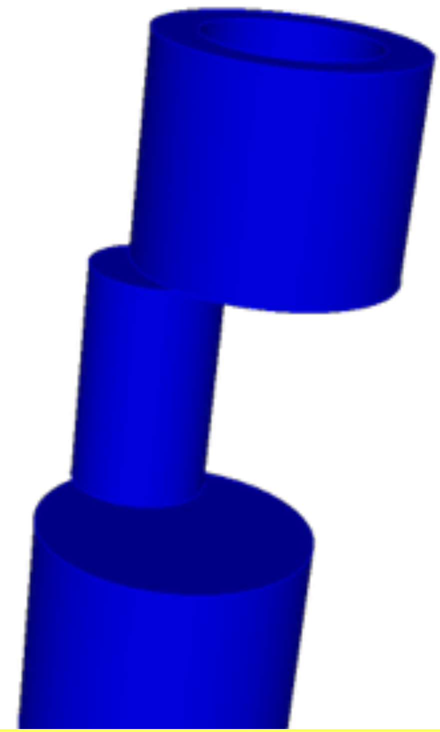
The fundamental strategy of multi-sweep is to convert an n-to-m sweepable volume into a number of n-to-1 sweepable volumes.

Sweeping

Computational Modeling Sciences Department

Sweep Direction





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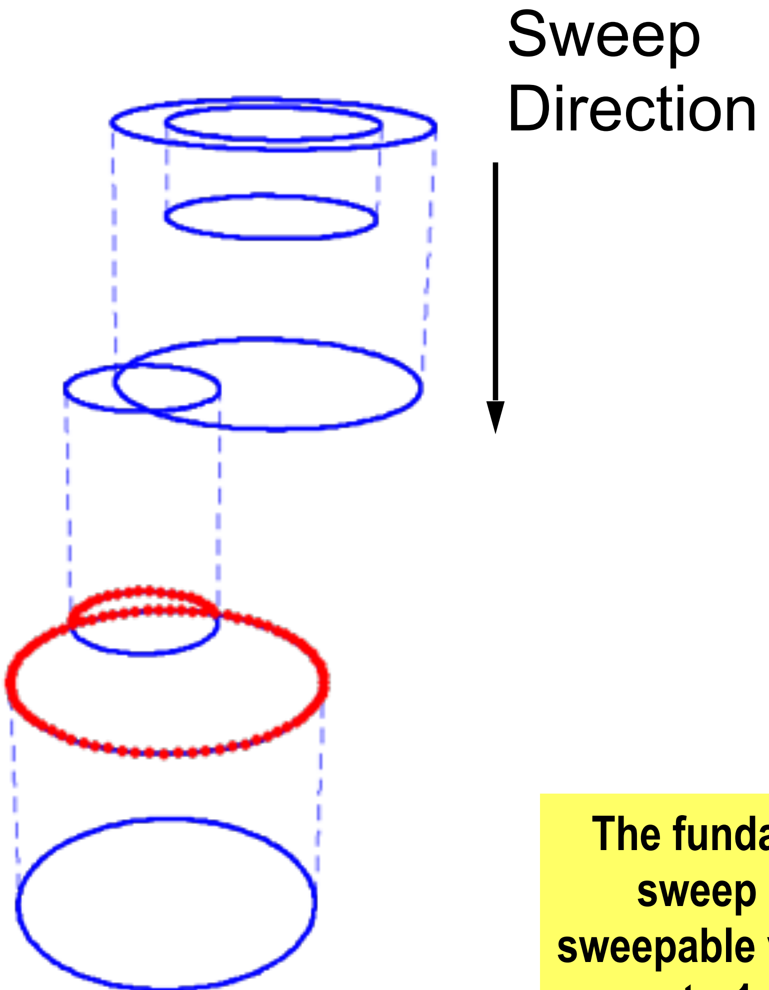
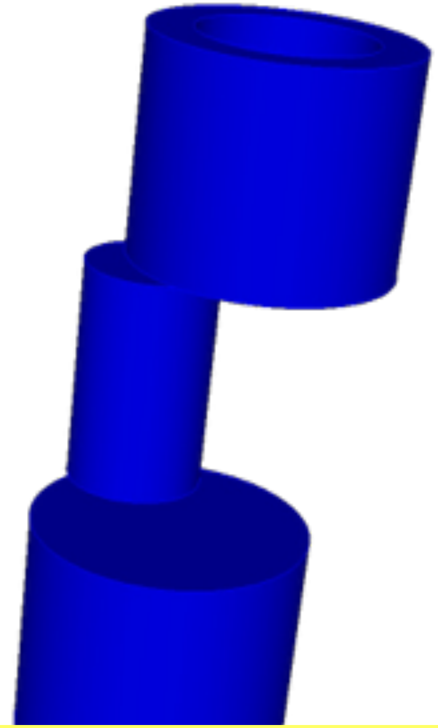
Sweeping

Computational Modeling Sciences Department





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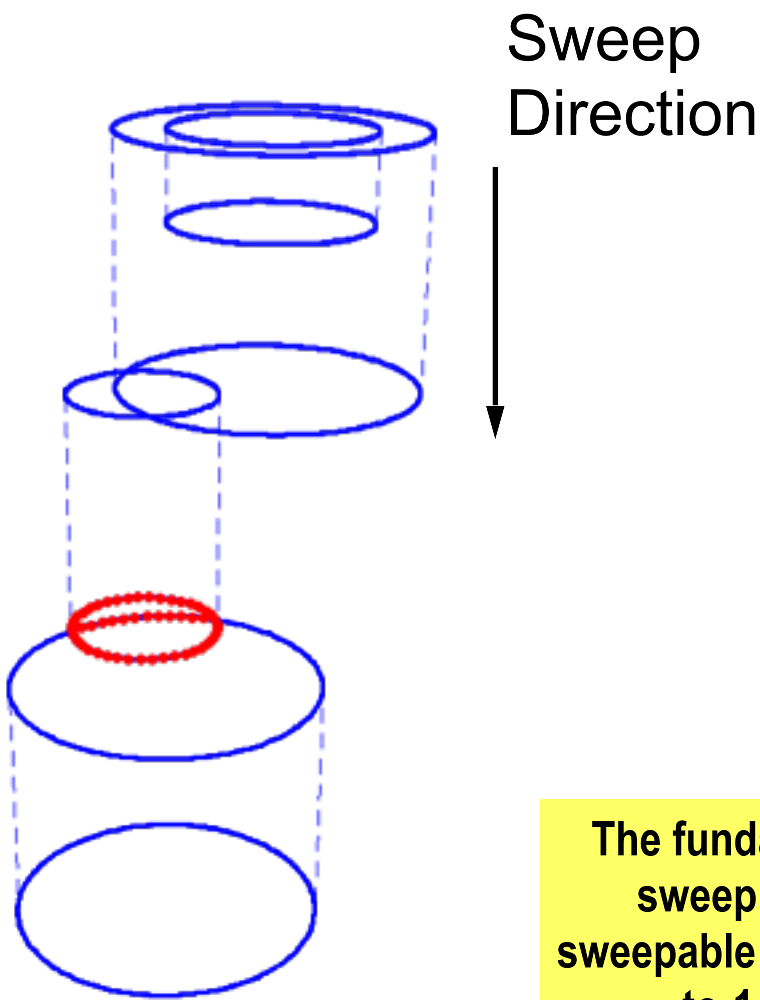
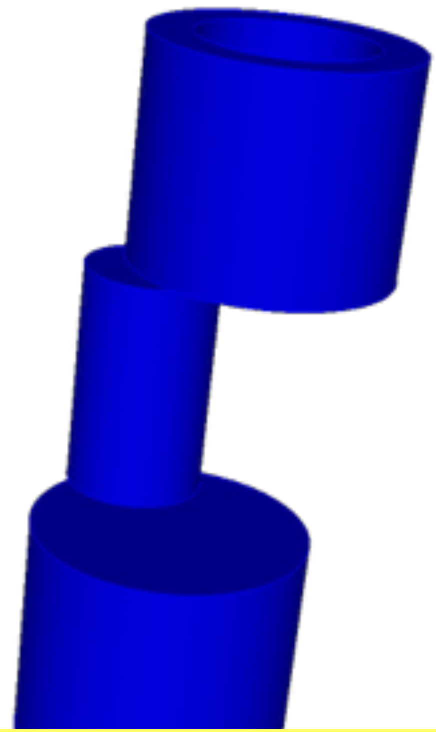
Sweeping

Computational Modeling Sciences Department





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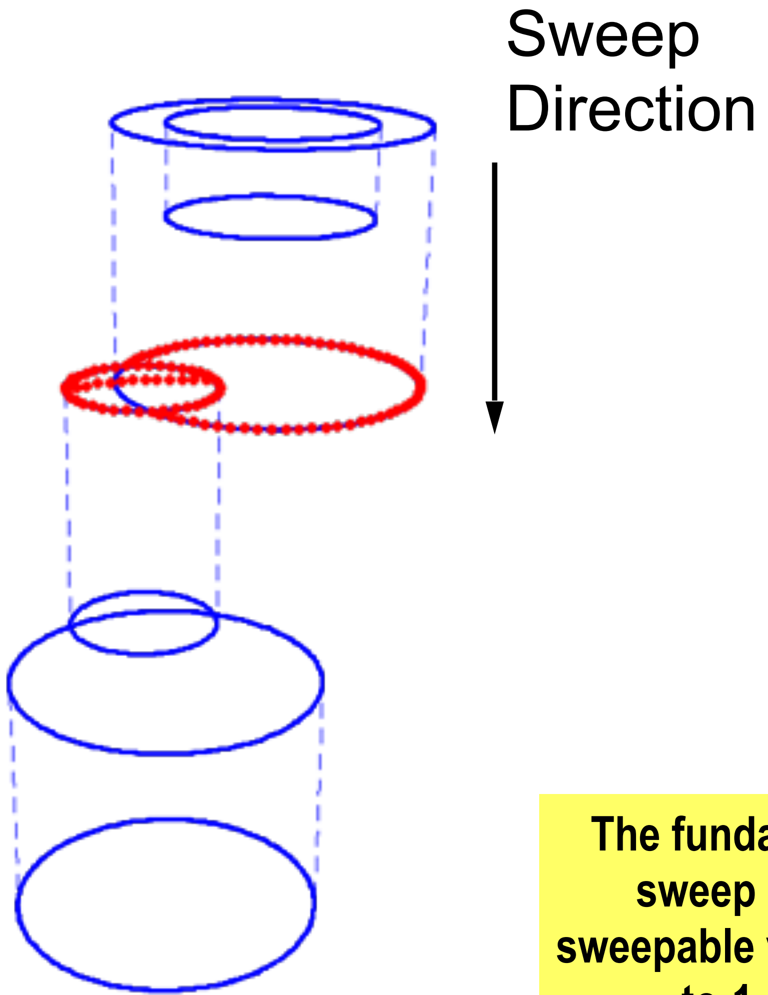
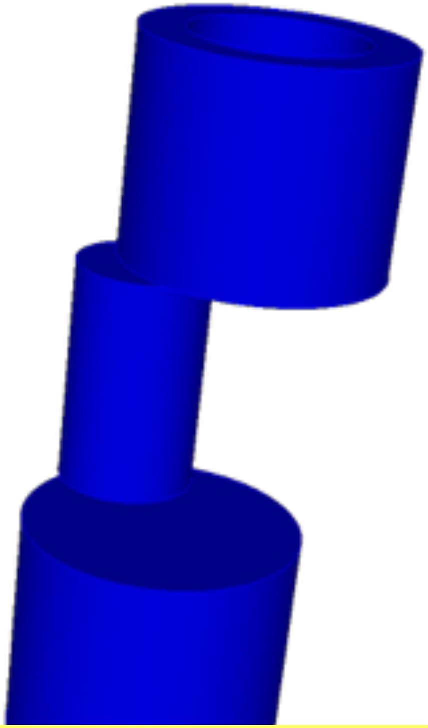
Sweeping

Computational Modeling Sciences Department





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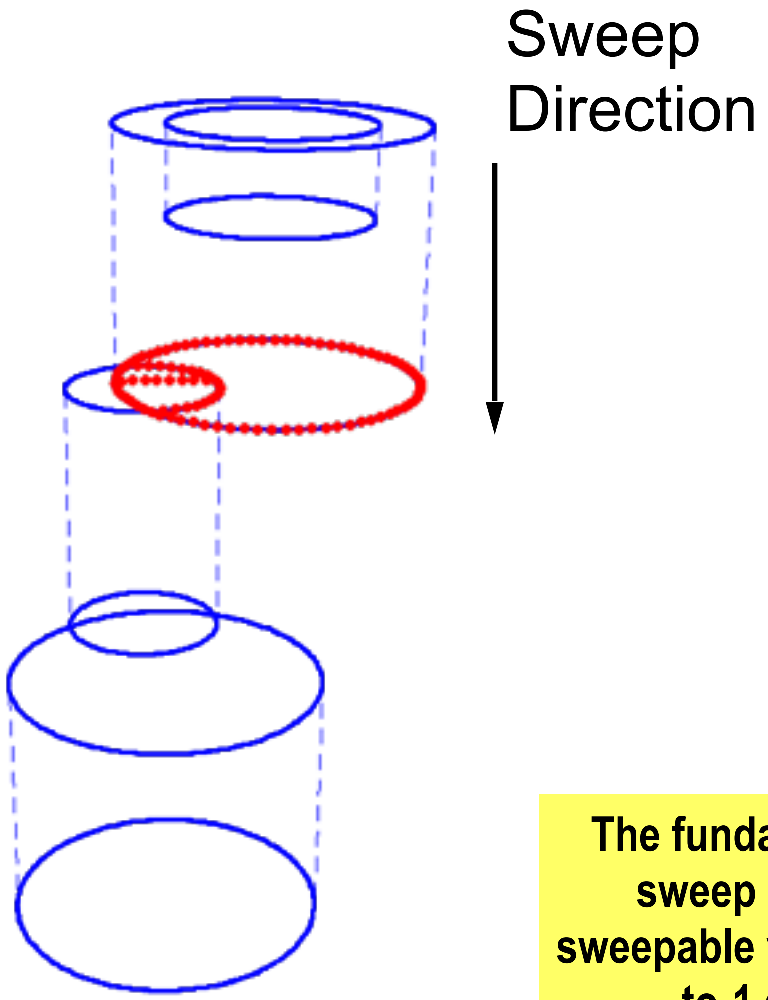

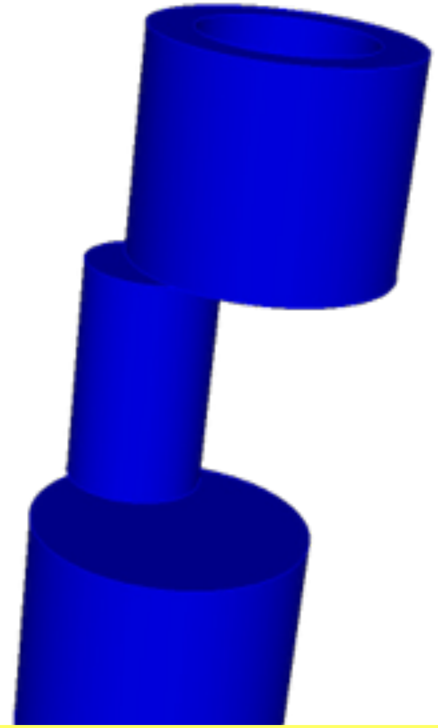

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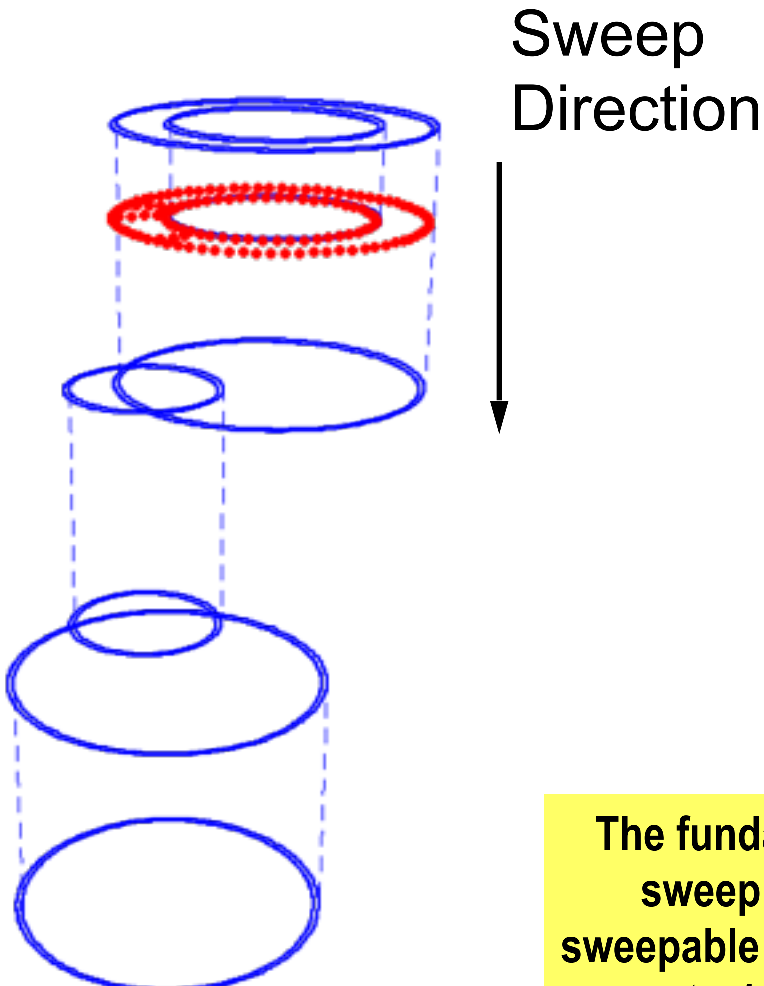
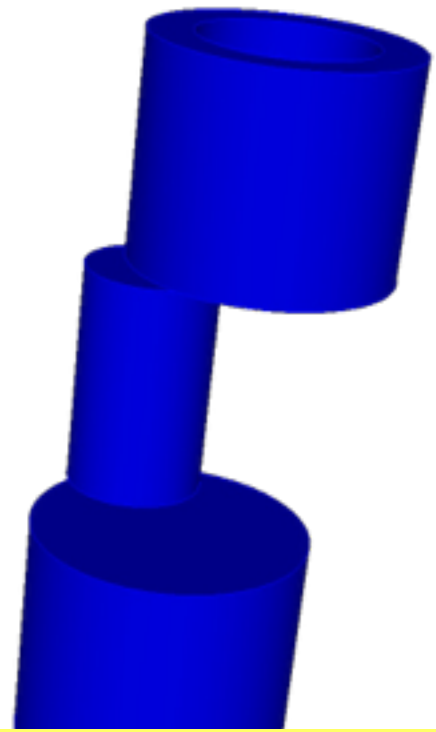
Sweeping





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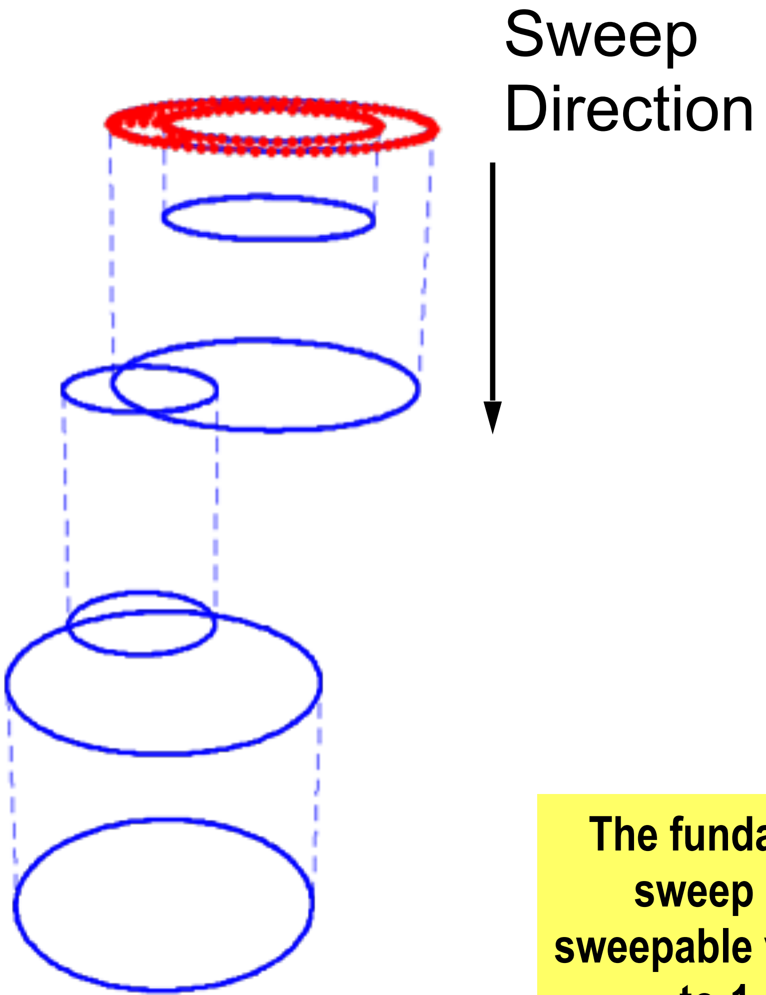
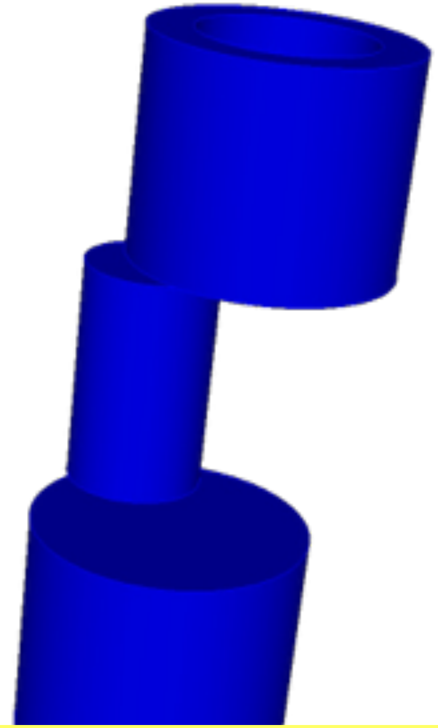
Sweeping

Computational Modeling Sciences Department





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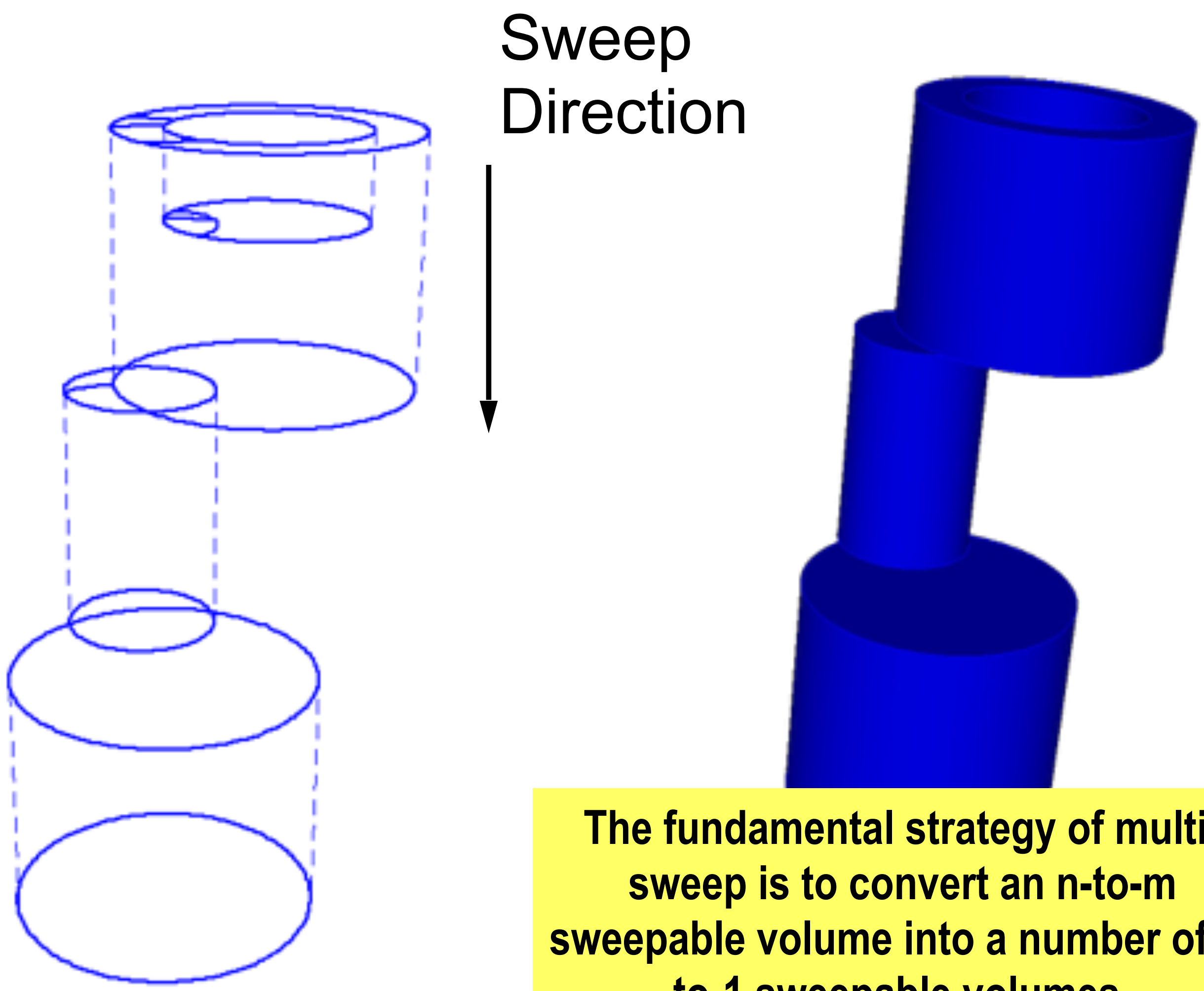
National Nuclear Security Administration

Sandia National Laboratories

Sweeping



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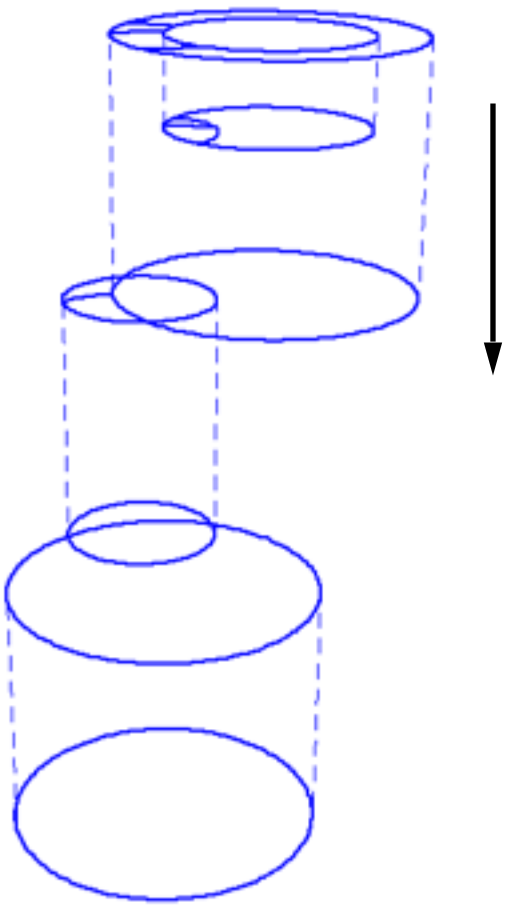
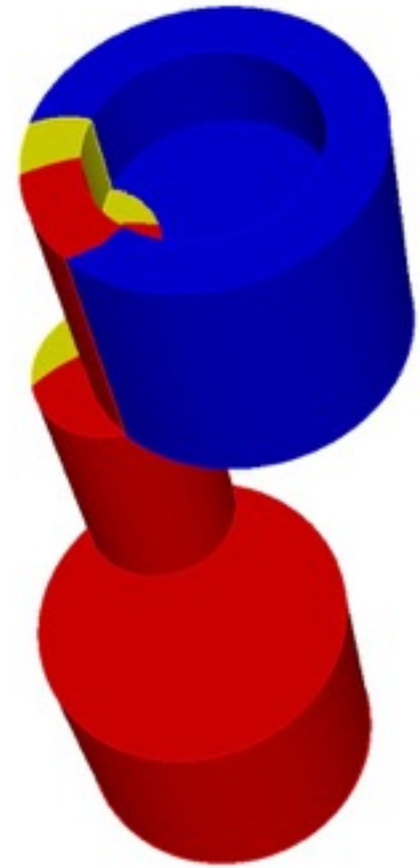
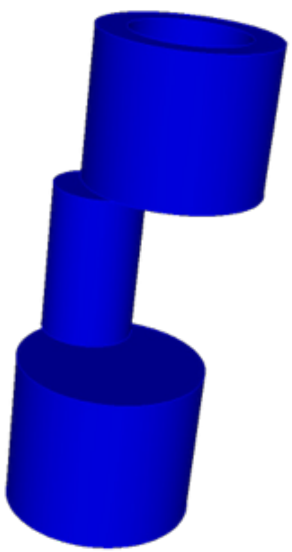
Decomp Sweep Overview

Computational Modeling Sciences Department



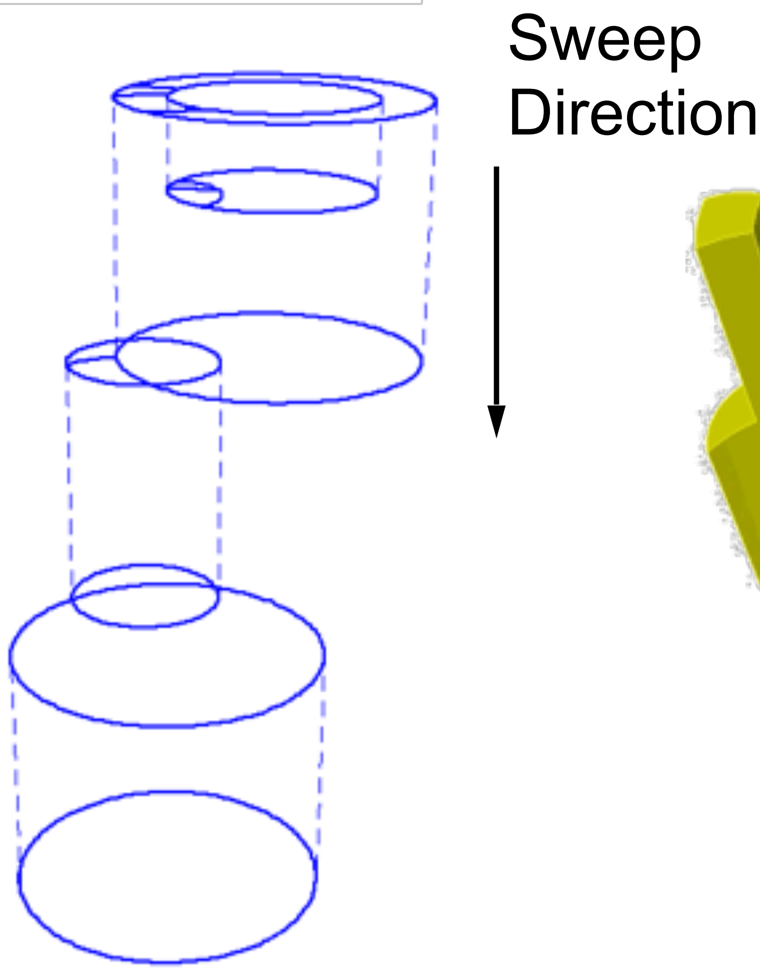
Sweep Direction

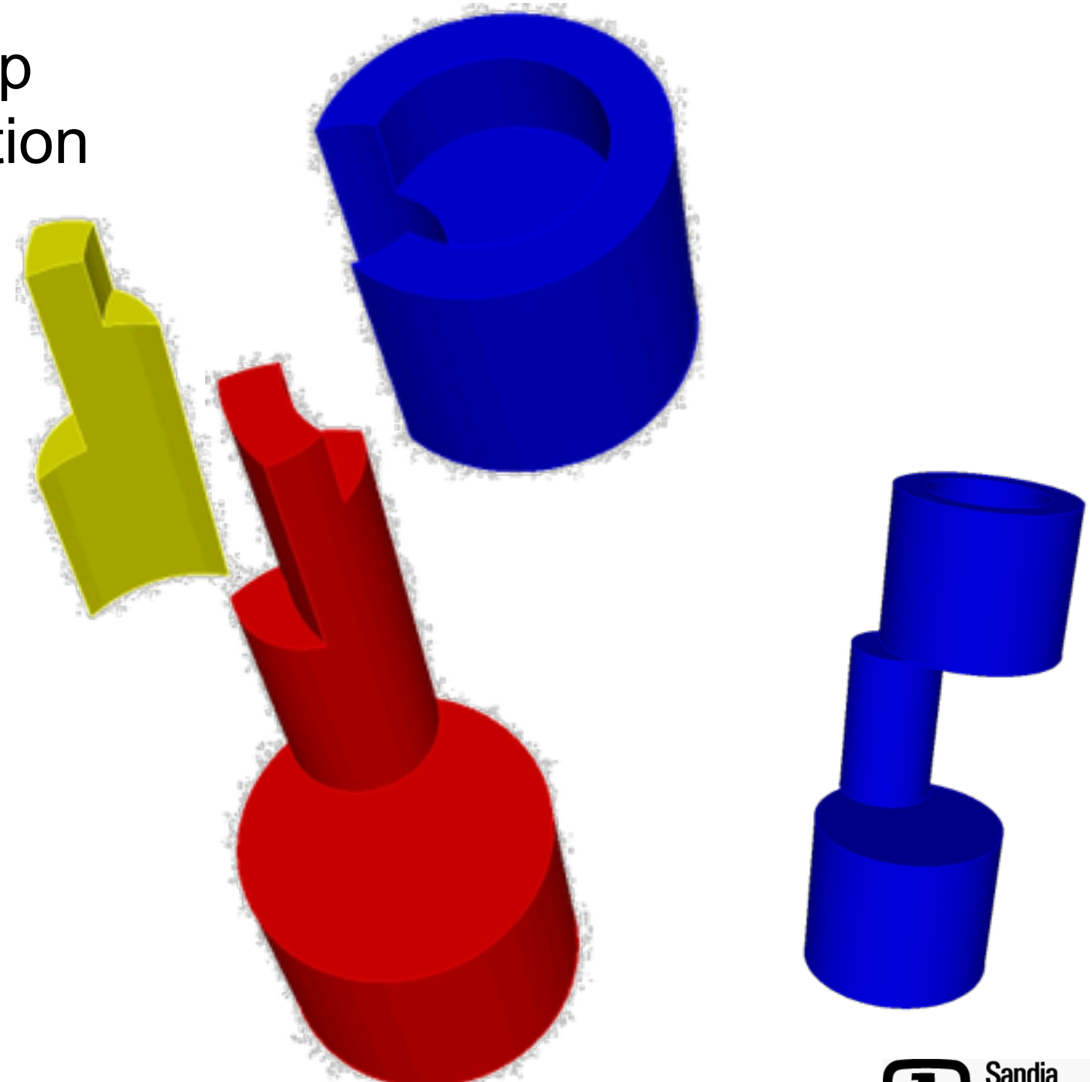




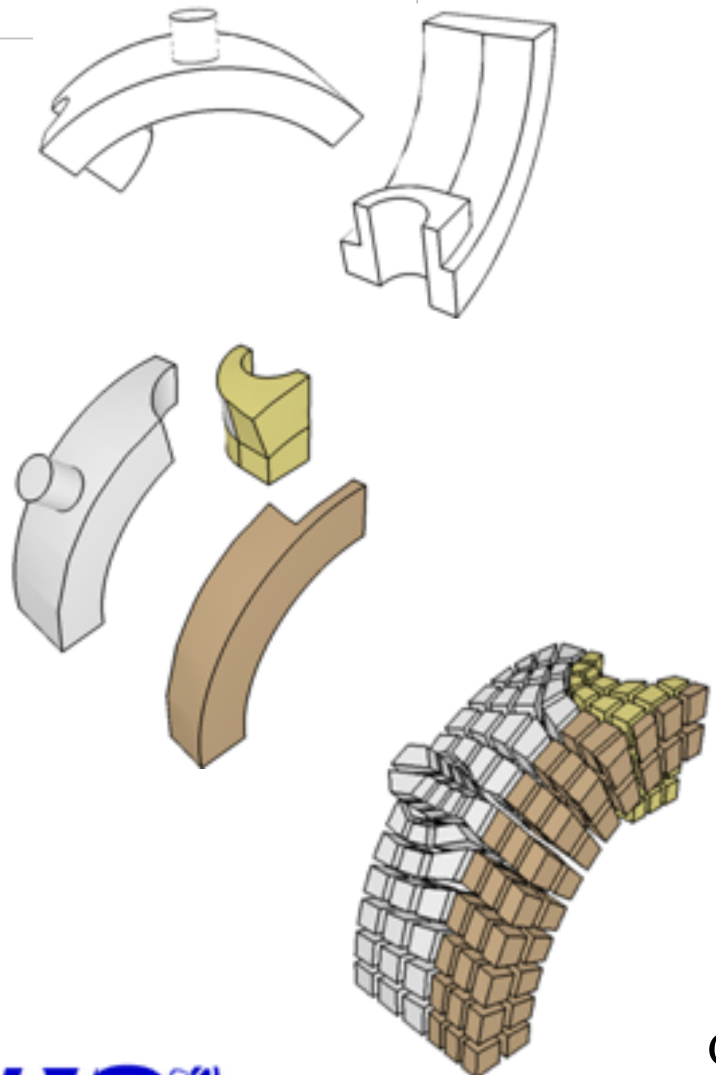
Decomp Sweep Overview

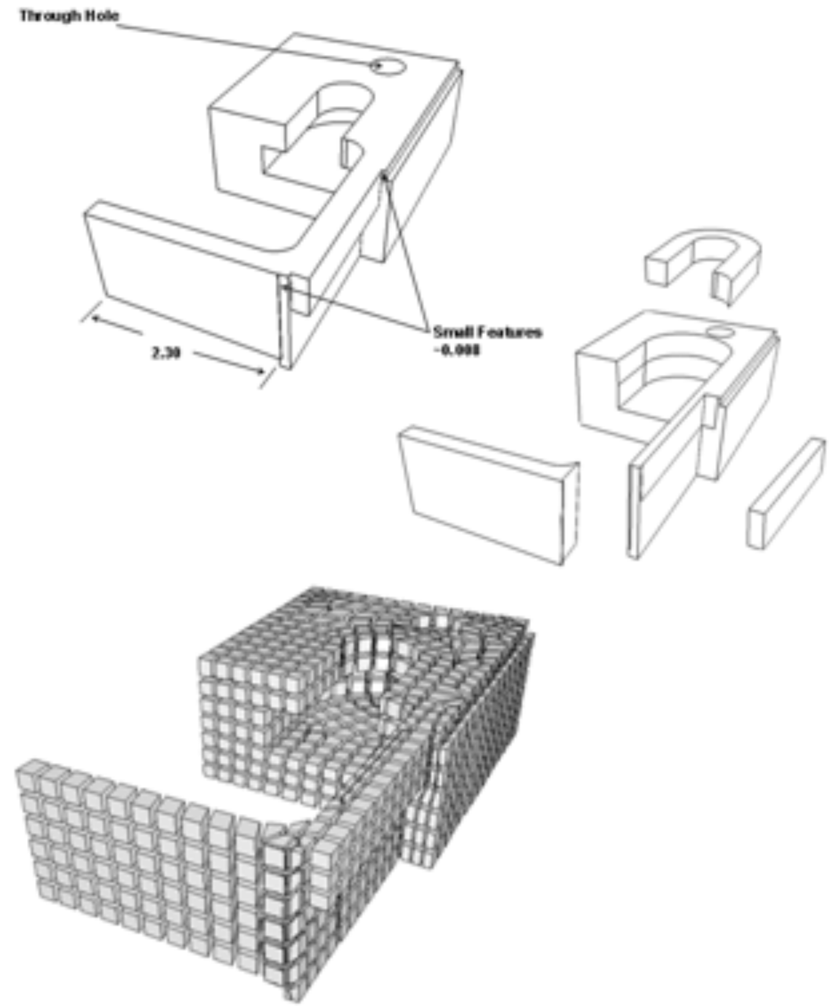
Computational Modeling Sciences Department





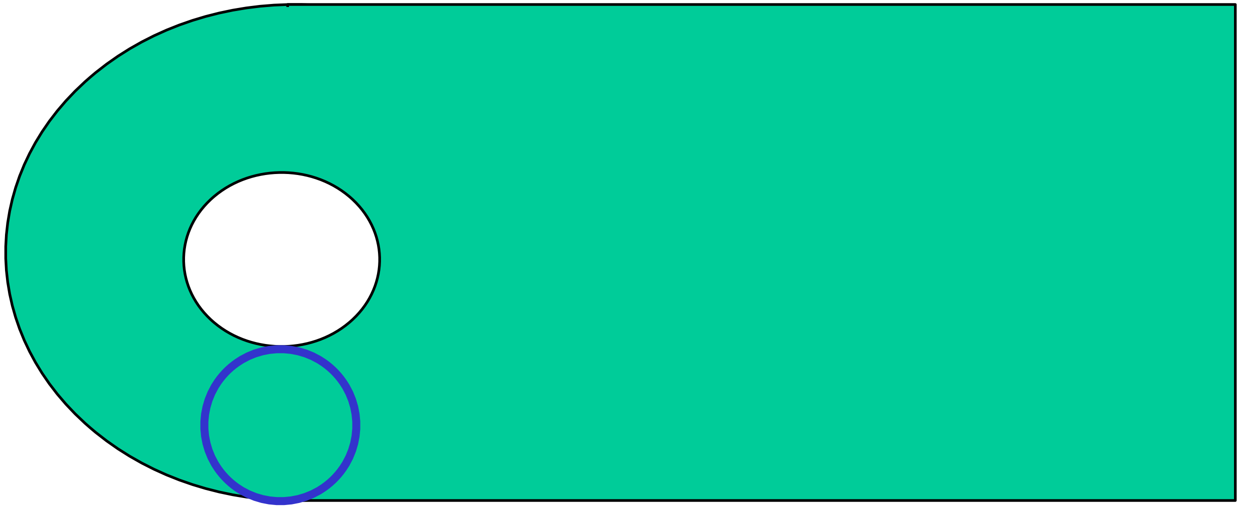
Sweeping





CCSweep
(White, 2004)

Medial Axis



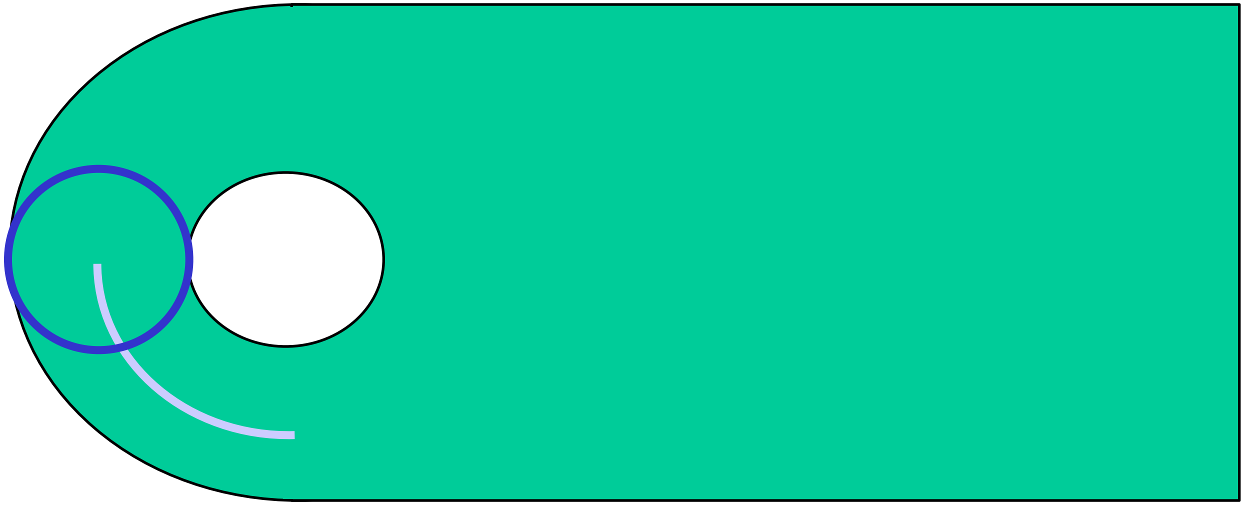
Medial Axis

•Medial Object - Roll a Maximal circle or sphere through the model. The center traces the medial object

•Medial Object used as a tool to automatically decompose model into simpler mapable or sweepable parts

(Price, 95;97)(Tam, 91)

Medial Axis



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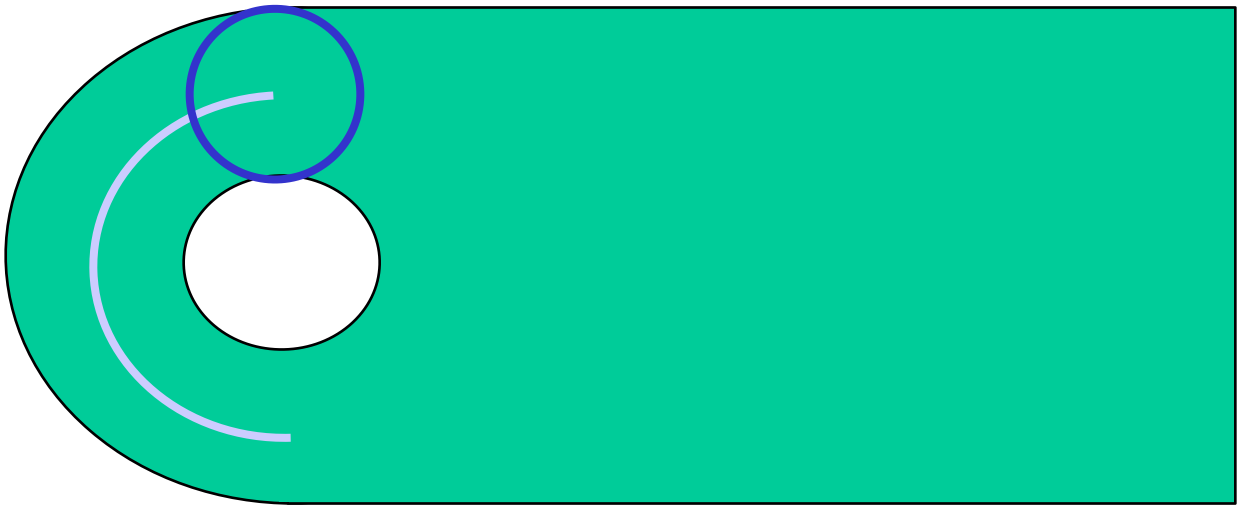

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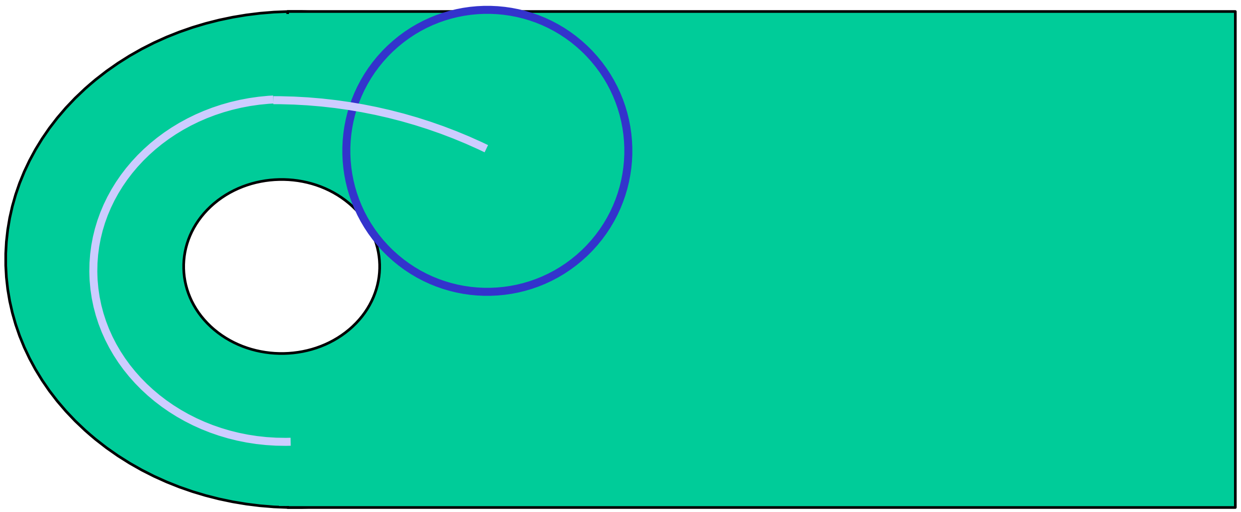

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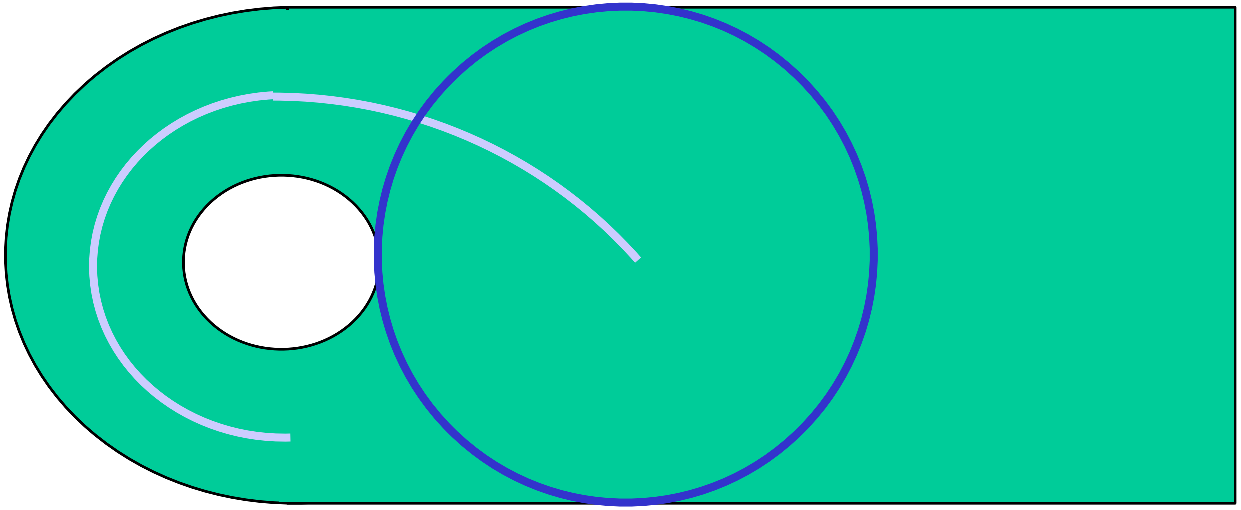

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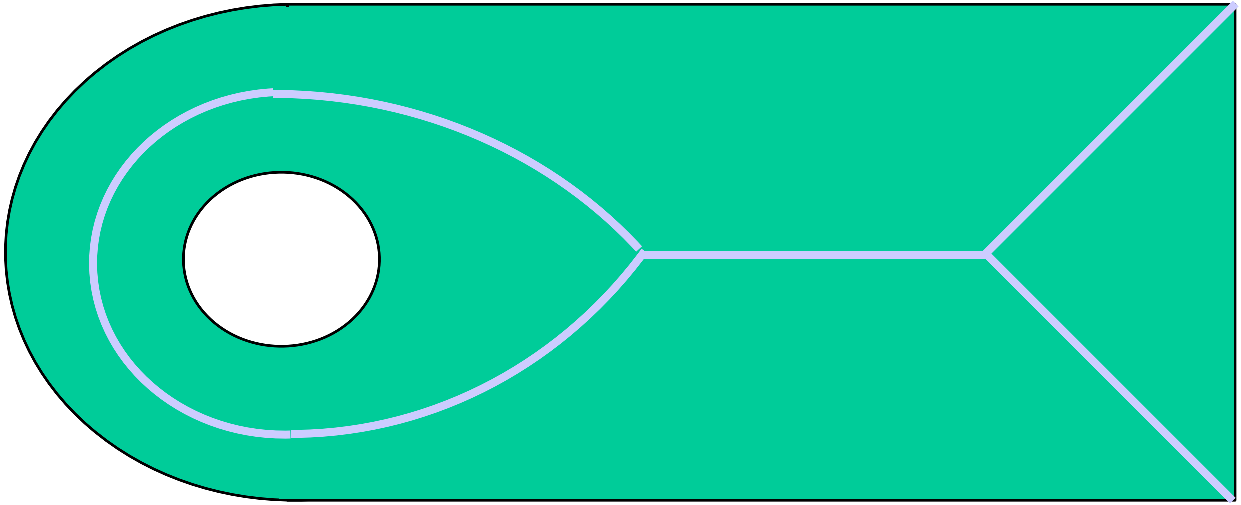

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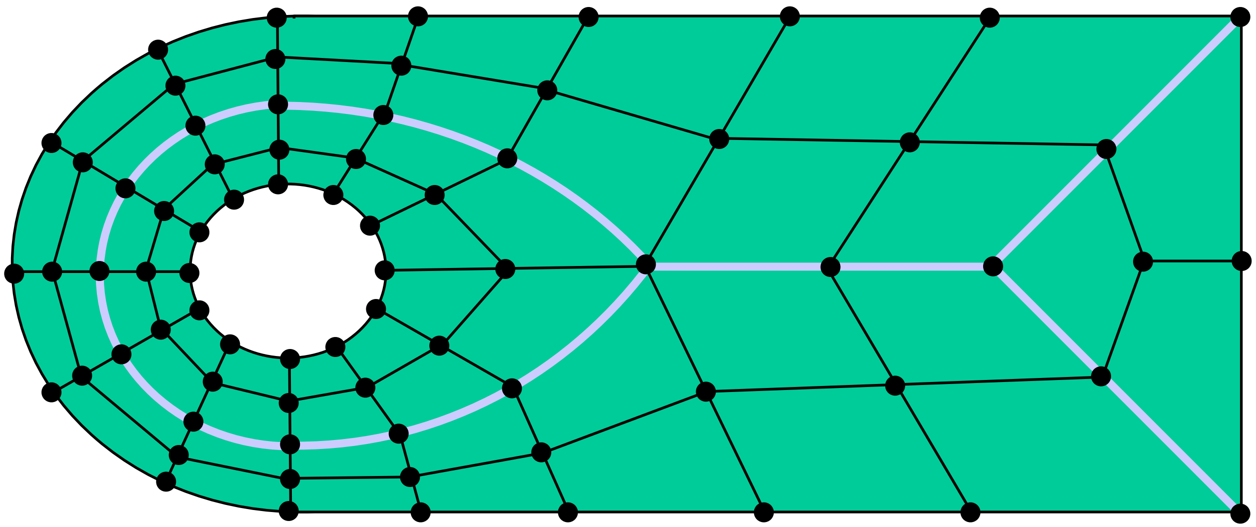

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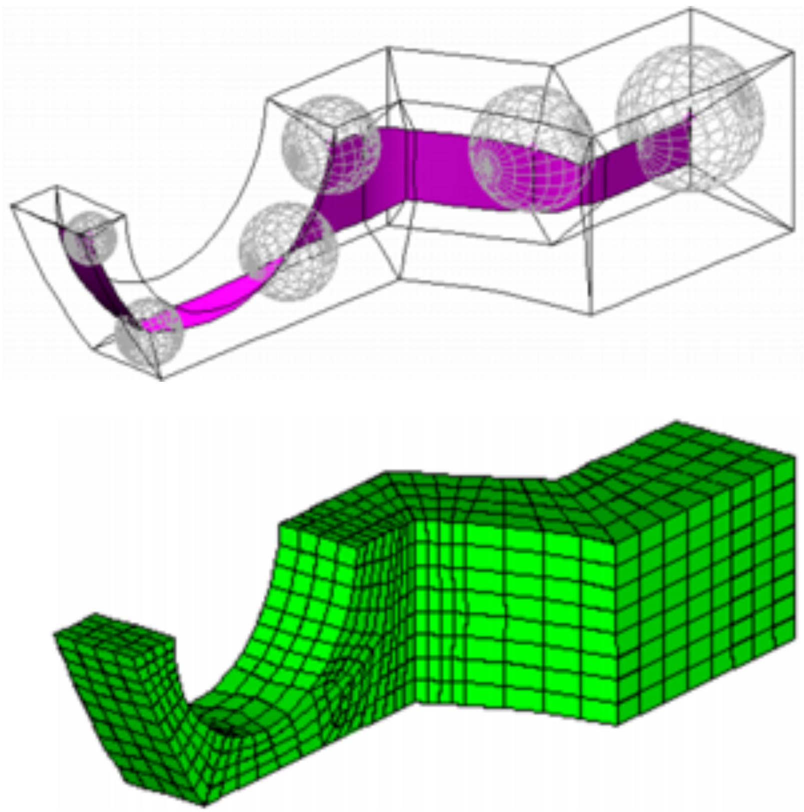
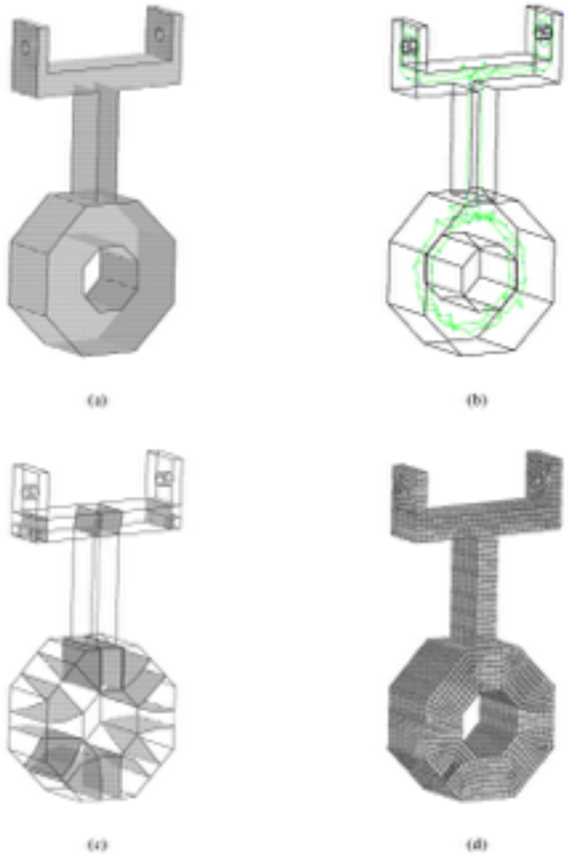
Computational Modeling Sciences Department



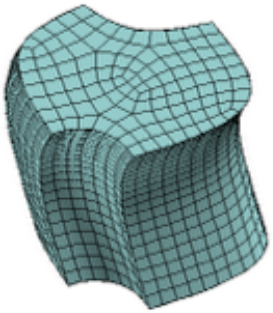
Medial Axis + Midpoint Subdivision
(Price, 95)

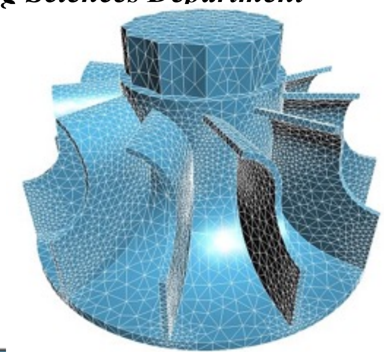


Embedded Voronoi Graph
(Sheffer, 98)

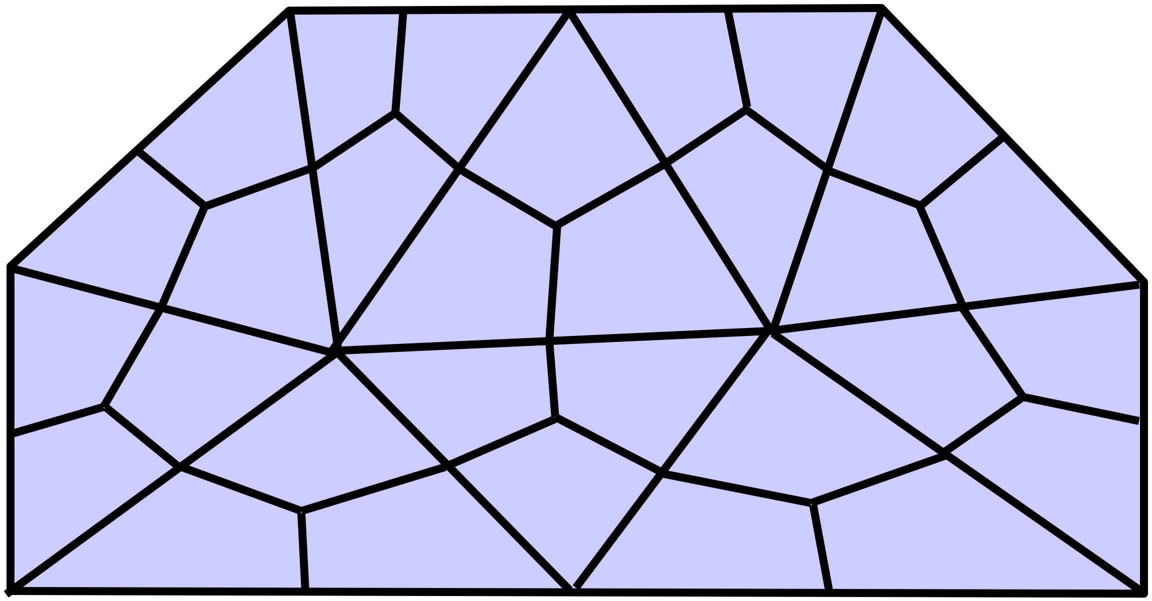
Meshing Algorithms

Computational Modeling Sciences Department





Indirect Quad

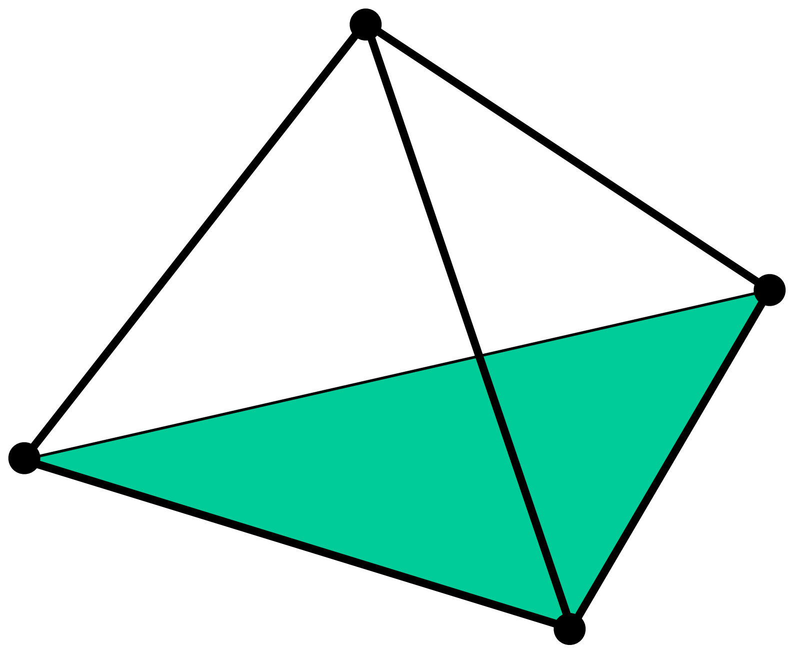


Triangle splitting

•Each triangle split into 3 quads

•Typically results in poor angles

Indirect Hex

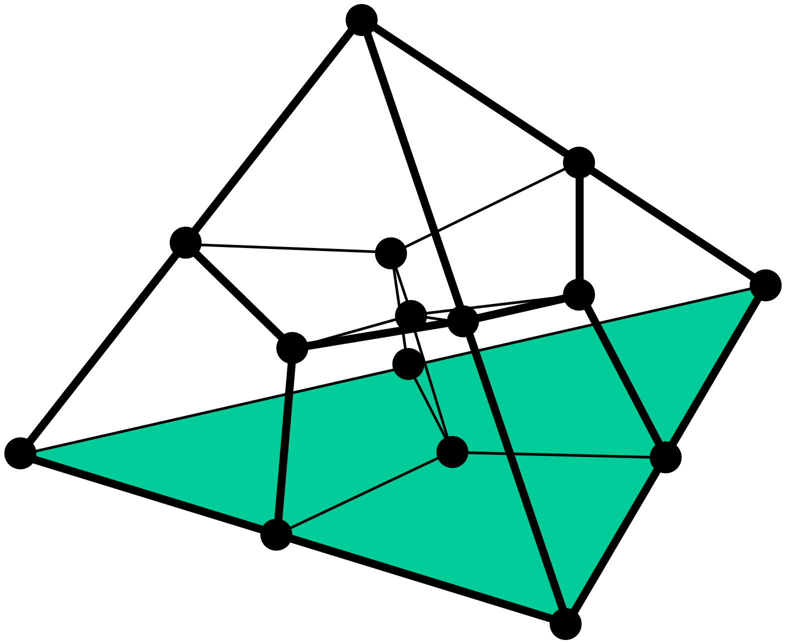


Tetrahedra splitting

•Each tetrahedtra split into 4 hexahedra

•Typically results in poor angles

Indirect Hex



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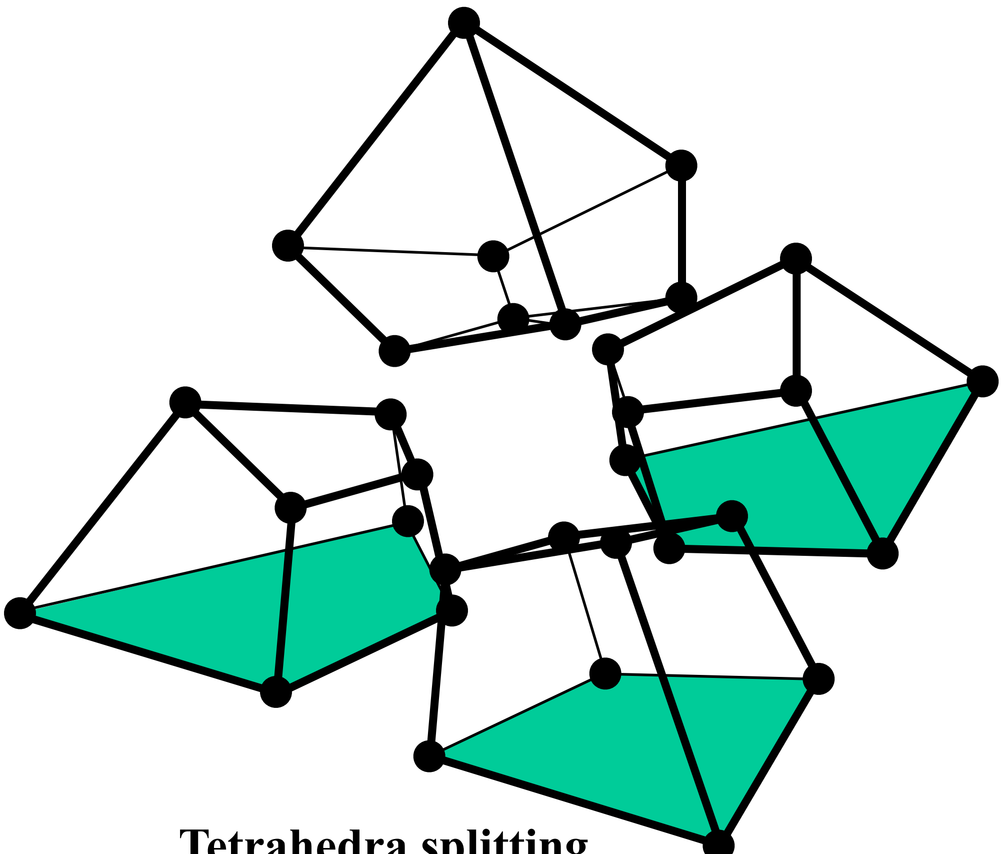

Tetrahedra splitting
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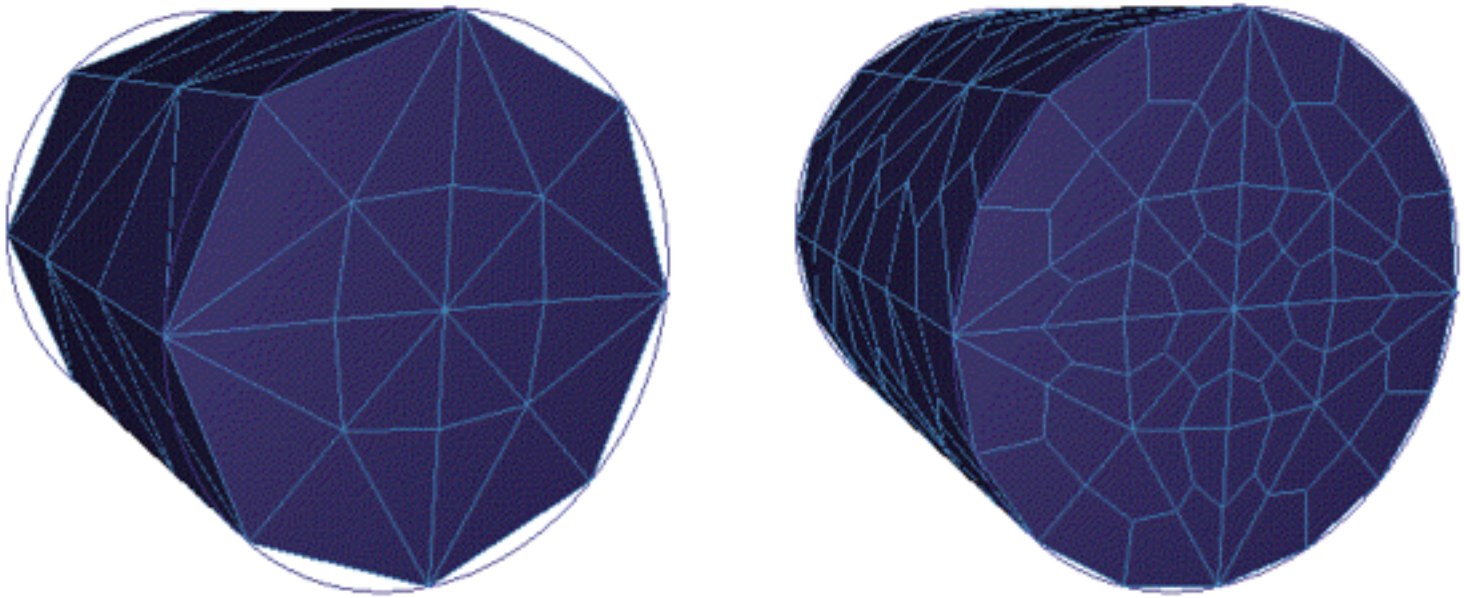
•Typically results in poor angles

(Taniguchi, 96)

Indirect Hex

Computational Modeling Sciences Department

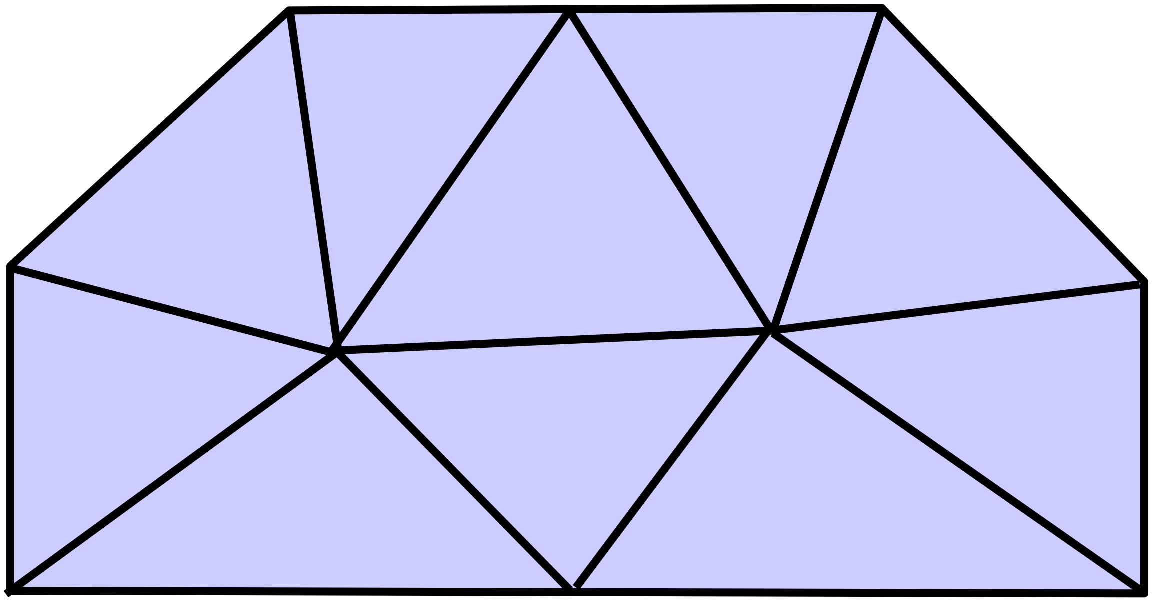


Example of geometry meshed by tetrahedra splitting

National Nuclear Security Administration

Sandia National Laboratories

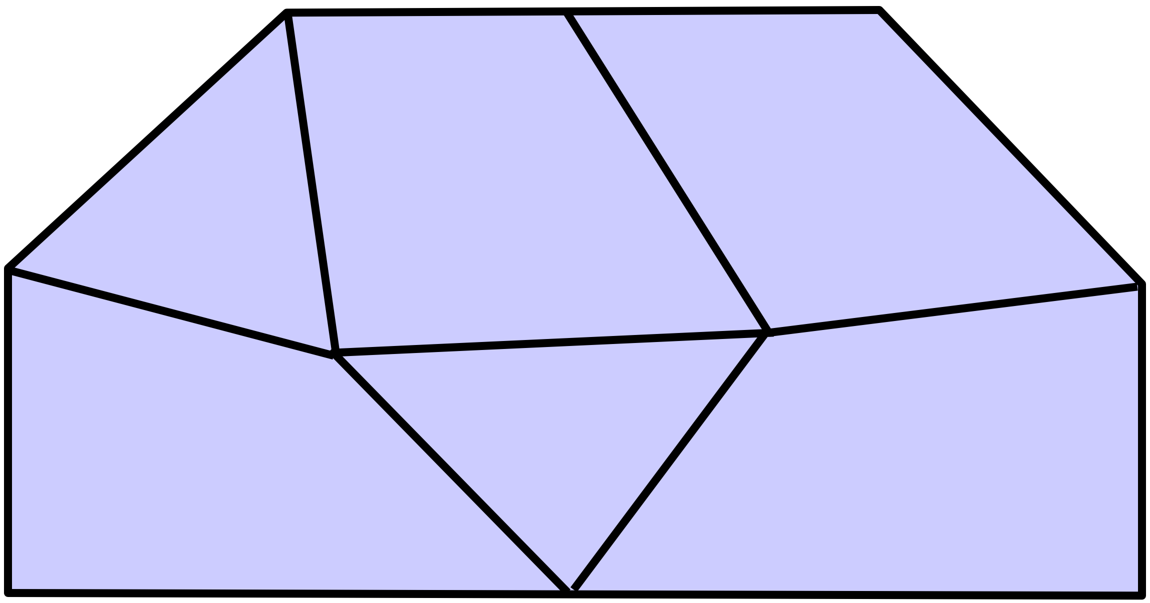
Indirect



Triangle Merge

- Two adjacent triangles combined into a single quad
- Test for best local choice for combination
- Triangles can remain if attention is not paid to order of combination

Indirect



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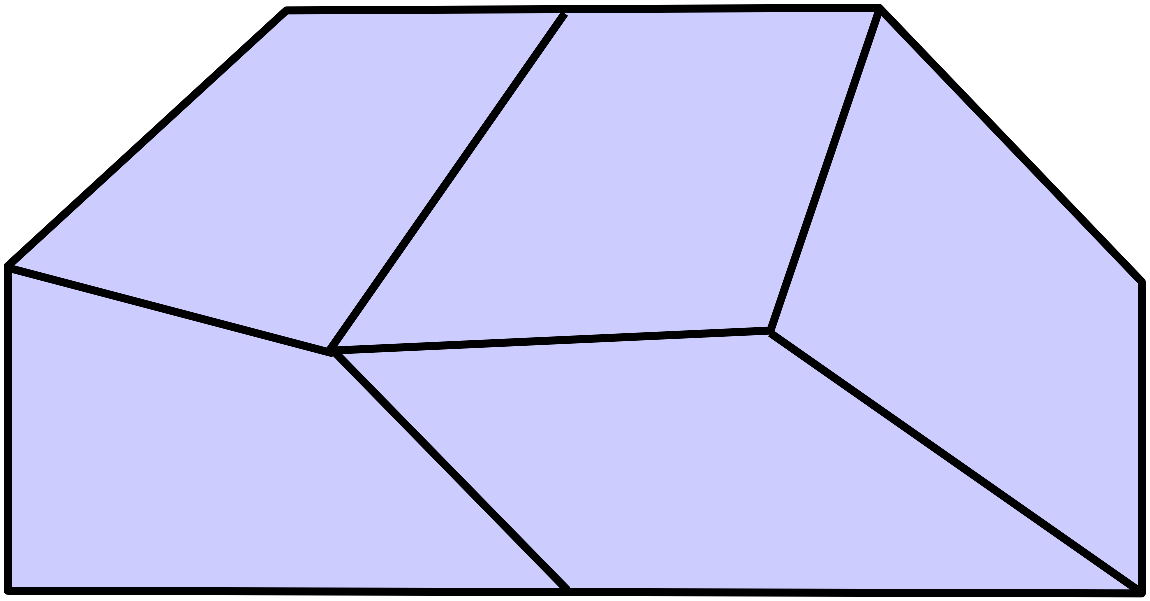


Triangle Merge

•Two adjacent triangles combined into a single quad

•Test for best local choice for combination

•Triangles can remain if attention is not paid to order of combination

Indirect

Directed Triangle Merge

•Merging begins at a boundary

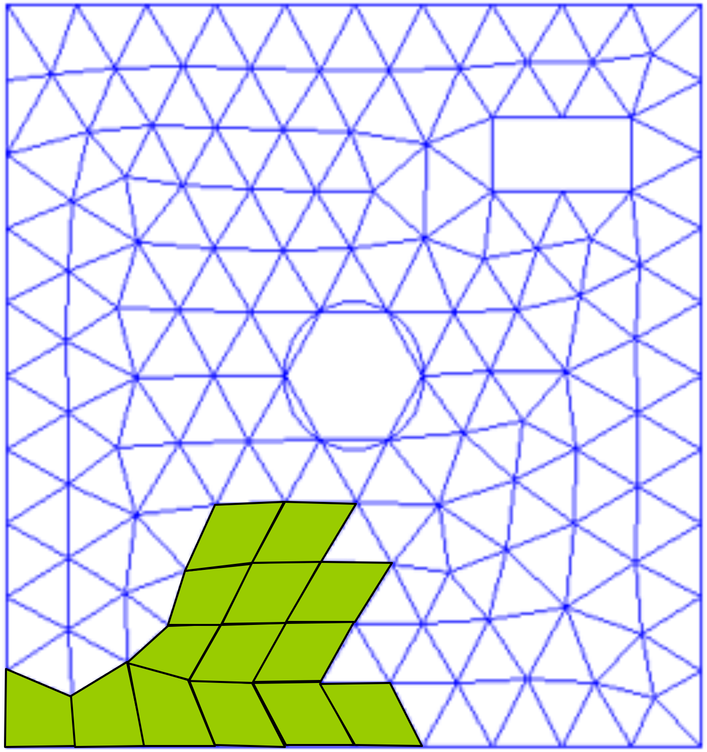
•Advances from one set of triangles to the next

•Attempts to maintain even number of intervals on any loop

•Can produce all-quad mesh

•Can also incorporate triangle splitting

•(Lee and Lo, 94)

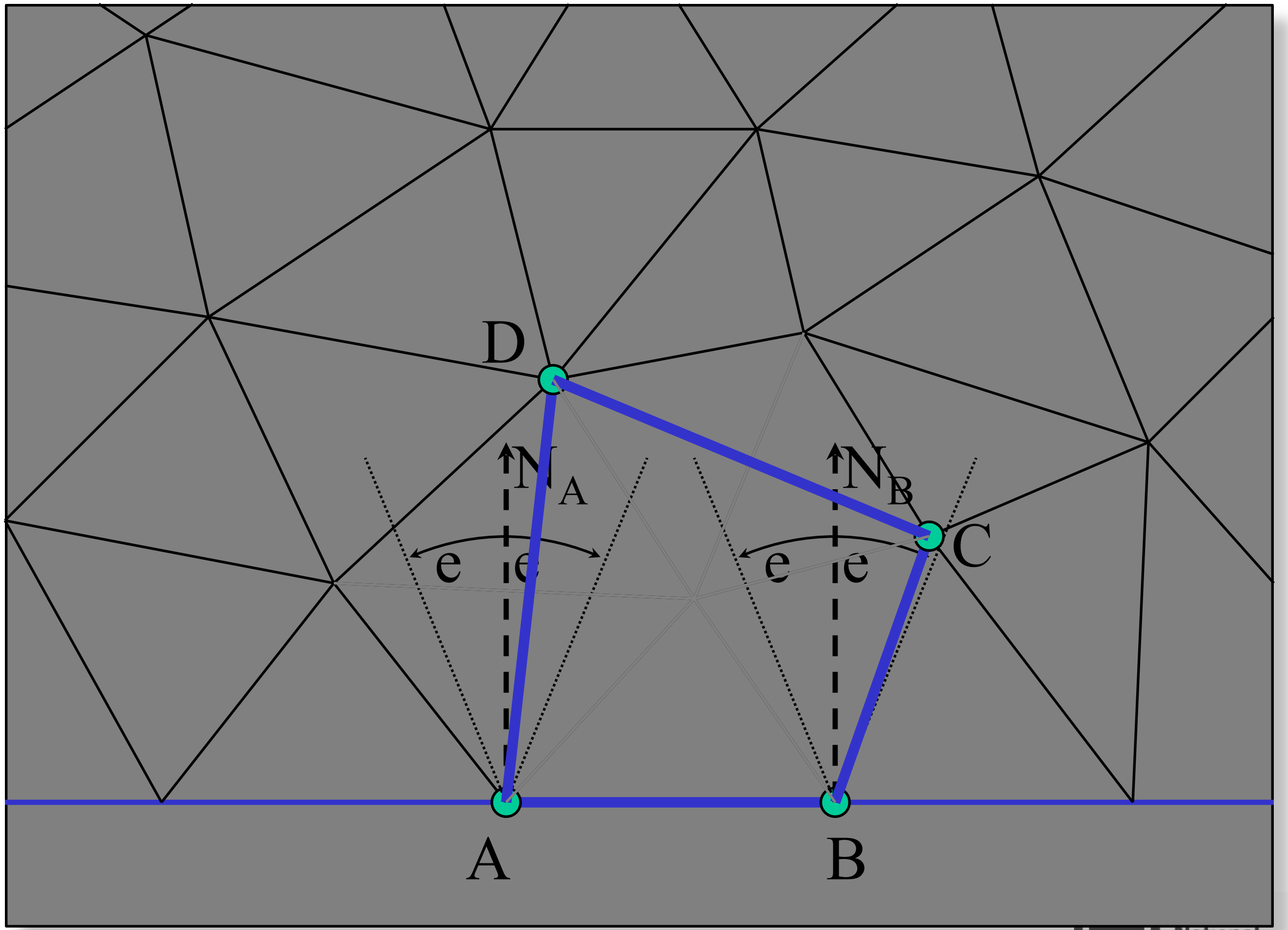


Indirect

Computational Modeling Sciences Department

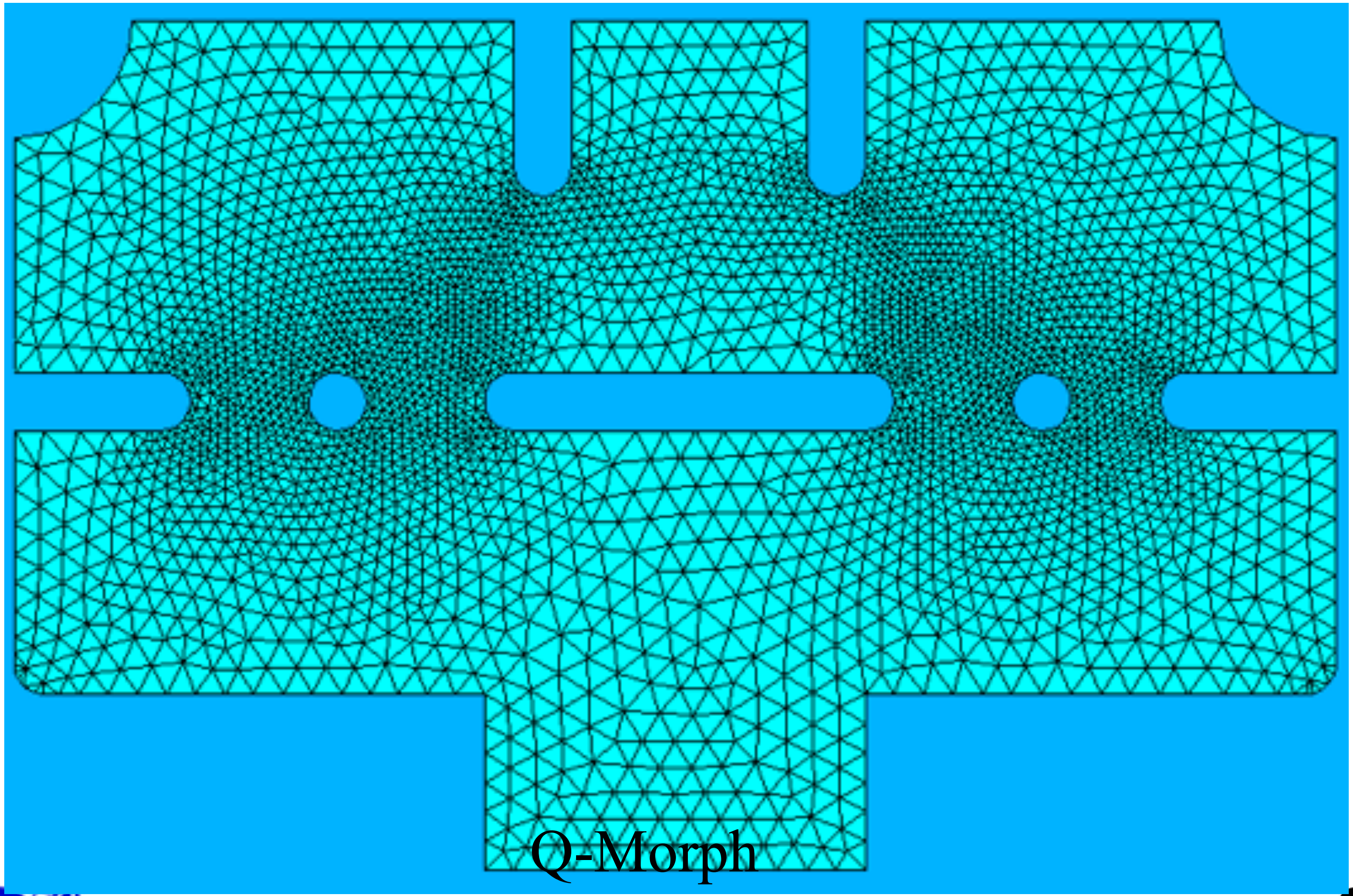
Triangle Merge w/ local transformations ("Q-Morph)

•Uses an advancing front approach
•Local swaps applied to improve resulting quad
•Any number of triangles merged to create a quad
•Attempts to maintain even number of intervals on any loop
•Produces all-quad mesh from even intervals
•(Owen, 99)



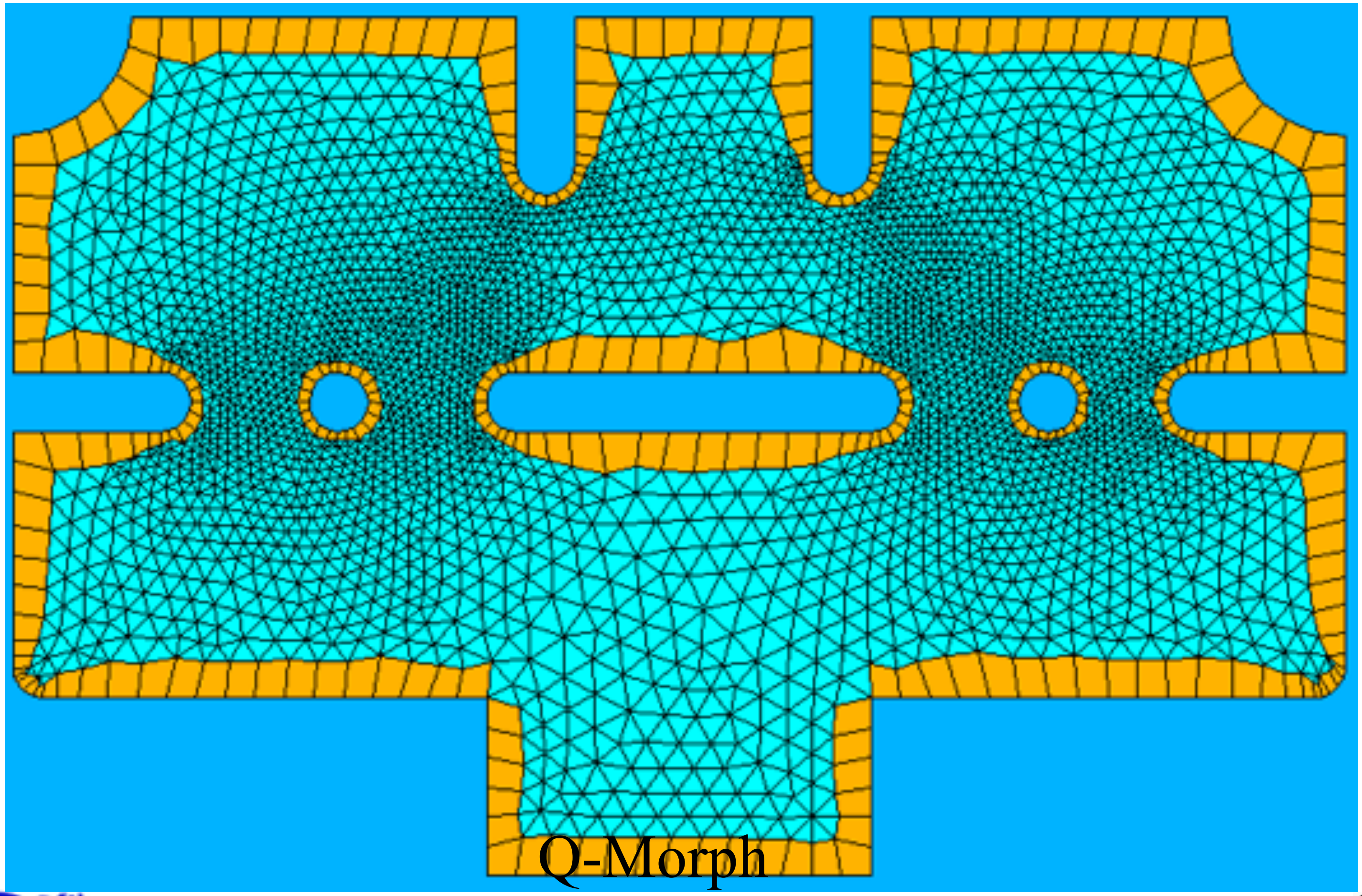
Unstructured-Quad

Computational Modeling Sciences Department



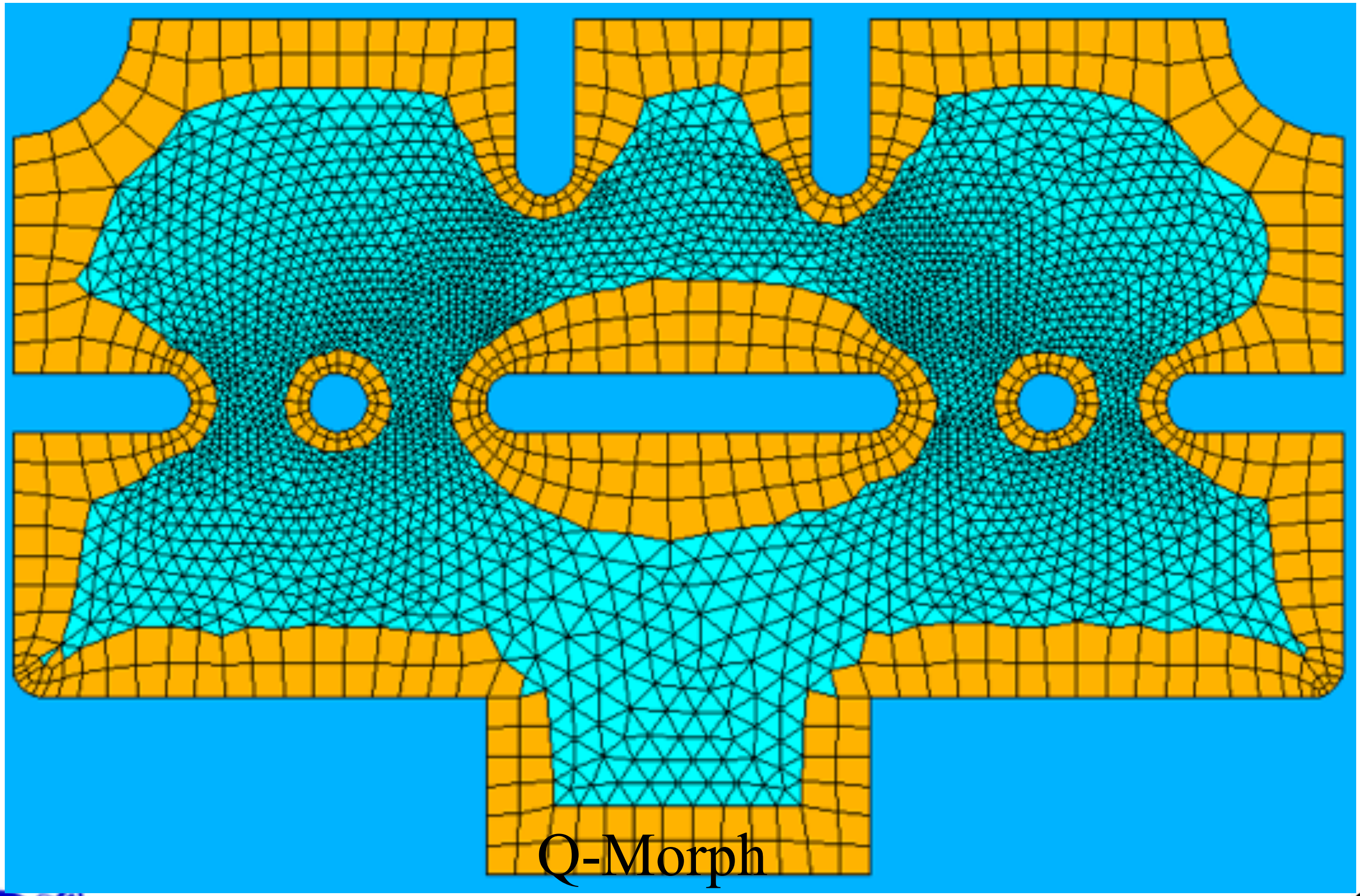
Unstructured-Quad

Computational Modeling Sciences Department

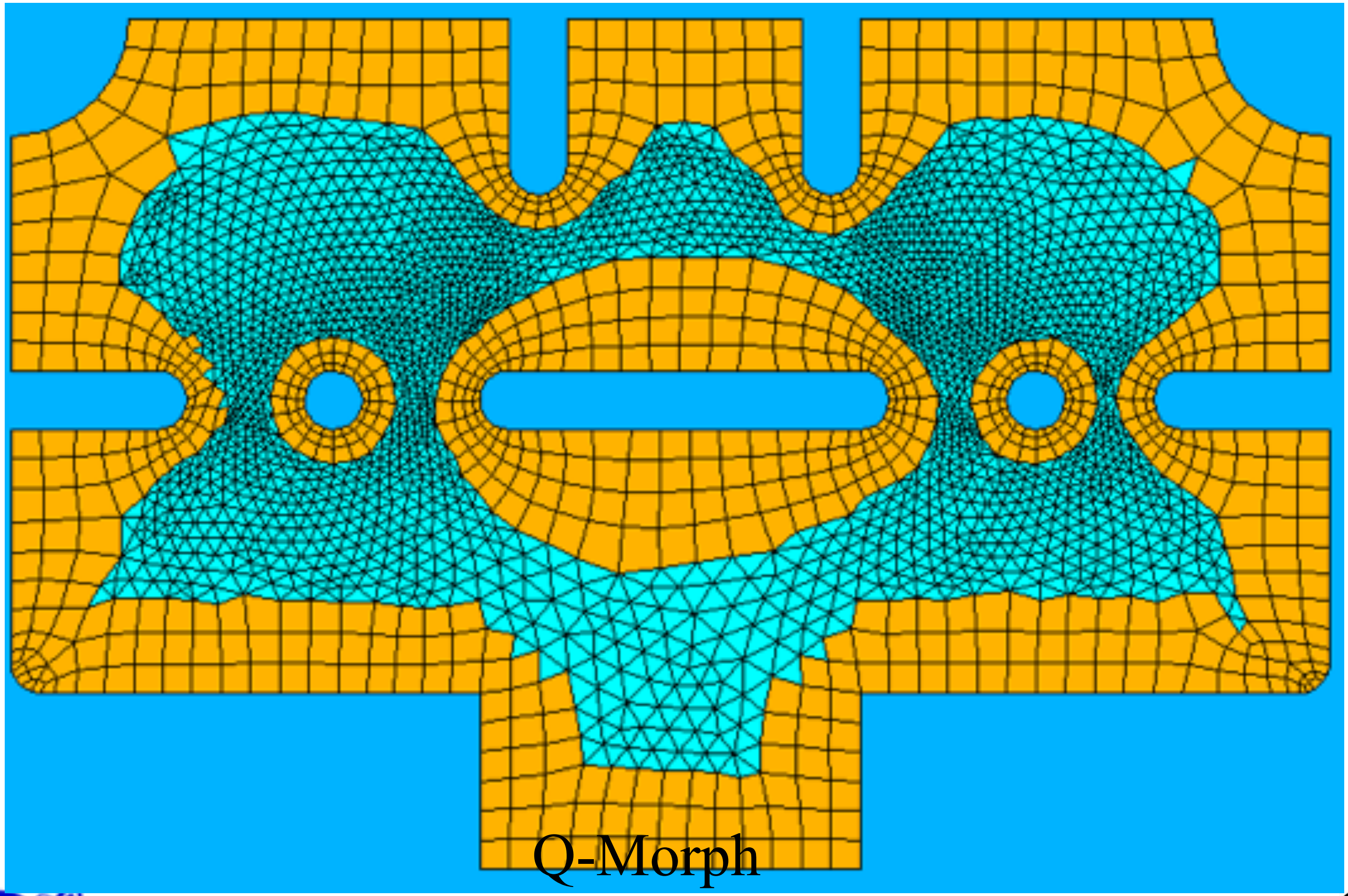


Q-Morph

Unstructured-Quad

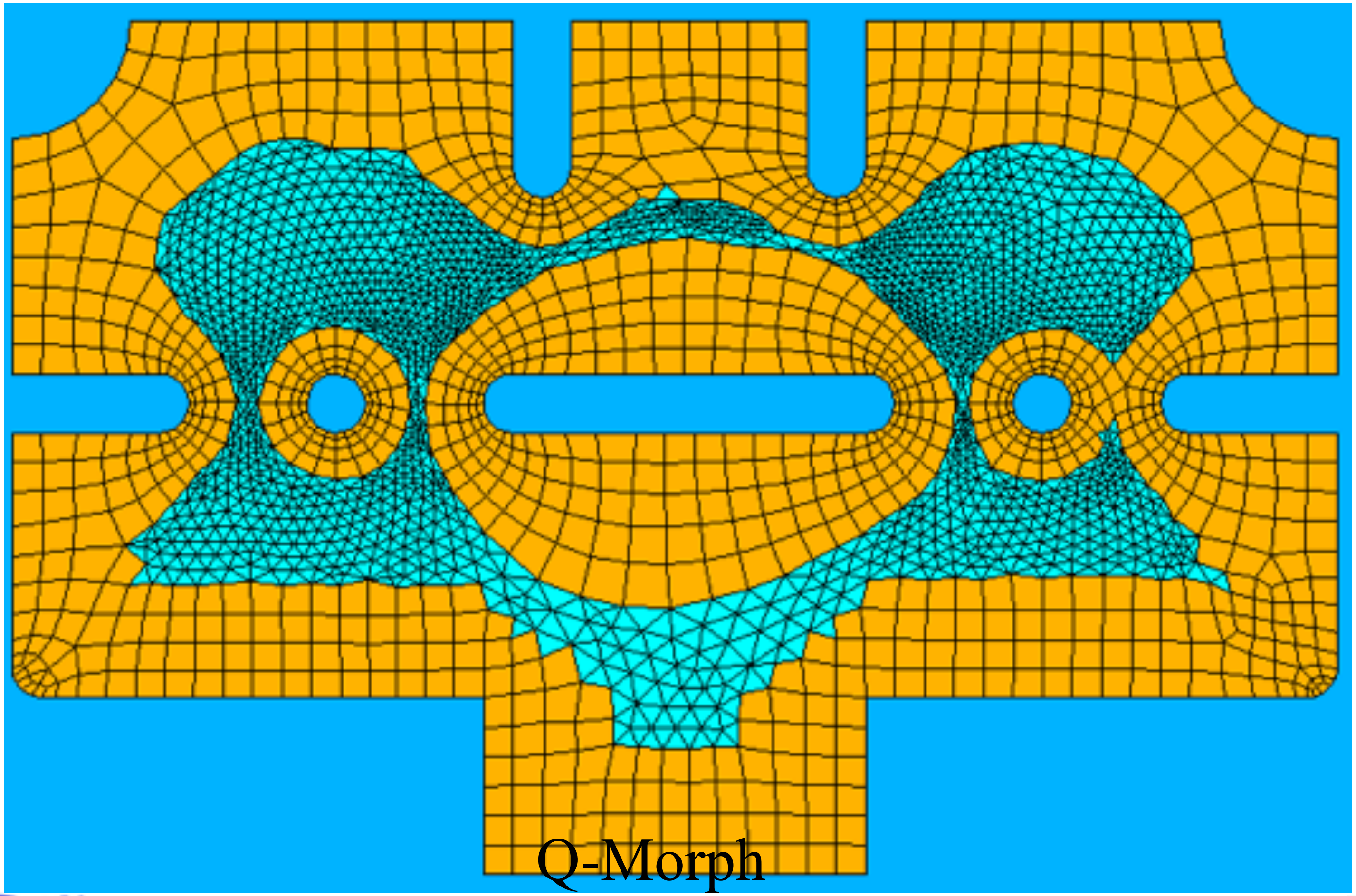


Unstructured-Quad



Q-Morph

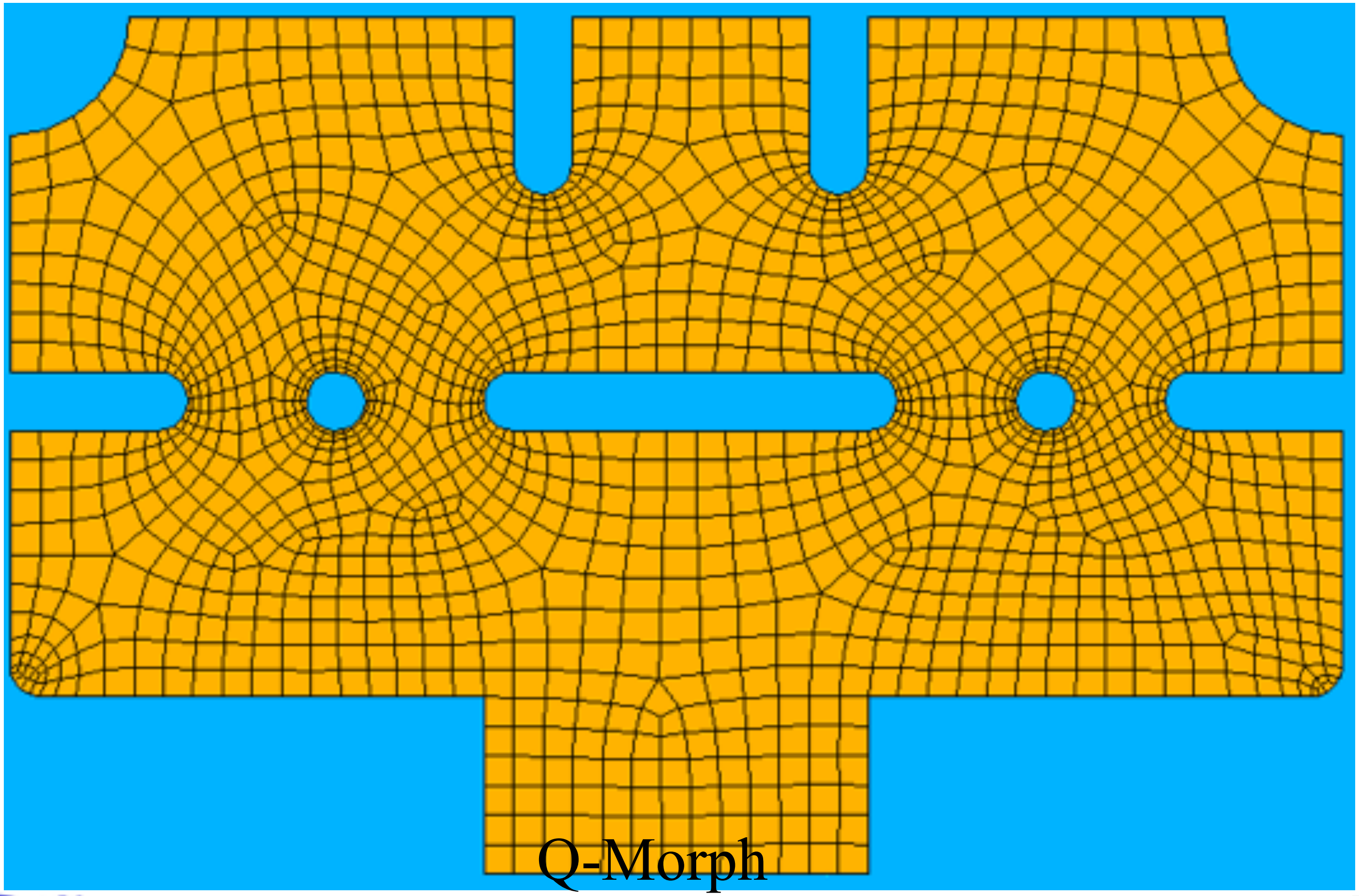
Unstructured-Quad



Q-Morph

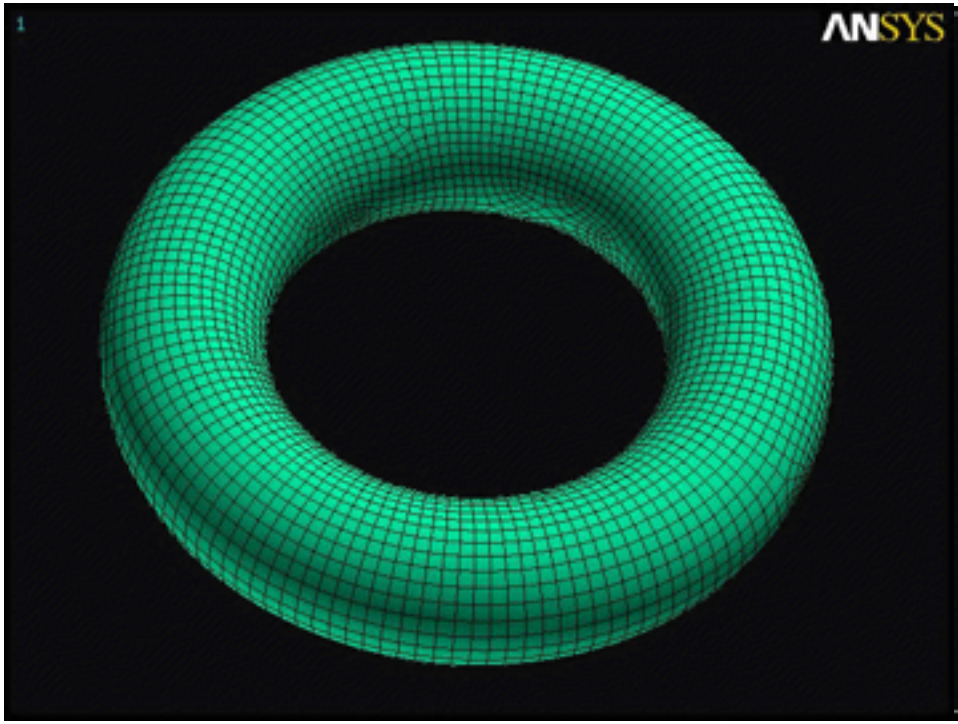
Unstructured-Quad

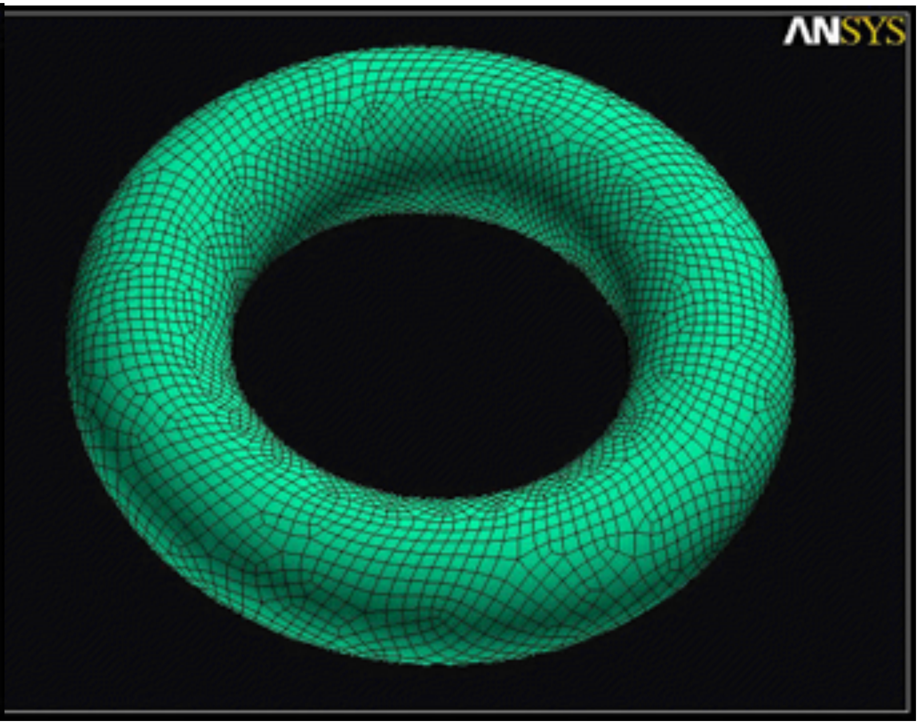
Computational Modeling Sciences Department



Q-Morph

Indirect

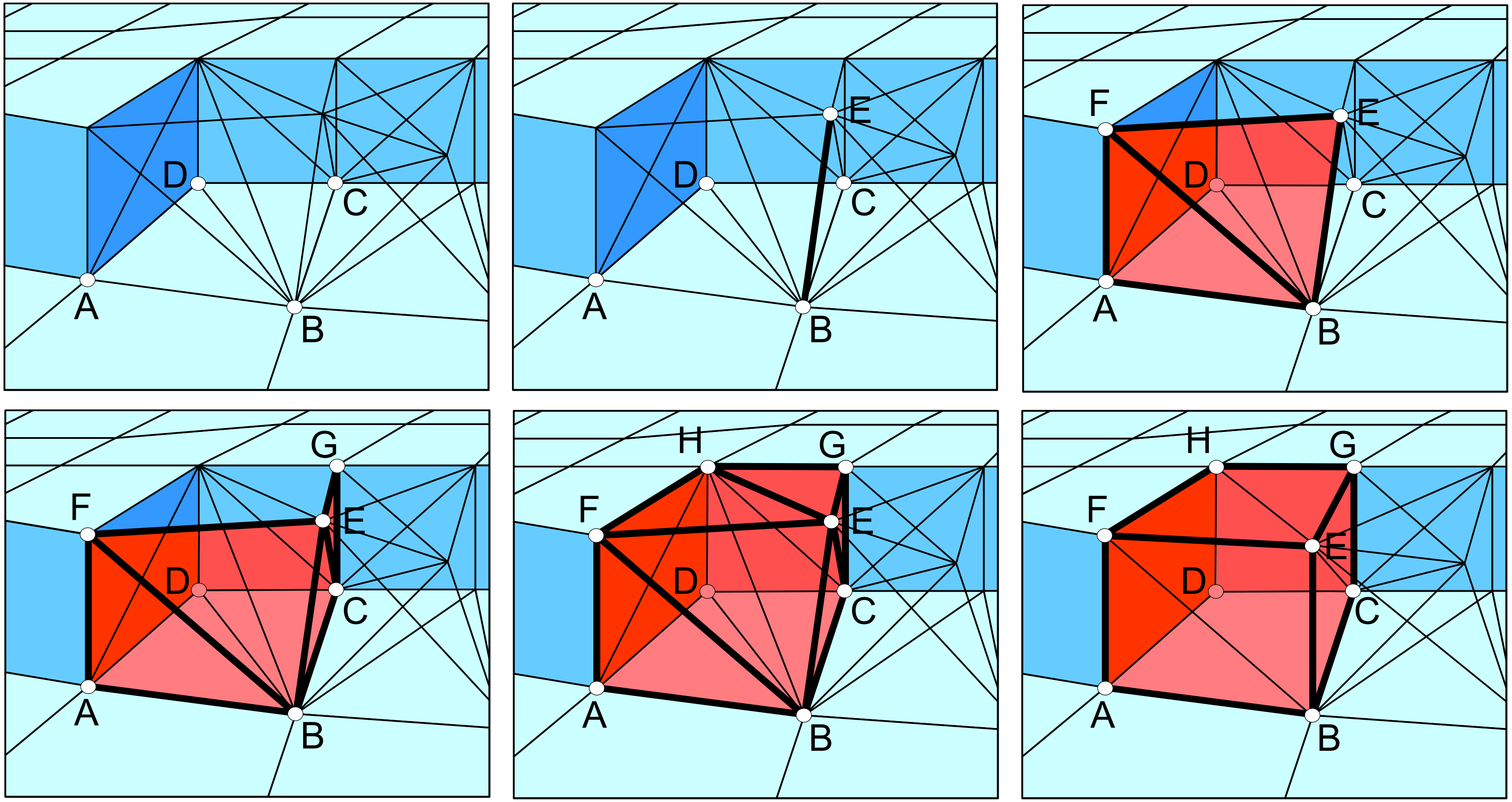




Q-Morph **Lee,Lo Method**

Indirect

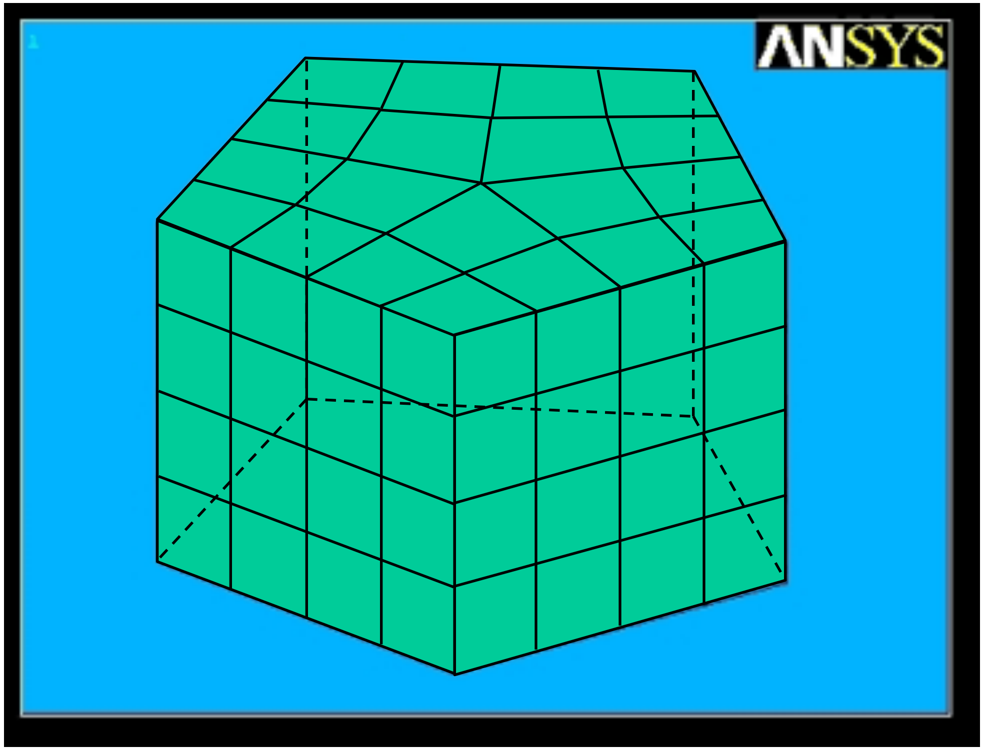
Computational Modeling Sciences Department



Tetrahedral Merge w/ local transformations ("H-Morph")

Unstructured-Hex

Computational Modeling Sciences Department

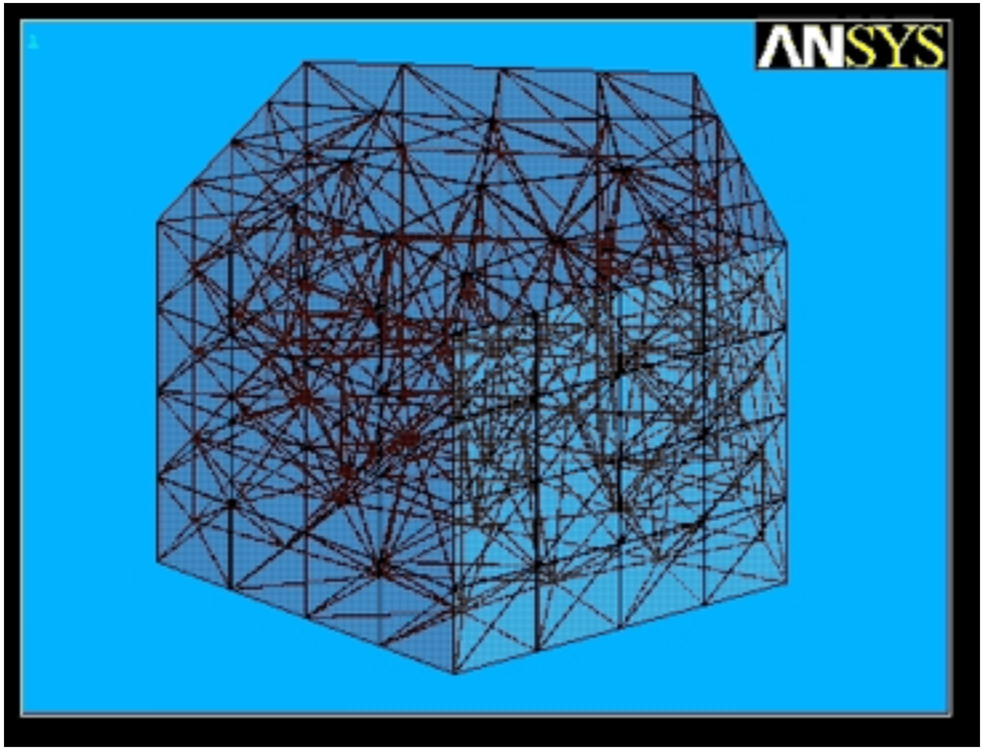


(Owen and Saigal, 00)

H-Morph

"Hex-Dominant Meshing"

Unstructured-Hex



(Owen and Saigal, 00)

H-Morph

"Hex-Dominant Meshing"

Unstructured-Hex

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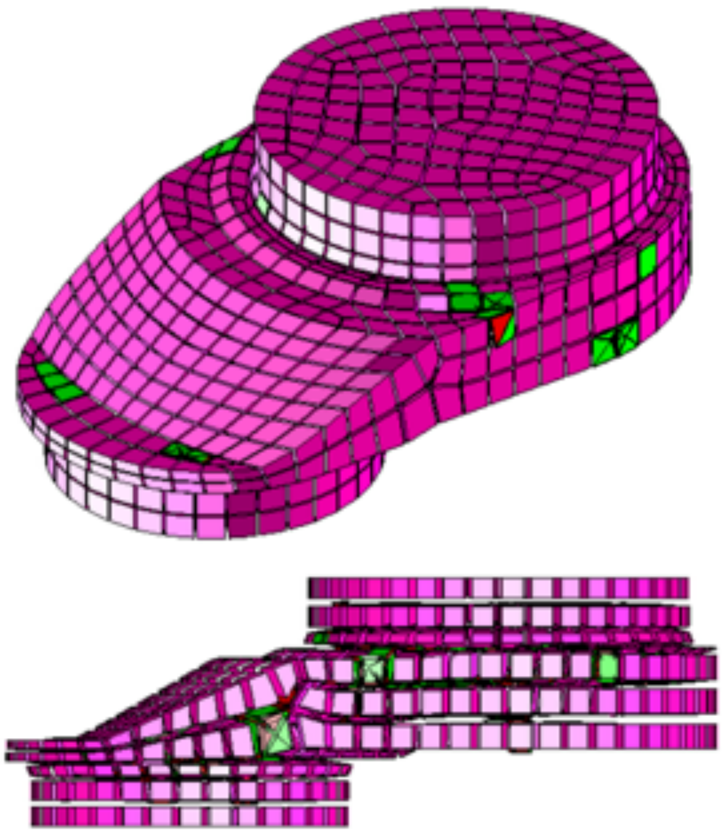
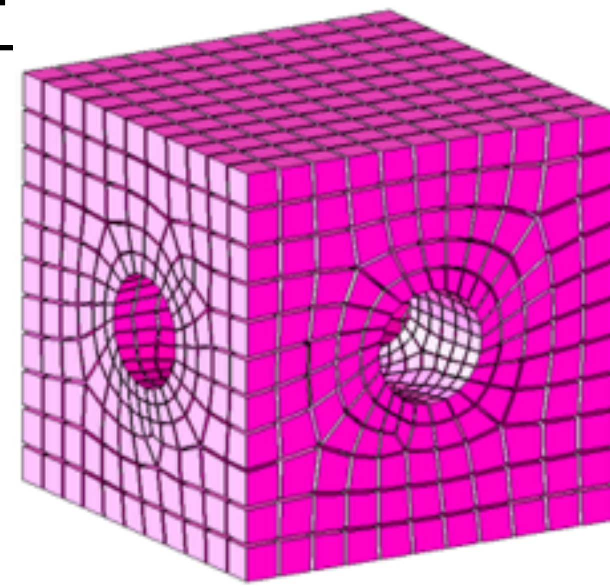
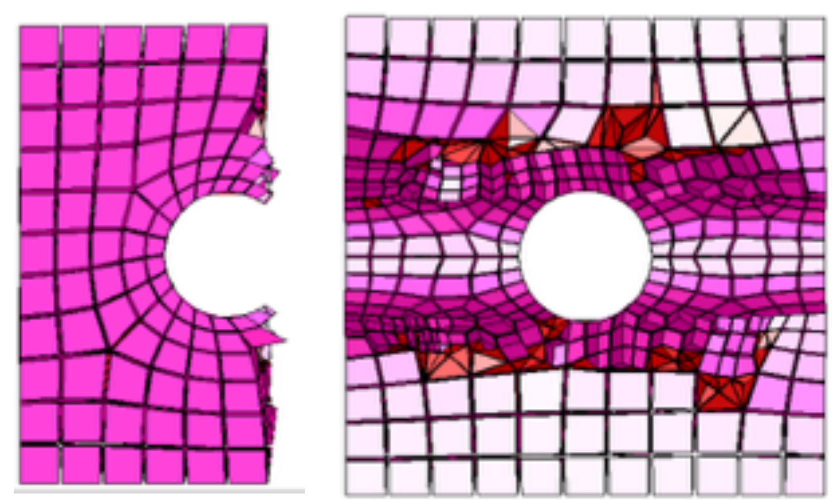
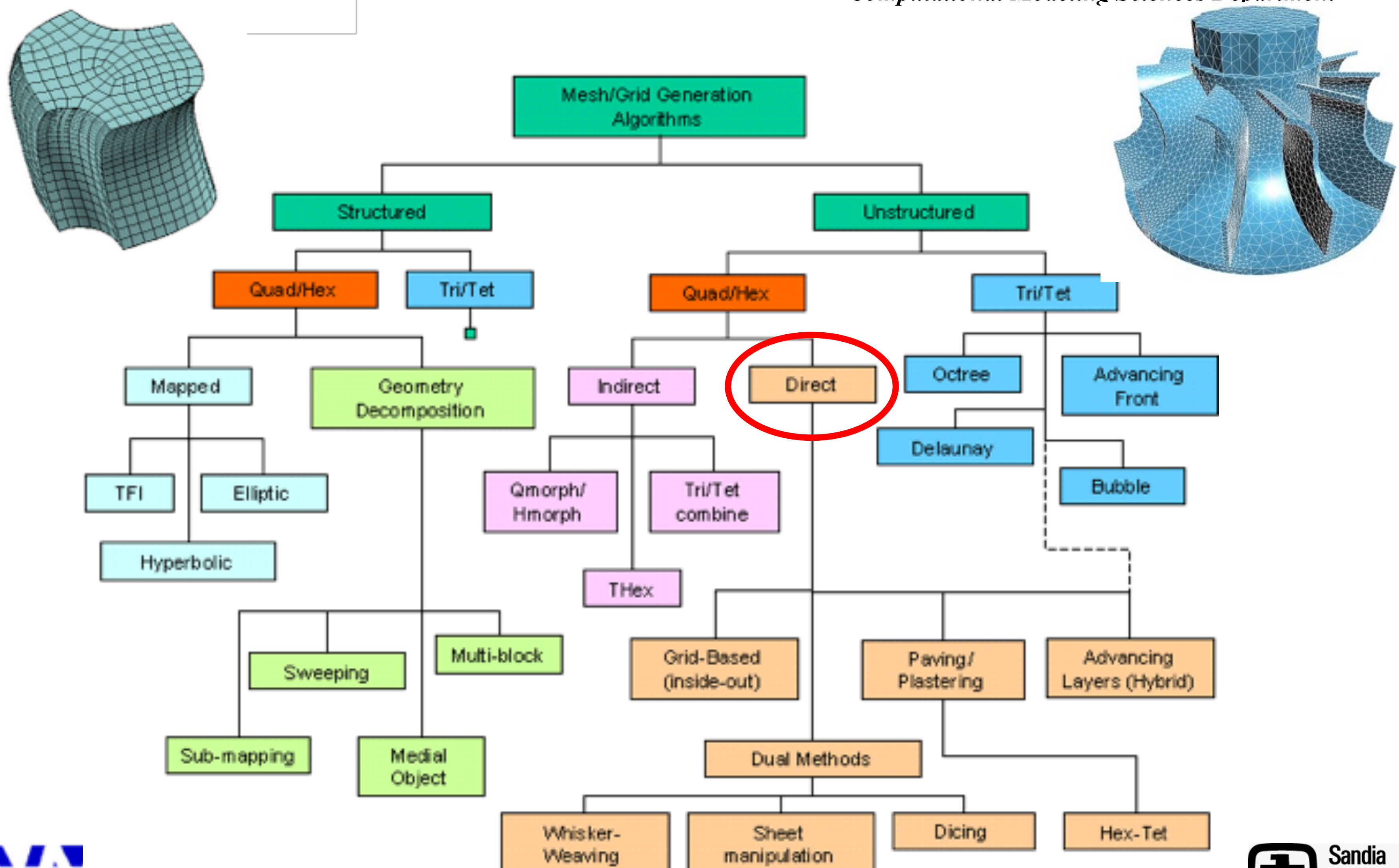




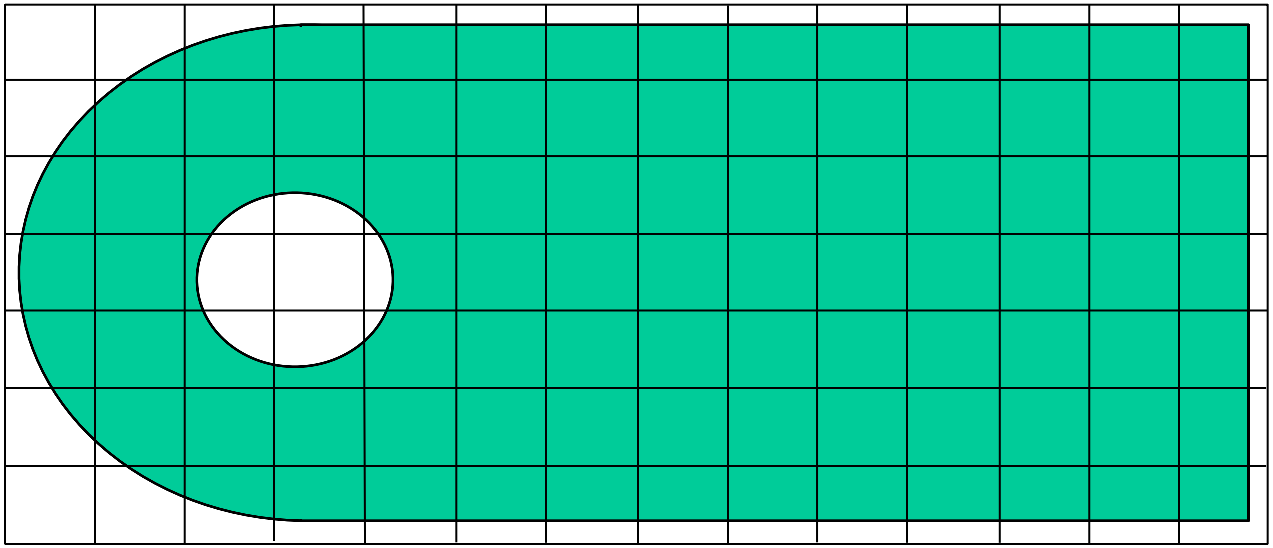


Meshing Algorithms



Unstructured-Hex

Computational Modeling Sciences Department

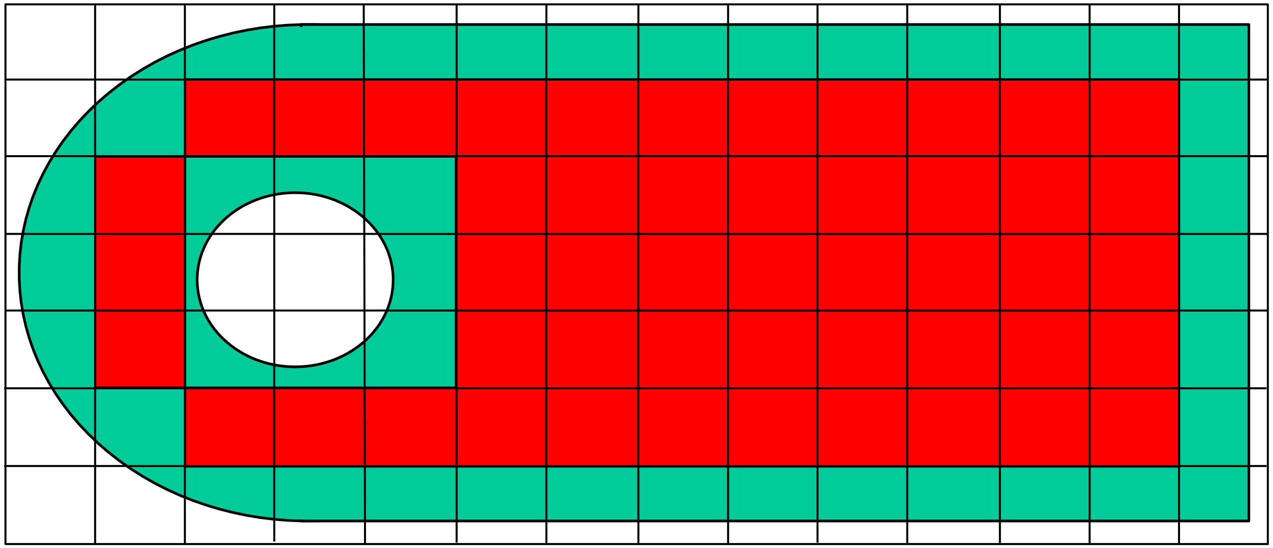


Grid-Based

•Generate regular grid of quads/hexes on the interior of model
•Fit elements to the boundary by projecting interior faces towards the surfaces
•Lower quality elements near boundary
•Non-boundary conforming

Unstructured-Hex

Computational Modeling Sciences Department



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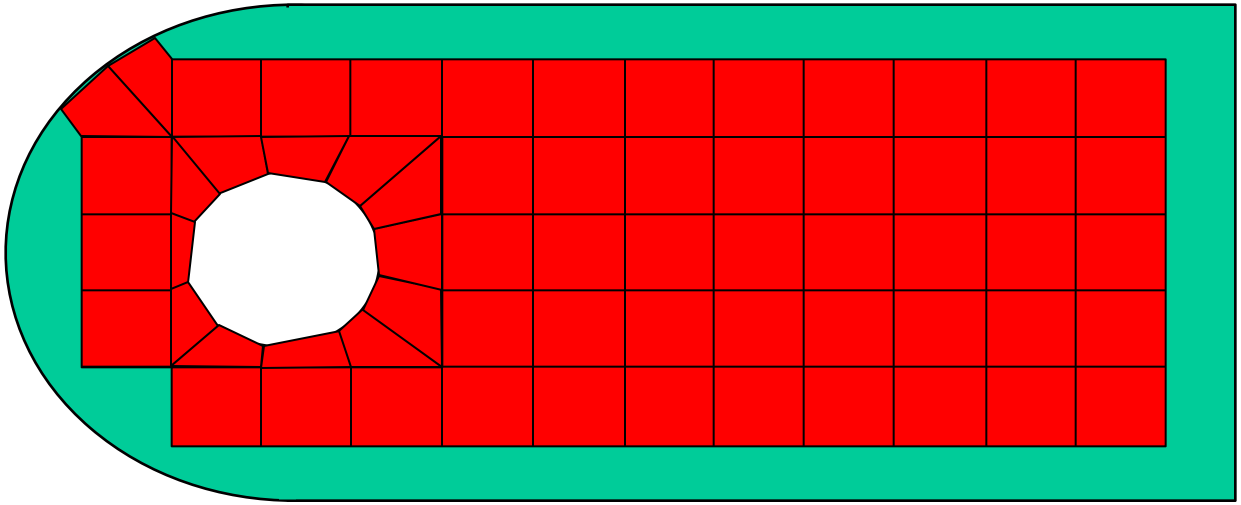
Computational Modeling Sciences Department



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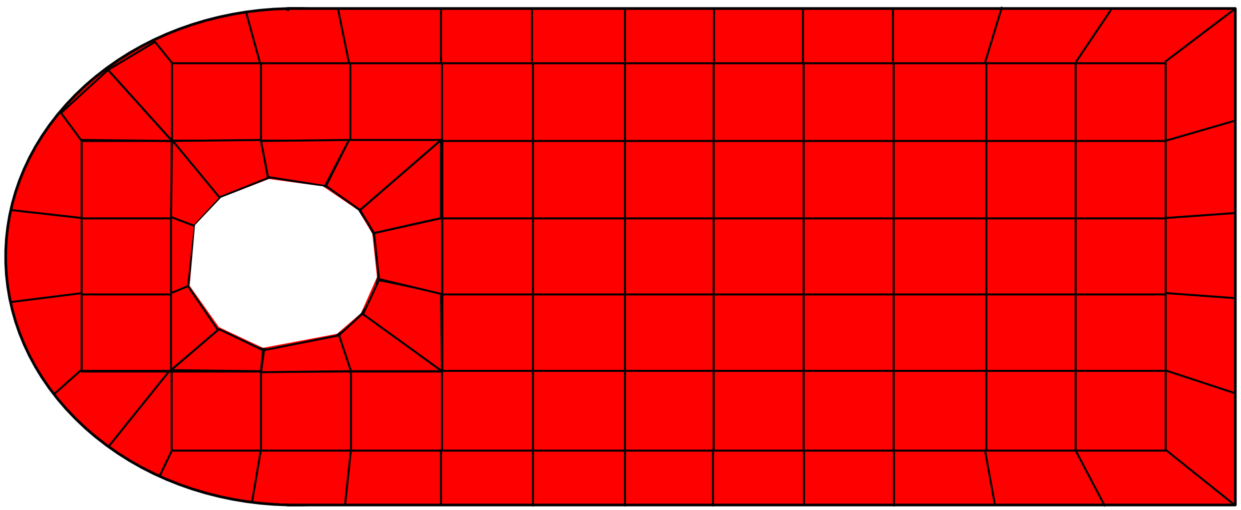

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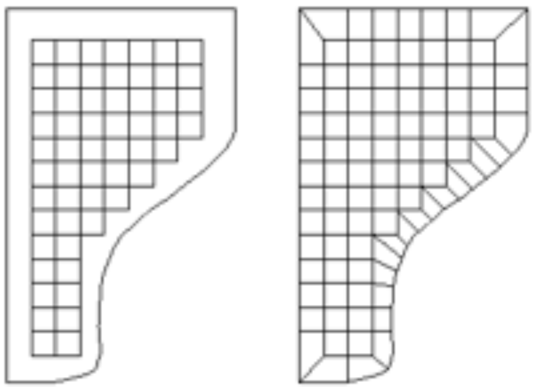
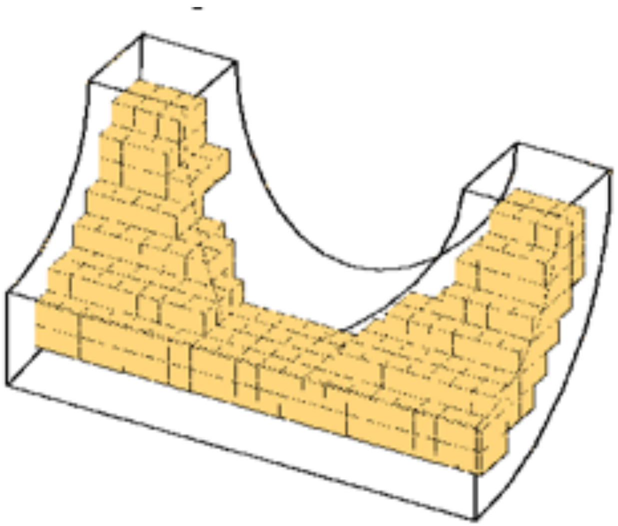
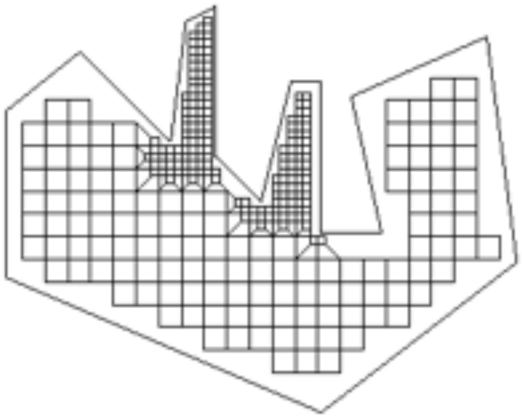
Unstructured-Hex

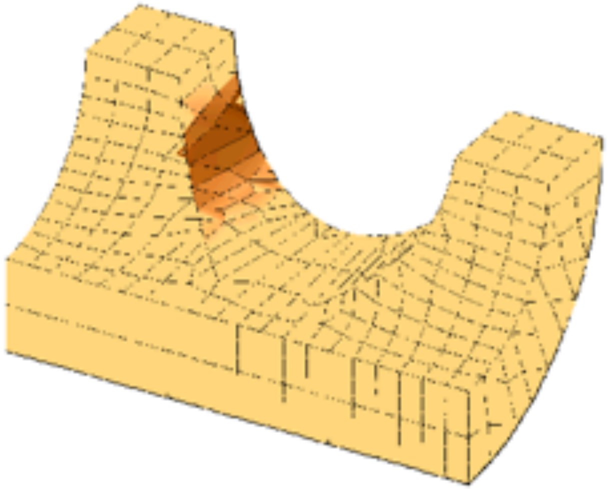
Computational Modeling Sciences Department





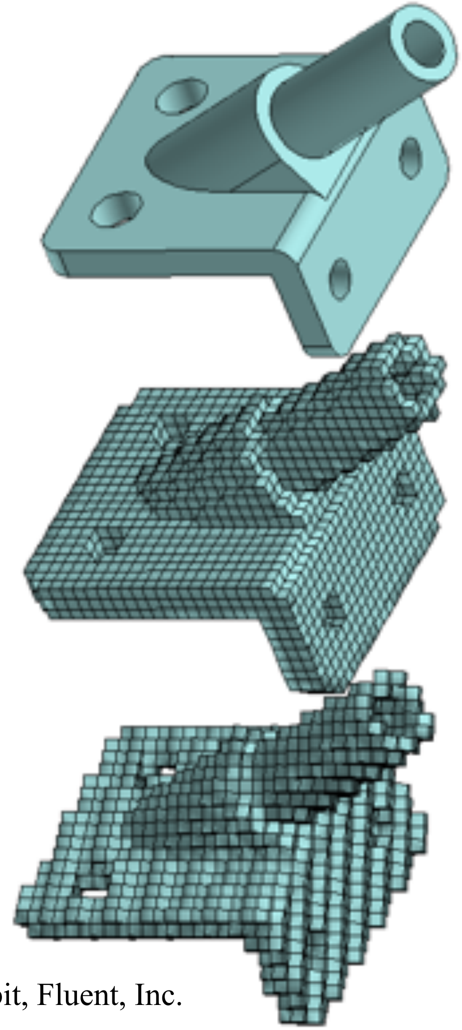
(Schneiders,96)



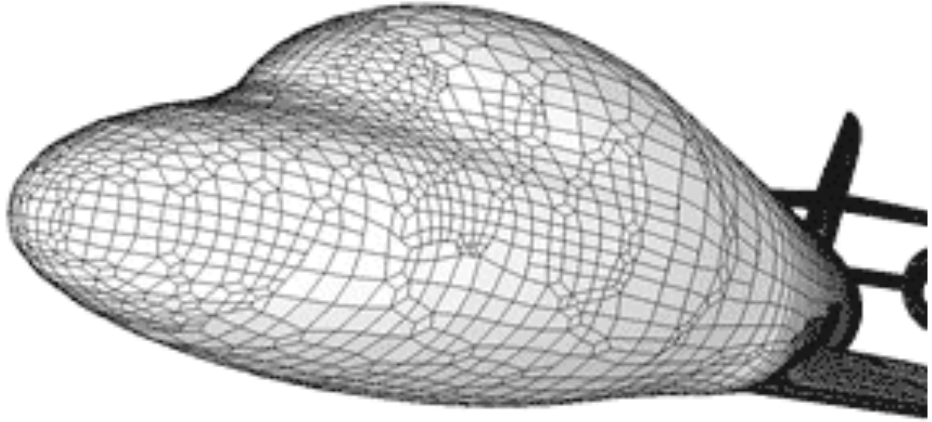


Grid-Based

Unstructured-Hex



Gambit, Fluent, Inc.

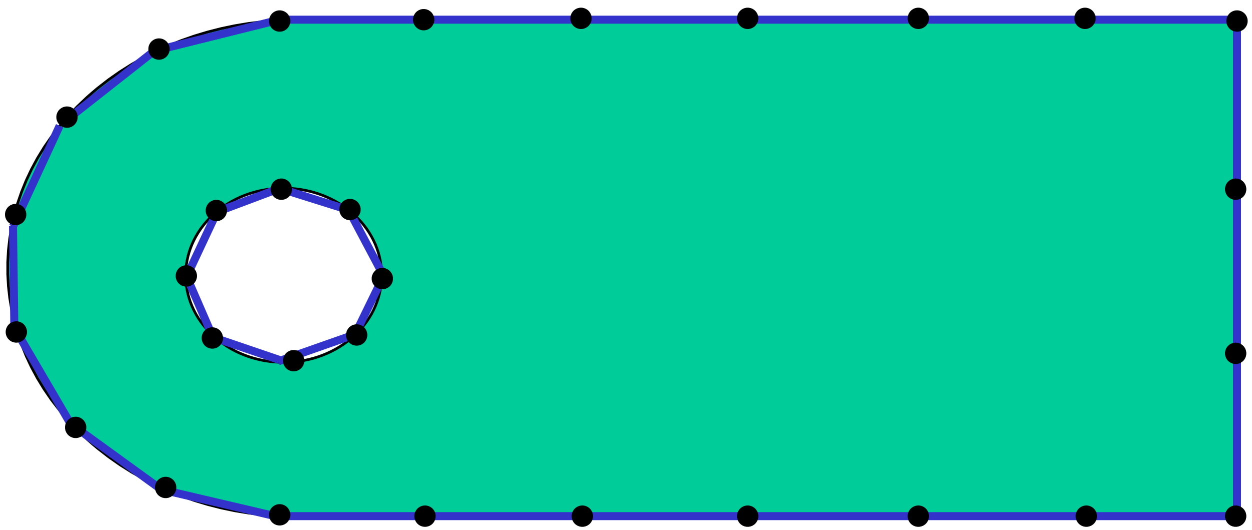


http://www.numeca.be/hexpress_home.html

Grid-Based

Direct Quad

Computational Modeling Sciences Department

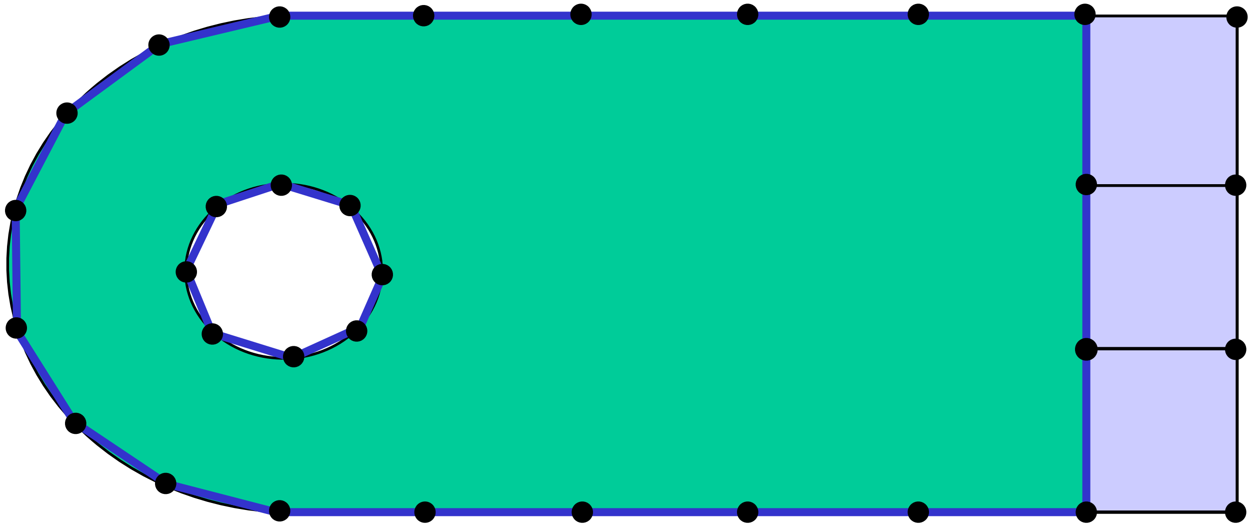


Paving

- Advancing Front: Begins with front at boundary
- Forms rows of elements based on front angles
- Must have even number of intervals for all-quad mesh

(Blacker,92)(Cass,96)

Unstructured-Quad



Paving

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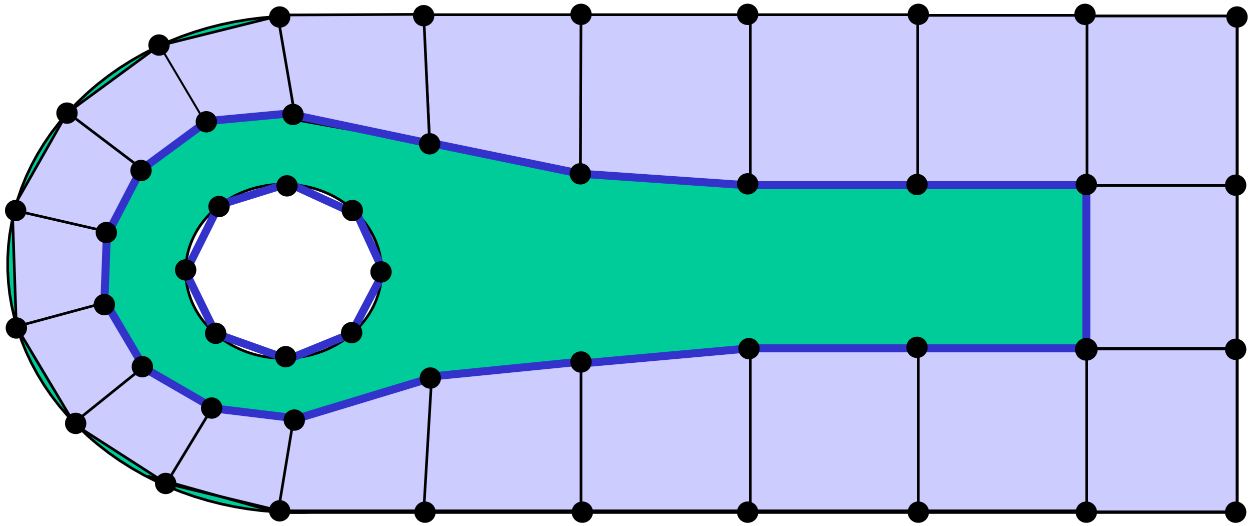
(Blacker,92)(Cass,96)

Unstructured-Quad

Computational Modeling Sciences Department



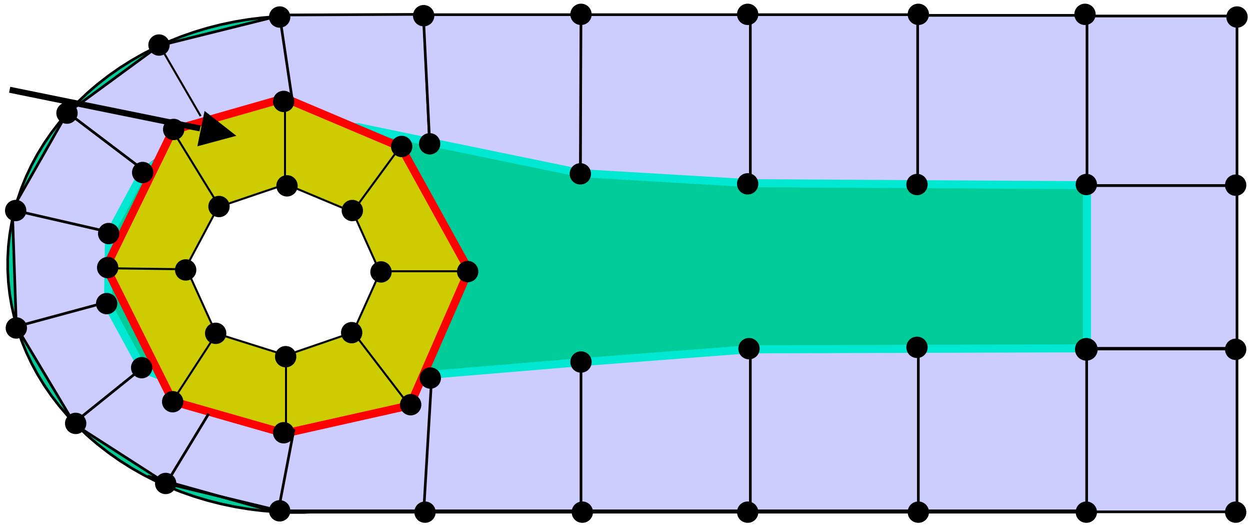
Paving

- Advancing Front: Begins with front at boundary
- Forms rows of elements based on front angles
- Must have even number of intervals for all-quad mesh

(Blacker,92)(Cass,96)

Unstructured-Quad

Form new row
and check for
overlap



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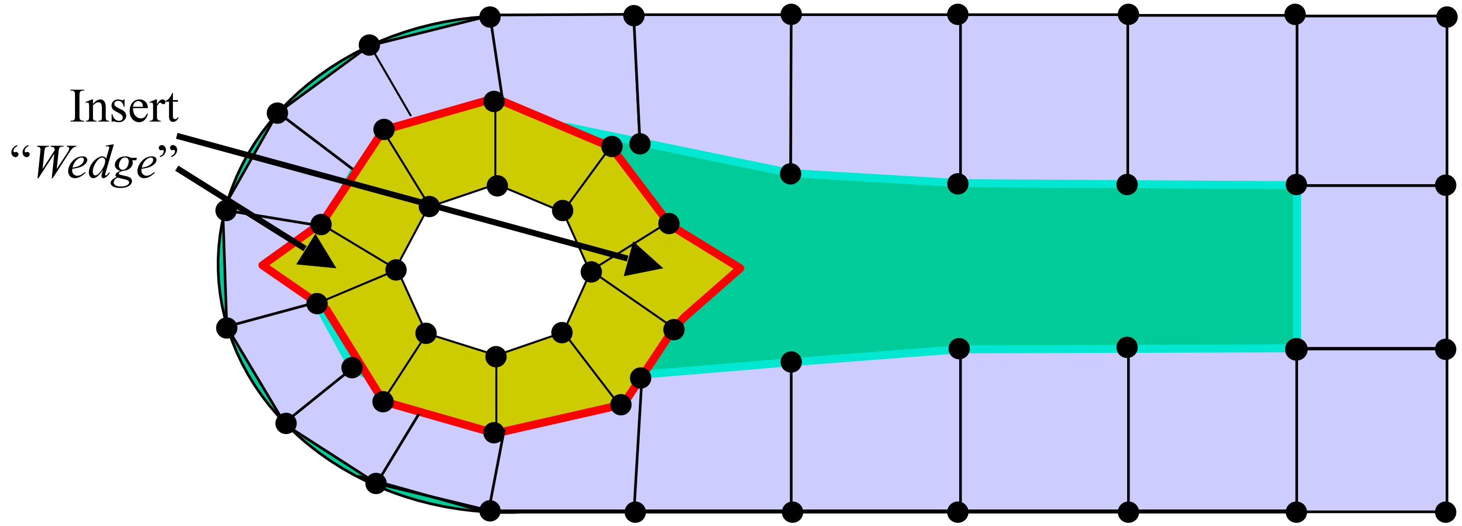
Unstructured-Quad



Paving

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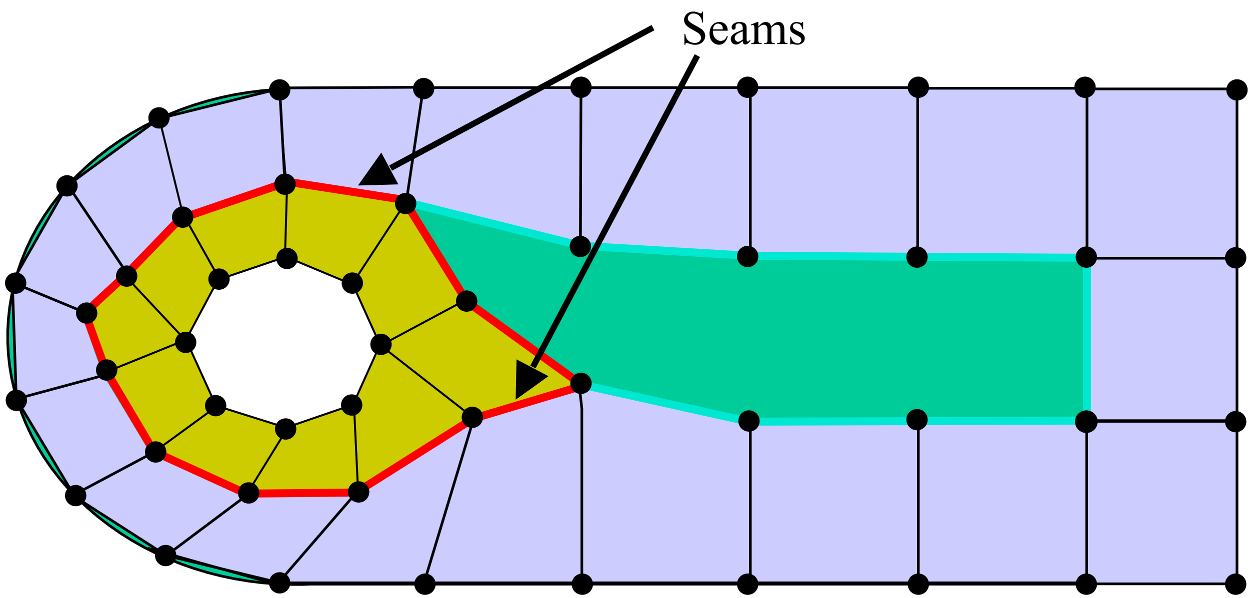
(Blacker,92)(Cass,96)

Unstructured-Quad

Computational Modeling Sciences Department



Paving

•Advancing Front: Begins with front at boundary
•Forms rows of elements based on front angles
•Must have even number of intervals for all-quad mesh

(Blacker,92)(Cass,96)

Unstructured-Quad

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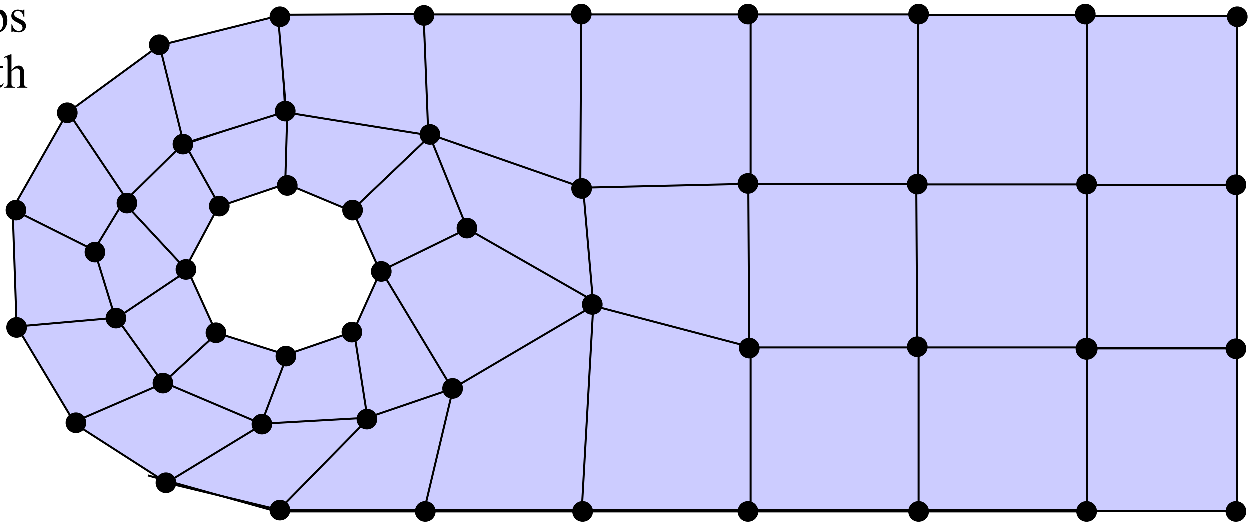
Close Loops
and smooth

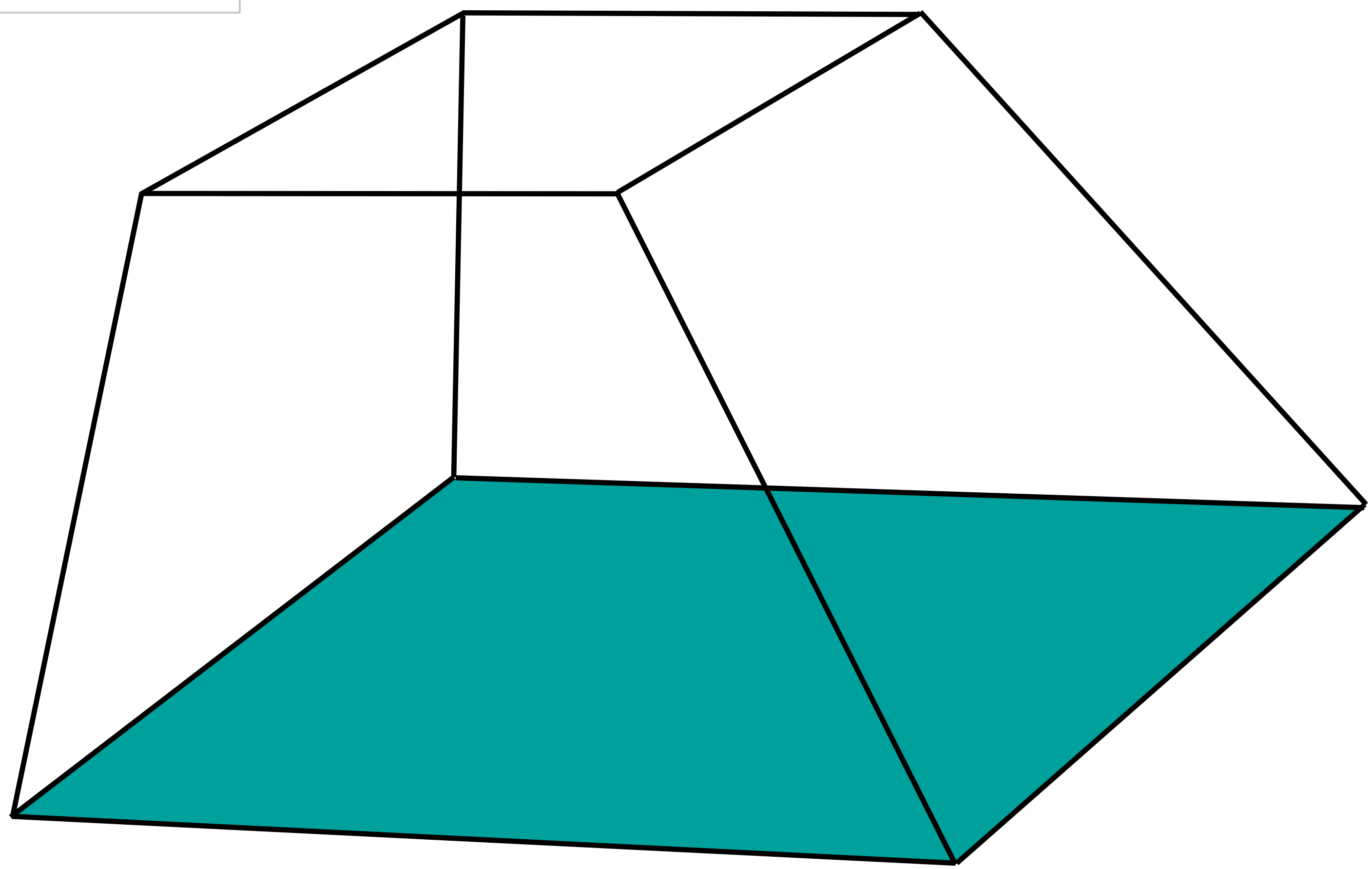


Paving

- Advancing Front: Begins with front at boundary
- Forms rows of elements based on front angles
- Must have even number of intervals for all-quad mesh

(Blacker,92)(Cass,96)

Unstructured-Hex

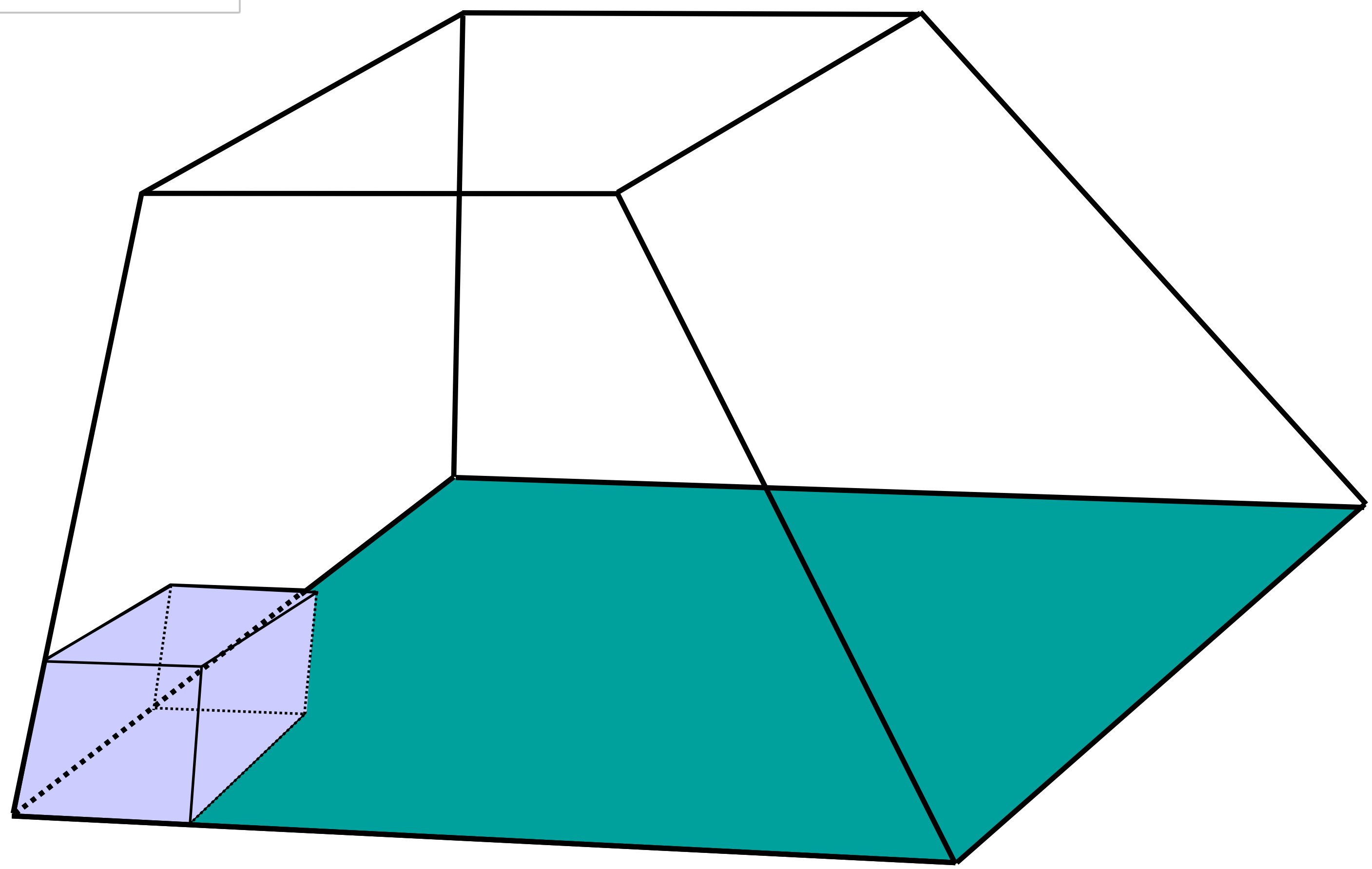


Plastering

- 3D extension of "paving"
- Row-by row or element-by-element

(Blacker, 93)

Unstructured-Hex



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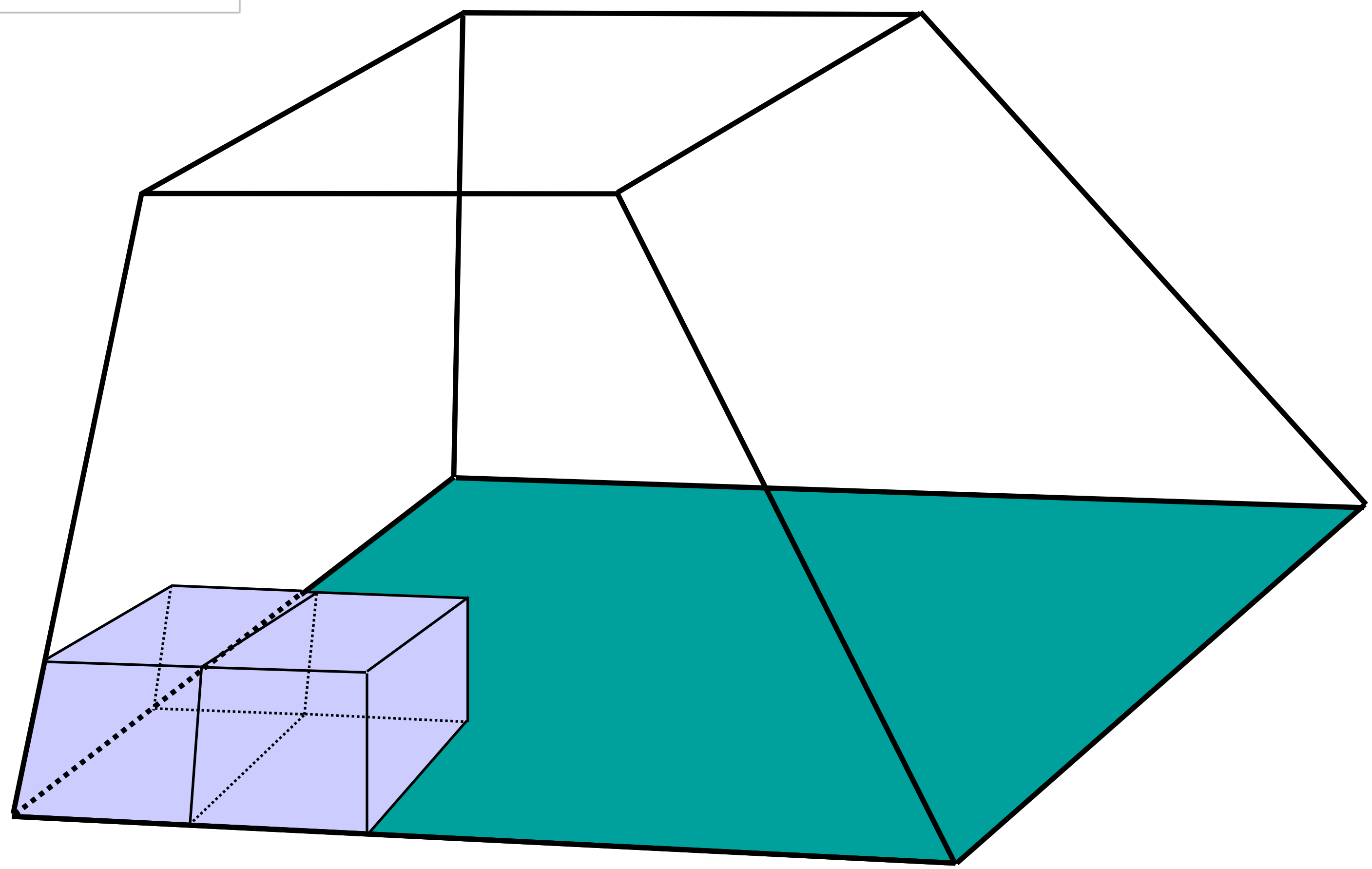
(Blacker, 93)

Unstructured-Hex



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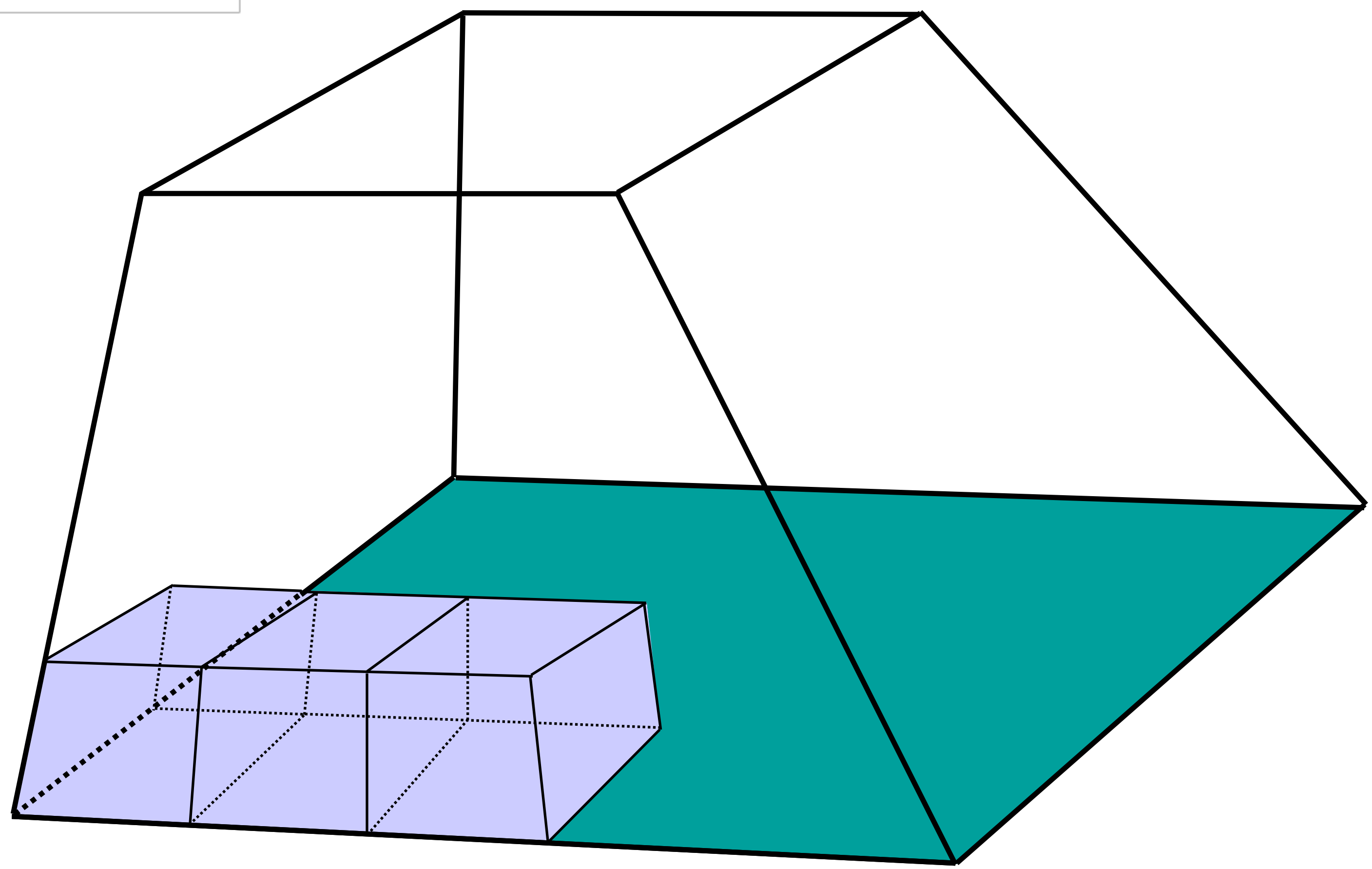

Plastering

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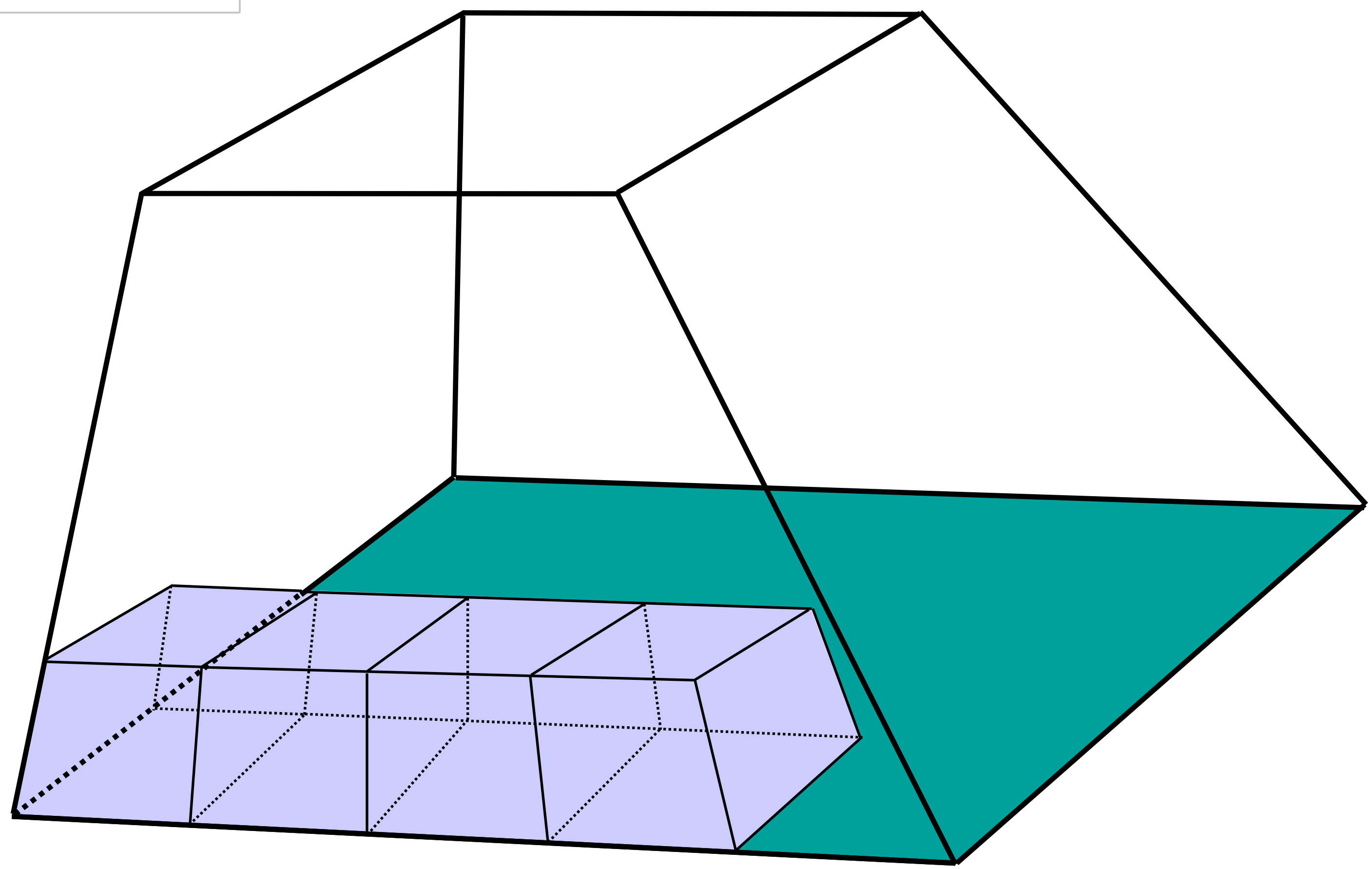

Plastering

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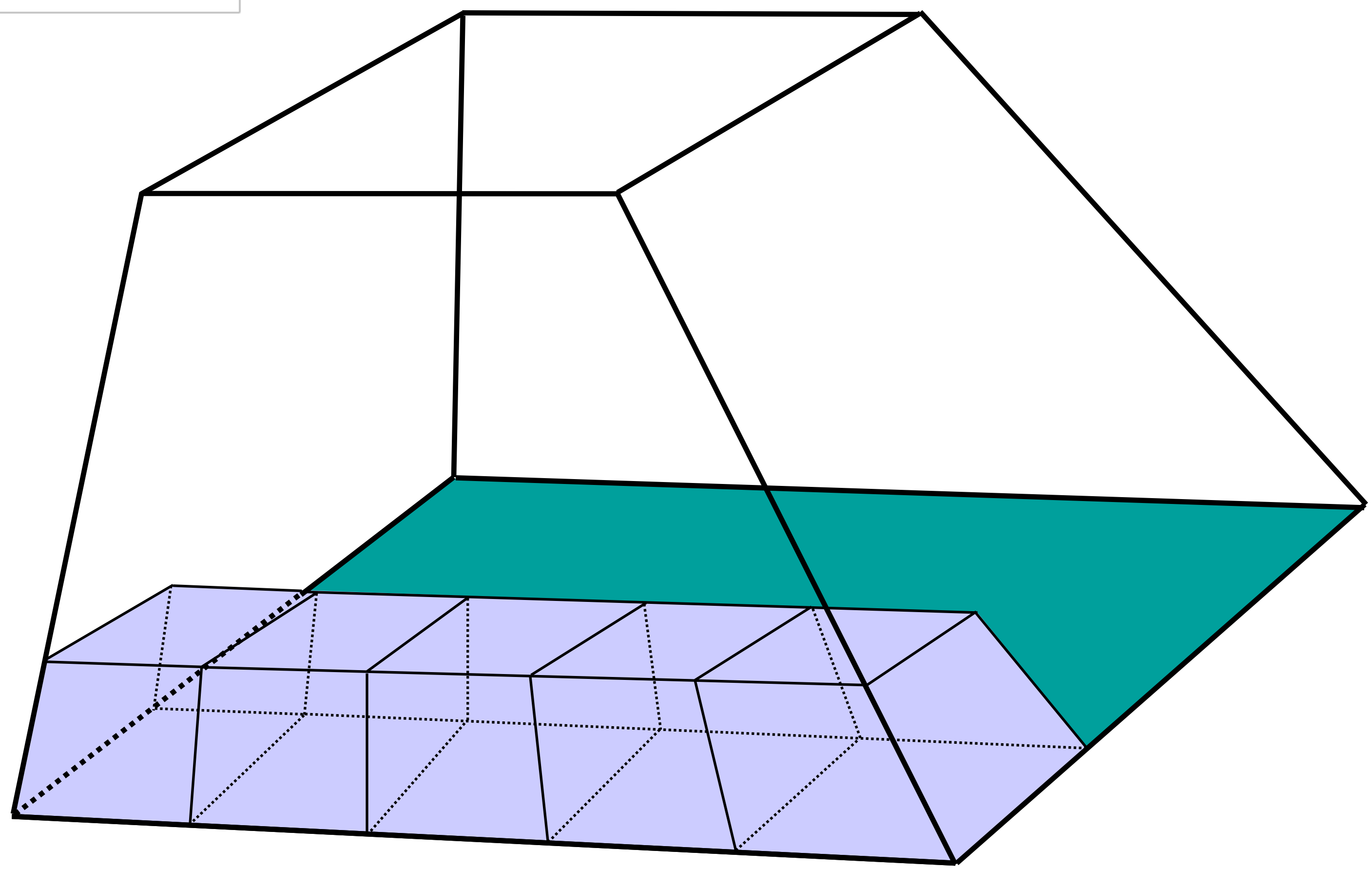
(Blacker, 93)

Unstructured-Hex

Computational Modeling Sciences Department



Plastering

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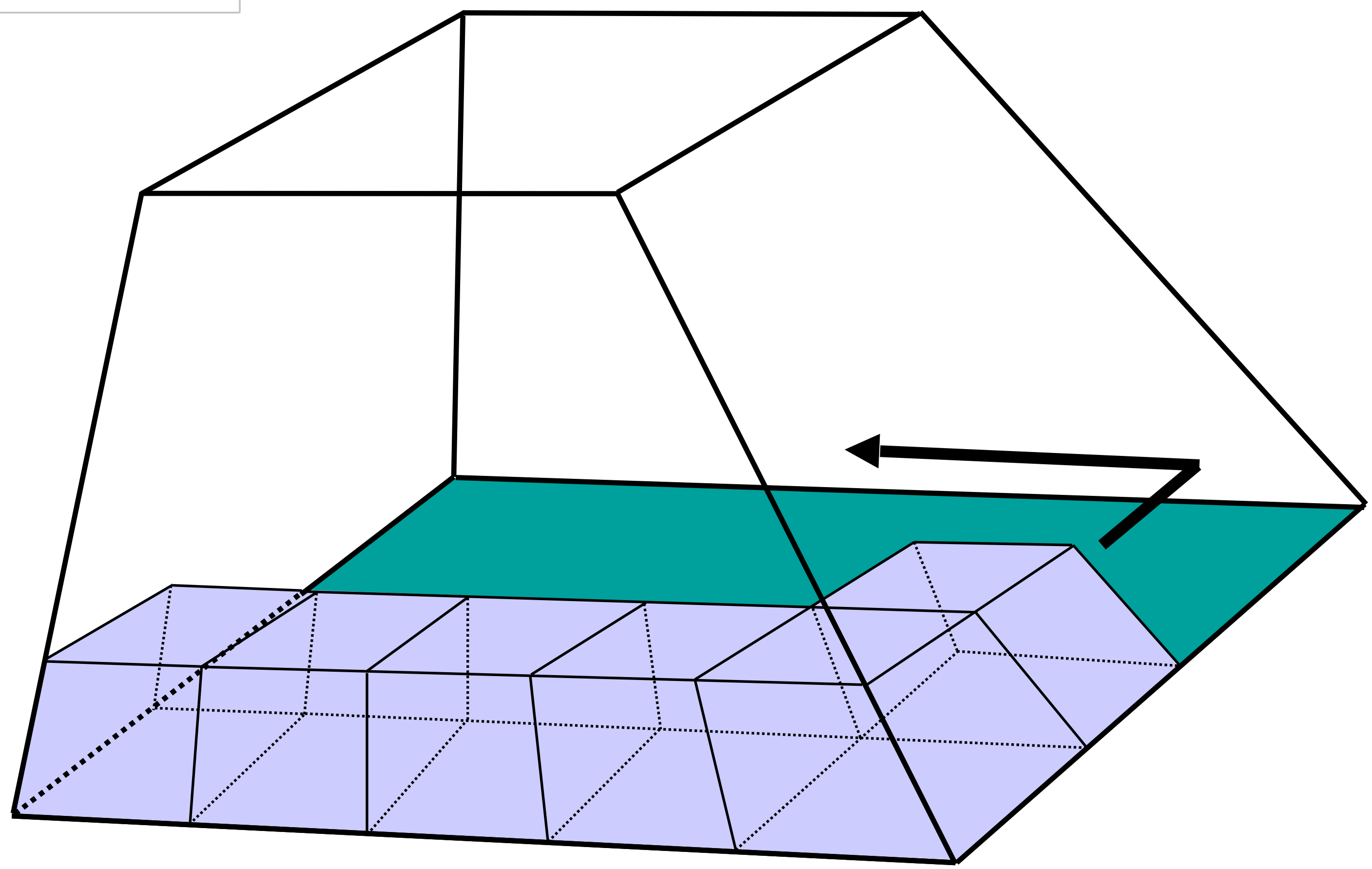
(Blacker, 93)

Unstructured-Hex



Plastering

- 3D extension of "paving"
- Row-by row or element-by-element

(Blacker, 93)

Unstructured-Hex

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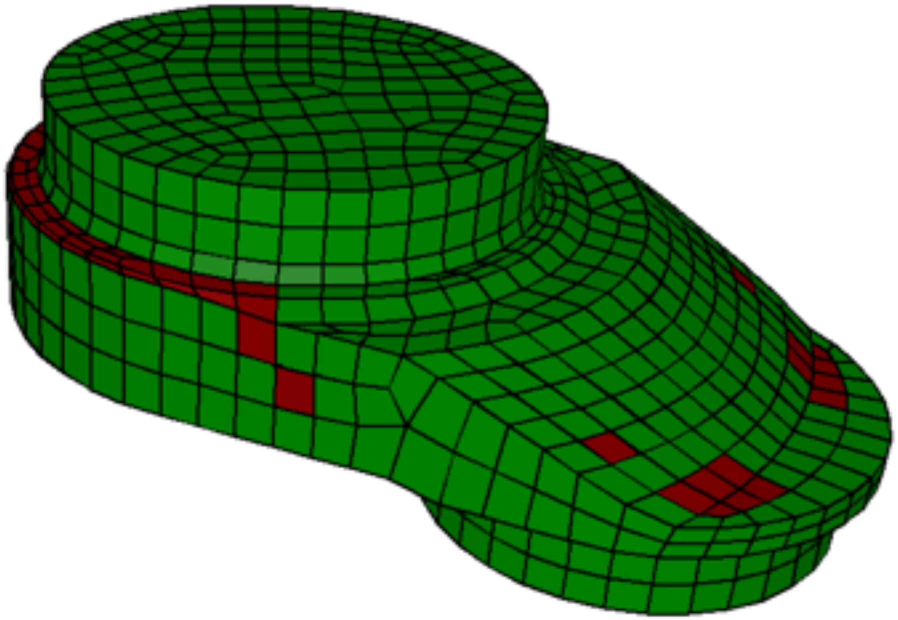
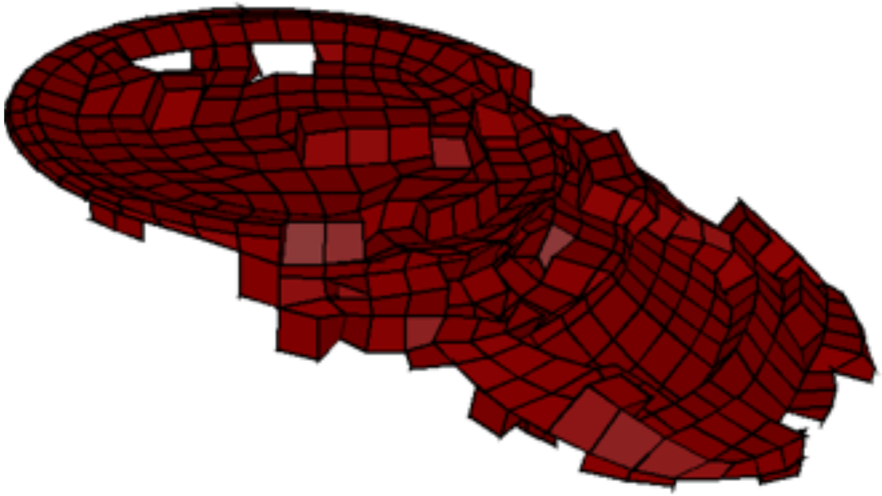
Exterior Hex mesh



Remaining Void

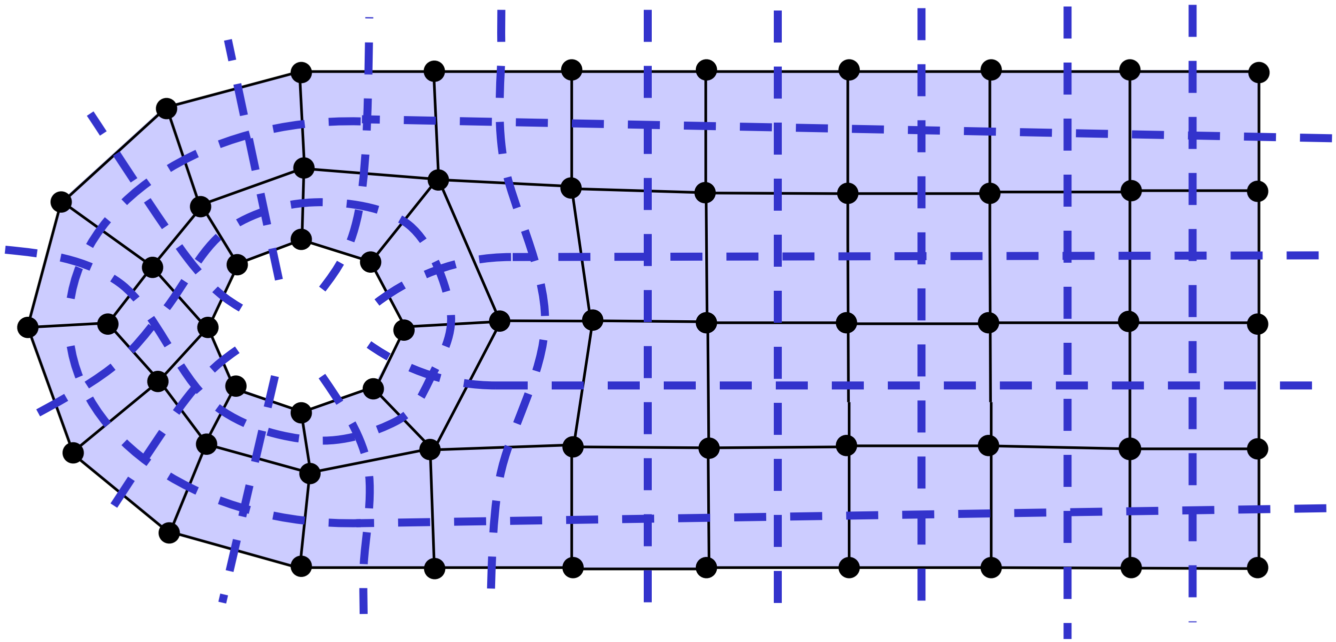


Ford Crankshaft

Plastering+Tet Meshing

"Hex-Dominant Meshing"

Direct



Whisker Weaving

- First constructs *dual* of the quad/hex mesh
- Inserts quad/hex at the intersections of the dual chords

Direct

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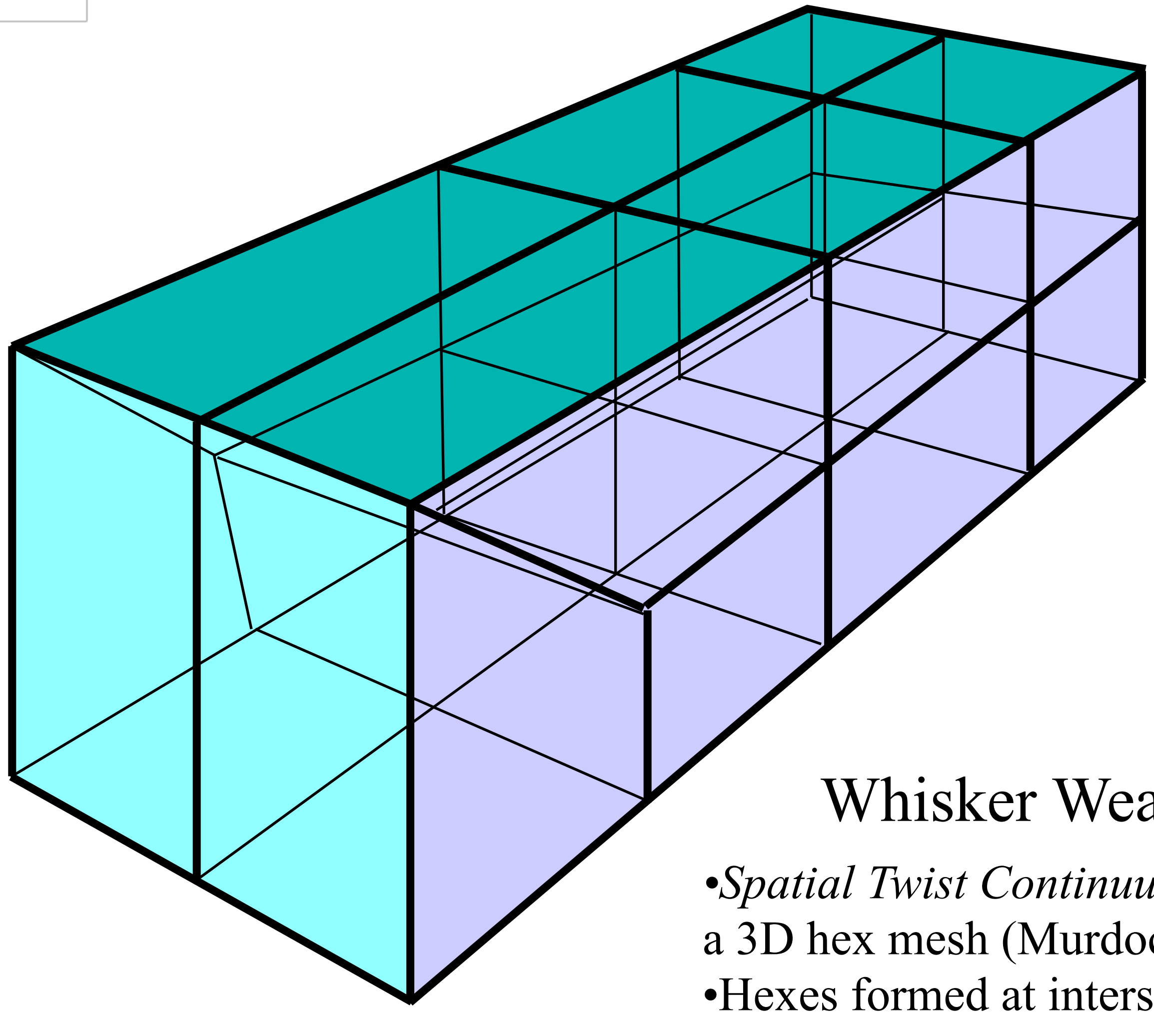

Whisker Weaving

- *Spatial Twist Continuum* - Dual of a 3D hex mesh (Murdoch, 96)
- Hexes formed at intersection of twist planes

Direct

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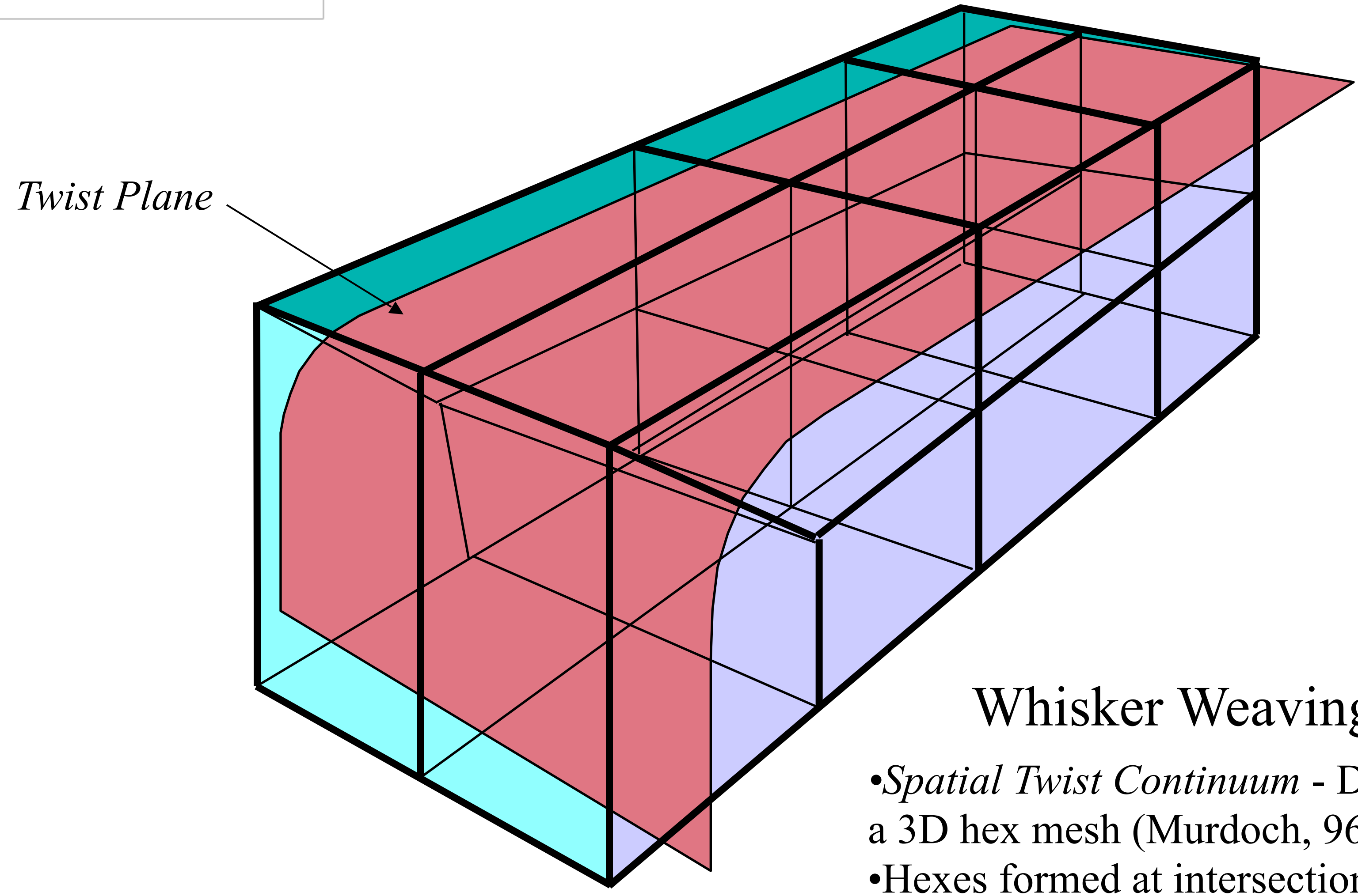

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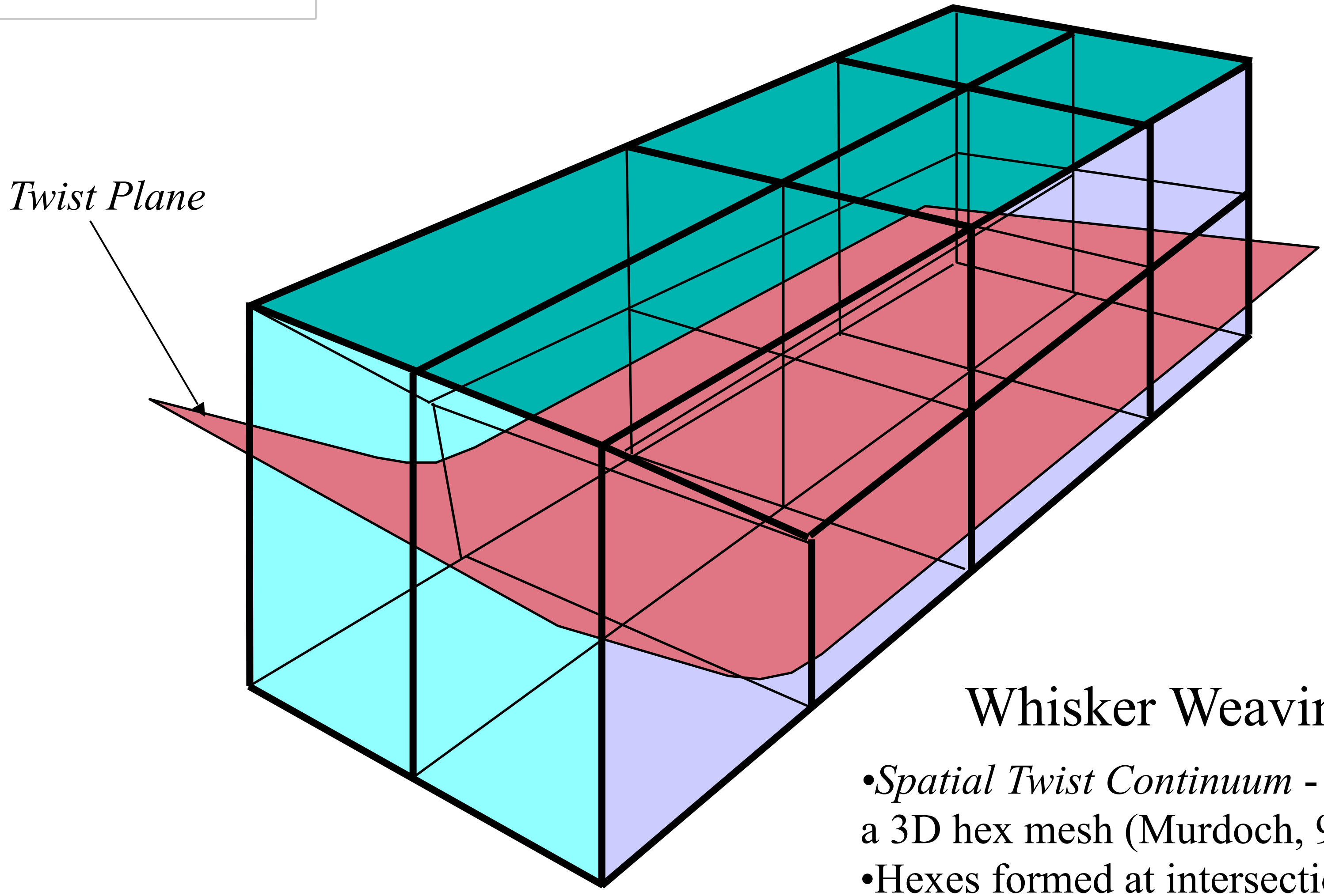

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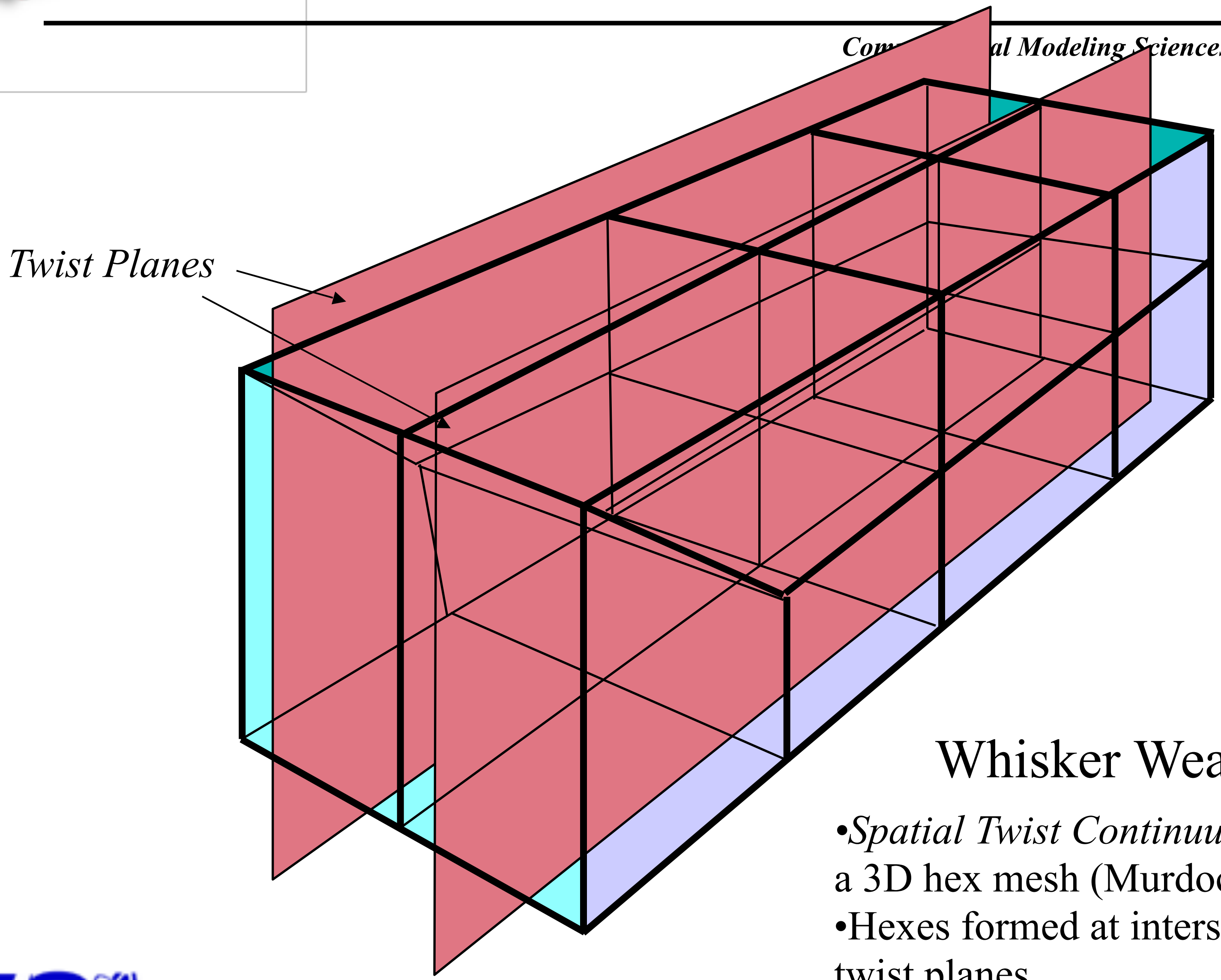

Whisker Weaving

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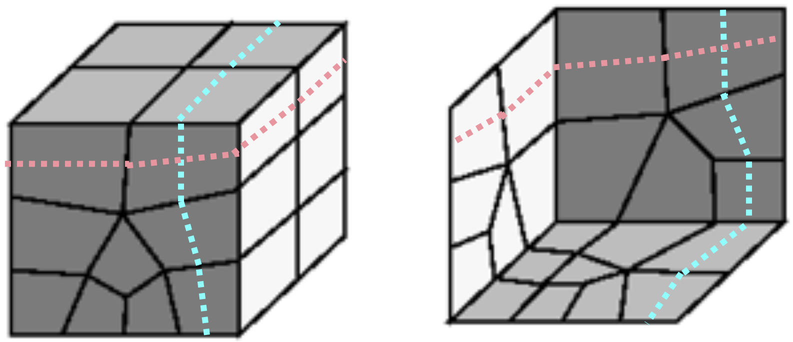
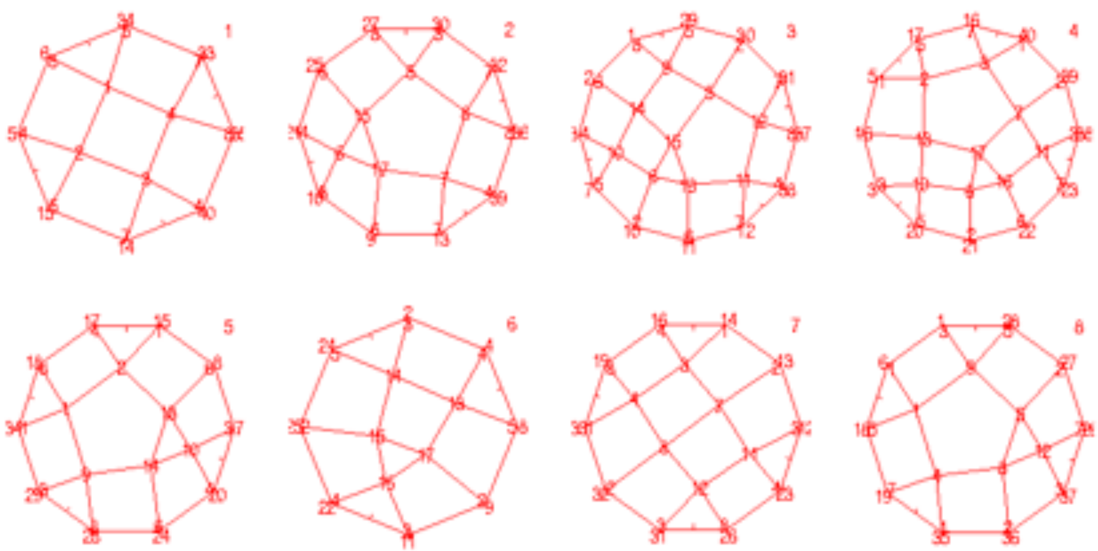




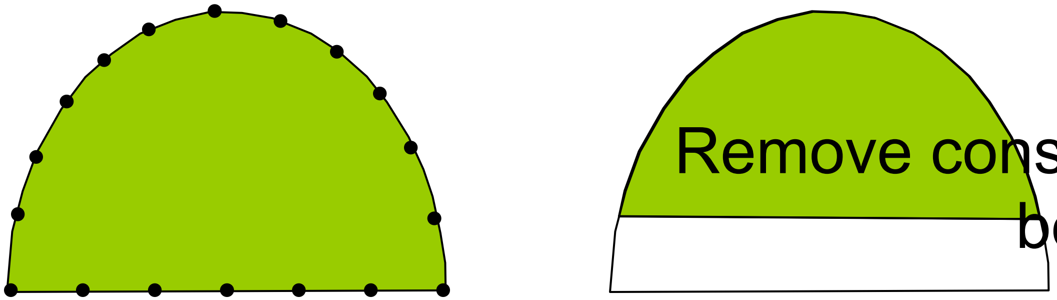
Whisker diagrams used to resolve hex mesh above

Whisker Weaving

- Define the topology of the twist planes using whisker diagrams
- Each whisker diagram represents a closed loop of the surface dual
- Each boundary vertex on the diagram represents a quad face on the surface
- Objective is to resolve internal connectivity by "weaving" the chords following a set of basic rules

(Tautges,95;96)

Unconstrained Paving

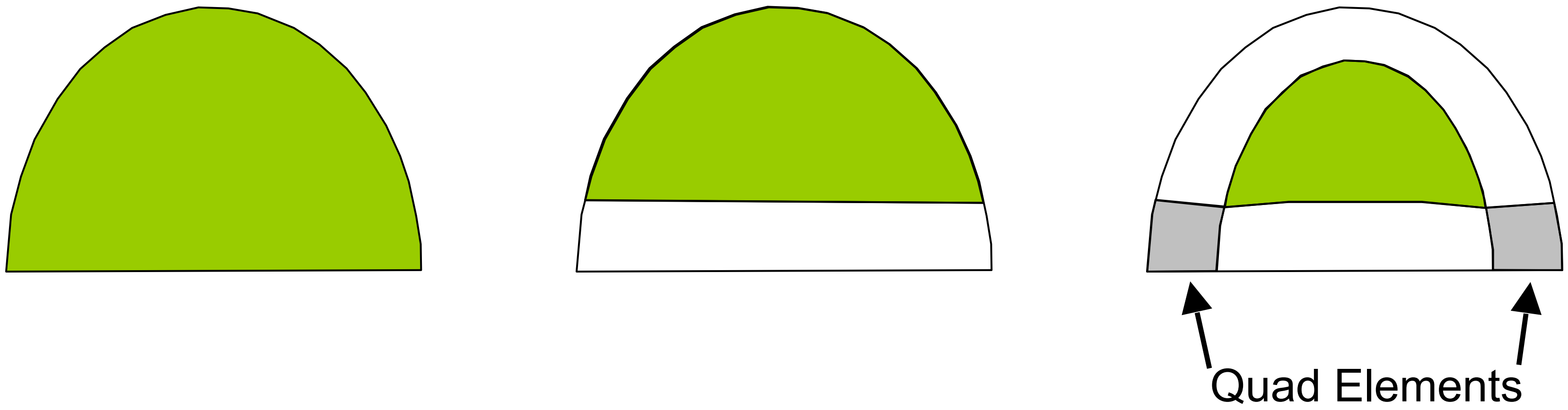


Remove constraint of pre-meshed boundary.

Remove constraint that we must define number of quad when row is advanced. This constrains only 1 DOF.

Unconstrained Paving

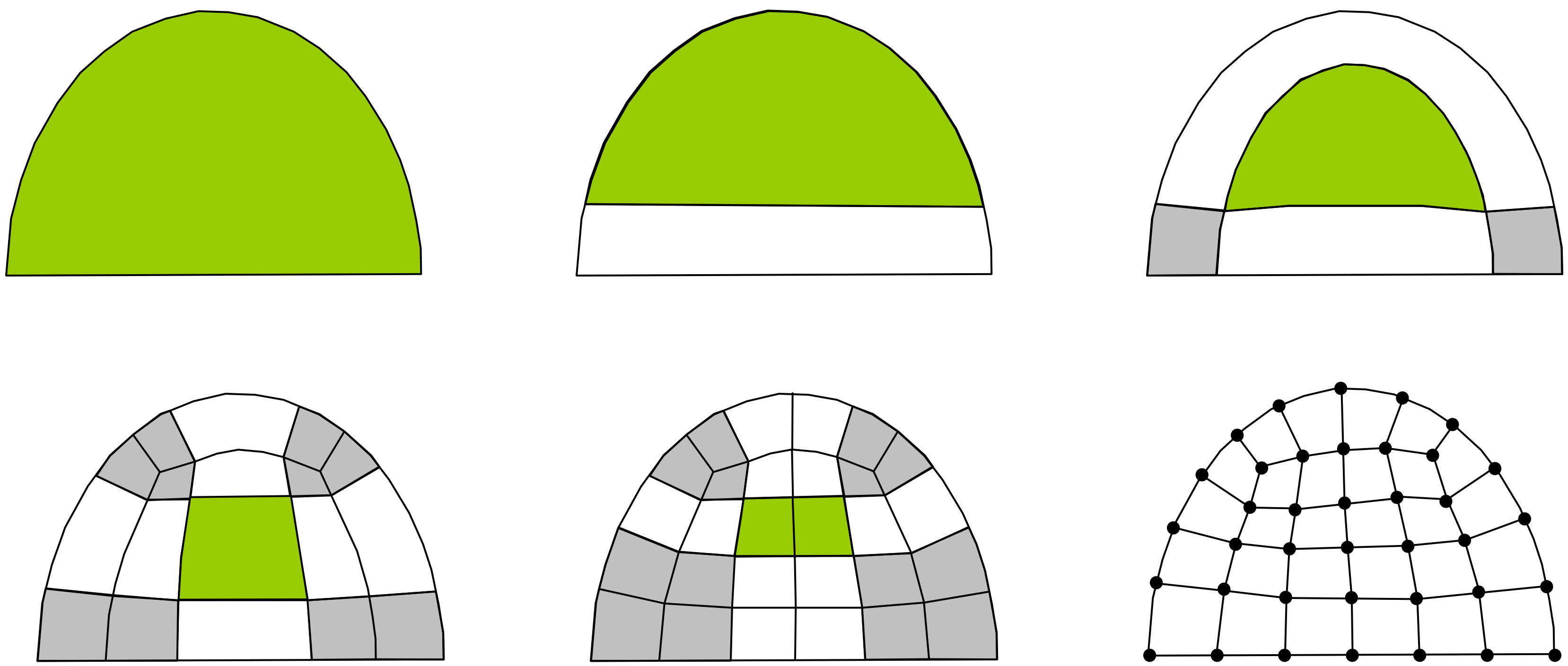
Each Row Advancement Constrains Only 1 DOF



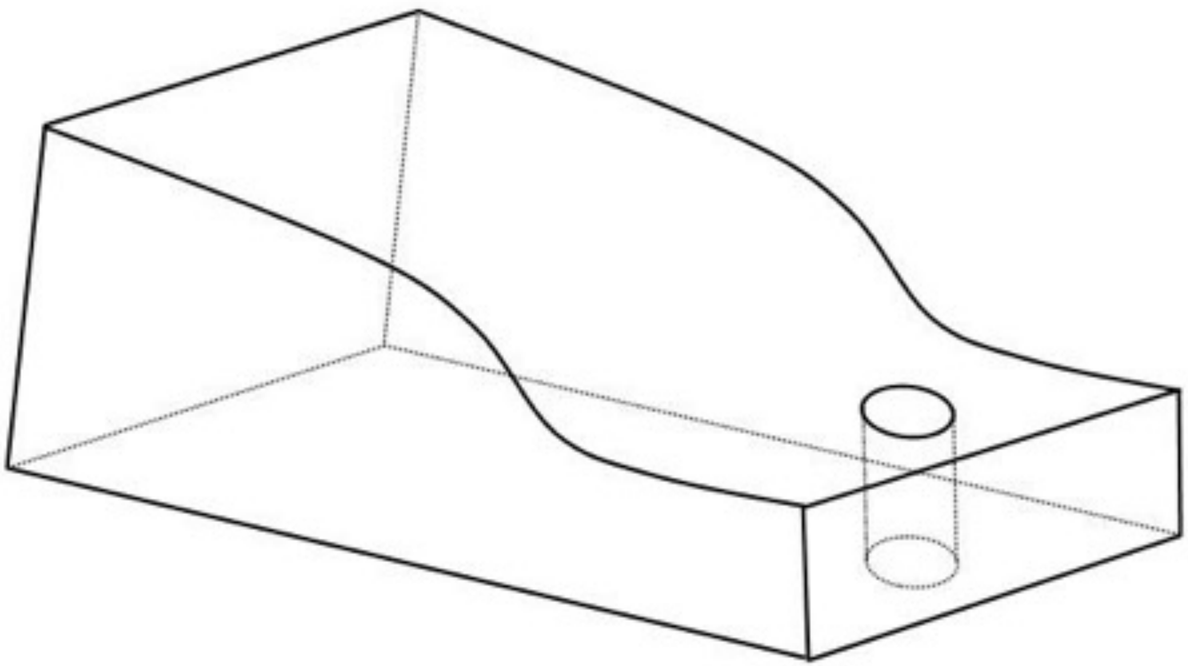
Quads are only completely defined when 2 unconstrained rows cross

Unconstrained Paving

Each Row Advancement Constrains Only 1 DOF

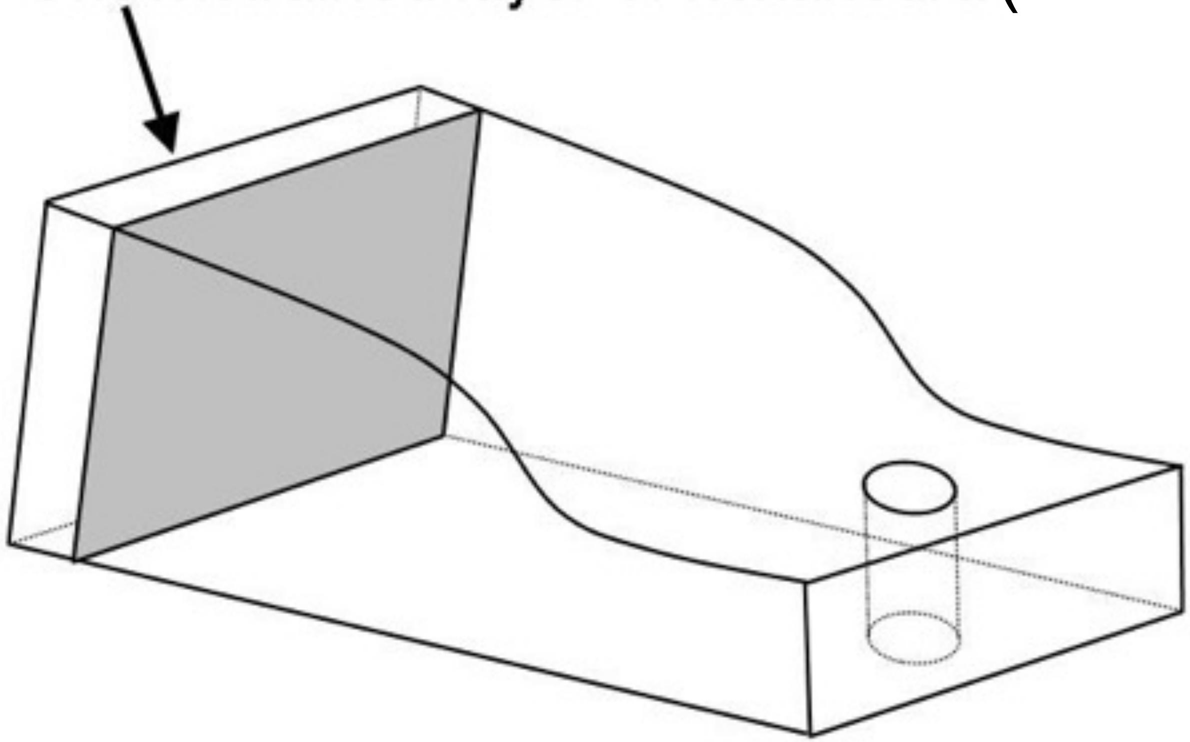


Unconstrained Plastering



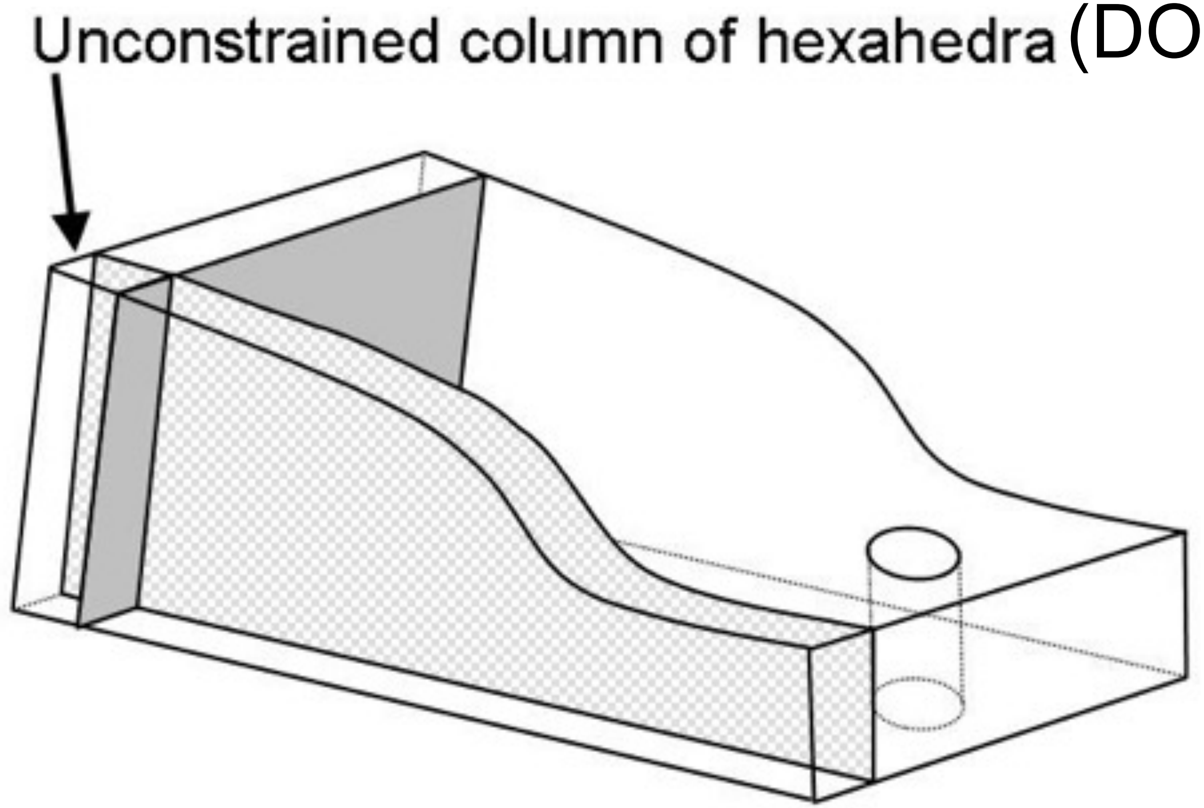
Unconstrained Plastering

Unconstrained layer of hexahedra (DOF = 2)

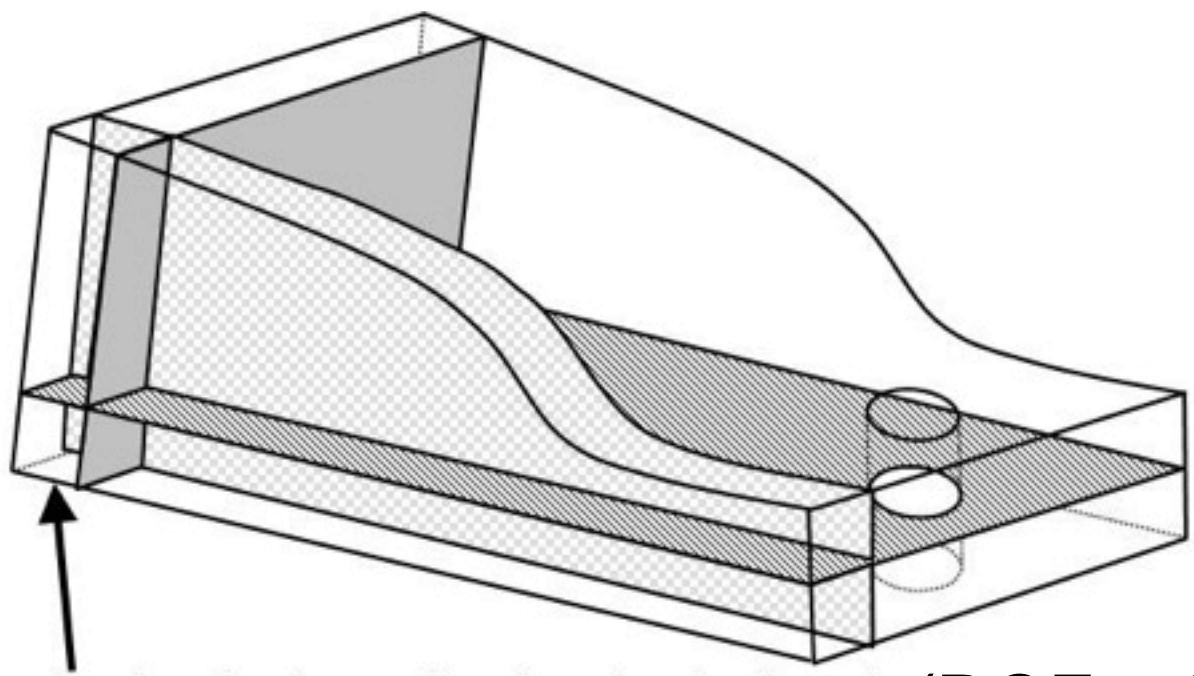


Unconstrained Plastering

(DOF = 1)

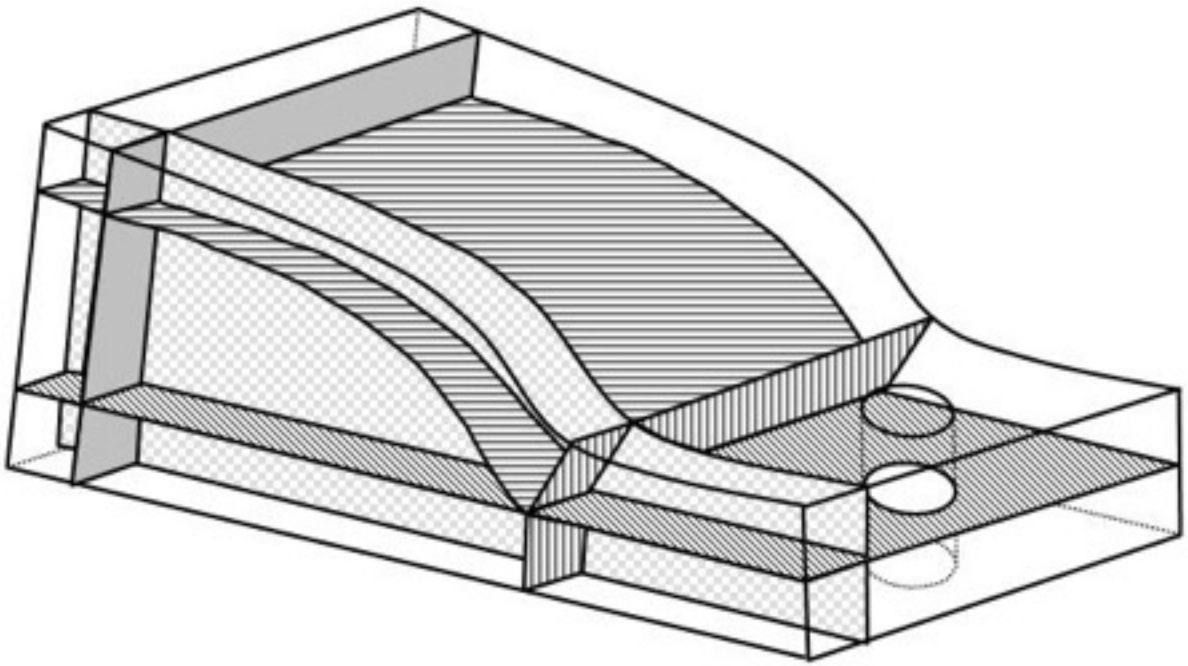


Unconstrained Plastering

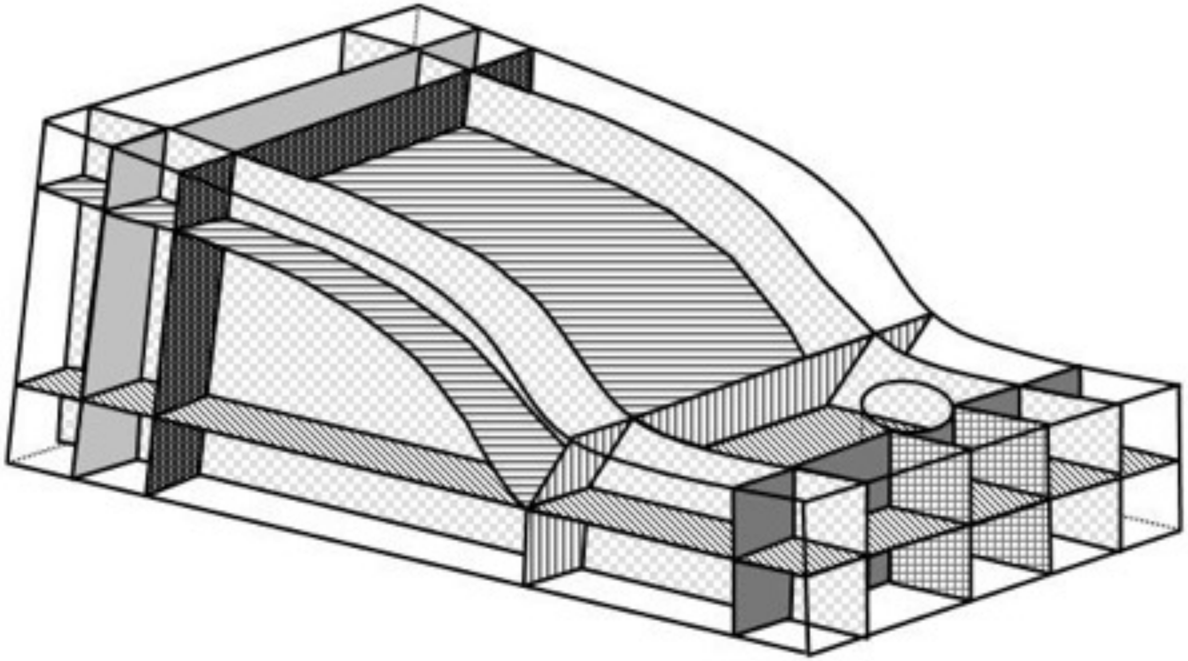


A single hexahedra is defined (DOF = 0)

Unconstrained Plastering



Unconstrained Plastering



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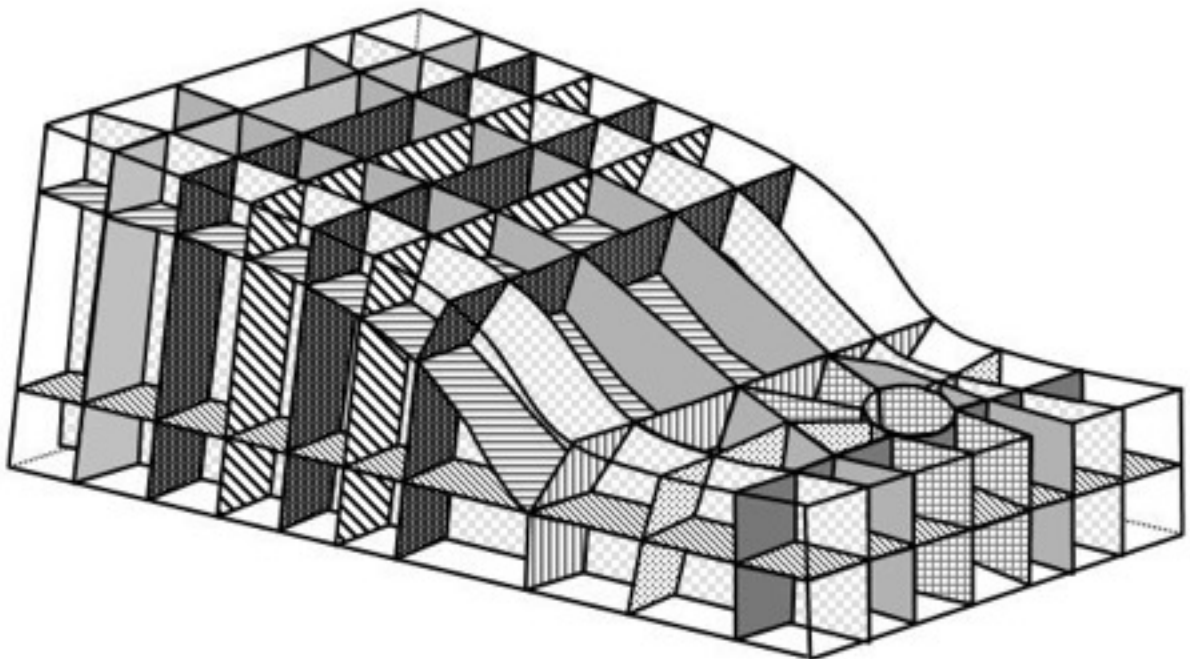

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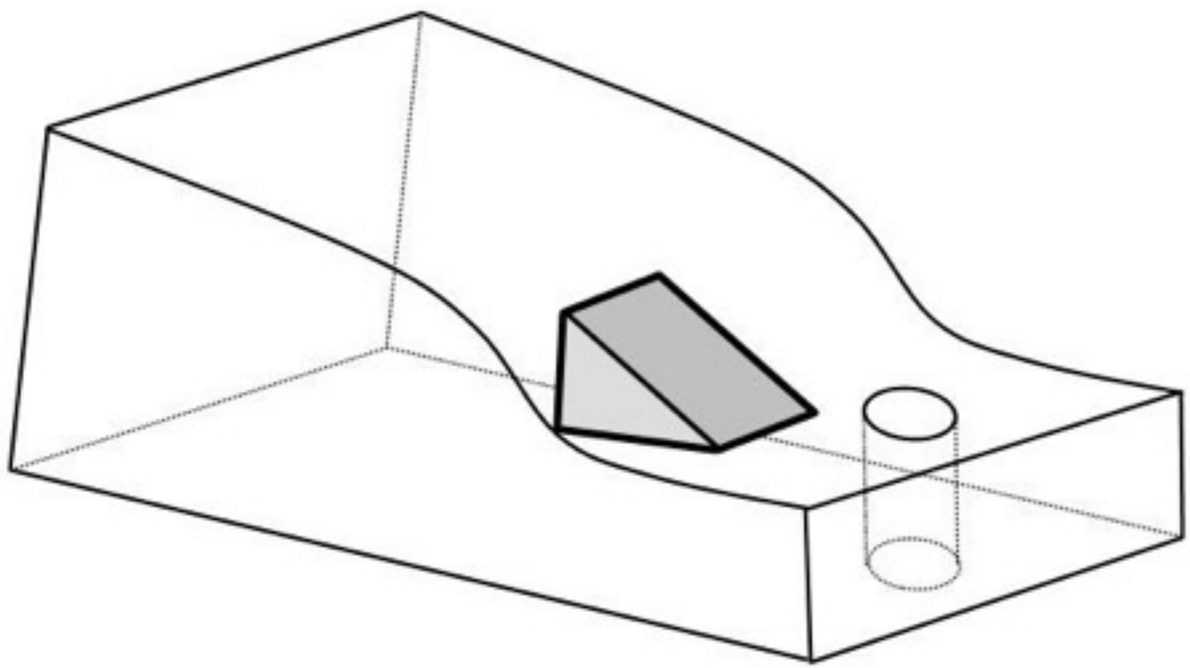

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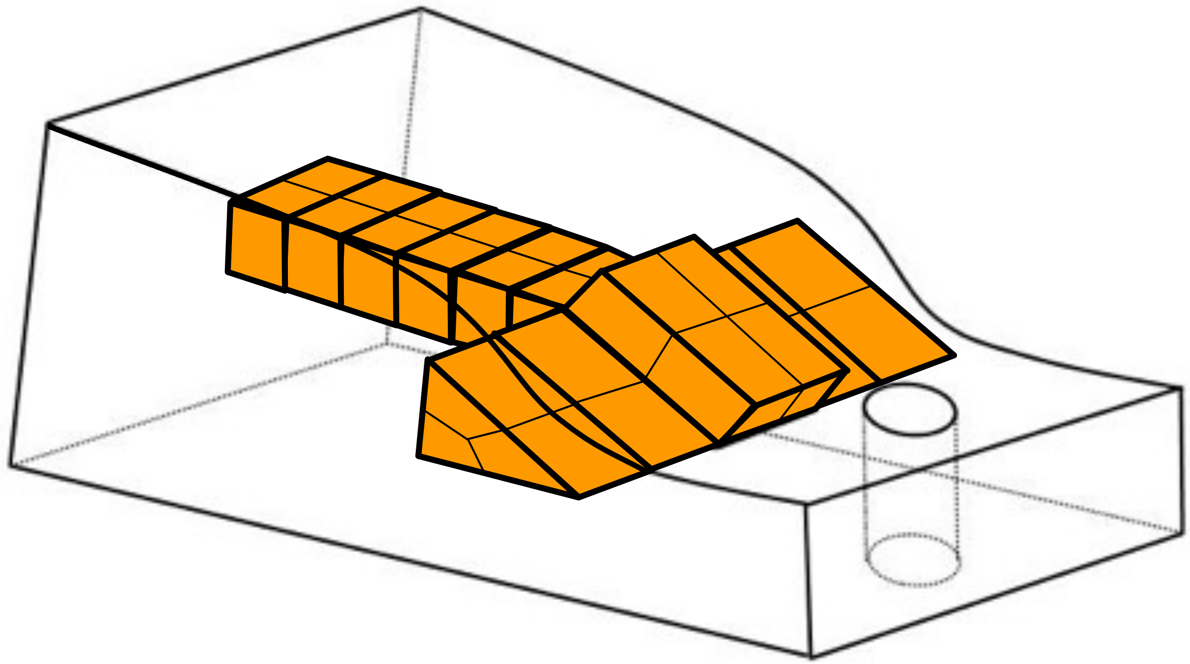
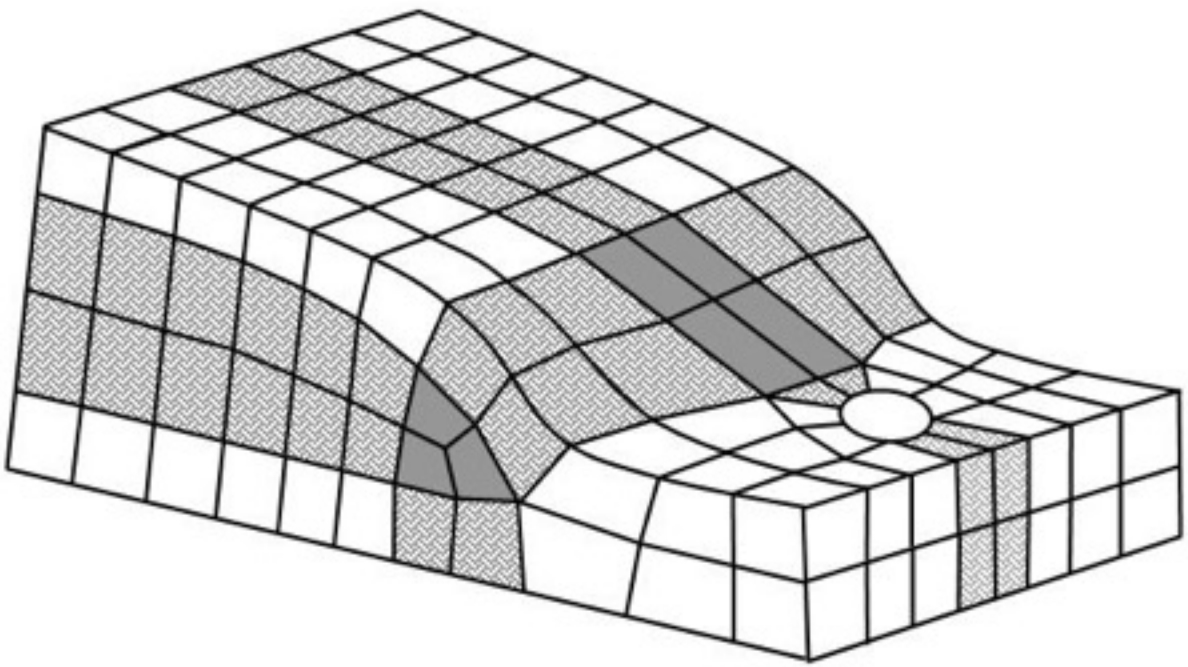
Unconstrained Plastering



Unconstrained Plastering



Hybrid Methods

CFD Meshing

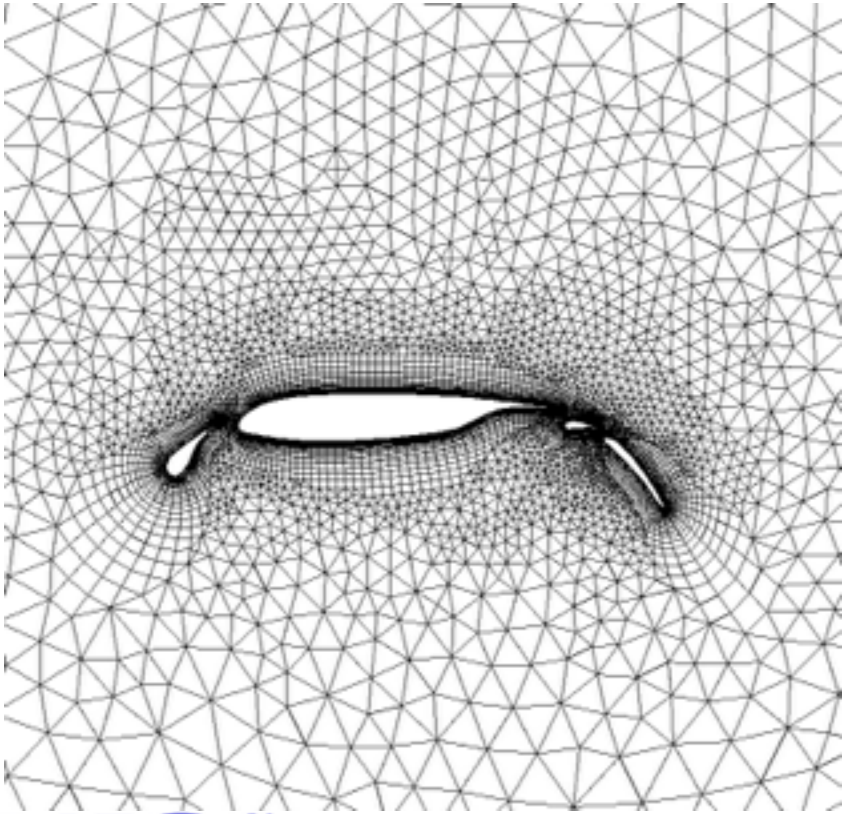


Image courtesy of Roy P. Koomullil, Engineering Research Center, Mississippi State University, http://www.erc.msstate.edu/~roy/

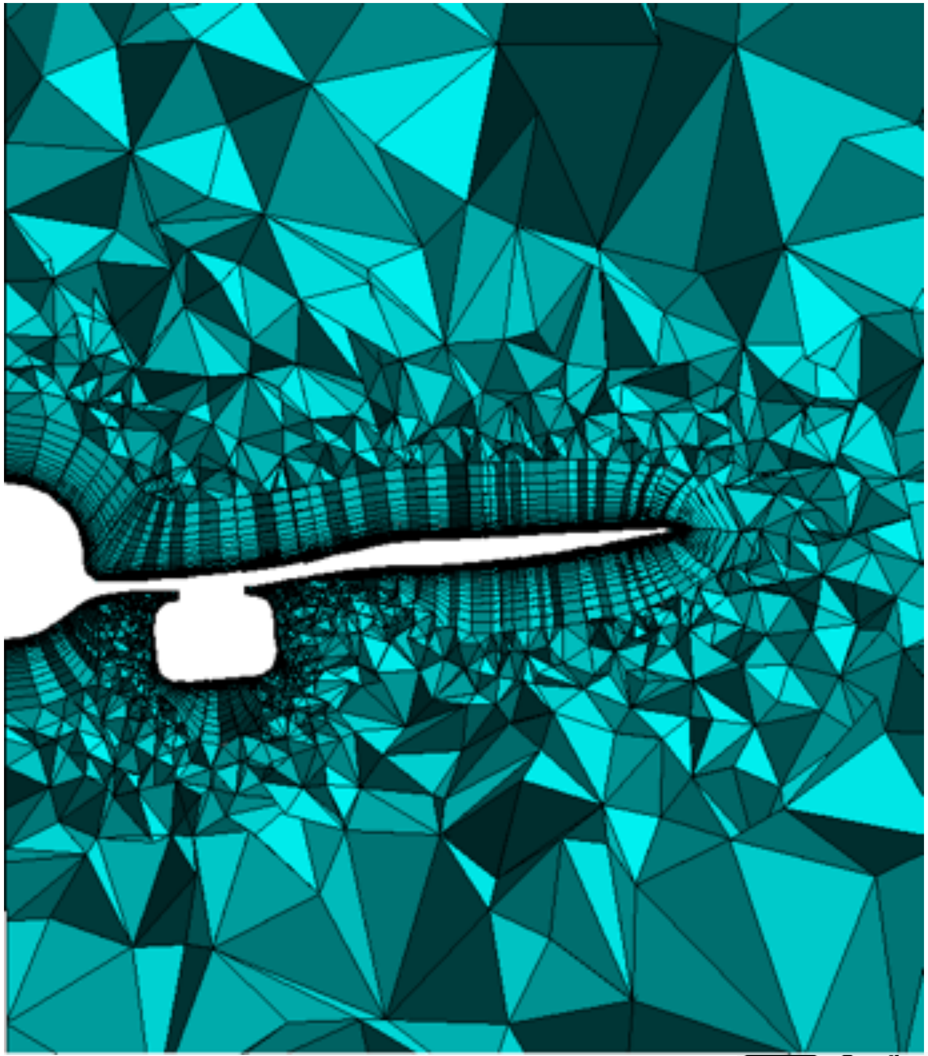
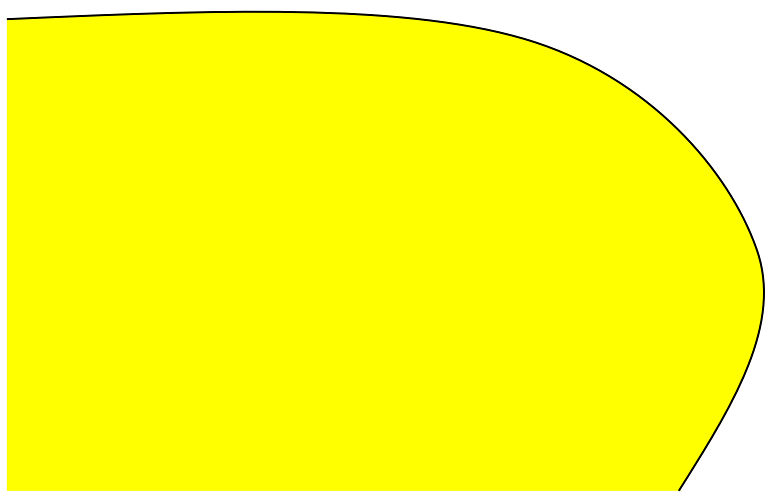


Image courtesy of acelab, University of Texas, Austin, http://acelab.ae.utexas.edu.

Hybrid Methods

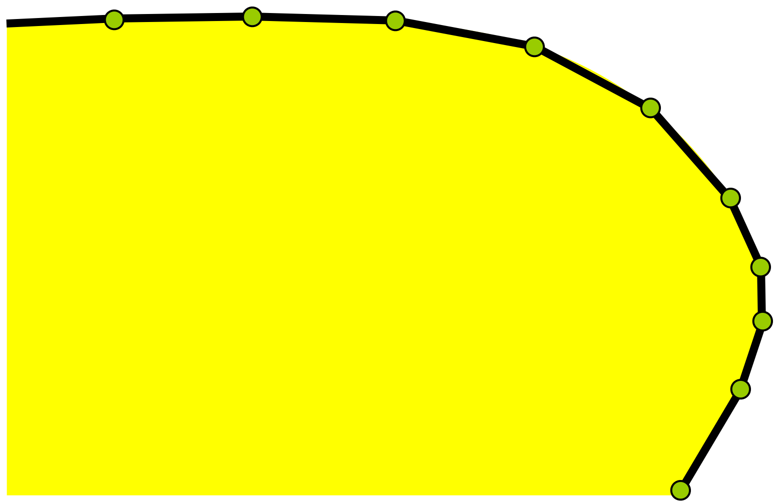
Computational Modeling Sciences Department

Advancing Layers Method



Hybrid Methods

Advancing Layers Method

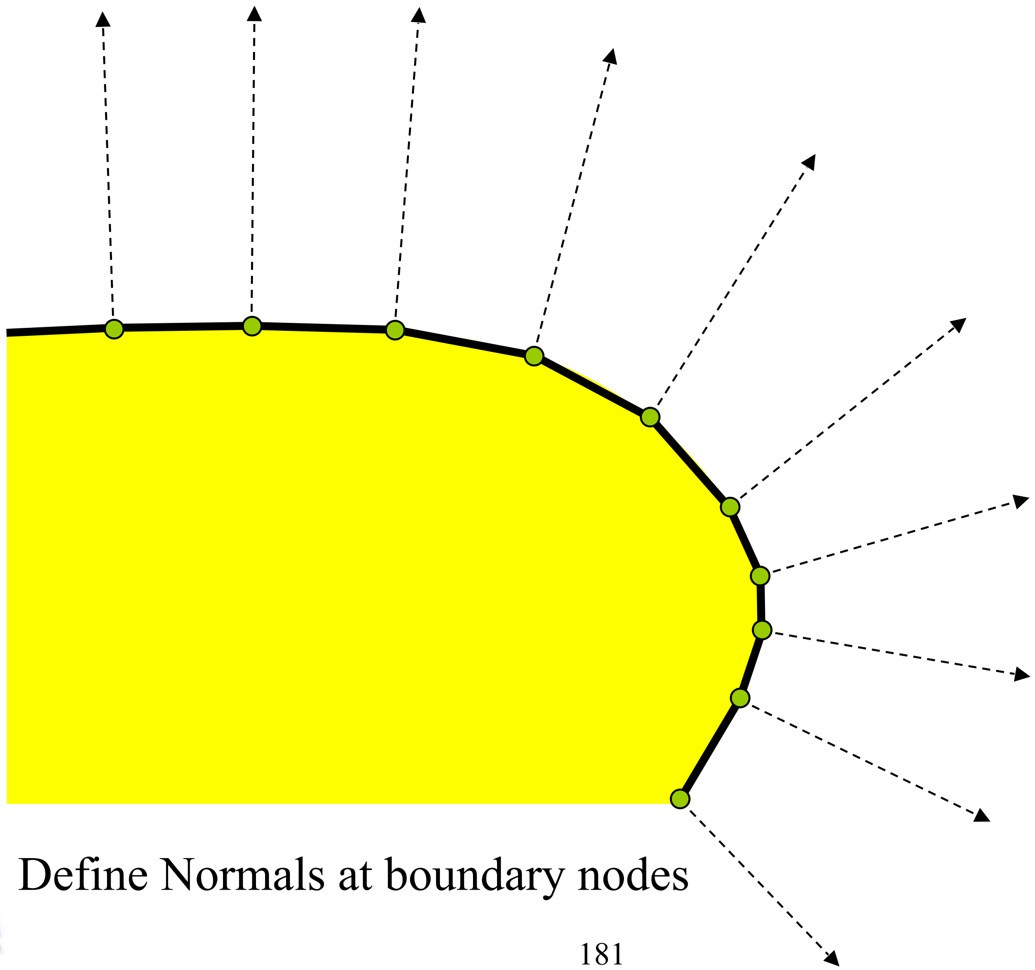


Discretize Boundary

Hybrid Methods

Computational Modeling Sciences Department

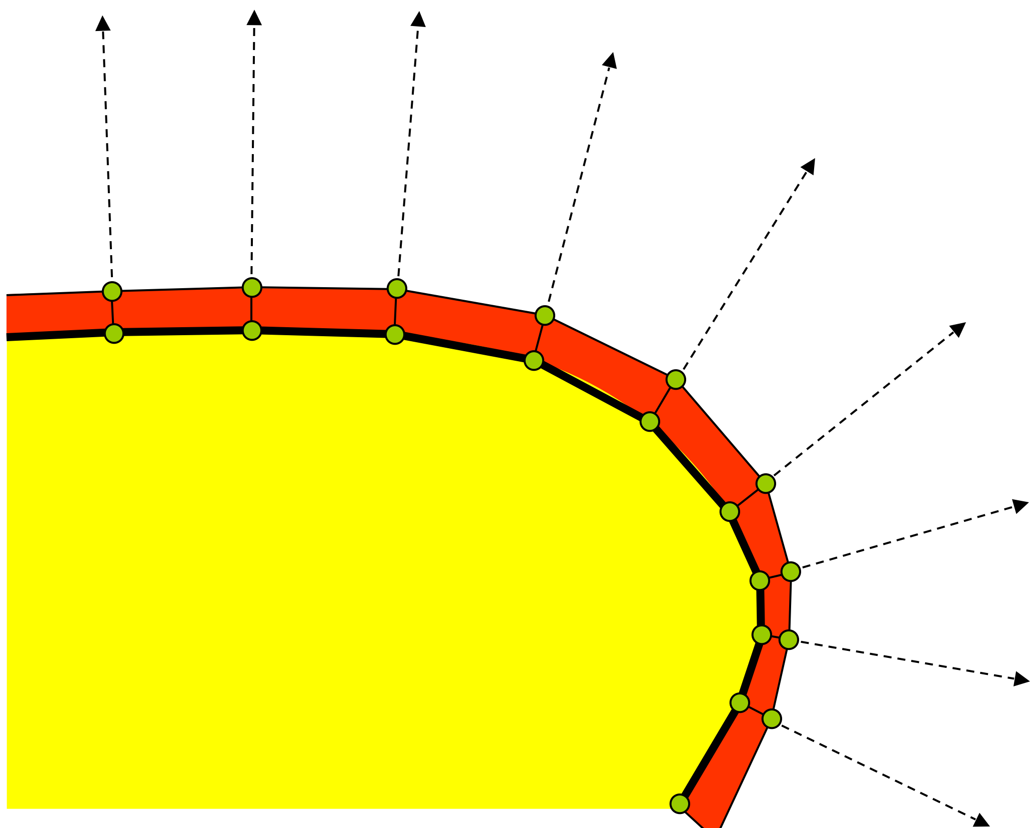
Advancing Layers Method
(Pirzadeh, 1994)

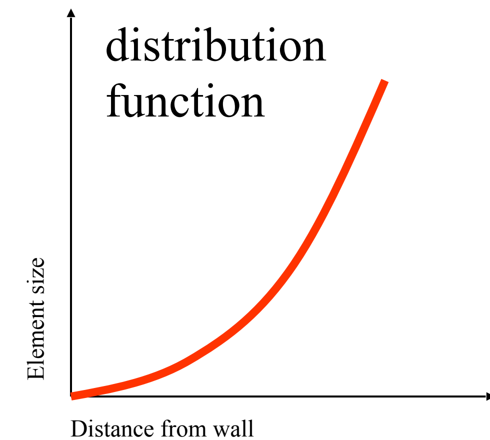


Define Normals at boundary nodes

Hybrid Methods

Advancing Layers Method



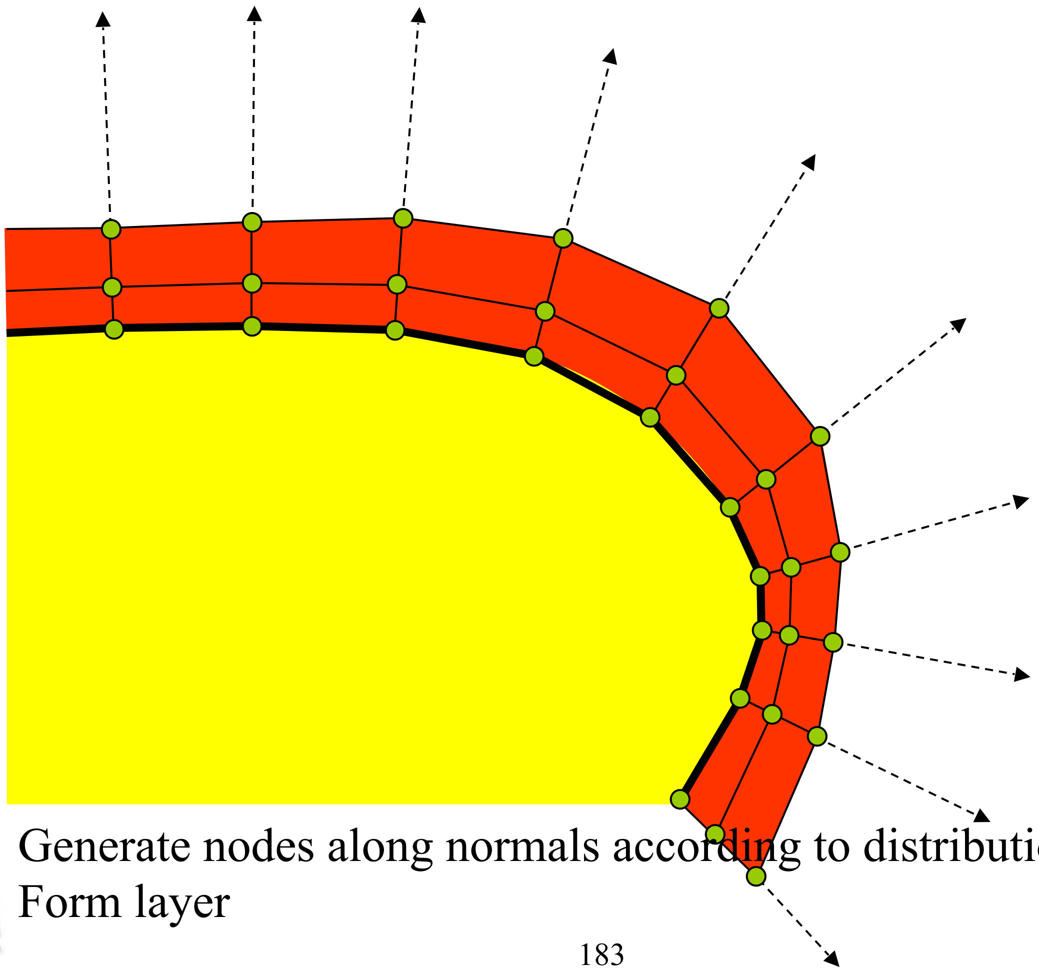


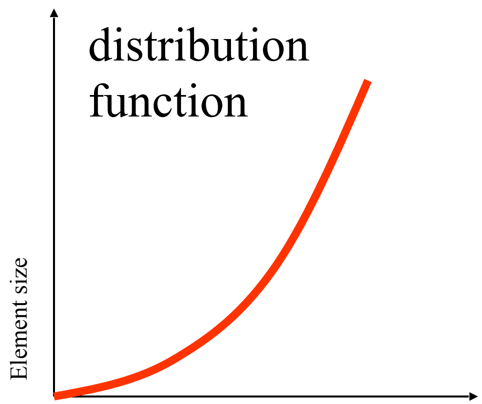
Generate nodes along normals according to distribution function
Form layer

Hybrid Methods

Computational Modeling Sciences Department

Advancing Layers Method





Generate nodes along normals according to distribution function
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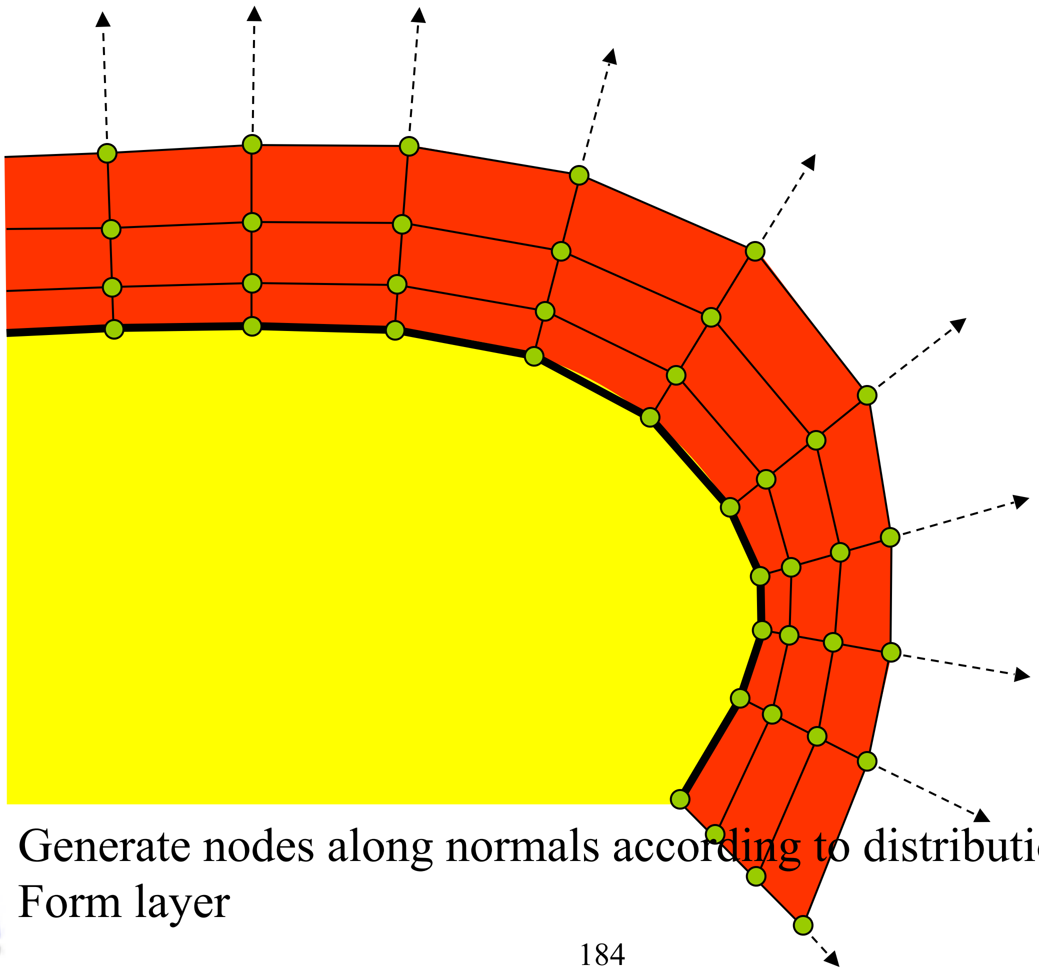
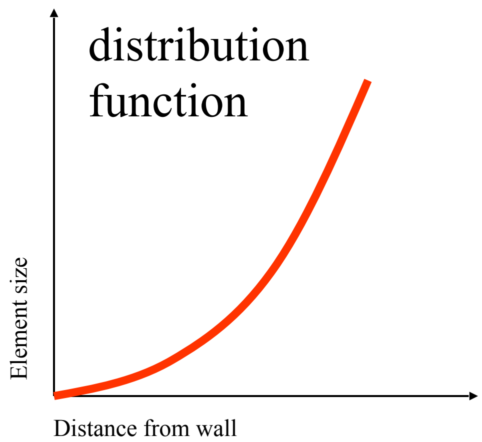
Hybrid Methods

Advancing Layers Method





Generate nodes along normals according to distribution function
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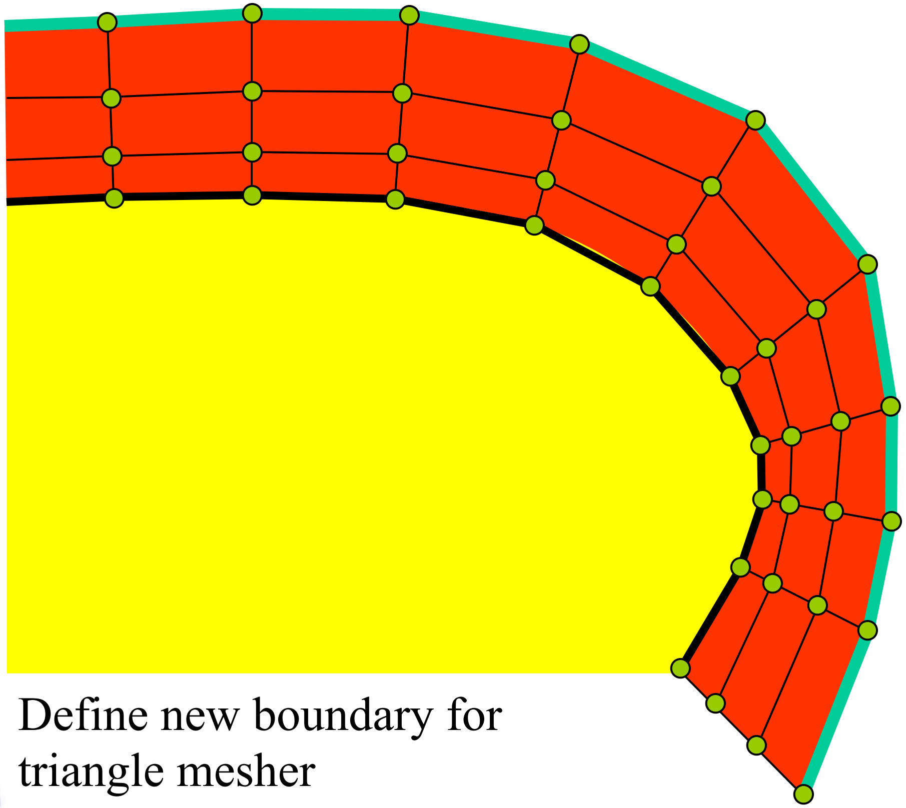
Hybrid Methods

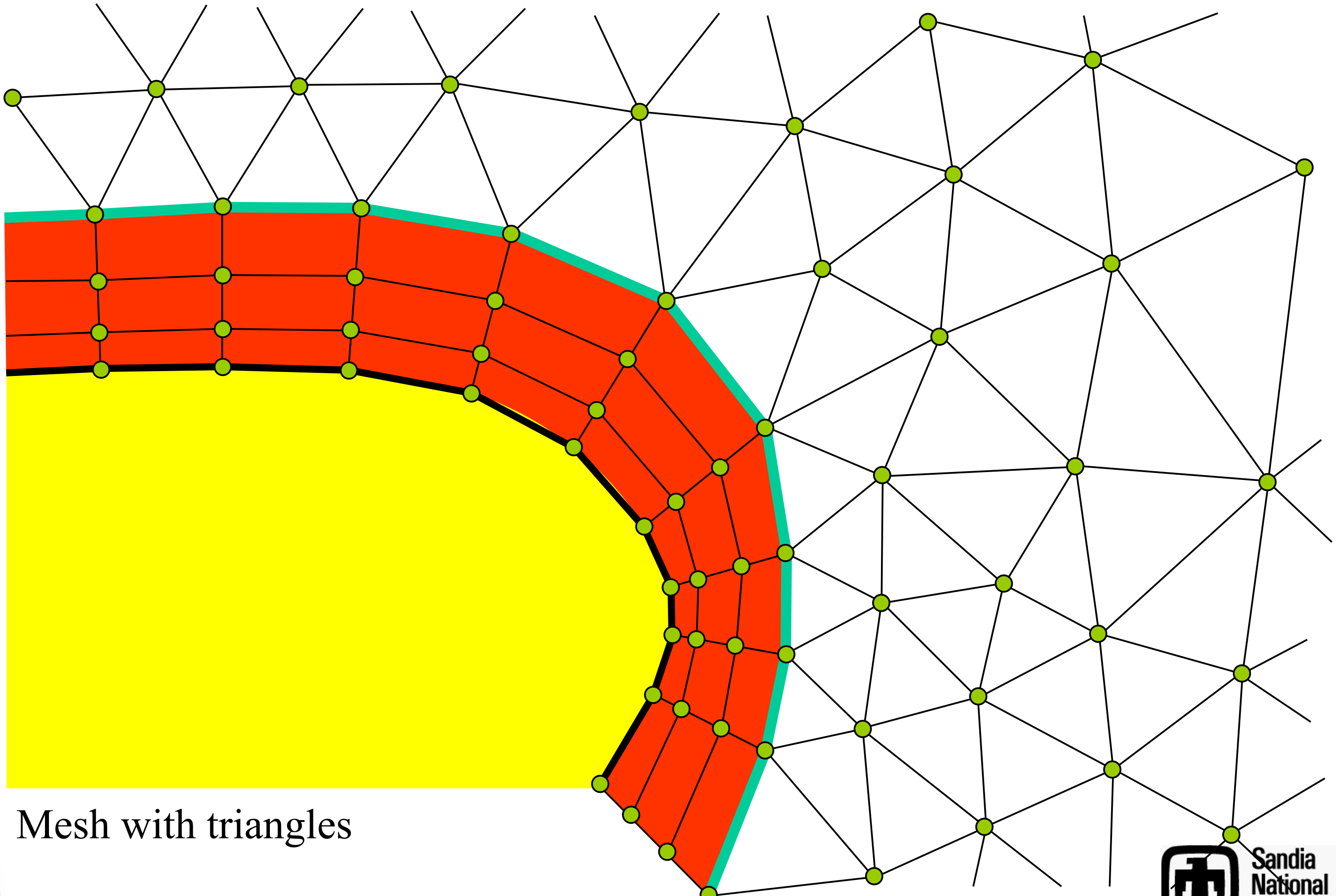
Computational Modeling Sciences Department

Advancing Layers Method



Define new boundary for triangle mesher

Hybrid Methods



Mesh with triangles

Hybrid Methods

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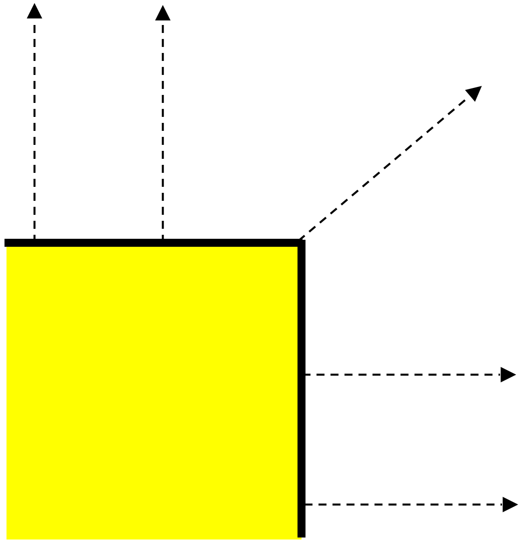
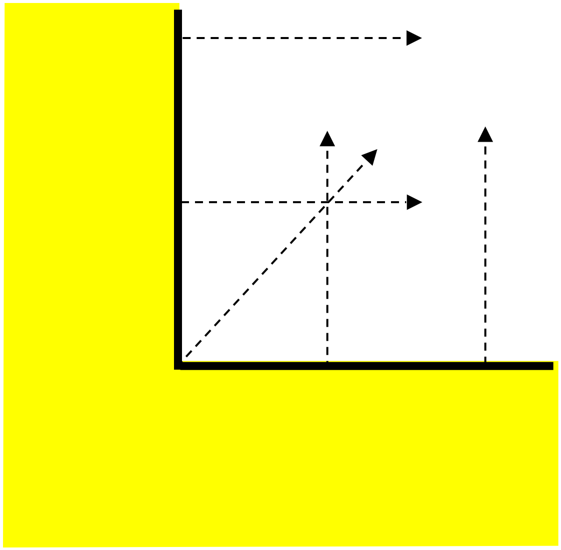


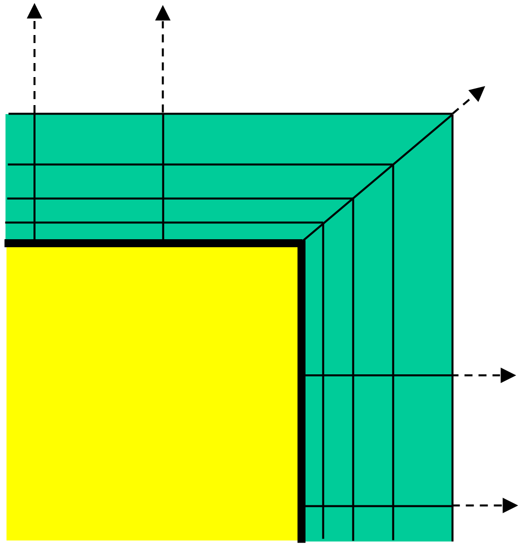
Convex Corner



Concave Corner

Hybrid Methods

Computational Modeling Sciences Department



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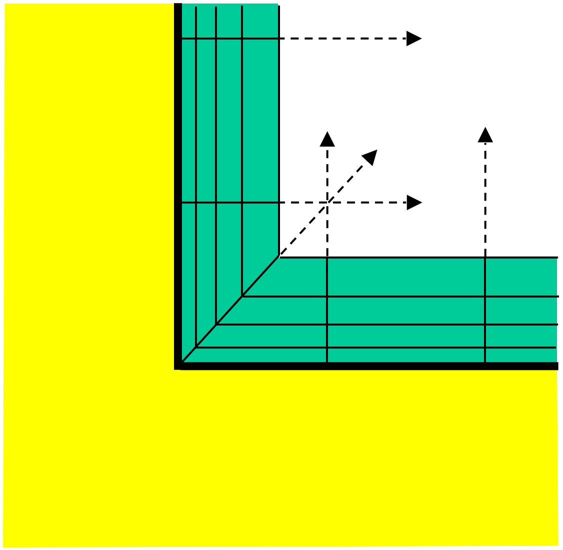
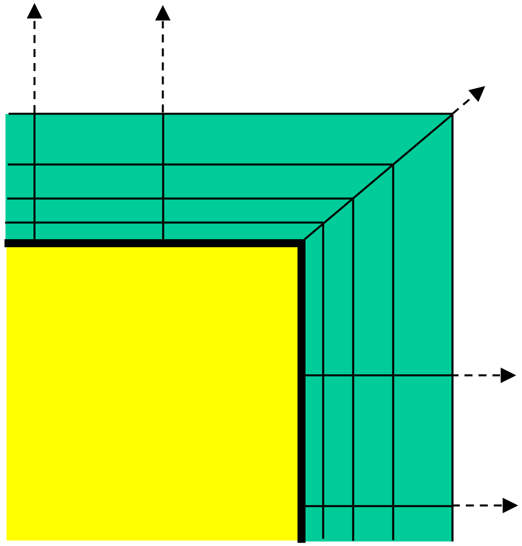


Concave Corner

Hybrid Methods

Computational Modeling Sciences Department



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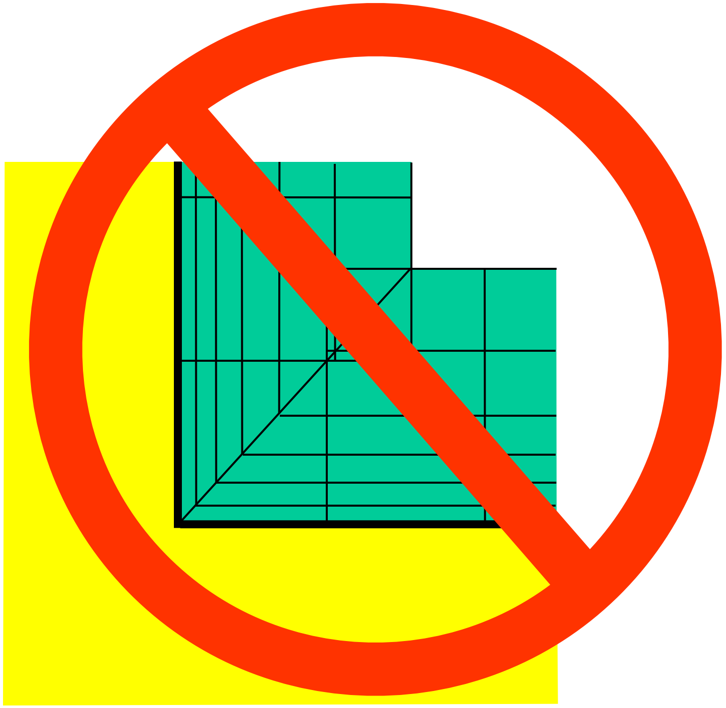
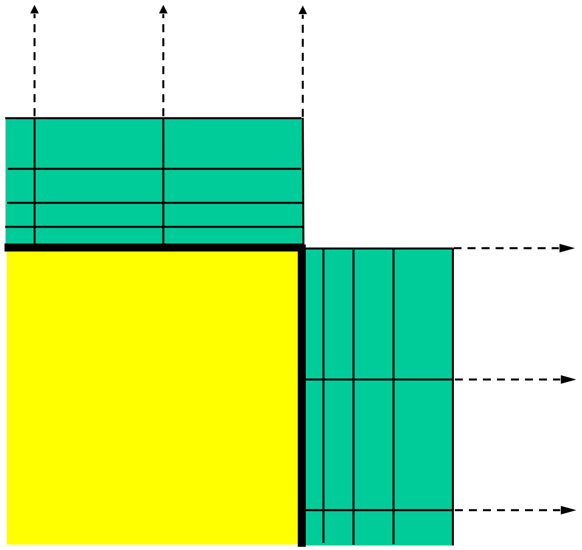


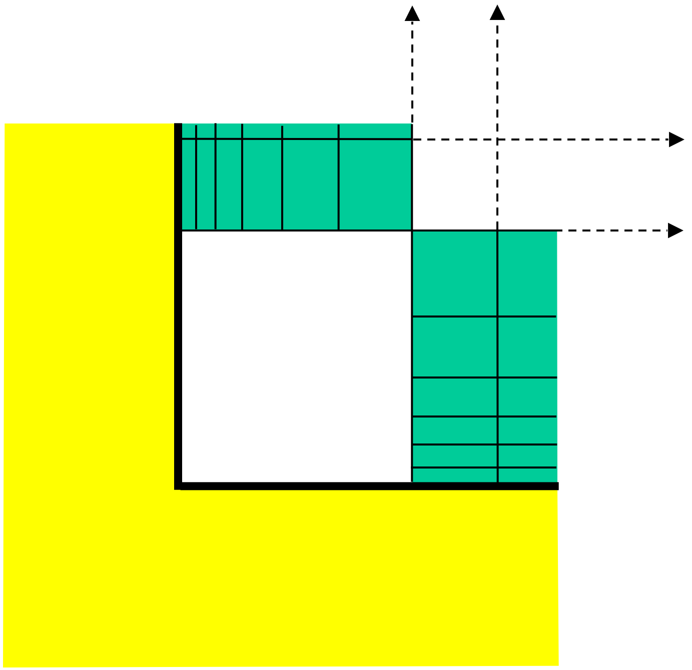
Concave Corner

Hybrid Methods

Computational Modeling Sciences Department



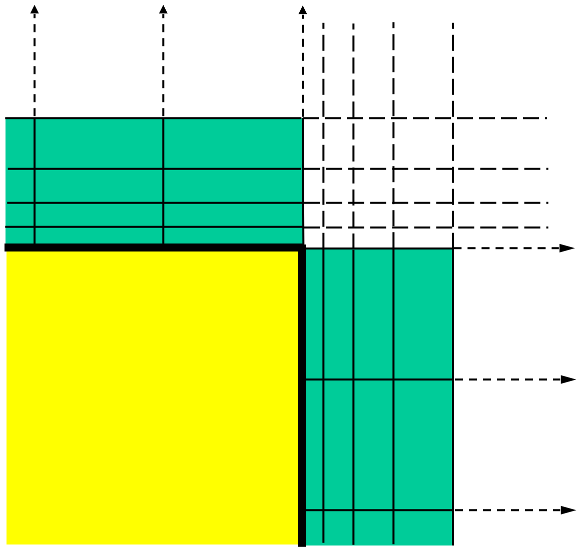
Convex Corner



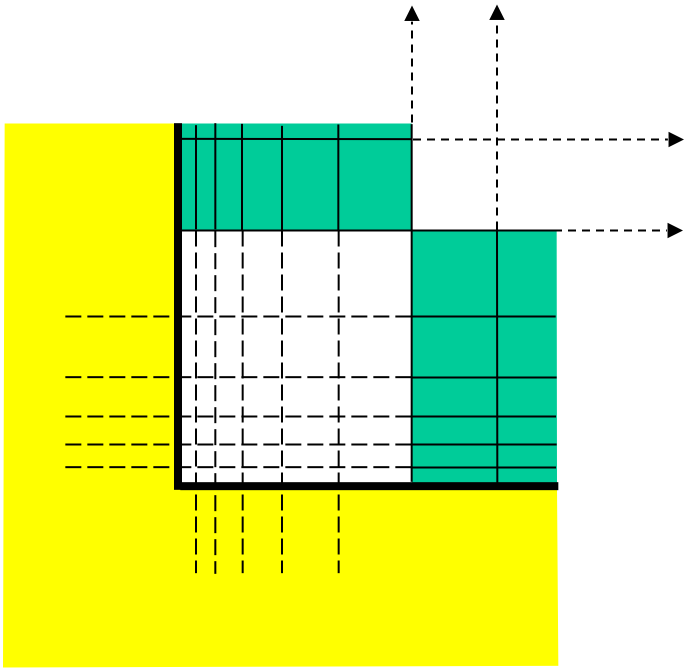
Concave Corner

Blend Regions

Hybrid Meshes



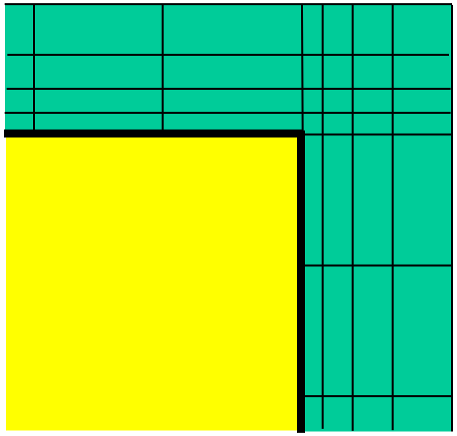
Convex Corner



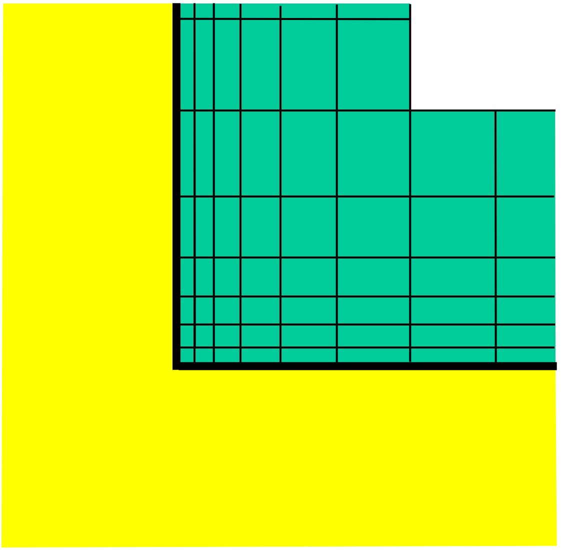
Concave Corner

Blend Regions

Hybrid Methods



Convex Corner

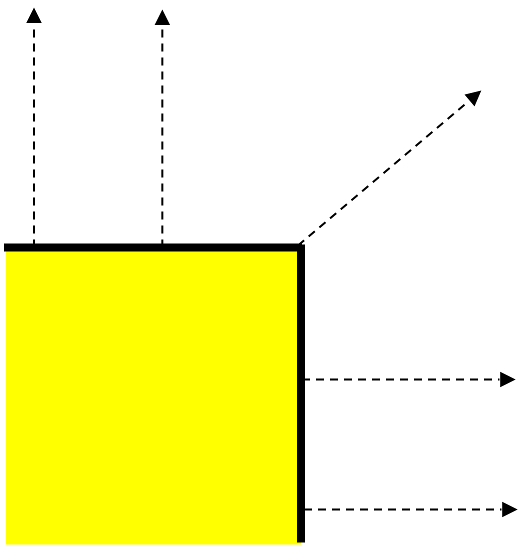


Concave Corner

Blend Regions

Hybrid Methods

Computational Modeling Sciences Department



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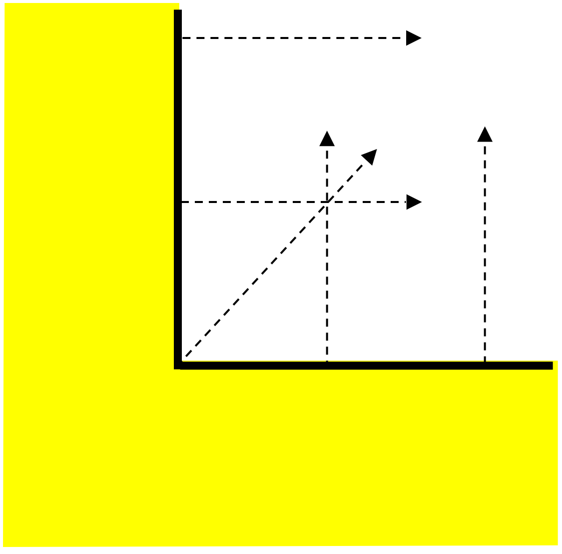


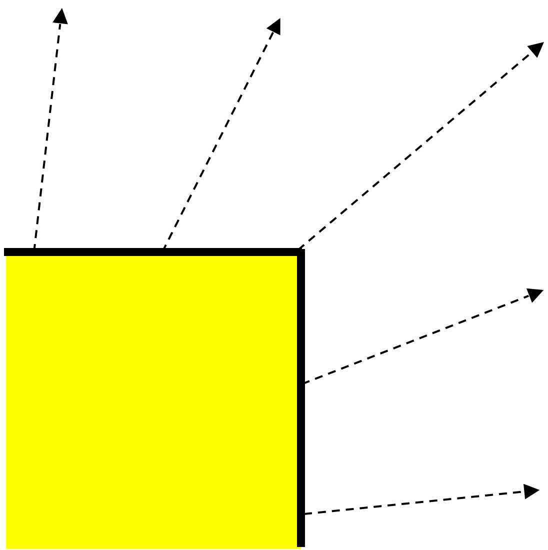
Concave Corner

Smoothed Normals

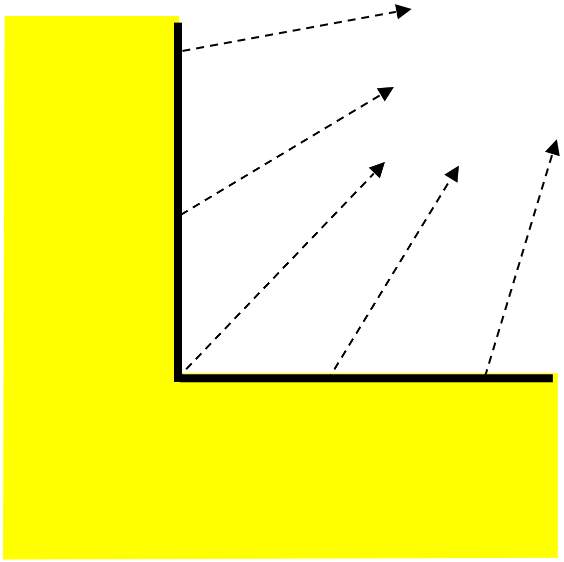
National Nuclear Security Administration

Sandia National Laboratories

Hybrid Methods



Convex Corner

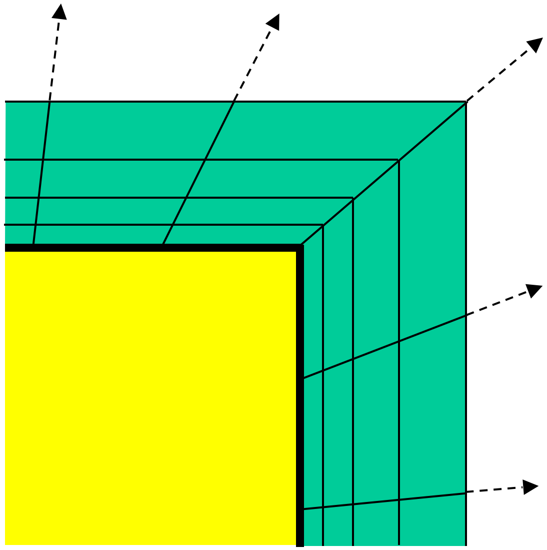


Concave Corner

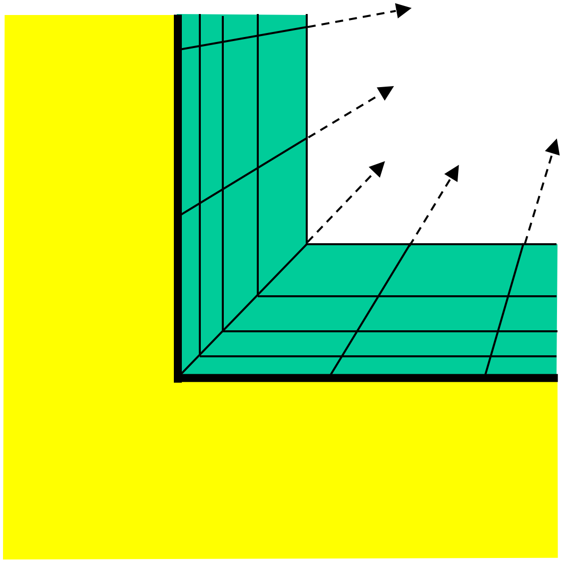
Smoothed Normals

Hybrid Methods

Computational Modeling Sciences Department



Convex Corner

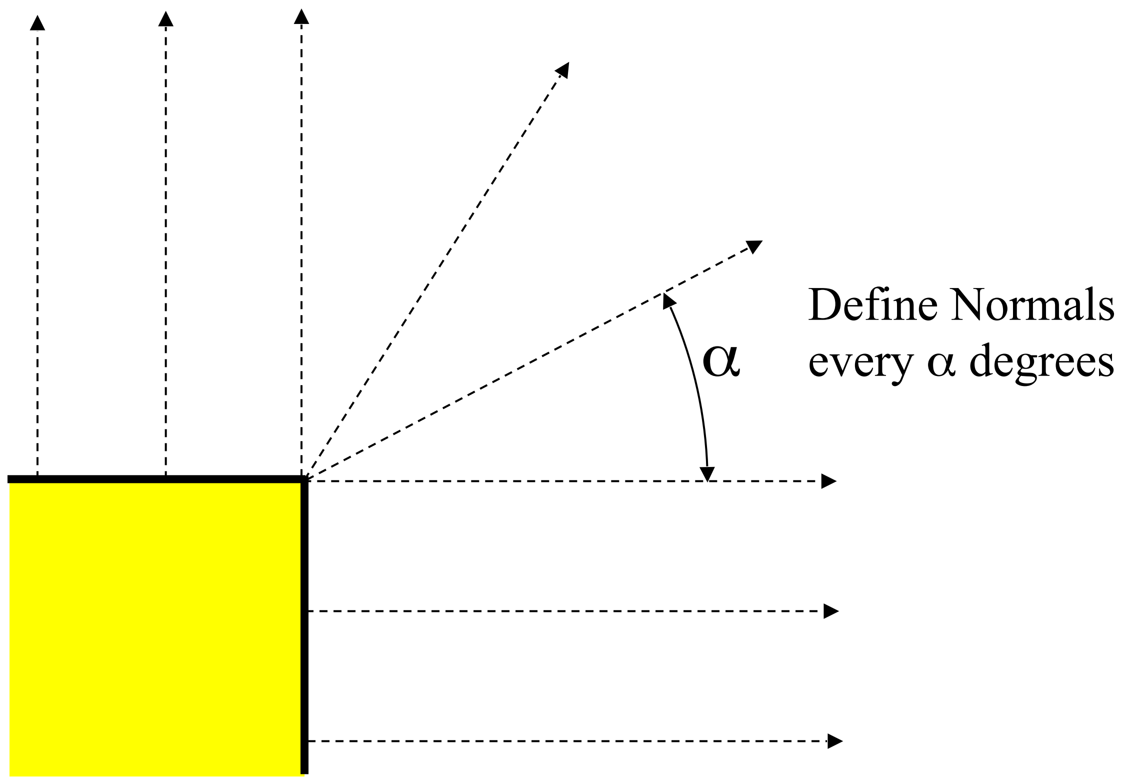


Concave Corner

Smoothed Normals

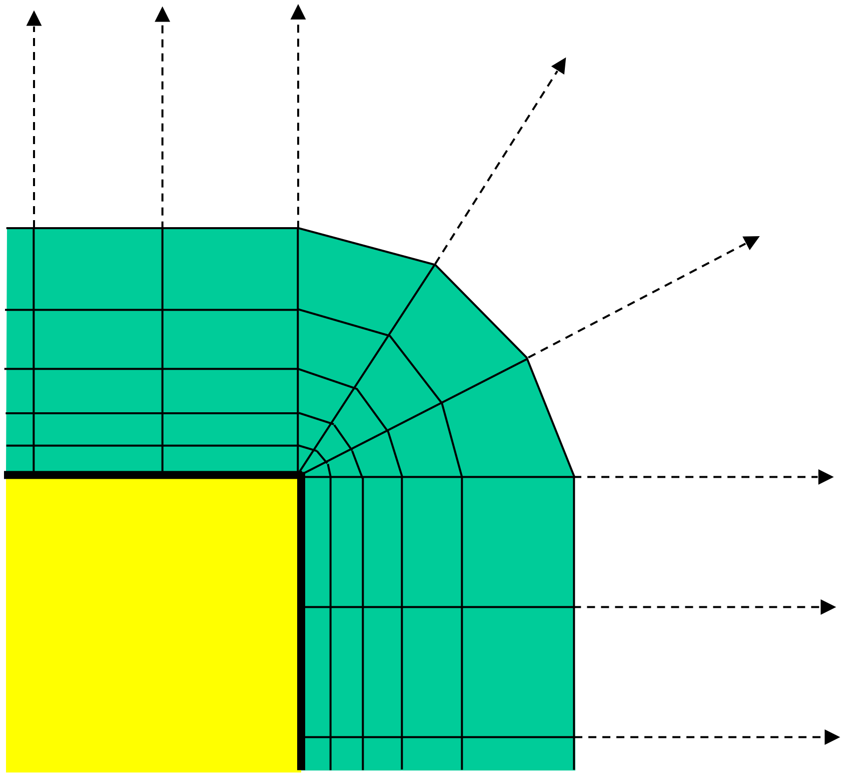
Hybrid Methods

Computational Modeling Sciences Department



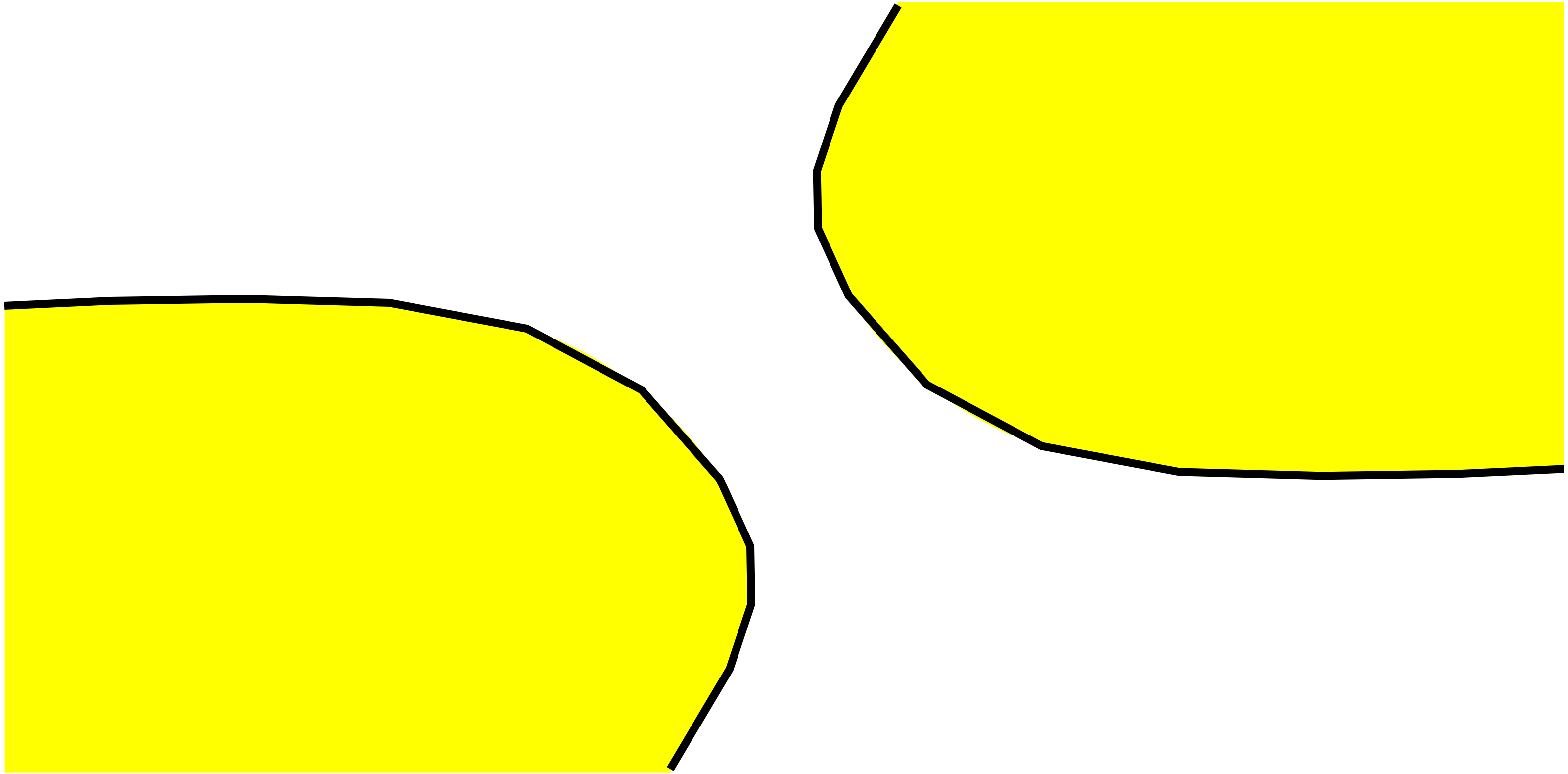
Multiple Normals

Hybrid Methods



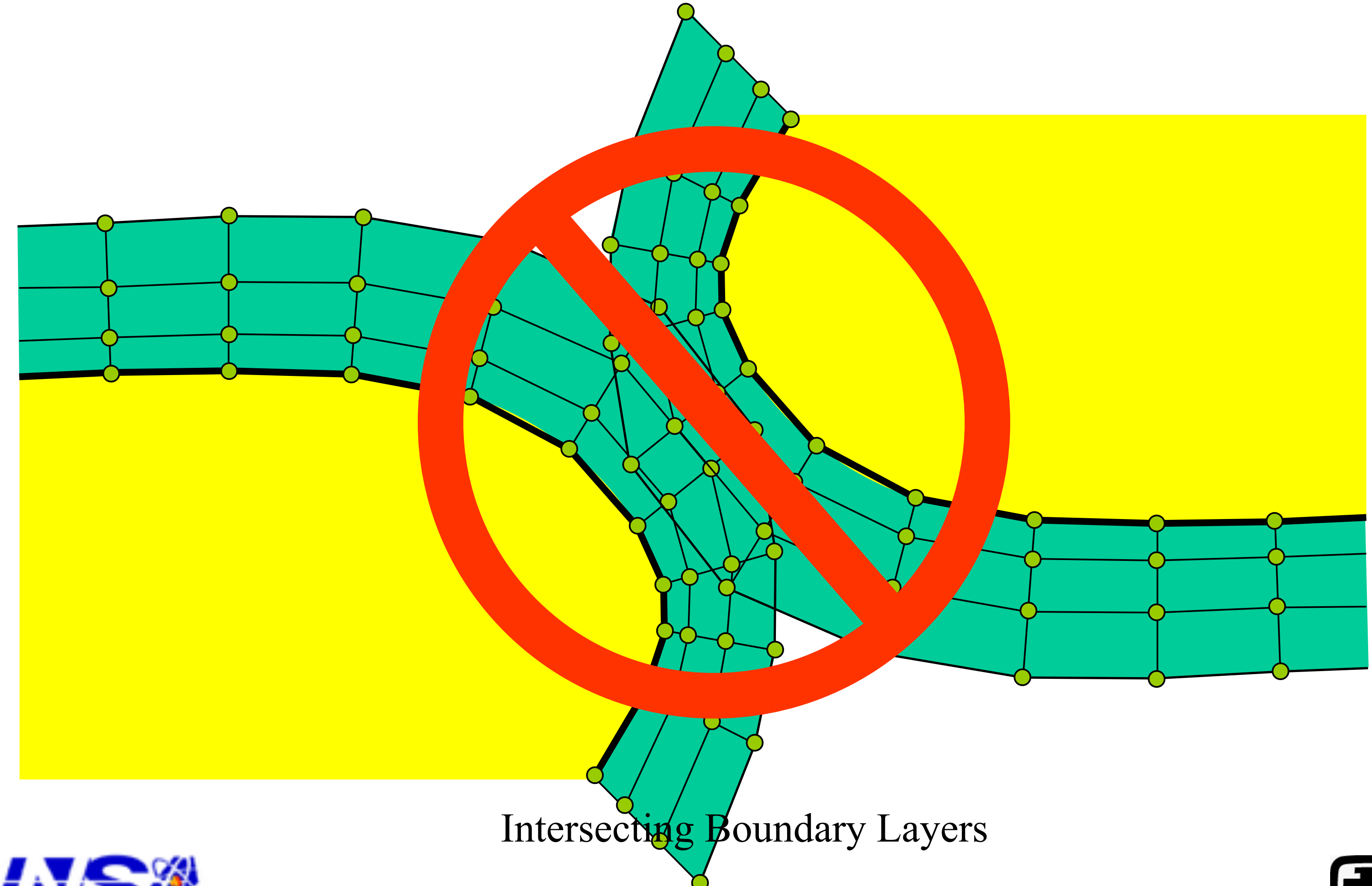
Multiple Normals

Hybrid Methods



Intersecting Boundary Layers

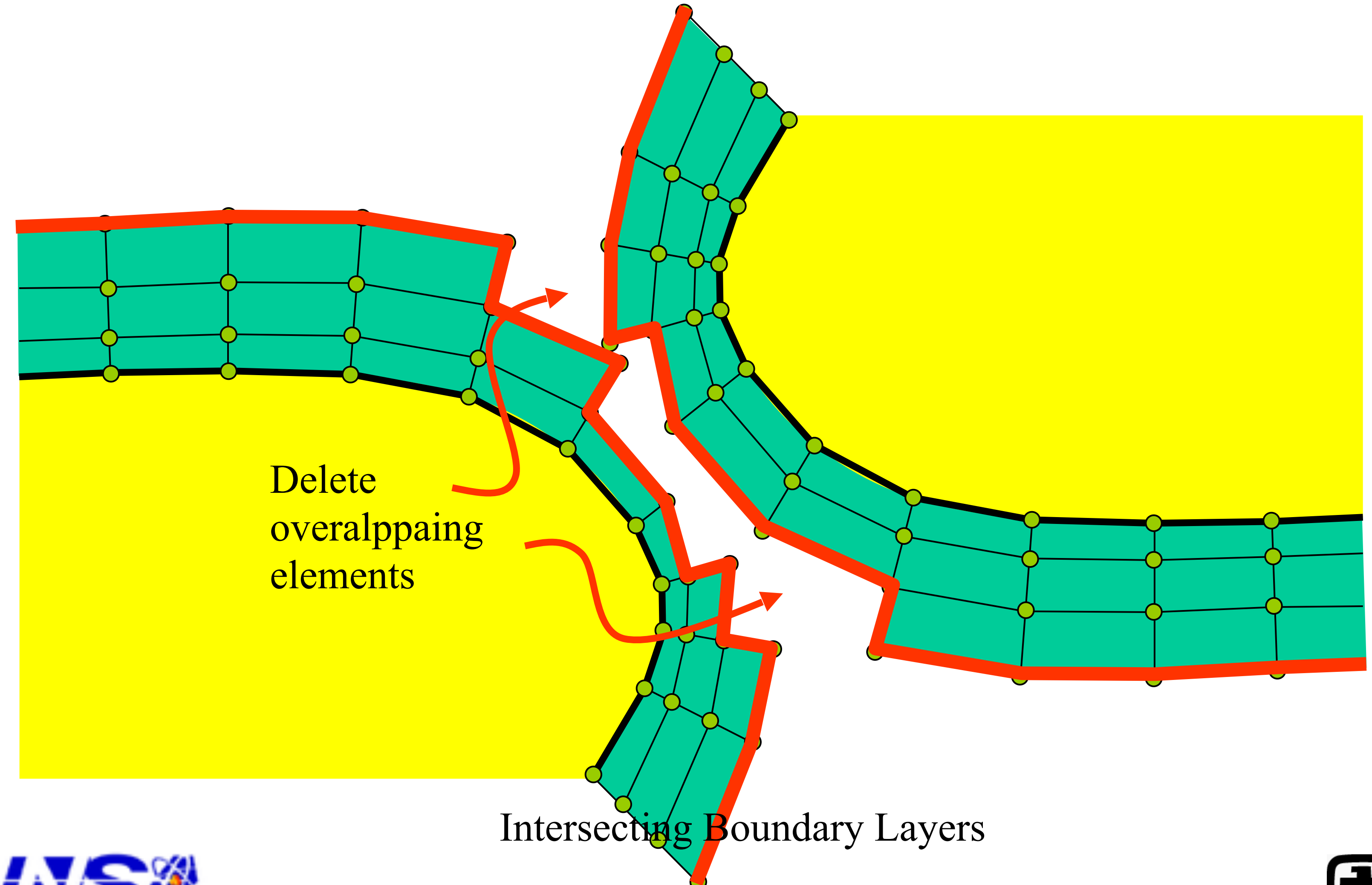
Hybrid Methods



Intersecting Boundary Layers

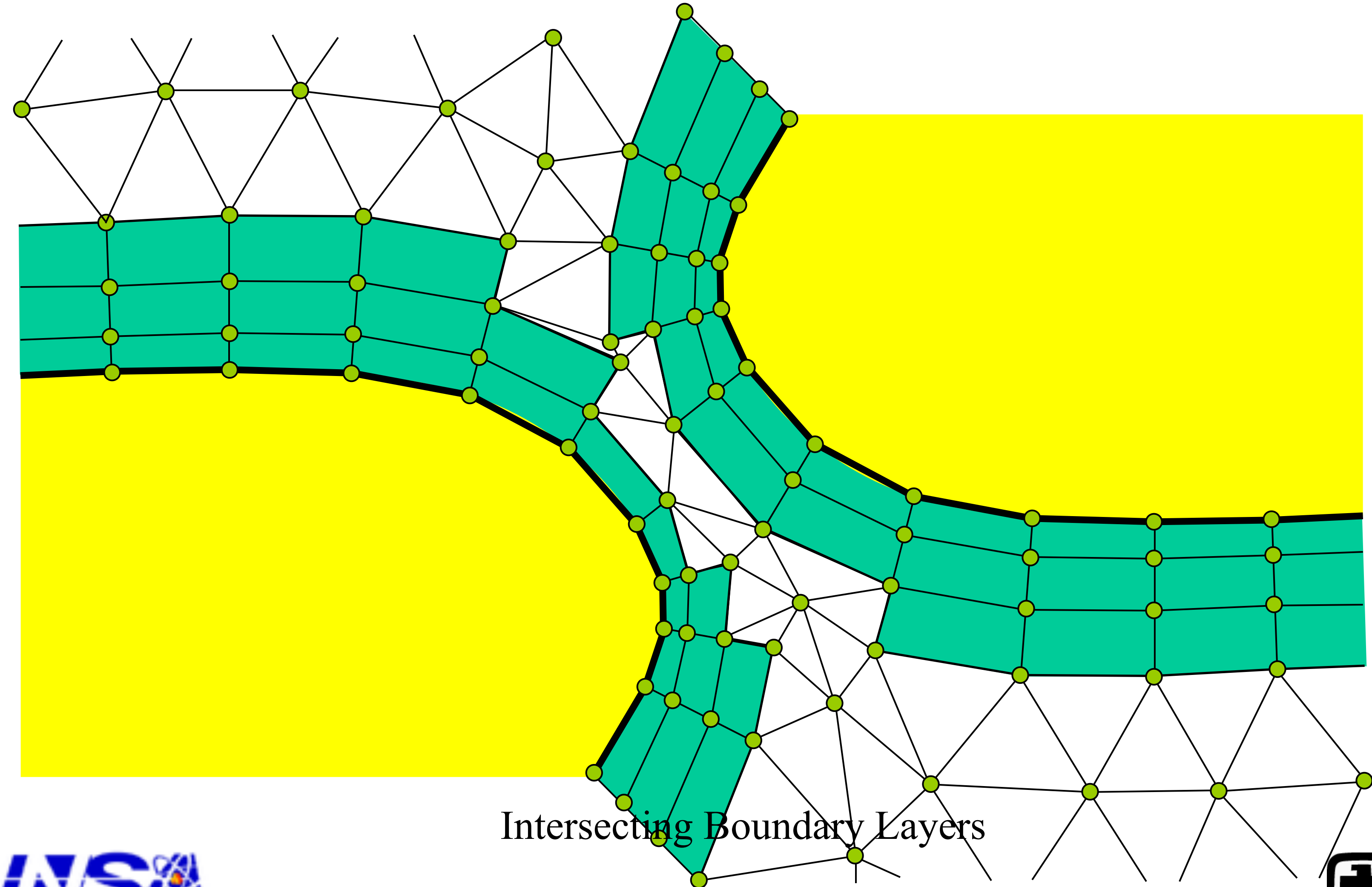
Hybrid Methods

Computational Modeling Sciences Department



Intersecting Boundary Layers

Hybrid Methods



Intersecting Boundary Layers

Hybrid Methods

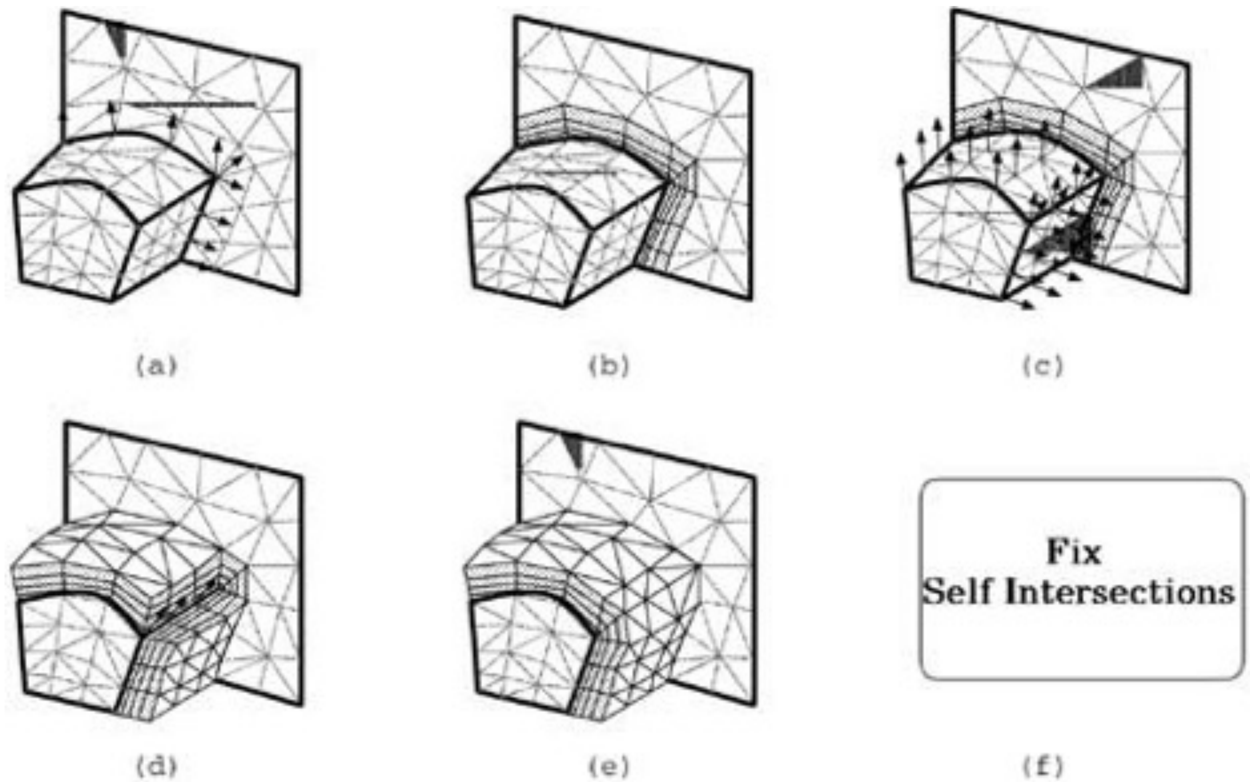
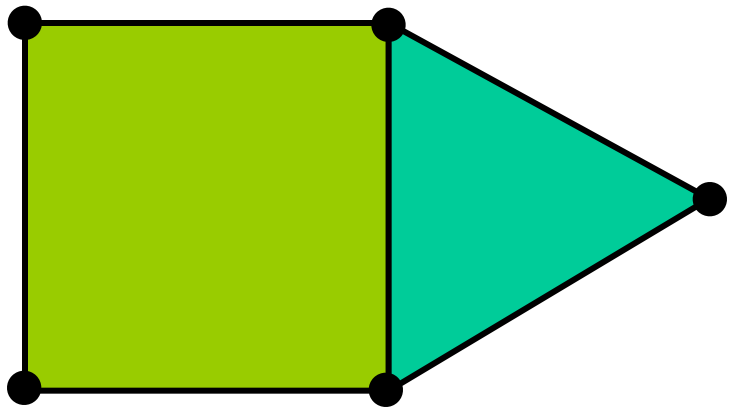


Image courtesy of SCOREC, Rensselaer Polytechnic Institute, http://www.scorec.rpi.edu/

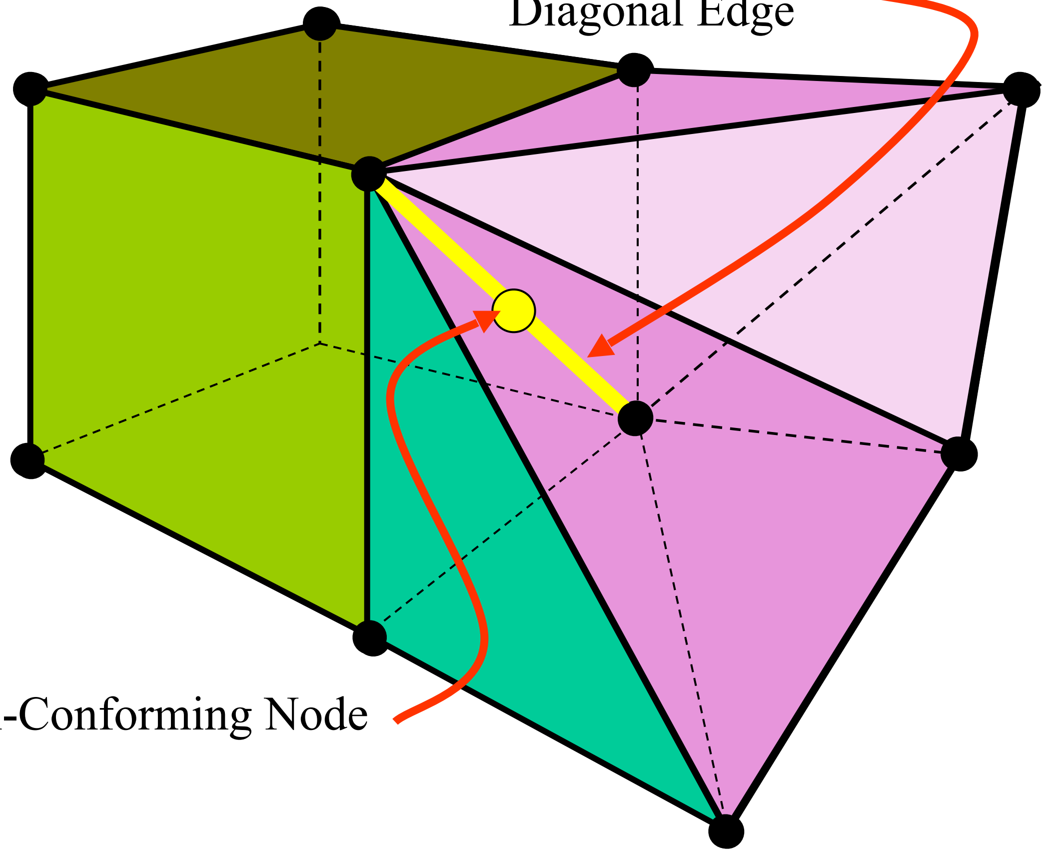
(Garimella, Shephard, 2000)

Hex-Tet Interface

Computational Modeling Sciences Department



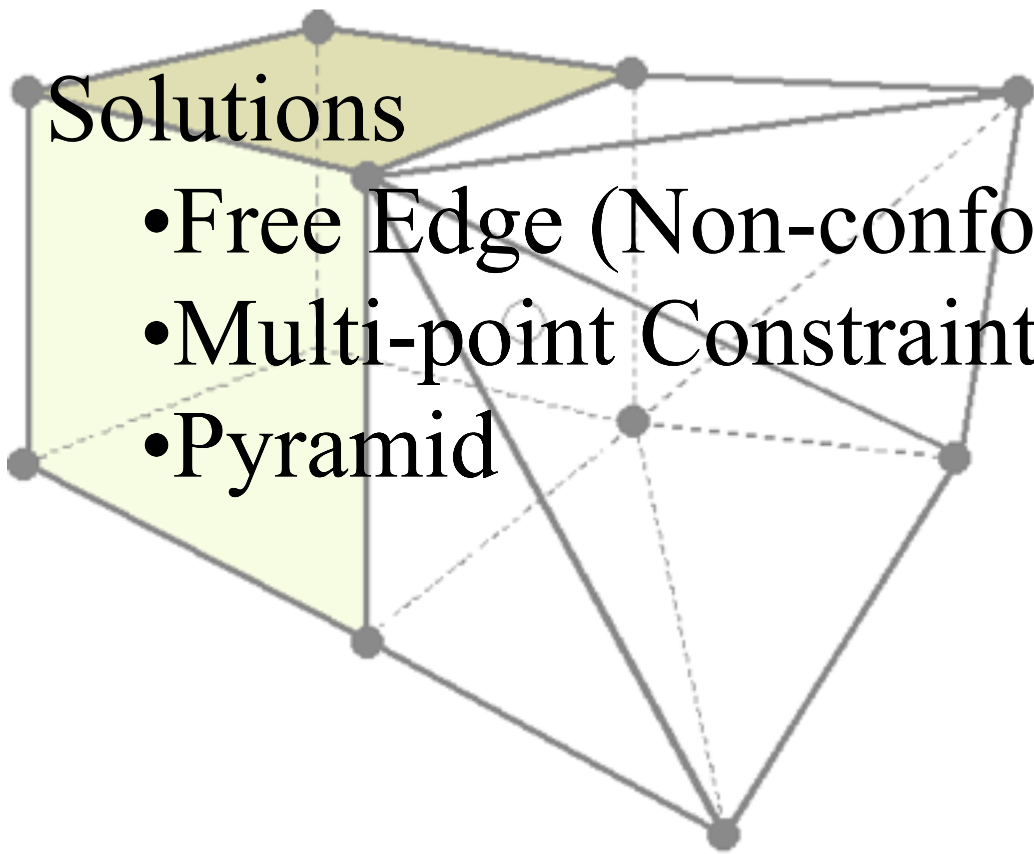
Conforming quad-triangle



Conforming hex-tet?

Hex-Tet Interface

Computational Modeling Sciences Department

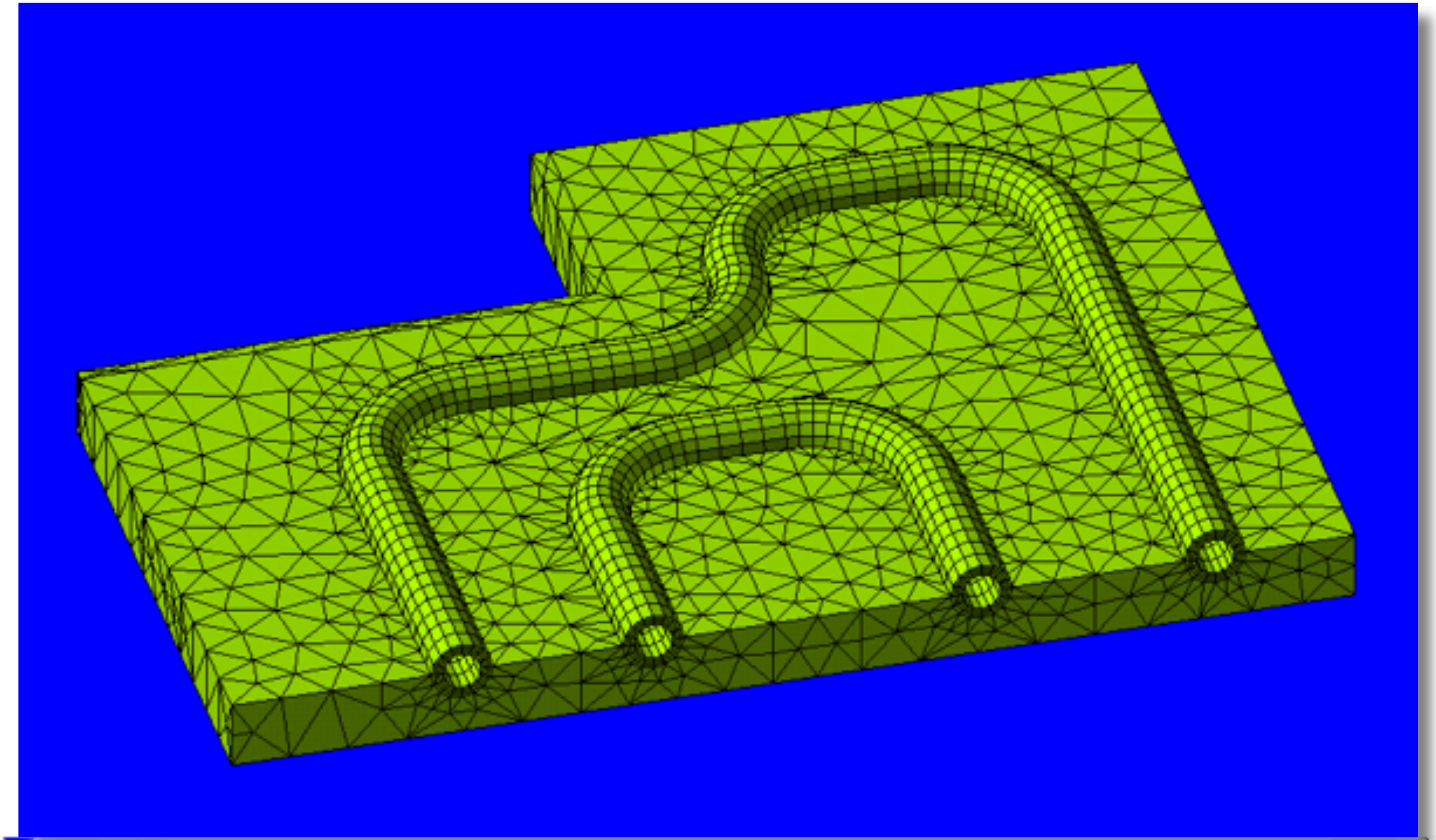


Solutions

- Free Edge (Non-conforming)
- Multi-point Constraint
- Pyramid

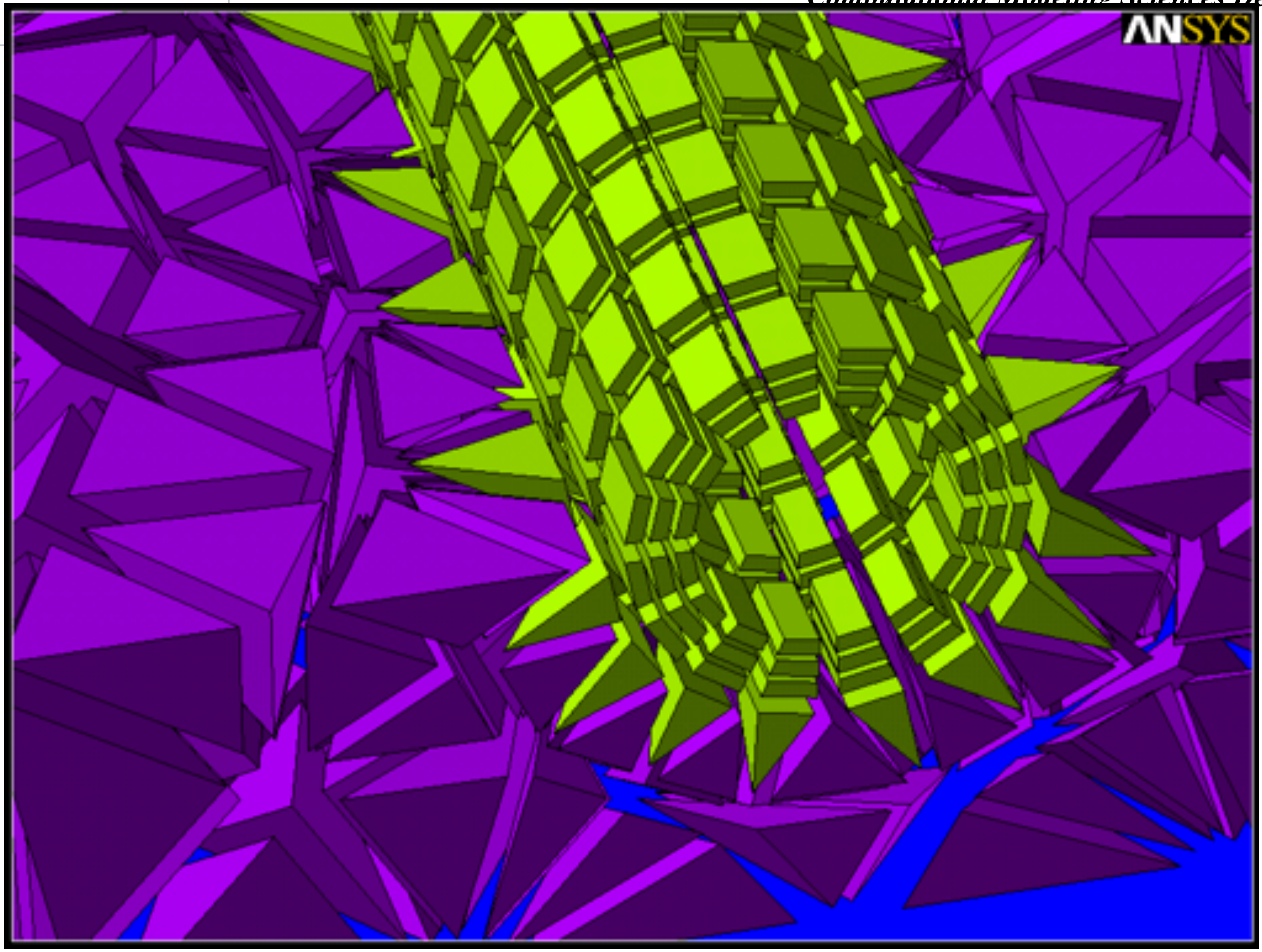
Hex-Tet Interface

Computational Modeling Sciences Department



Heat sink meshed with hexes, tets and pyramids

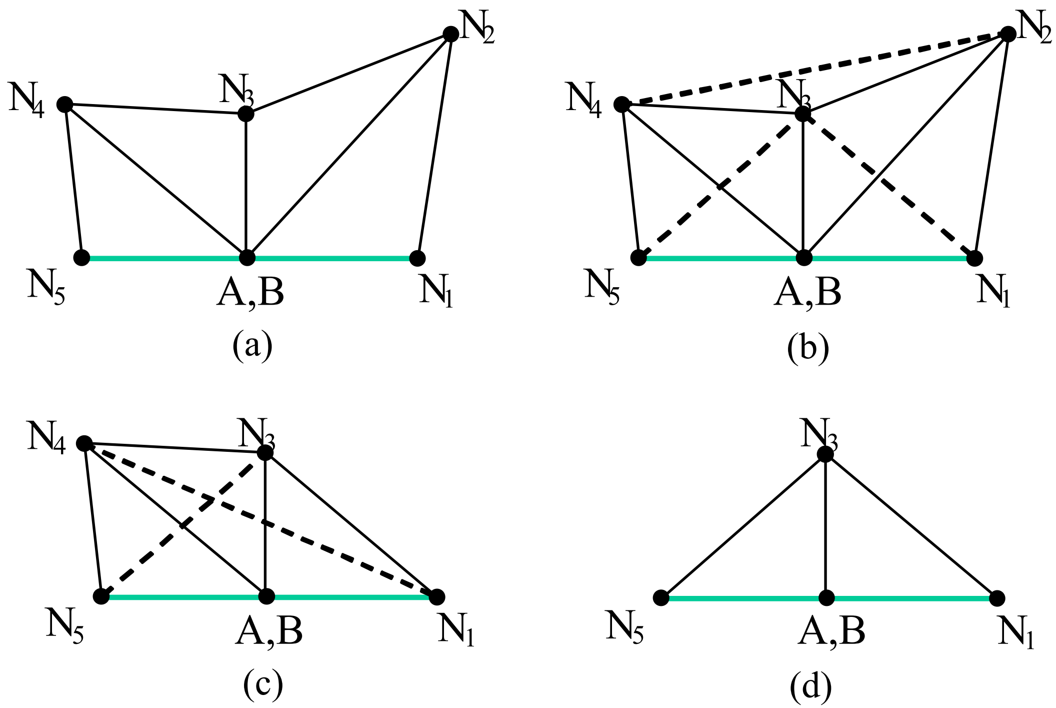
Hex-Tet Interface

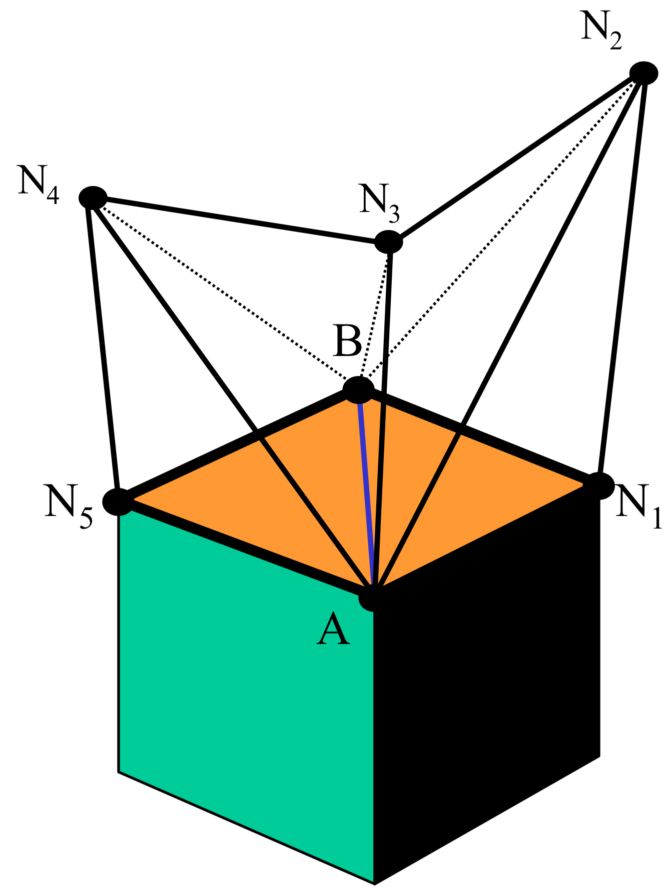


Pyramid Elements for maintaining compatibility between hex and tet elements (Owen 00)

Hex-Tet Interface

Computational Modeling Sciences Department



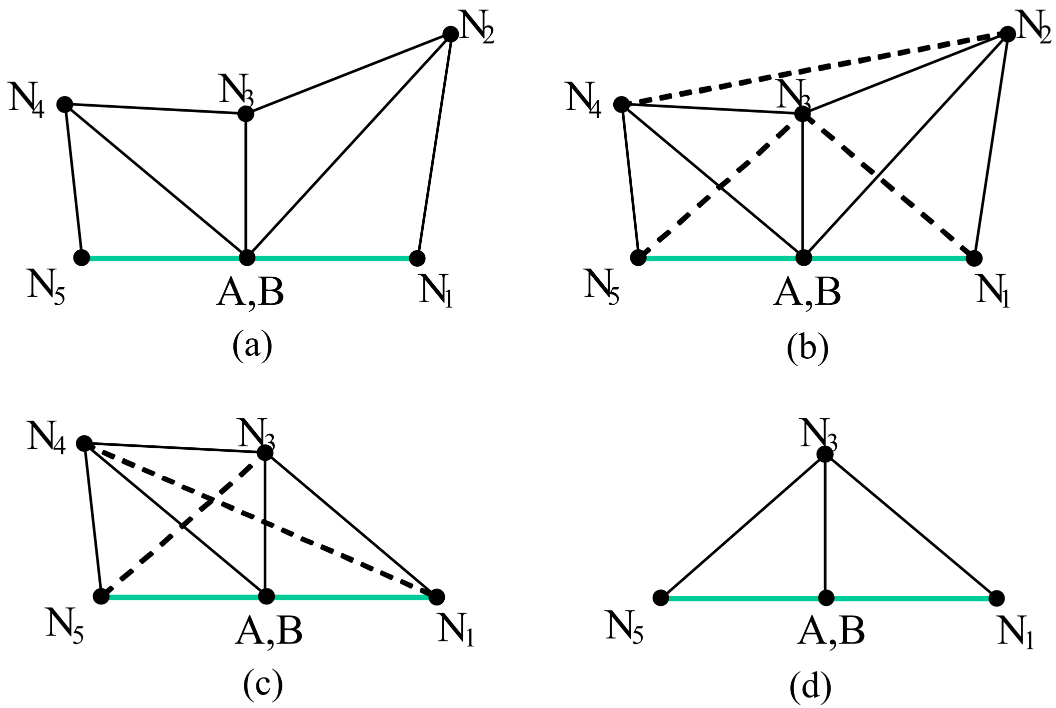


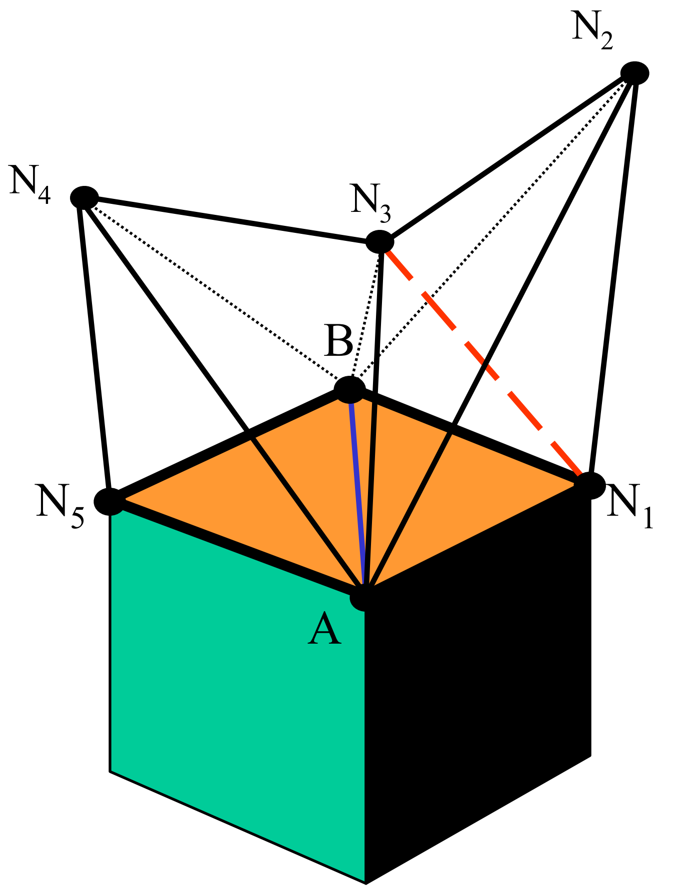
Tetrahedral transformations to form Pyramids

- Use 2-3 swaps to obtain 2 tets at diagonal
- combine 2 tets to form pyramid

Hex-Tet Interface

Computational Modeling Sciences Department





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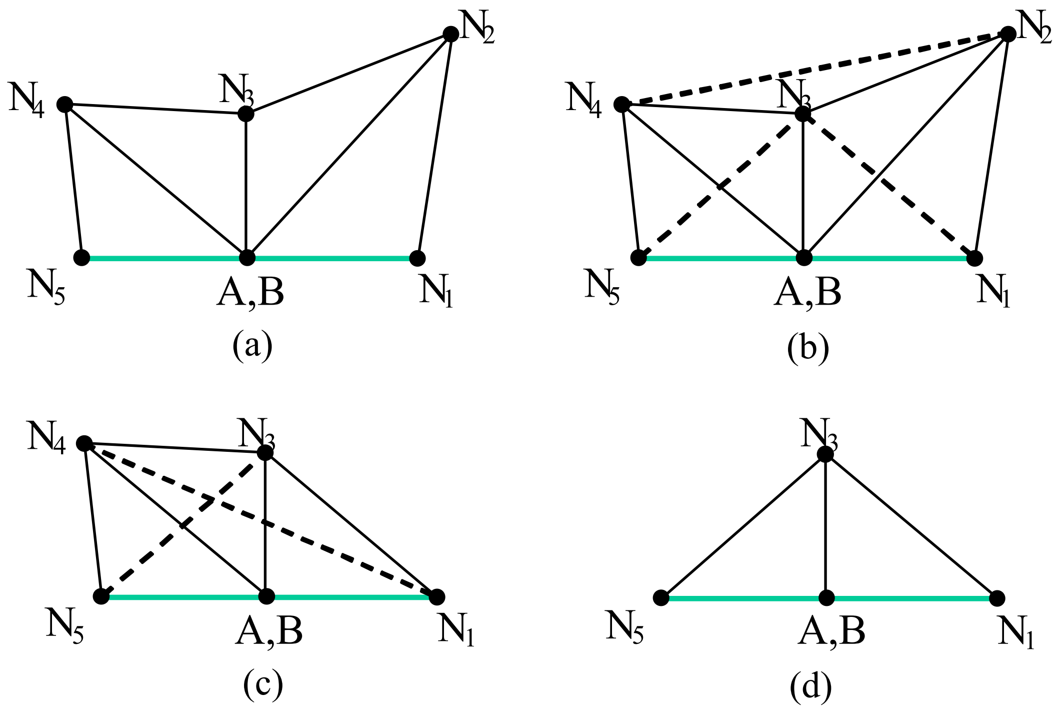
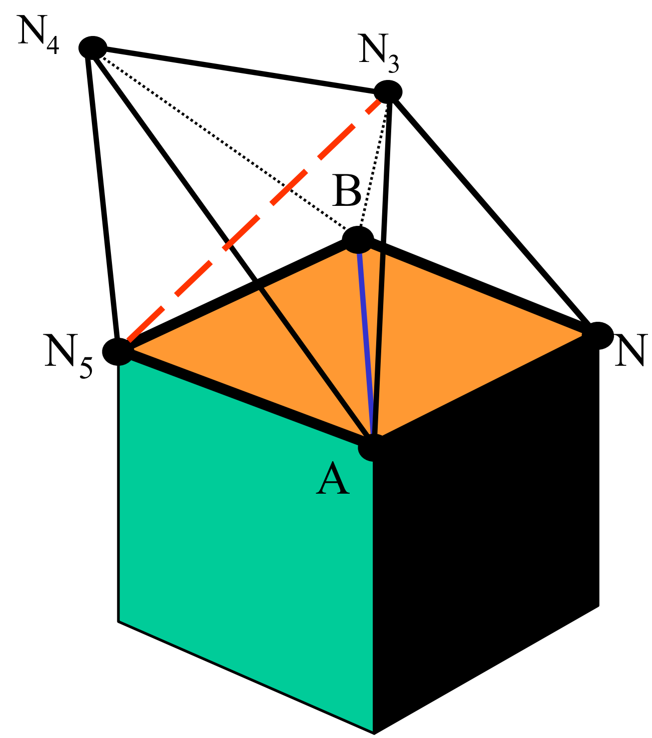
Hex-Tet Interface





Tetrahedral transformations to form Pyramids

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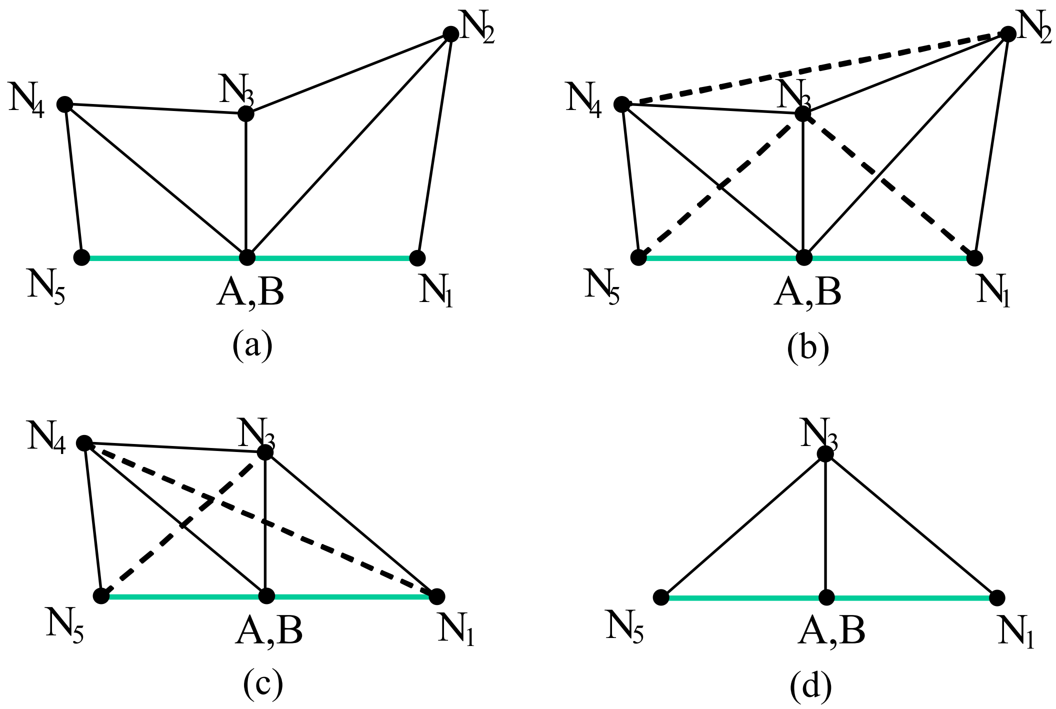
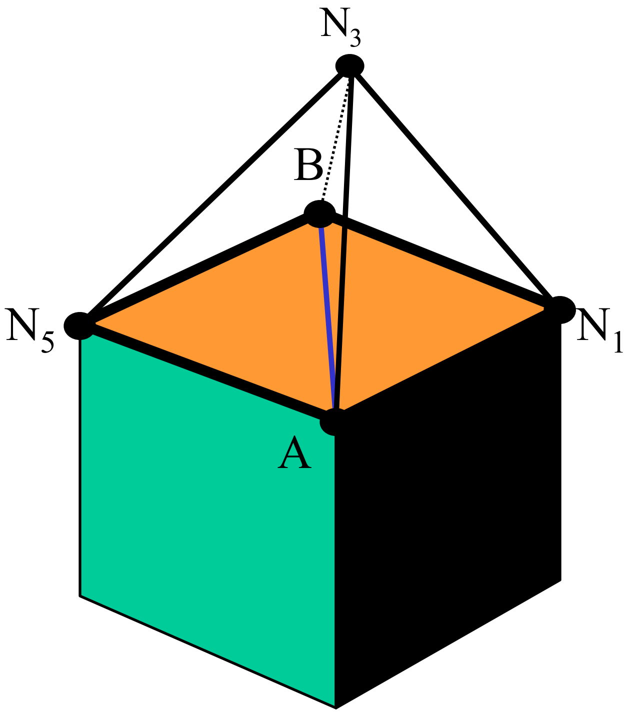
Hex-Tet Interface





Tetrahedral transformations to form Pyramids

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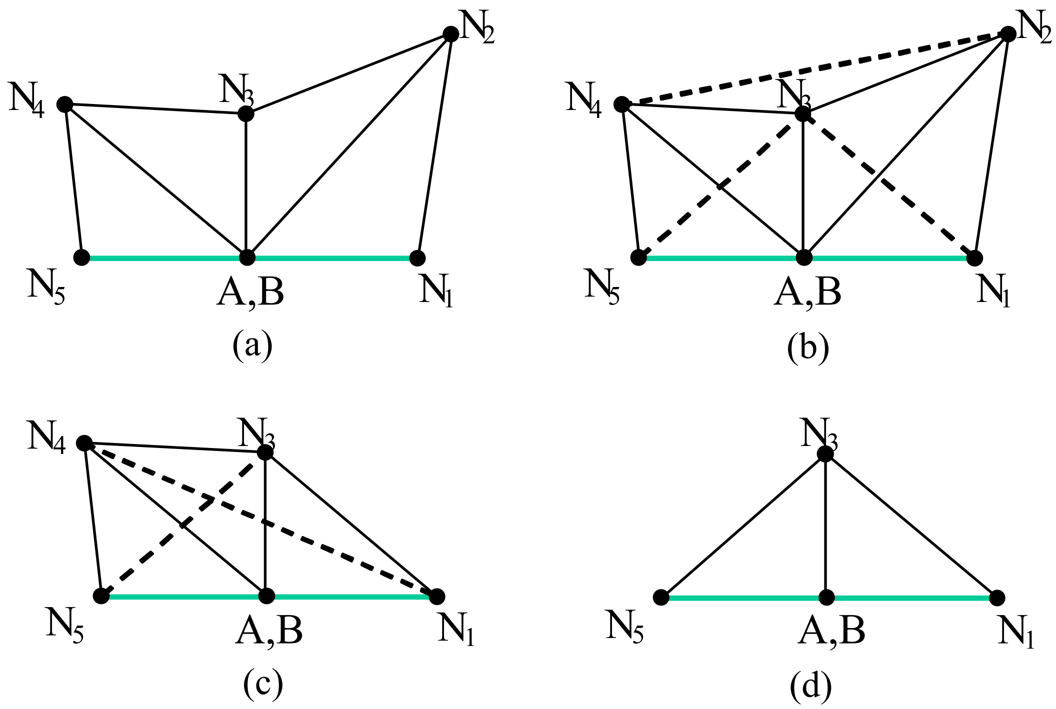
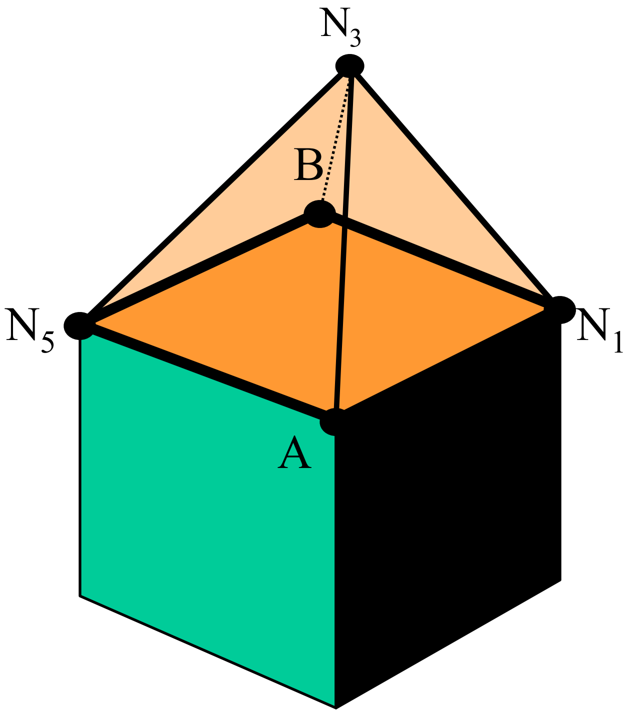
Hex-Tet Interface

Computational Modeling Sciences Department





Tetrahedral transformations to form Pyramids

- Use 2-3 swaps to obtain 2 tets at diagonal
- combine 2 tets to form pyramid

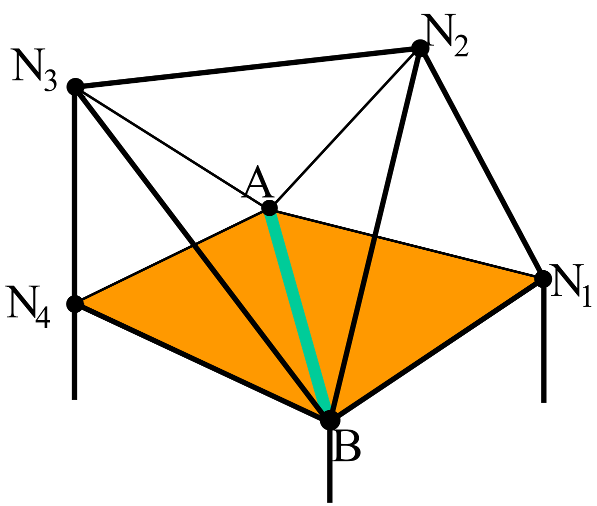
National Nuclear Security Administration

Sandia National Laboratories

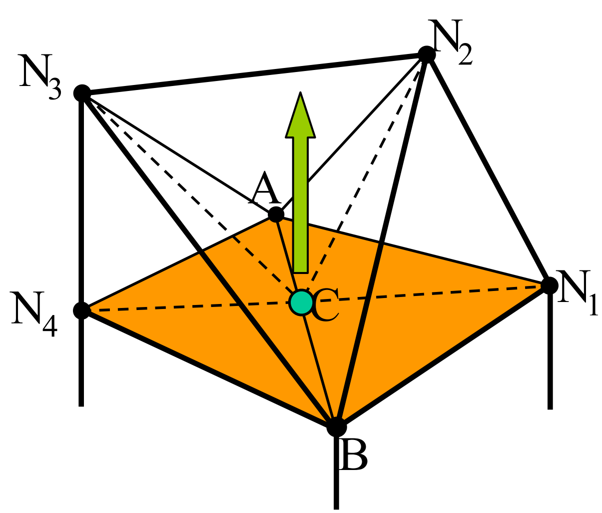
Hex-Tet Interface

Computational Modeling Sciences Department

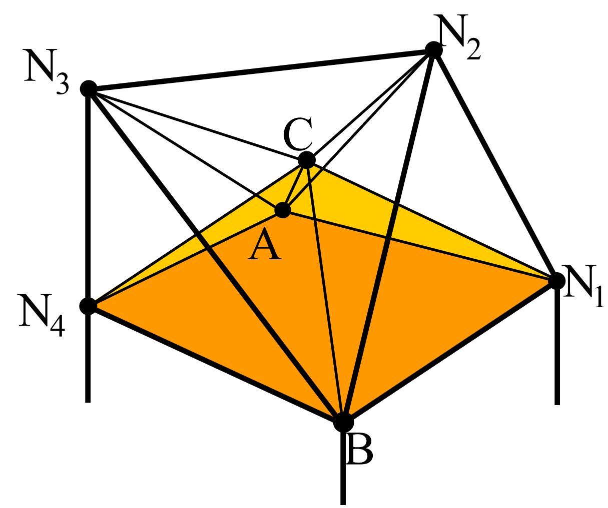
Pyramid Open Method



Non-Conforming Condition:
Tets at quad diagonal A-B



•Insert C at midpoint AB:
•Split all tets at edge AB

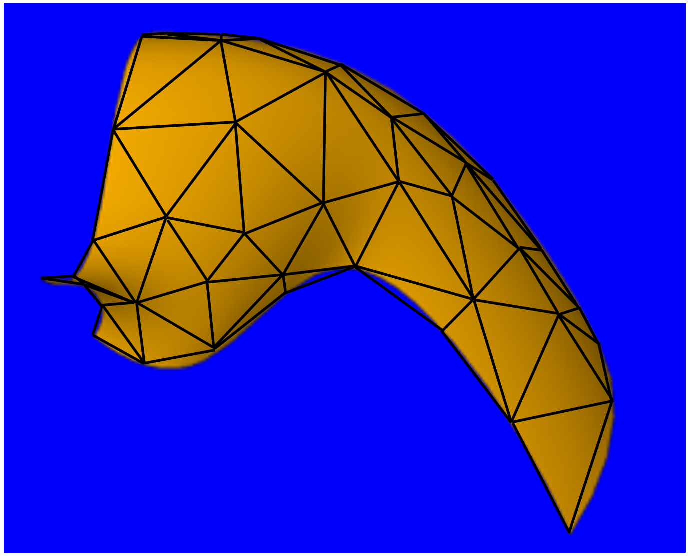


•Move C to average $N_1, N_2 \ldots N_n$
•Create New Pyramid A, N_n, B, N_1, C

Surface Meshing

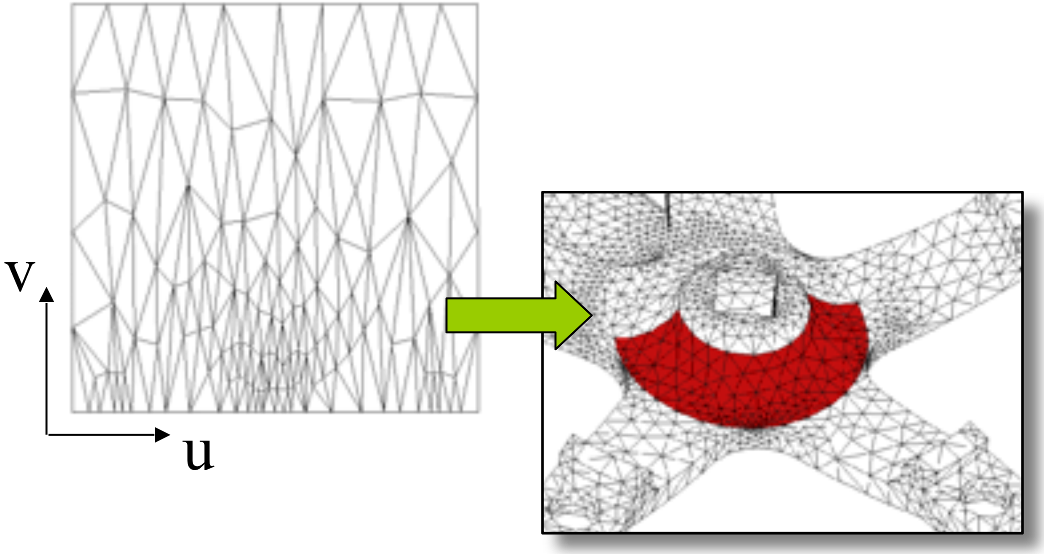
Computational Modeling Sciences Department

Direct 3D Meshing



- Elements formed in 3D using actual x-y-z representation of surface

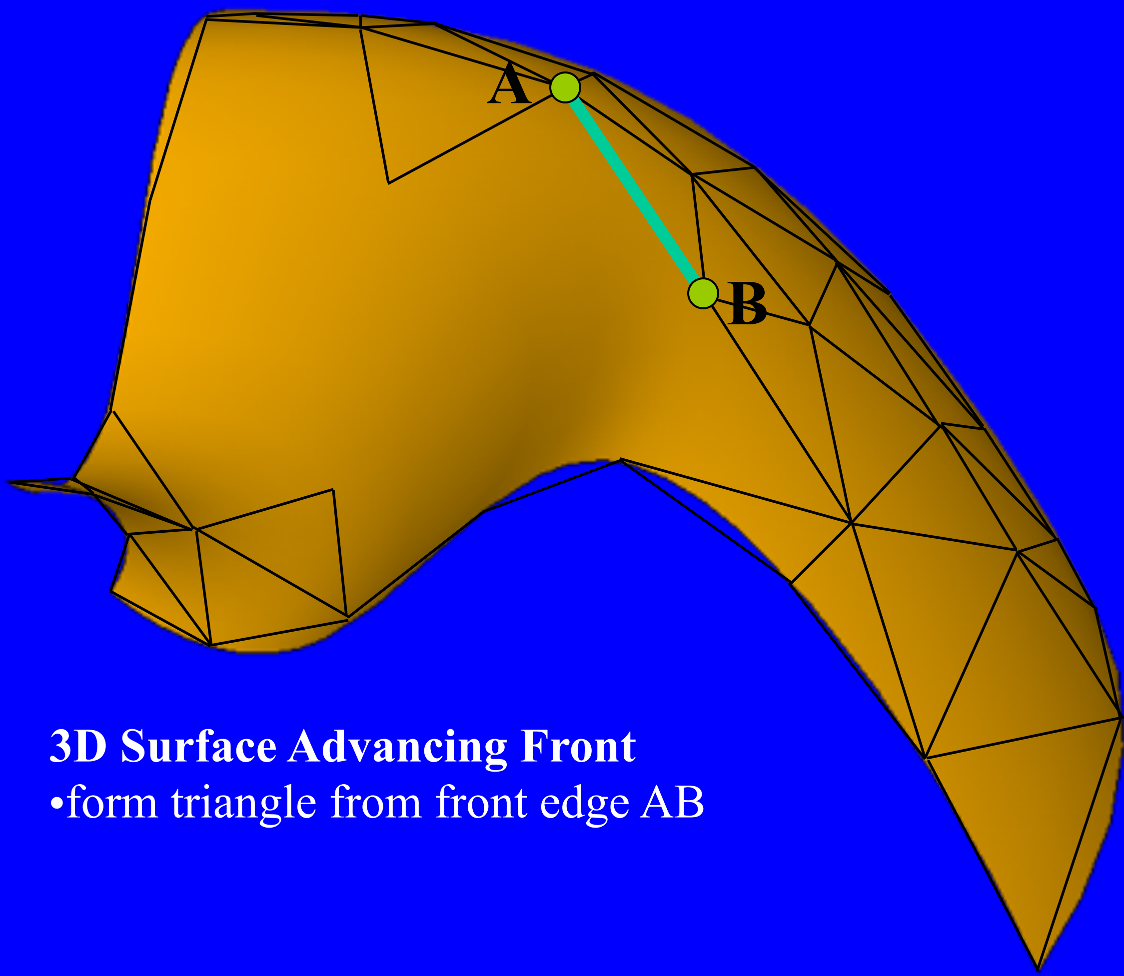
Parametric Space Meshing



- Elements formed in 2D using parametric representation of surface
- Node locations later mapped to 3D

Surface Meshing

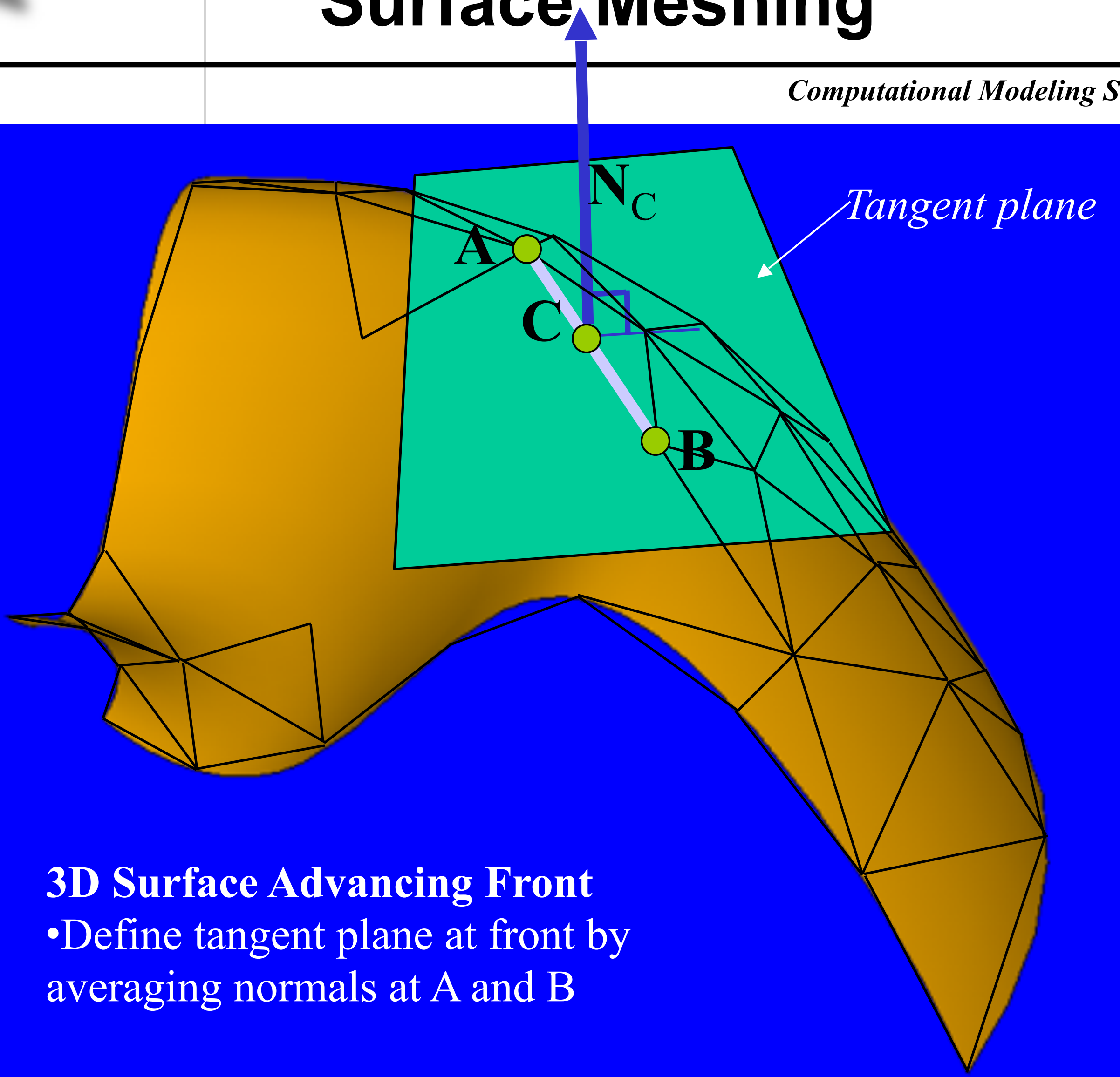
Computational Modeling Sciences Department



3D Surface Advancing Front
•form triangle from front edge AB

Surface Meshing

Computational Modeling Sciences Department

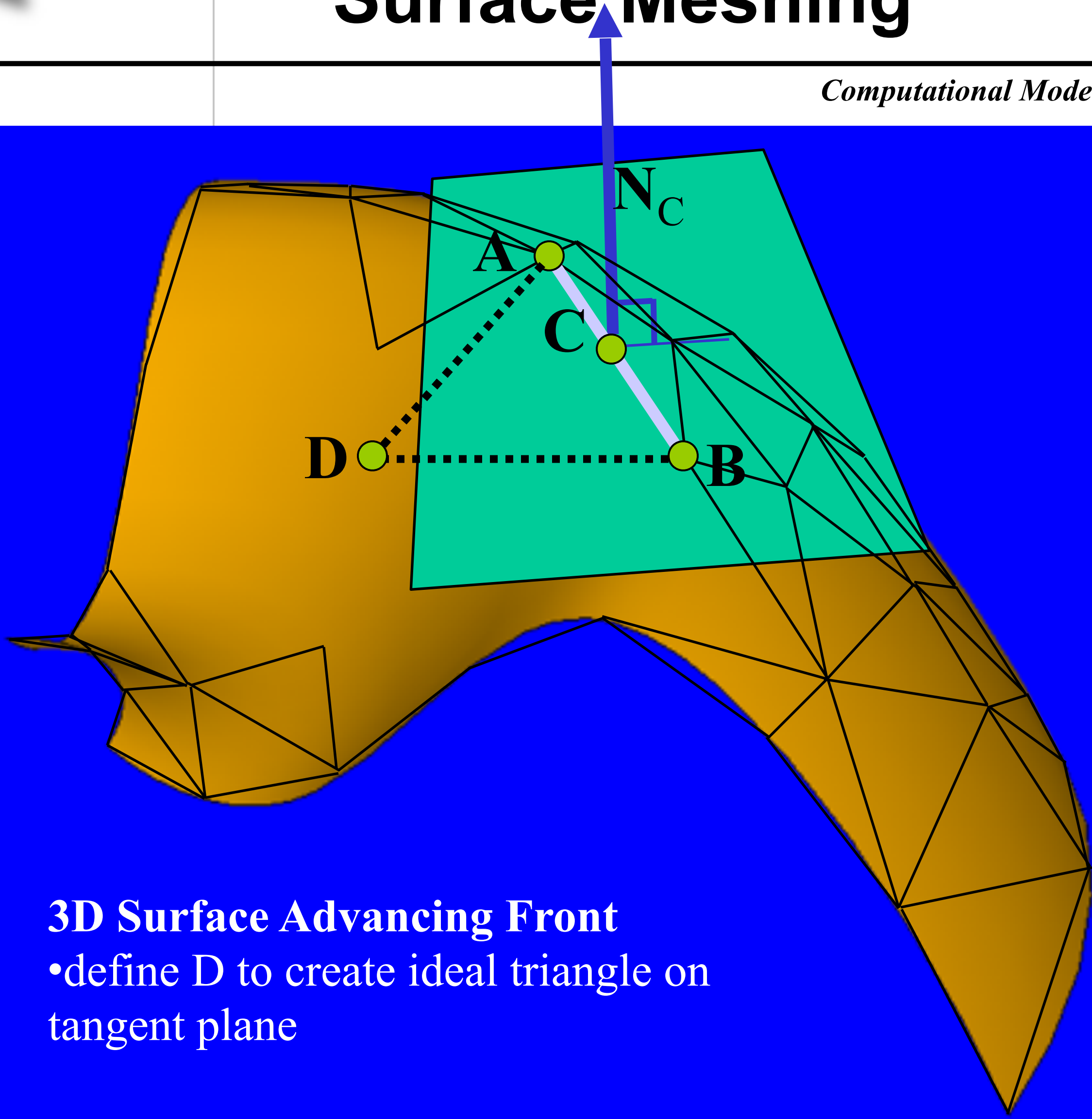


3D Surface Advancing Front

•Define tangent plane at front by averaging normals at A and B

Surface Meshing

Computational Modeling Sciences Department

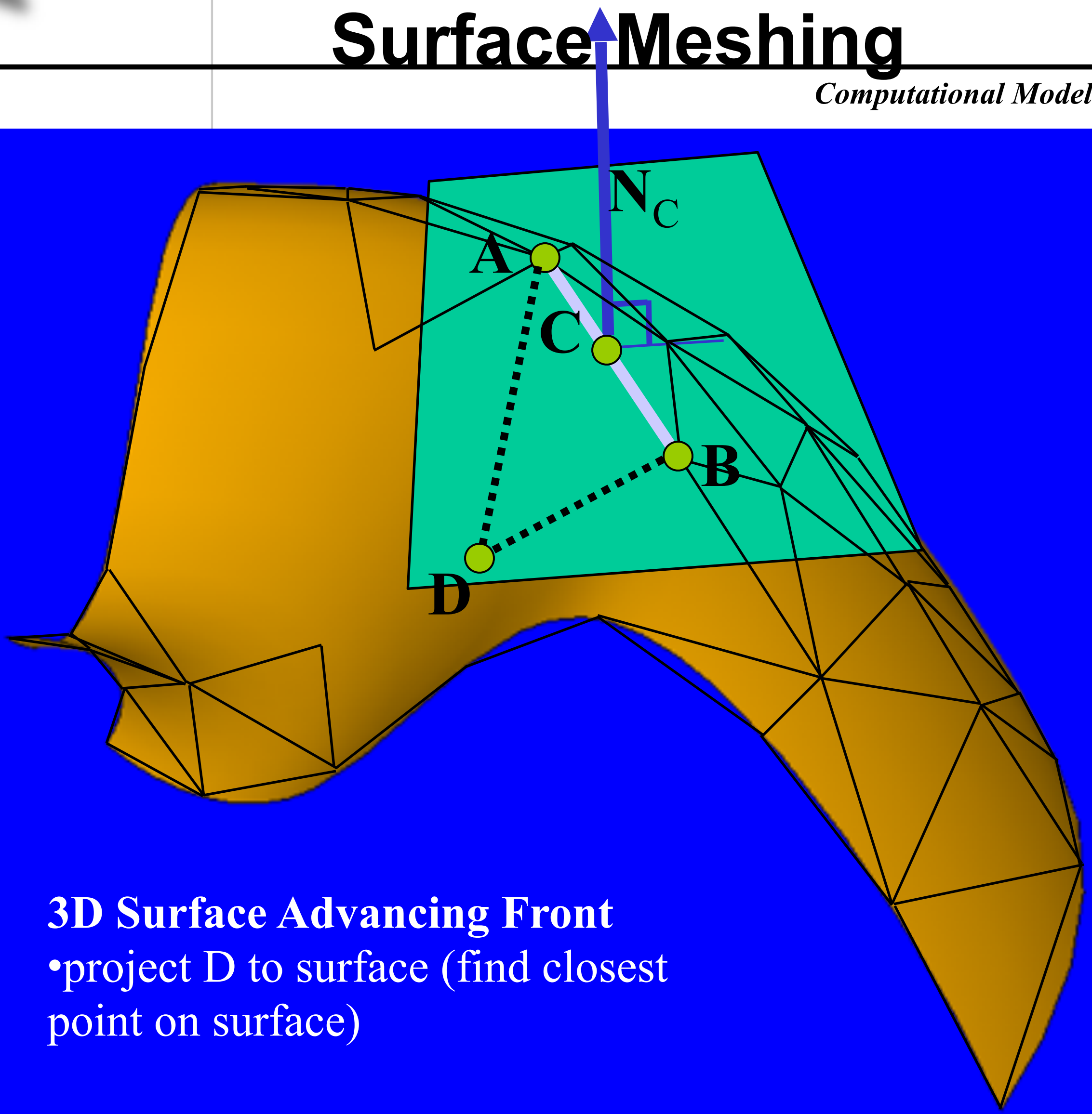


3D Surface Advancing Front

- define D to create ideal triangle on tangent plane

Surface Meshing

Computational Modeling Sciences Department

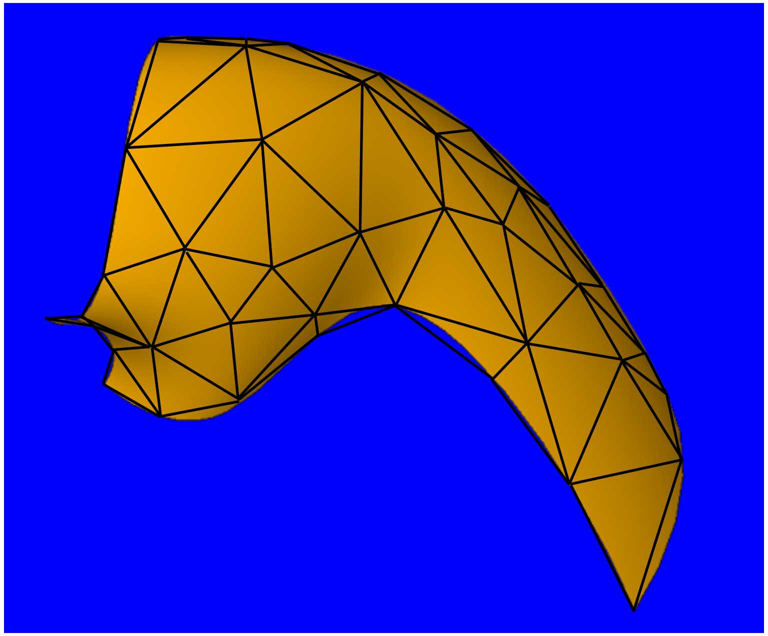


3D Surface Advancing Front
•project D to surface (find closest point on surface)

Surface Meshing

3D Surface Advancing Front

- Must determine overlapping or intersecting triangles in 3D. (Floating point robustness issues)
- Extensive use of geometry evaluators (for normals and projections)
- Typically slower than parametric implementations
- Generally higher quality elements
- Avoids problems with poor parametric representations (typical in many CAD environments)
- (Lo,96;97); (Cass,96)

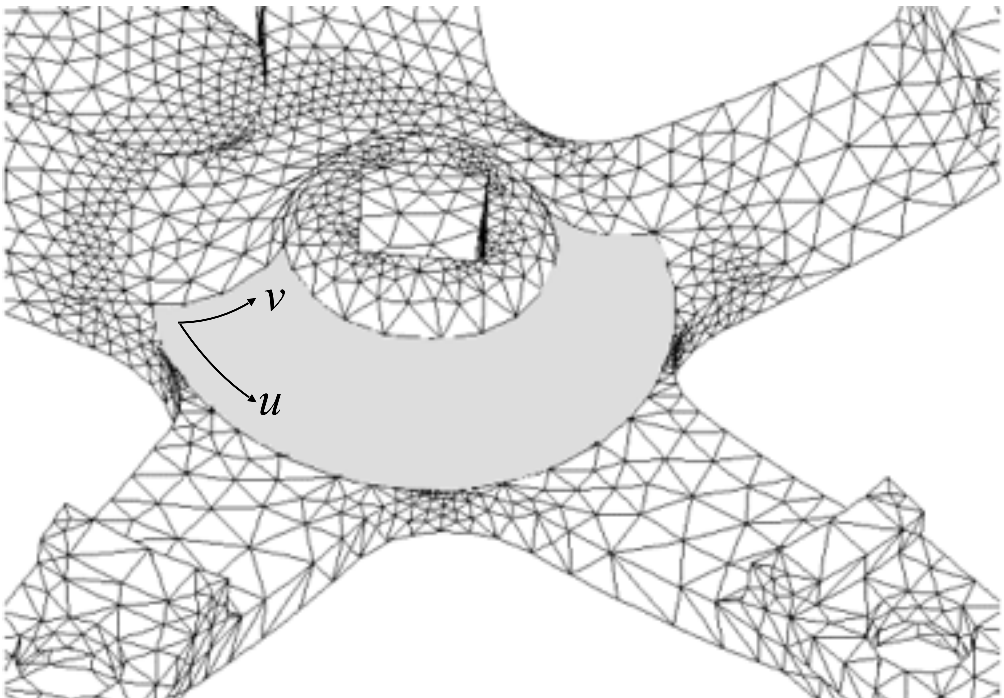


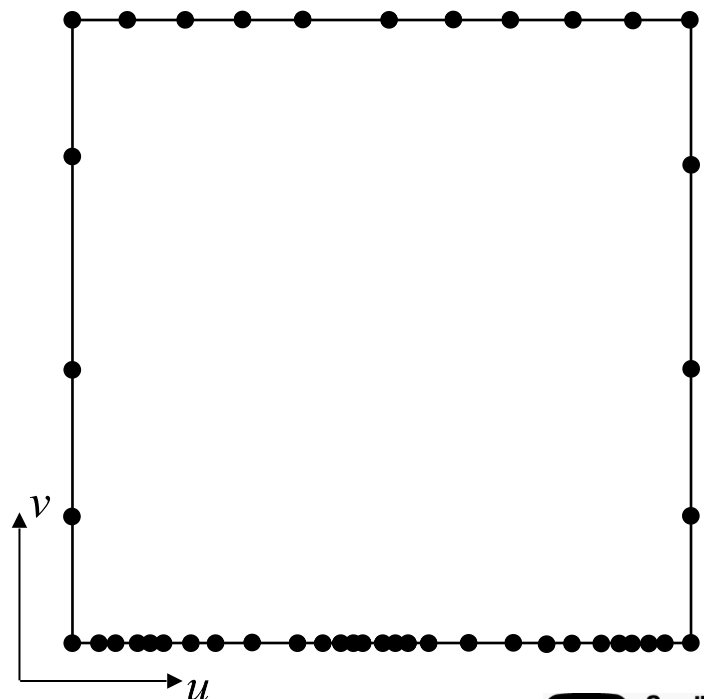
Surface Meshing

Computational Modeling Sciences Department

Parametric Space Mesh Generation

•Parameterization of the NURBS provided by the CAD model can be used to reduce the mesh generation to 2D

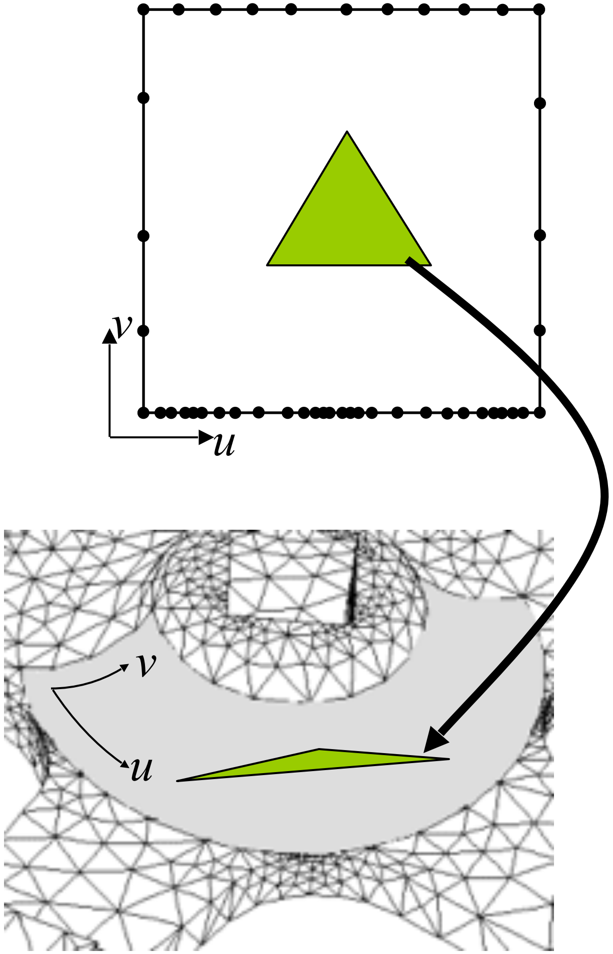




Surface Meshing

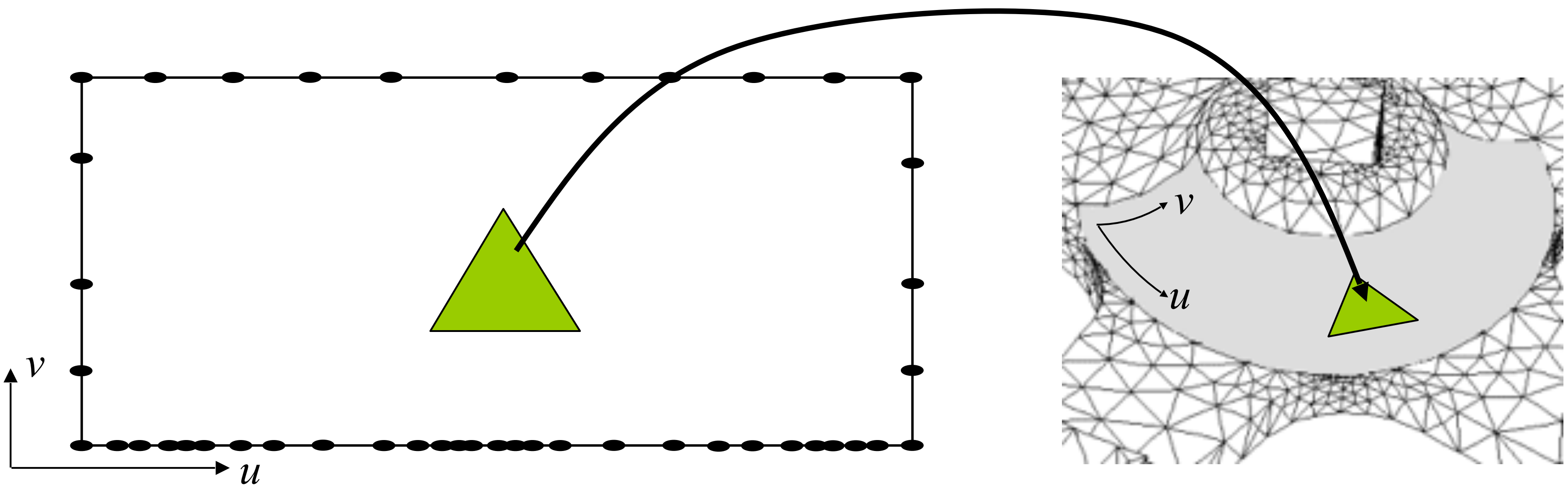
Parametric Space Mesh Generation

- Isotropic: Target element shapes are equilateral triangles
  - Equilateral elements in parametric space may be distorted when mapped to 3D space.
  - If parametric space resembles 3D space without too much distortion from *u*-*v* space to *x*-*y*-*z* space, then isotropic methods can be used.



Parametric Space Mesh Generation

- Parametric space can be “customized” or *warped* so that isotropic methods can be used.
- Works well for many cases.
- In general, isotropic mesh generation does not work well for parametric meshing



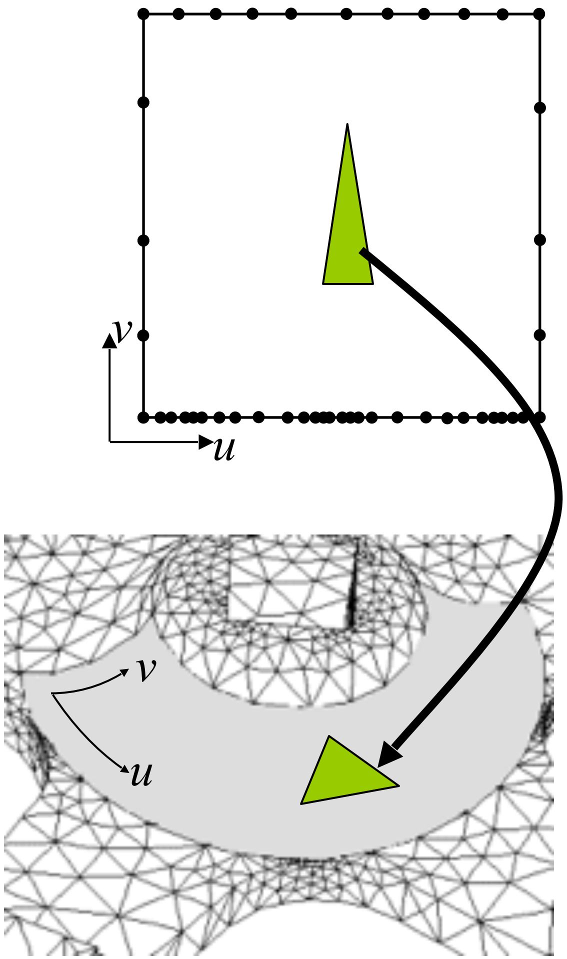
Warped parametric space

Surface Meshing

Computational Modeling Sciences Department

Parametric Space Mesh Generation

- Anisotropic: Triangles are stretched based on a specified vector field
    - Triangles appear stretched in 2d (parametric space), but are near equilateral in 3D



Parametric Space Mesh Generation

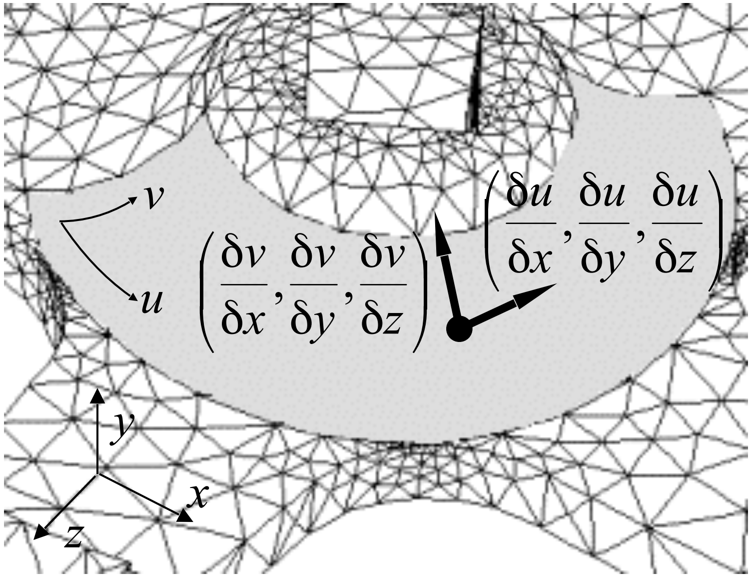
- Stretching is based on field of surface derivatives

$$\Delta\mathbf{u} = \left(\frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}, \frac{\delta u}{\delta z}\right) \qquad \Delta\mathbf{v} = \left(\frac{\delta v}{\delta x}, \frac{\delta v}{\delta y}, \frac{\delta v}{\delta z}\right)$$

- Metric, **M** can be defined at every location on surface. Metric at location **X** is:

$$\mathbf{M}(\mathbf{X}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

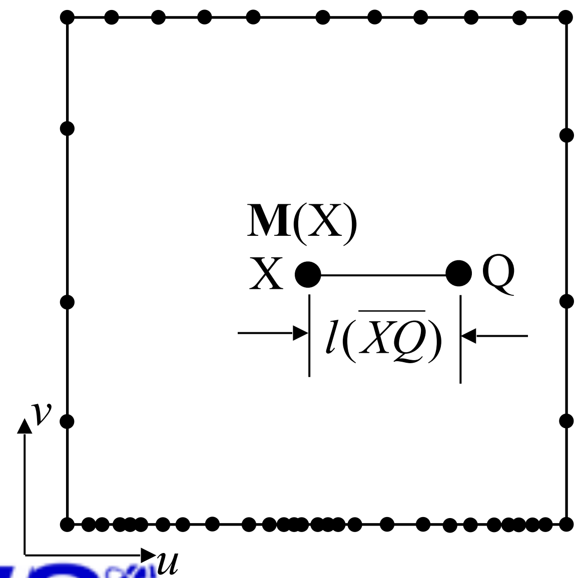
$$E = \Delta\mathbf{u}\cdot\Delta\mathbf{u} \qquad F = \Delta\mathbf{u}\cdot\Delta\mathbf{v} \qquad G = \Delta\mathbf{v}\cdot\Delta\mathbf{v}$$

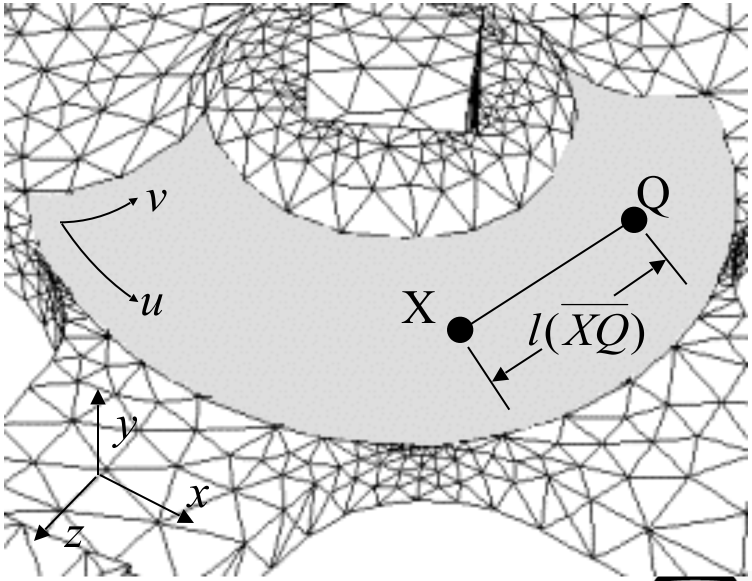


Parametric Space Mesh Generation

•Distances in parametric space can now be measured as a function of direction and location on the surface. Distance from point X to Q is defined as:

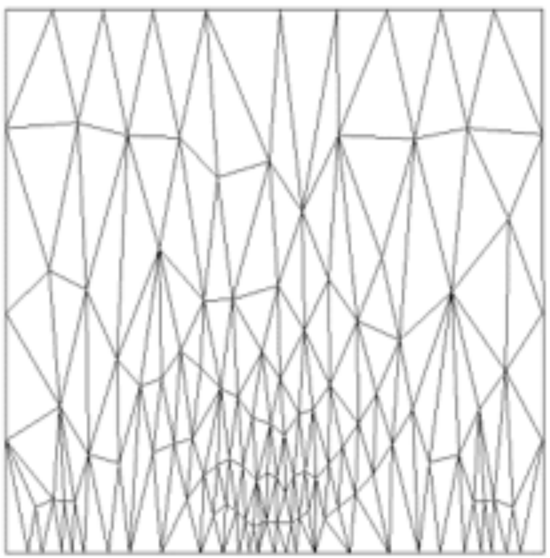
$$l(\overline{XQ}) \approx \sqrt{\overline{XQ}^T \mathbf{M}(\mathbf{X}) \overline{XQ}}$$

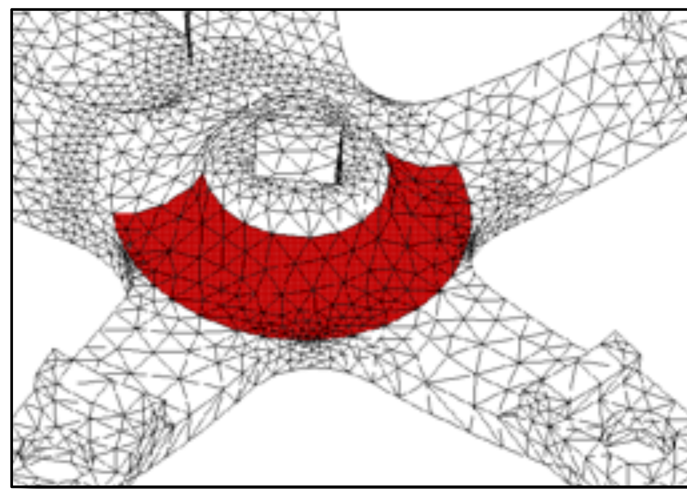




Parametric Space Mesh Generation

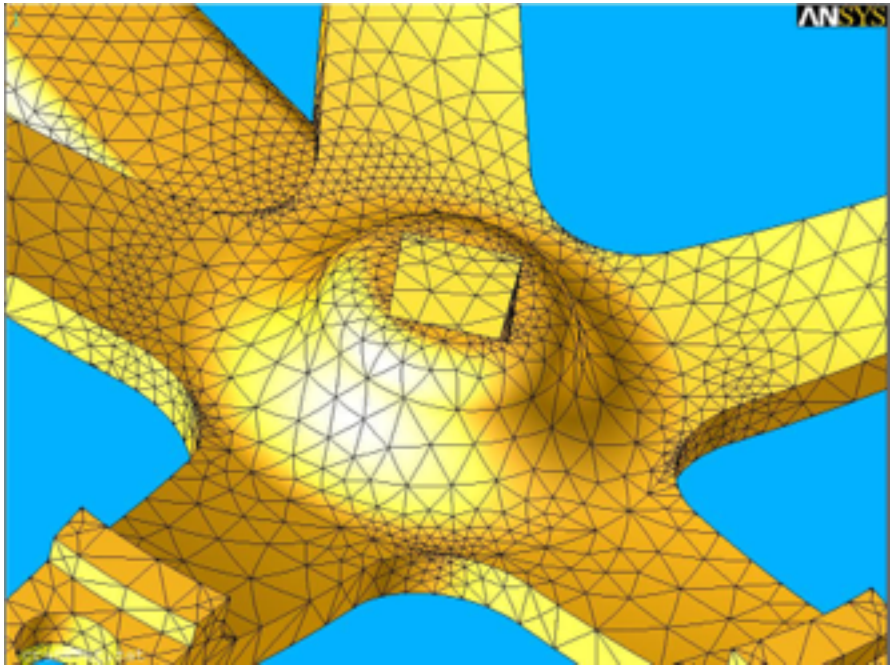
•Use essentially the same isotropic methods for 2D mesh generation, except distances and angles are now measured with respect to the local metric tensor **M**(X).
•Can use Delaunay (George, 99) or Advancing Front Methods (Tristano,98)





Parametric Space Mesh Generation

- Is generally faster than 3D methods
- Is generally more robust (No 3D intersection calculations)
- Poor parameterization can cause problems
- Not possible if no parameterization is provided
    - Can generate your own parametric space (Flatten 3D surface into 2D) (Marcum, 99) (Sheffer,00)



Algorithm Characteristics

- 1. Conforming Mesh
  - •Elements conform to a prescribed surface mesh
- 2. Boundary Sensitive
  - •Rows/layers of elements roughly conform to the contours of the boundary
- 3. Orientation Insensitive
  - •Rotating/Scaling geometry will not change the resulting mesh
- 4. Regular Node Valence
  - •Inherent in the algorithm is the ability to maintain (nearly) the same number of elements adjacent each node)
- 5. Arbitrary Geometry
  - •The algorithm does not rely on a specific class/shape of geometry
- 6. Commercial Viability (Robustness/Speed)
  - •The algorithm has been used in a commercial setting

Algorithm Characteristics

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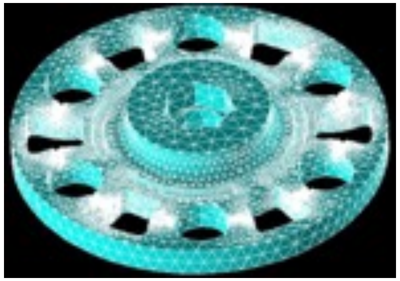
		Tris			Tets			Quads								Hexes									
		Quadtree	Delaunay	Adv. Front	Octree	Delaunay	Adv. Front	Mapped	Sub-map	Tri Split	Tri Merge	Q-Morph	Grid-Based	Medial Axis	Paving	Mapped	Sub-map	Sweeping	Tet Split	Tet Merge	H-Morph	Grid-Based	Medial Surf.	Plastering	Whisker W.
1	Conforming Mesh		●	●		●	●	●	●	●	●	●			●	●	●	●			●			●	●
2	Boundary Sensitive			●			●	●	●			●		●	●	●	●	●			●		●	●	●
3	Orientation Insensitive		●	●		●	●	●	●	●	●	●		●	●	●	●	●	●	●	●		●	●	●
4	Regular Node Valence			●			●	●	●			●	●	●	●	●	●	●			●	●	●	●	●
5	Arbitrary Geometry	●	●	●	●	●	●			●	●	●	●	●	●				●	●	●	●	●	●	●
6	Commercially Viability	●	●	●	●	●	●	●	●		●	●	●	●	●	●	●	●		●		●			

More Info



Meshing Research Corner

http://www.andrew.cmu.edu/~sowen/mesh.html

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