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The Real Multiple Dual

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Abstract In this paper we present a dual representation for the multiple stopping problem, hence multiple exercise options. As such it is a natural generalization of the method in Rogers (2002) and Haugh and Kogan (2004) for the standard stopping problem for American options. We consider this representation as the *real dual* as it is solely expressed in terms of an infimum over martingales rather than an infimum over martingales *and* stopping times as in Meinshausen and Hambly (2004). For the multiple dual representation we present three Monte Carlo simulation algorithms which require only one degree of nesting.

Keywords Optimal stopping · Dual representations · Multiple callable derivatives

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1 Introduction

The key issue in valuation of financial derivatives with several exercise rights is solving a multiple stopping problem. Such derivatives encounter, for example, in electricity markets (swing options) and interest rate markets (chooser caps). Typically, the dimension of the underlying financial object is rather high, for instance a Libor interest rate model, and therefore Monte Carlo based methods are called for. In this respect the last decades have seen several breakthroughs for standard American (or Bermudan style) derivatives, hence the standard stopping problem. Among the most popular ones are the regression based methods of Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999), and alternative approaches by Andersen (1999), Broadie and Glasserman (2004) and others. These methods alow for computation of a lower approximation of

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the product under consideration by straightforward (non-nested) Monte Carlo simulation when the underlying dimension is not too high. More recently, Kolodko and Schoenmakers (2006) proposed a policy improvement procedure and it is demonstrated in Bender et al. (2006) and Bender et al. (2008) that this method can be effectively combined with Longstaff and Schwartz (2001) for very high-dimensional products. In Bender and Schoenmakers (2006) this policy iteration method is extended to multiple stopping problems. Evaluation of products with multiple exercise rights (on a low dimensional underlying) is also possible by using trinomial forests (Jaillet et al., 2004). In Carmona and Touzi (2006) a Malliavin related approach for the valuation of swing options is presented.

In Rogers (2002) and Haugh and Kogan (2004) a dual approach is developed (inspired by Davis and Karatzas (1994)) which allows for computing tight upper bounds for American style products. Jamshidian (2007) proposed a multiplicative version of the dual representation, Belomestny and Milstein (2006), and Belomestny et al. (2006) proposed to compute upper bounds based on the concept of consumption processes. Effective algorithms for dual upper bounds are proposed in Andersen and Broadie (2004), Kolodko and Schoenmakers (2004), and Belomestny et al. (2009). For products with multiple exercise possibilities Meinshausen and Hambly (2004) found a dual representation for the marginal excess value of the product due to one additional exercise right. In this representation an infimum over a family of stopping times and a family of martingales is involved. Generalizations of this method to multiple exercise products under volume constraints are developed in Bender (2008) and Aleksandrov and Hambly (2008). While the mentioned methods for multiple exercise products have shown to be feasible in practice, the question whether a 'real' dual representation for the multiple stopping problem exists as a natural extension of the dual representation for the single exercise case, in terms of an infimum over martingales (only), was still open. The main result in this paper is such a dual representation and so fills this gap. Moreover we present three Monte Carlo algorithms for this representation, which require at most one degree of nesting, just as in the one-exercise case. The procedures are spelled out in detail and, in particular, one of them (Algorithm 2) may be seen as a natural generalization of the algorithm in Andersen and Broadie (2004). As such the presented algorithms are natural extensions of the corresponding ones for the single exercise case. So it is more or less obvious that their numerical potential is inherited from the numerical qualities of simulation algorithms for the standard additive dual extensively documented in the literature. Therefore, the author prefers to communicate the new multiple dual representation together with the three respective simulation algorithms in this paper, and considers an in depth numerical study to be more suitable for subsequent work. The main result, Theorem 2, is derived in Section 2, and the corresponding algorithms are given in Section 3.

2 The Multiple Stopping Problem and its Dual Representation

Let $(Z_i : i = 0, 1, \ldots, T)$ be a non-negative stochastic process in discrete time on a filtered probability space $(\Omega, \mathcal{F}, P)$ adapted to some filtration $\mathbb{F} := (\mathcal{F}_i : 0 \le i \le T)$ which satisfies

$$\sum_{i=1}^{T} E|Z_i| < \infty.$$

The process Z may be seen as a (discounted) cash-flow, which an investor may exercise L times, subjected to the additional constraint that it is not allowed to exercise more than one right at the same date. The goal of the investor is to maximize his expected gain by making optimal use of his L exercise rights. This goal may be formalized as a multiple stopping problem. For notational convenience we extend the cash-flow process in a trivial way by $Z_i = 0$ and $\mathcal{F}_i = \mathcal{F}_T$ for $i > T$.

Let us define for each fixed $0 \leq i \leq T$ and L, $\mathcal{S}_i(L)$ as the set of $\mathbb{F}$-stopping vectors $\tau := (\tau^{(1)}, \ldots, \tau^{(L)})$ such that $i \leq \tau^{(1)}$ and, for all l, $1 < l \leq L$, $\tau^{(l-1)} + 1 \leq \tau^{(l)}$. The multiple stopping problem then comes down to find a family of stopping vectors $\tau_i^* \in \mathcal{S}_i(L)$ such that for $0 \leq i \leq T$,

$$E_i \sum_{l=1}^{L} Z_{\tau_i^{*l}} = \sup_{\tau \in \mathcal{S}_i(L)} E_i \sum_{l=1}^{L} Z_{\tau^{(l)}} \tag{1}$$

with $E_i := E_{\mathcal{F}_i}$ denoting conditional expectation with respect to the σ-algebra $\mathcal{F}_i$, and sup is to be understood as the *essential supremum*.

Remark 1 Henceforth the operators sup and inf are to be understood as ess.sup and ess.inf, respectively, if they range over an uncountable family of random variables.

The process on the right hand of (1) is called the *Snell envelope* of Z under L exercise rights and we denote it by Y_i^{*L}. In the case of one exercise right we usually write $Y_i^* := Y_i^{*1}$. We recall from Bender and Schoenmakers (2006) that the multiple stopping problem can be reduced to L nested stopping problems with one exercise right in the following way. $Y^{*0} := 0$, Y^{*1} is the Snell envelope of Z. For general L, $L \geq 1$, Y^{*L} is the Snell envelope of the process $Z_i + E_i\, Y_{i+1}^{*L-1}$ (seen as generalized cash-flow) under one exercise right. It is thus natural to define (as in Bender and Schoenmakers (2006)) for each $L = 1, 2, \ldots$, the stopping family

$$\sigma_i^{*L} = \inf\{j \geq i : \ Z_j + E_j\, Y_{j+1}^{*L-1} \geq Y_j^{*L}\}, \qquad i \geq 0, \tag{2}$$

i.e. the first optimal stopping family for exercising the first of L exercise rights. The family of optimal stopping vectors $\tau_i^{*L} \in \mathcal{S}_i(L)$ for the multiple stopping problem with L exercise rights and cash-flow Z is connected with (2) via

$$\begin{aligned} \tau_i^{*1,L} &= \sigma_i^{*L} \\ \tau_i^{*l+1,L} &= \tau_{\sigma_i^{*L}+1}^{*l,L-1}, \quad 1 \leq l < L. \end{aligned} \tag{3}$$

The reduction (2), (3) is intuitively clear: It basically says, that the investor has to choose the first stopping time of the stopping vector in the following way: Decide, at time i whether it is better to take the cash-flow Z_i and enter a new contract with $L-1$ exercise rights starting at $j+1$, or to keep the L exercise rights. Then, after entering the stopping problem with $L-1$ exercise rights, he proceeds in the same (optimal) way.

2.1 Case $L = 1$: The Standard Stopping Problem

In the case of one exercise right $L = 1$ we have the standard stopping problem. Let us recall some well-known facts (e.g. see Neveu (1975)).

1. The Snell envelope Y^* of Z is the smallest super-martingale that dominates Z.
2. A family of optimal stopping times is given by

$$\tau_i^* = \inf\{j : j \geq i, \quad Z_j \geq Y_j^*\}, \quad 0 \leq i \leq T.$$

   In particular, the above family is the family of first optimal stopping times if several optimal stopping families exist.

2.2 Dual Representation for the Standard Stopping Problem

For the standard stopping problem, that is one exercise right $L = 1$, we have the (additive) dual representation theorem which we state in a form suitable for our purposes:

Theorem 1 *Rogers (2002), Haugh and Kogan (2004)*
If $\mathcal{M}$ is the set of all $\mathbb{F}$-martingales, it holds

$$Y_i^{*,1} = Y_i^* = \inf_{M \in \mathcal{M}} E_i \max_{i \leq j \leq T} \left(Z_j + M_i - M_j\right) \tag{4}$$

$$= \max_{i \leq j \leq T} \left(Z_j + M_i^* - M_j^*\right) \;\; a.s. \tag{5}$$

with M^* being the unique Doob martingale of Y^*, that is $Y^* = Y_0^* + M^* - A^*$ where M^* is a martingale, A^* is predictable, and $M_0^* = A_0^* = 0$.

For the results in this paper the almost sure statement (5) is very important. Therefore, and because of its appealing simplicity, let us shortly recall the proof:

Proof For any martingale M we have

$$\begin{aligned} Y_i^* &= \sup_{i \leq \tau \leq T} E_i Z_\tau = \sup_{i \leq \tau \leq T} E_i \left[Z_\tau + M_i - M_\tau\right] \\ &\leq E_i \max_{i \leq j \leq T} \left(Z_j + M_i - M_j\right). \end{aligned}$$

For the martingale M^* it then holds

$$\begin{aligned} Y_i^* &\leq E_i \max_{i \leq j \leq T} \left(Z_j + M_i^* - M_j^*\right) \\ &\leq E_i \max_{i \leq j \leq T} \left(Z_j + Y_i^* + A_i^* - Y_j^* - A_j^*\right) \\ &\leq Y_i^* + E_i \max_{i \leq j \leq T} \left(A_i^* - A_j^*\right) = Y_i^*, \end{aligned}$$

since for all j, $0 \leq j \leq T$, $Y_i^* - E^i Y_{i+1}^* = A_{i+1}^* - A_i^* \geq 0$, and thus

$$Y_i^* = E_i \max_{i \leq j \leq T} \left(Z_j + M_i^* - M_j^*\right). \tag{6}$$

Moreover, by

$$\begin{aligned} \max_{i \leq j \leq T} \left(Z_j + M_i^* - M_j^*\right) &= \max_{i \leq j \leq T} \left(Z_j + Y_i^* + A_i^* - Y_j^* - A_j^*\right) \\ &\leq Y_i^* + \max_{i \leq j \leq T} \left(A_i^* - A_j^*\right) = Y_i^* \end{aligned}$$

and (6) we have (5).

The corner stone for generalizing Theorem 1 to the multiple stopping problem is the following simple proposition which is a slight extension of (5) in a sense.

Proposition 1 *Let* $(Z_i : 0 \leq i \leq T)$ *be a nonnegative integrable cash-flow process with Snell envelope* Y^* *and let* $Y^* = Y_0^* + M^* - A^*$ *be its Doob decomposition as in Theorem 1. It then holds for each* j, $0 \leq j < T$,

$$E_j \max_{j<l\leq T} \left(Z_l - M_l^* + M_j^*\right) = \max_{j<l\leq T} \left(Z_l - M_l^* + M_j^*\right) \quad a.s.$$

Proof For fixed $0 \leq j < T$, we have by (5) on the one hand

$$Y_{j+1}^* = \max_{j<l\leq T} \left(Z_l - M_l^* + M_{j+1}^*\right), \tag{7}$$

and from the Doob decomposition of Y^* and using (7),

$$\begin{aligned} E_j Y_{j+1}^* &= Y_{j+1}^* + M_j^* - M_{j+1}^* = \max_{j<l\leq T} \left(Z_l - M_l^* + M_{j+1}^*\right) + M_j^* - M_{j+1}^* \\ &= \max_{j<l\leq T} \left(Z_l - M_l^* + M_j^*\right) \end{aligned}$$

on the other hand.

Remark 2 It is not difficult to see that a further generalization of Proposition 1 is not possible in the sense that in general

$$E_j \max_{p<l\leq T} \left(Z_l - M_l^* + M_j^*\right) \overset{\text{a.s.}}{\neq} \max_{p<l\leq T} \left(Z_l - M_l^* + M_j^*\right)$$

if $p > j$.

2.3 Dual Representation for the Multiple Stopping Problem

We are now ready for proving the following theorem which is a natural generalization of Theorem 1 and Proposition 1 to the multiple exercise case.

Theorem 2 *It holds for all* $0 \leq i \leq T$, $L = 1, 2, ...$

$$\begin{aligned} Y_i^{*L} &= \inf_{M^{(1)},\ldots,M^{(L)}\in\mathcal{M}} E_i \max_{i\leq j_1<\cdots<j_L\leq T} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(k)} - M_{j_k}^{(k)}\right) &(8) \\ &= \max_{i\leq j_1<\cdots<j_L\leq T} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1}\right) \quad a.s., &(9) \end{aligned}$$

with $j_0 := 0$, *and, in addition*

$$\begin{aligned} E_i &\max_{i<j_1<\cdots<j_L\leq T} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1}\right) &(10) \\ &= \max_{i<j_1<\cdots<j_L\leq T} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1}\right) \quad a.s. \end{aligned}$$

where for $k = 1, ..., L$, M^{*k} is the Doob martingale of the Snell envelope for k exercise rights. That is

$$Y^{*k} = Y_0^{*k} + M^{*k} - A^{*k}$$

with A^* predictable, and $M_0^{*k} = A_0^{*k} = 0$. In particular, for each k, A_i^{*k} is nondecreasing in i.

Proof For any set of martingales $M^{(1)}, ..., M^{(L)} \in \mathcal{M}$ we have,

$$\begin{aligned}
Y_i^{*L} &= \sup_{i \le \tau^{(1)} < ... < \tau^{(L)} \le T} E_i \sum_{k=1}^{L} Z_{\tau^{(k)}} \\
&= \sup_{i \le \tau^{(1)} < ... < \tau^{(L)} \le T} E_i \sum_{k=1}^{L} \left(Z_{\tau_k} + M_{\tau_{k-1}}^{(k)} - M_{\tau_k}^{(k)} \right) \\
&\le \sup_{i \le \tau^{(1)} < ... < \tau^{(L)} \le T} E_i \max_{i \le j_1 < \cdots < j_L \le T} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(k)} - M_{j_k}^{(k)} \right) \\
&= E_i \max_{i \le j_1 < \cdots < j_L \le T} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(k)} - M_{j_k}^{(k)} \right),
\end{aligned}$$

from which it follows that Y_i^{*L} is less than or equal to the right-hand-side of (8). We will now show that this inequality is sharp and that moreover (9) and (10) hold, by induction to the number of exercise rights L. For $L = 1$ the statements collapse to the statements of Theorem 1 and Proposition 1 . Suppose the Theorem holds for L exercise rights. By the Bellman principle,

$$Y_i^{*L+1} = \max \left[Z_i + E_i Y_{i+1}^{*L},\ E_i Y_{i+1}^{*L+1} \right],$$

hence Y_i^{*L+1} may be seen as the Snell envelope of the cash-flow $Z_i + E_i Y_{i+1}^{*L}$ under one exercise right. So by the standard dual representation Theorem 1,

$$Y_i^{*L+1} = \max_{i \le j_1 \le T} \left(Z_{j_1} + E_{j_1} Y_{j_1+1}^{*L} + M_i^{*L+1} - M_{j_1}^{*L+1} \right), \tag{11}$$

where M^{*L+1} is the Doob martingale of Y_i^{*L+1} satisfying $M_0^{*L+1} = 0$. By the induction hypothesis it now follows using (9) and (10) respectively,

$$\begin{aligned}
Y_i^{*L+1} &= \max_{i \le j_1 \le T} \Big(Z_{j_1} + M_i^{*L+1} - M_{j_1}^{*L+1} \\
&\quad + E_{j_1} \max_{j_1+1 \le p_1 < \cdots < p_L \le T} \sum_{k=1}^{L} \left(Z_{p_k} + M_{p_{k-1}}^{*L-k+1} - M_{p_k}^{*L-k+1} \right) \Big) \\
&= \max_{i \le j_1 \le T} \Big(Z_{j_1} + M_i^{*L+1} - M_{j_1}^{*L+1} \\
&\quad + \max_{j_1 < p_1 < \cdots < p_L \le T} \sum_{k=1}^{L} \left(Z_{p_k} + M_{p_{k-1}}^{*L-k+1} - M_{p_k}^{*L-k+1} \right) \Big) \\
&= \max_{i \le j_1 < j_2 < \cdots < j_{L+1} \le T} \sum_{k=1}^{L+1} \left(Z_{j_k} + M_{j_{k-1}}^{*L+1-k+1} - M_{j_k}^{*L+1-k+1} \right),
\end{aligned} \tag{12}$$

hence (9) for $L+1$ rights. Next, using (12), (11), and applying (5) of Theorem 1, Proposition 1, and again the induction hypothesis for L exercise rights yields

$$
\begin{aligned}
&E_i \max_{i<j_1<\cdots<j_{L+1}\le T} \sum_{k=1}^{L+1} \left(Z_{j_k} + M_{j_{k-1}}^{*L+1-k+1} - M_{j_k}^{*L+1-k+1} \right) = E_i Y_{i+1}^{*L+1} \\
&= E_i \max_{i<j_1\le T} \left(Z_{j_1} + E_{j_1} Y_{j_1+1}^{*L} + M_i^{*L+1} - M_{j_1}^{*L+1} \right) \\
&\overset{\text{a.s.}}{=} \max_{i<j_1\le T} \left(Z_{j_1} + E_{j_1} Y_{j_1+1}^{*L} + M_i^{*L+1} - M_{j_1}^{*L+1} \right) \\
&= \max_{i<j_1\le T} \Bigg(Z_{j_1} + M_i^{*L+1} - M_{j_1}^{*L+1} \\
&+ E_{j_1} \max_{j_1+1\le p_1<\cdots<p_L\le T} \sum_{k=1}^{L} \left(Z_{p_k} + M_{p_{k-1}}^{*L-k+1} - M_{p_k}^{*L-k+1} \right) \Bigg) \\
&= \max_{i<j_1\le T} \Bigg(Z_{j_1} + M_i^{*L+1} - M_{j_1}^{*L+1} \\
&\max_{j_1+1\le p_1<\cdots<p_L\le T} \sum_{k=1}^{L} \left(Z_{p_k} + M_{p_{k-1}}^{*L-k+1} - M_{p_k}^{*L-k+1} \right) \Bigg) \\
&= \max_{i\le j_1<\cdots<j_{L+1}\le T} \sum_{k=1}^{L+1} \left(Z_{j_k} + M_{j_{k-1}}^{*L+1-k+1} - M_{j_k}^{*L+1-k+1} \right),
\end{aligned}
$$

hence (10) for $L+1$ rights. Finally, A^{*L+1} is nondecreasing as it is the predictable part of the Snell envelope of the generalized cashflow $Z_i + E_i Y_{i+1}^{*L}$.

3 Monte Carlo Algorithms for the Multiple Dual

In this section we show how well known dual algorithms for the one exercise case such as the primal-dual algorithm of Andersen and Broadie (2004) may be generalized to the multiple exercise cases. In this context we assume that the cash-flow process Z is of the form (with slight abuse of notation)

$$Z_i = Z_i(X_i) \quad 0 \le i \le T, \tag{13}$$

for some underlying (possibly high-dimensional) Markovian process X. Moreover it is assumed that we are given approximations $Y_i^{(k)}$ of Y_i^{*k}, $k = 1, ..., L$, which are of the form

$$Y_i^{(k)} = Y_i^{(k)}(X_i), \quad 0 \le i \le T, \quad 1 \le k \le L. \tag{14}$$

Remark 3 An immediate consequence of (13) is that for the Snell envelopes

$$Y_i^{*k} = Y_i^{*k}(X_i), \quad k = 1, ..., L, \tag{15}$$

so (14) is a quite natural assumption.

It is meanwhile industrial standard to obtain approximations of the form (14) by regression methods. For the single exercise case (Bermudan derivatives for example) the methods of Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (1999) are quite popular, and for the multiple exercise case (e.g. swing options) one may apply these methods recursively as explained in Section 3.1. One generally obtains in this way sub-optimal exercise strategies, hence lower bounds for the optimal value. In Section 3.2 it is described how to incorporate (e.g. regression based) approximations for the Snell envelope in a Monte Carlo algorithm for dual upper bounds.

3.1 Recap of Regression Based Approaches

Let us recap briefly how well-known regression methods such as the method of Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (1999) may be recursively applied to the multiple exercise problem. As these methods are broadly known, we do not explain them here in detail but merely recall that for the single exercise case, both methods end up with an expansion of the continuation function in terms of a properly chosen and 'rich enough' system of basis functions on the state space. That is, for an approximation of the (single exercise) Snell envelope one obtains formally

$$C_i^*(X_i) := E_i Y_{i+1}^*(X_{i+1}) \approx \sum_{r=0}^{R} \beta_{ir} \psi_r(X_i) =: C_i(X_i), \quad 0 \leq i < T,$$

where ($\psi_r : \mathbb{R}^d \to \mathbb{R}^d$, $r = 0, 1, 2, ...$) is a (countable) set of basis functions and R some number which determines the number of basis functions involved in the regression. (Note that due to our conventions in Section 2, $C_T :\equiv 0$.) The coefficients (β_{ir}) are obtained by a regression procedure applied to a Monte Carlo sample of trajectories of X. In Clement et al. (2002) it is analyzed that for a suitable set of basis functions under suitable conditions $C_i \to C_i^*$, when the number of trajectories and the number of basis functions involved go to infinity in a suitable relationship.

Application of the above regression method to the multiple exercise problem is described by the following inductive scheme:

- Step 1 : Construct with our favorite regression method for $0 \leq i \leq T$ the (approximative) continuation functions $C_i^{(1)}(\cdot)$ of the single exercise problem.
- Step k : Let the continuation functions $C_i^{(p)}(\cdot)$, $0 \leq i \leq T$, of the (approximative) multiple exercise problem for p exercise rights be constructed, for all $1 \leq p \leq k \leq L$, that is,

$$C_i^{*p}(X_i) := E_i Y_{i+1}^{*p}(X_{i+1}) \approx \sum_{r=0}^{R} \beta_{ir}^{(p)} \psi_r(X_i) =: C_i^{(p)}(X_i)$$

  (with $C_T^{(p)} :\equiv 0$). Then,
  - If $k < L$, define the cash-flow process

$$\widetilde{Z}_i(X_i) := Z_i(X_i) + C_i^{(k)}(X_i)$$

    with $C_T^{(k)} :\equiv 0$, and apply our favorite regression method to obtain the (approximative) continuation function $\widetilde{C}_i(X_i)$ corresponding to the Snell envelope of $\widetilde{Z}_i$ under one exercise right. Then set

$$C_i^{(k+1)}(X_i) := \widetilde{C}_i(X_i), \quad 0 \leq i \leq T.$$

  - if $k = L$, then stop.

Inductive application of the above scheme thus yields a system of (approximative) continuation functions $C_i^{(k)}(\cdot)$, $1 \le k \le L$, $1 \le i \le T$. At this stage one may take as approximations for the Snell envelopes $(Y_i^{*k} : 1 \le i \le T, 1 \le k \le L)$,

$$Y_i^{(k)}(X_i) := \max[Z_i(X_i) + C_i^{(k-1)}(X_i),\ C_i^{(k)}(X_i)] \tag{16}$$

with $C^{(0)} :\equiv 0$.

It is important to note that while approximations (16) may be accurate, they can be biased from above or below. Nonetheless, (16) can be used for constructing upper bounds via Theorem 2 as explained below. For bounding the Snell envelopes from above *and* below we also need lower bounds however. For this we construct for each k, $1 \le k \le L$, a system of (sub-optimal) exercise policies $(\tau_i^{p,k} : 1 \le i \le T,\ 1 \le p \le k)$ as follows. Define $\tau_i^{0,k} := i - 1$, and

$$\tau_i^{p,k} = \inf\{j : \tau_i^{p-1,k} < j \le T,\ Z_j(X_j) + C_j^{(k-p)}(X_i) \ge C_j^{(k-p+1)}(X_j)\}. \tag{17}$$

Then the process defined by

$$\underline{Y}_i^{(k)} := E_i \sum_{p=1}^{k} Z_{\tau_i^{p,k}} \tag{18}$$

due to the stopping family (17) is X_i measurable and is a lower bound process, i.e. $\underline{Y}_i^{(k)}(X_i) \le Y_i^{*k}(X_i)$. Obviously, the stopping family $\tau_i^{1,k}$ satisfies for each k the consistency relation

$$\tau_i^{1,k} > i \Longrightarrow \tau_i^{1,k} = \tau_{i+1}^{1,k}. \tag{19}$$

Because of (19) we additionally have

$$\begin{aligned}
\underline{Y}_i^{(k)} 1_{\tau_i^{1,k} > i} &= 1_{\tau_i^{1,k} > i} E_i \sum_{p=1}^{k} Z_{\tau_i^{p,k}} = 1_{\tau_i^{1,k} > i} E_i 1_{\tau_i^{1,k} > i} \sum_{p=1}^{k} Z_{\tau_i^{p,k}} \\
&= 1_{\tau_i^{1,k} > i} E_i \sum_{p=1}^{k} Z_{\tau_{i+1}^{p,k}} = 1_{\tau_i^{1,k} > i} E_i E_{i+1} \sum_{p=1}^{k} Z_{\tau_{i+1}^{p,k}} \\
&= 1_{\tau_i^{1,k} > i} E_i \underline{Y}_{i+1}^{(k)}.
\end{aligned} \tag{20}$$

which is in the case $L = 1$ a corner stone of the primal-dual algorithm (Andersen and Broadie, 2004). The lower bounds $\underline{Y}$ may be constructed by a standard (non-nested) Monte Carlo simulation using (17).

3.2 Dual Algorithms for the Multiple Exercise Problem

For constructing dual upper bounds we have two options: the first one is the multiple dual based on (16) and the second is the dual based on (18). For any set of approximations $Y^{(k)}$, for example (16), we may construct the Doob martingale $M^{(k)}$ of $Y^{(k)}$, via

$$M_i^{(k)} - M_{i-1}^{(k)} = Y_i^{(k)} - E_{i-1} Y_i^{(k)}$$

and consider the upper bound

$$
\begin{aligned}
Y_i^{\mathrm{up},L} &:= E_i \max_{i\le j_1<\cdots<j_L\le T} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(L-k+1)} - M_{j_k}^{(L-k+1)} \right) \\
&= E_i \max_{i\le j_1<\cdots<j_L\le T} \sum_{k=1}^{L} \left(Z_{j_k} - \sum_{r=j_{k-1}+1}^{j_k} Y_r^{(L-k+1)} + \sum_{r=j_{k-1}+1}^{j_k} E_{r-1} Y_r^{(L-k+1)} \right) \\
&= E_i \max_{i\le j_1<\cdots<j_L\le T} \sum_{k=1}^{L} \left(Z_{j_k} + Y_{j_{k-1}}^{(L-k+1)} - Y_{j_k}^{(L-k+1)} \right. \\
&\qquad \left. + \sum_{r=j_{k-1}}^{j_k-1} \left(E_r Y_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)} \right) \right) \\
&= Y_i^{(L)} + E_i \max_{i\le j_1<\cdots<j_L\le T} \sum_{k=1}^{L} \left(Z_{j_k} + Y_{j_k}^{(L-k)} - Y_{j_k}^{(L-k+1)} \right. \\
&\qquad \left. + \sum_{r=j_{k-1}}^{j_k-1} \left(E_r Y_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)} \right) \right)
\end{aligned} \tag{21}
$$

(note that $Y^{(0)} \equiv 0$). We now have the following proposition which provides an estimate for the gap between the lower and upper bound.

Proposition 2 *If the approximate solution satisfies $Z_j + E_j Y_{j+1}^{(k)} \le Y_j^{(k+1)}$ for all j and k, it then holds,*

$$
\underline{Y}_i^{up,L} - \underline{Y}_i^{(L)} \le E_i \sum_{k=1}^{L} \sum_{r=i}^{T-1} \left(E_r Y_{r+1}^{(k)} - Y_r^{(k)} \right)^+ ,
$$

with equality if $Y^{(k)} = Y^{*k}$ for all k.

Proof From (21) we obtain by rearranging terms,

$$
\begin{aligned}
\underline{Y}_i^{\mathrm{up},L} - \underline{Y}_i^{(L)} &= E_i \max_{i\le j_1<\cdots<j_L\le T} \left(\sum_{k=1}^{L} \left(Z_{j_k} + E_{j_k} Y_{j_k+1}^{(L-k)} - Y_{j_k}^{(L-k+1)} \right) \right. \\
&\quad + \sum_{r=i}^{j_1-1} \left(E_r Y_{r+1}^{(L)} - Y_r^{(L)} \right) + \sum_{k=1}^{L} \left(Y_{j_k}^{(L-k)} - E_{j_k} Y_{j_k+1}^{(L-k)} \right) \\
&\quad \left. \sum_{k=2}^{L} \left(E_{j_{k-1}} Y_{j_{k-1}+1}^{(L-k+1)} - Y_{j_{k-1}}^{(L-k+1)} \right) + \sum_{k=2}^{L} \sum_{r=j_{k-1}+1}^{j_k-1} \left(E_r Y_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)} \right) \right) \\
&= E_i \max_{i\le j_1<\cdots<j_L\le T} \left(\sum_{k=1}^{L} \left(Z_{j_k} + E_{j_k} Y_{j_k+1}^{(L-k)} - Y_{j_k}^{(L-k+1)} \right) \right. \\
&\quad \left. + \sum_{r=i}^{j_1-1} \left(E_r Y_{r+1}^{(L)} - Y_r^{(L)} \right) + \sum_{k=1}^{L-1} \sum_{r=j_{k-1}+1}^{j_k-1} \left(E_r Y_{r+1}^{(L-k)} - Y_r^{(L-k)} \right) \right),
\end{aligned}
$$

and then the statement easily follows.

Proposition 2 may be considered as a generalization of a similar result for the case $L = 1$ in Andersen and Broadie (2004) and Kolodko and Schoenmakers (2004).

Obviously, the implementation of (21), based on (16) for instance, generally leads to a (one-degree) nested Monte Carlo simulation. Just as in the single exercise case it is effective to simulate the term $Y_0^{(L)}$ in (21) accurately with a standard (non-nested) Monte Carlo simulation and a relatively large sample size. As a rule in the case $L = 1$, this takes out about 90% of the variance depending on the quality of the approximation of course. What is left is the simulation of the gap term

$$\Delta_i^{(L)} := Y_i^{\mathrm{up},L} - Y_i^{(L)}.$$

This can be carried out by the following algorithm (we take $i = 0$ w.l.o.g.).

Algorithm 1

- ***for*** $n = 1$ ***to*** N ***do:***
  - *Simulate and store an 'outer' trajectory* $({}_nX_r : 0 \le r \le T)$ *which starts in* x_0 *say. Hence* ${}_nX_0 = x_0$.
  - *Simulate for each* r, $0 \le r < T$, *independently a set of* M *'one-step inner trajectories'* $({}_n^mX_{r+1} : 1 \le m \le M)$ *which start at* ${}_nX_r$. *That is, each inner simulation* ${}_n^mX_{r+1}$ *is independently distributed according to* $P_r(X_{r+1} \in dx \mid {}_nX_r)$.
  - ***for*** $r = 0$ ***to*** $T - 1$ ***do :***
    - ***for*** $l = 1$ ***to*** L ***do:*** *Evaluate and store*

$$\left[E_r Y_{r+1}^{(l)}\right]({}_nX_r) \approx \frac{1}{M}\sum_{m=1}^{M} Y_{r+1}^{(l)}({}_n^mX_{r+1}) =: \ {}_nE_r^{(l)},$$

where the $Y_r^{(l)}(\cdot)$ *are given by (16) for example.*

  - *Evaluate and store*

$$\xi_n := \max_{0 \le j_1 < \cdots < j_L \le T} \sum_{k=1}^{L} \Bigg(Z_{j_k}({}_nX_{j_k}) + Y_{j_k}^{(L-k)}({}_nX_{j_k}) - Y_{j_k}^{(L-k+1)}({}_nX_{j_k}) + \sum_{r=j_{k-1}}^{j_k - 1} \left({}_nE_r^{(L-k+1)} - Y_r^{(L-k+1)}({}_nX_r) \right) \Bigg).$$

  - *Give the memory used for intermediate quantities other than* ξ_n *(outer trajectory, inner trajectories, etc.) free.*
- *Evaluate*

$$\widehat{\Delta}_0^{(L)} := \frac{1}{N}\sum_{n=1}^{N} \xi_n \approx \Delta_0^{(L)}.$$

Finally, if $\widehat{Y}_0^{(L)}$ is a Monte Carlo estimate of $Y_0^{(L)}$, we set $\widehat{Y}_0^{\mathrm{up},L} = \widehat{Y}_0^{(L)} + \widehat{\Delta}_0^{(L)}$.

Since for any (finite) set of random variables $(\varsigma_i : i \in I)$ it holds $E \max_{i \in I} \varsigma_i \ge \max_{i \in I} E\varsigma_i$, it follows analogously to Andersen and Broadie (2004) (and also Kolodko and Schoenmakers (2004)) that $\widehat{Y}_0^{\mathrm{up},L}$ is a Monte Carlo estimate of $Y_0^{\mathrm{up},L}$ which is biased up, hence an upper bound. We underline that Algorithm 1 is essentially different from the one in Meinshausen and Hambly (2004) as it may be applied to any set of approximations to (Y_i^{*k}) and so may be regarded as 'stopping time free' in this respect. In contrast, the Meinshausen and Hambly (2004) method always involves a set of 'good' stopping times for the multiple stopping problem besides a set of 'good' martingales.

As an alternative we may consider the dual based on (18) and extend the primal-dual algorithm of Andersen and Broadie (2004) to the dual representation in Theorem 2. We thus construct the Doob martingale $\underline{M}^{(k)}$ of $\underline{Y}^{(k)}$, via

$$\underline{M}_i^{(k)} - \underline{M}_{i-1}^{(k)} = \underline{Y}_i^{(k)} - E_{i-1}\underline{Y}_i^{(k)}$$

and analogue to (21) we consider the upper bound

$$\underline{Y}_i^{\mathrm{up},L} := \underline{Y}_i^{(L)} + E_i \max_{i \le j_1 < \cdots < j_L \le T} \sum_{k=1}^{L} \Bigg(Z_{j_k} + \underline{Y}_{j_k}^{(L-k)} - \underline{Y}_{j_k}^{(L-k+1)} \tag{22}$$

$$+ \sum_{r=j_{k-1}}^{j_k - 1} 1_{\tau_r^{1,L-k+1}=r} \left(E_r \underline{Y}_{r+1}^{(L-k+1)} - \underline{Y}_r^{(L-k+1)} \right) \Bigg),$$

using (20).

Similar as for (21) the implementation of (22) leads to a nested upper biased Monte Carlo algorithm as spelled out below. The lower bound $\underline{Y}_i^{(L)}$ in (22) can be accurately computed using a standard (non-nested) Monte Carlo simulation using stopping rule (17) and a relatively large sample size. Typically, as in Algorithm 1, the gap term

$$\underline{\Delta}_i^{(L)} := \underline{Y}_i^{\mathrm{up},L} - \underline{Y}_i^{(L)}.$$

has relatively low variance, and may be computed by the following algorithm (we take $i = 0$ w.l.o.g.).

Algorithm 2

- ***for*** $n = 1$ ***to*** N ***do***
  - *Simulate and store an 'outer' trajectory* $({}_nX_r : 0 \le r \le T)$ *which starts in* x_0 *say. Hence* ${}_nX_0 = x_0$.
  - *Simulate and store for each fixed* r, $0 \le r < T$, *independently a set of* M *'inner trajectories'* $({}_n^mX_s : r \le s \le T,\ 1 \le m \le M)$ *which all start at* ${}_nX_r$. *That is, each inner simulation* $({}_n^mX_s : r \le s \le T)$ *is independently distributed according to* $P_r(X_\cdot \in B \mid {}_nX_r)$ *with* $B \subset \mathbb{R}^d$ *being an arbitrary Borel set.*
  - ***for*** $r = 0$ ***to*** $T - 1$ ***do***
    - ***for*** $l = 1$ ***to*** L ***do*** *evaluate and store*

$$\underline{Y}_r^{(l)}({}_nX_r) \approx \frac{1}{M} \sum_{m=1}^{M} \sum_{p=1}^{l} Z_{{}_n^m\tau_r^{p,l}}(X_{{}_n^m\tau_r^{p,l}}) =: y_{n,r}^{(l)}.$$

    - ***for*** $l = 1$ ***to*** L ***do*** *evaluate and store*

$$Bool_{n,r,l} := \left[Z_r({}_nX_r) + C_r^{(l-1)}({}_nX_r) \ge C_r^{(l)}({}_nX_r) \right] = [{}_n\tau_r^{1,l} = r]$$

      where the functions $C_r^{(p)}(\cdot)$ *are constructed by the regression procedure in Section 3.1.*

- ***for*** $l = 1$ ***to*** L ***do if*** $Bool_{n,r,l}$ ***then*** *evaluate and store*

$$\left[E_r \underline{Y}_{r+1}^{(l)}\right]({}_nX_r) - \left[\underline{Y}_r^{(l)}\right]({}_nX_r) \approx$$
$$-y_{n,r}^{(l)} + \frac{1}{M}\sum_{m=1}^{M}\sum_{p=1}^{l} Z_{{}_n^m\tau_{r+1}^{p,l}}(X_{{}_n^m\tau_{r+1}^{p,l}}) =: \vartheta_{n,r}^{(l)}$$

else $\vartheta_{n,r}^{(l)} := 0$.

– *Evaluate and store (with* $y^{(0)} \equiv 0$*)*

$$\xi_n :=$$
$$\max_{0\le j_1<\cdots<j_L\le T}\sum_{k=1}^{L}\left(Z_{j_k}({}_nX_{j_k}) + y_{n,j_k}^{(L-k)} - y_{n,j_k}^{(L-k+1)} + \sum_{r=j_{k-1}}^{j_k-1}\vartheta_{n,r}^{(l)}\right).$$

– *Give the memory used for intermediate quantities other than* ξ_n *(outer trajectory, inner trajectories, etc.) free.*

– *Evaluate*

$$\widehat{\underline{\Delta}}_0^{(L)} := \frac{1}{N}\sum_{n=1}^{N}\xi_n \approx \underline{\Delta}_0^{(L)} = \underline{Y}_0^{up,L} - \underline{Y}_0^{(L)}.$$

Similar to Algorithm 1, it can be seen by the same argument that replacing the inner conditional expectations with their Monte Carlo estimates leads to un upper biased estimator for $\underline{Y}_0^{\mathrm{up},L}(X_0)$ (when $i = 0$).

Remark 4 For $L = 1$ (and $i = 0$) representation (22) and Algorithm 2 collapses to the well-known Andersen-Broadie representation and Andersen-Broadie algorithm, respectively (note that $j_0 := 0$). Indeed, for $L = 1$ we get

$$\underline{Y}_0^{\mathrm{up},L}(X_0) = \underline{Y}_0(X_0) + E_0 \max_{i\le j\le T}\left(Z_j - \underline{Y}_j + \sum_{r=0}^{j-1} 1_{\tau=r}\left(E_r\underline{Y}_{r+1} - \underline{Y}_r\right)\right) \tag{23}$$

with the well demonstrated advantage that the term $\underline{Y}_0(X_0)$ may be computed using an accurate non-nested Monte Carlo simulation, and that the remaining gap term has typically low variance.

It would be interesting to investigate a comparison between Algorithm 2 and Algorithm 1. For the case $L = 1$ however, it is generally known that Algorithm 2, carried out with a stopping family generated from (16) via (17), usually gives better bounds than Algorithm 1, although the latter procedure may require less computational costs.

Algorithm 3 If approximations $Y^{(k)}$ (for example (16)) of the Snell envelopes satisfy the condition in Proposition 2 we may modify Algorithm 1 in an obvious way to obtain an algorithm which estimates the gap

$$\widetilde{\Delta}_0^{(L)} := E_0\sum_{k=1}^{L}\sum_{r=0}^{T-1}\left(E_rY_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)}\right)^+ \ge \Delta_0^{(L)}$$

and therefore a detailed description is omitted. If the condition in Proposition 2 is not fulfilled one may take recursively from $k = 1$ up to L (with $\widetilde{Y}_i^{(0)} \equiv 0$),

$$\widetilde{Y}_i^{(k)} := \max(Z_i + E_i \widetilde{Y}_{i+1}^{(k-1)}, Y_i^{(k)}), \quad 0 \leq i < T,$$

instead of the $Y^{(k)}$. Clearly, in this algorithm no maximization procedure is involved. The price one may have to pay however is a larger gap, but, nonetheless, due to Proposition 2 this gap may be still small if the input approximations of the Snell envelopes are good enough.

Remark 5 Given the success of the dual representation in the single exercise case as reported in the literature, it will be obvious that the analogue algorithms for the multiple case presented here are potentially promising as well. A meaningful extensive numerical study and comparison with the method of Meinshausen and Hambly (2004), for example, should be carried out for a complex enough product. However, we already note that it is not difficult to see that the complexity of Algorithms 1–3 is comparable with the complexity of the procedure in Meinshausen and Hambly (2004) while the implementation of these algorithms looks more transparent and straightforward.

Remark 6 In the case where the process X is adapted to a Brownian filtration it looks feasible to construct a linear Monte Carlo algorithm for the multiple dual in a similar way as presented in Belomestny et al. (2009). This might be done in future work.

Remark 7 One may wonder whether it is possible to generalize, like in this article, also the multiplicative dual approach of Jamshidian (2007) to the multiple exercise case. In this respect we found a multiplicative dual representation for the multiple exercise problem indeed, but, this representation consists of nested conditional expectations where the degree of nesting is equal to the number of exercise possibilities. As such this is of no practical use of course. In particular the construction of the additive multiple dual as in this article relies on the nice almost sure properties of the standard additive dual representation when the optimal martingale is plugged in. The multiplicative dual fails to have this property, see also the discussion in Chen and Glasserman (2007) on this. It therefore seems not possible to find a multiplicative dual representation for the multiple stopping problem which allows in general for Monte Carlo simulation with only one degree of nesting.

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$$Y_0^{*L} = \inf_{\substack{M \in \mathcal{M}, \\ M_0 = 0}} E_0 \max_{0 \leq j_1 < \cdots < j_L \leq T} \sum_{k=1}^{L} \left(Z_{j_k} - M_{j_k} \right).$$

Inspired to examine this guess the present paper resulted.

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