Holomorphic transforms with application to affine processes☆,☆☆

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Abstract

In a rather general setting of Itô-Lévy processes we study a class of transforms (Fourier for example) of the state variable of a process which are holomorphic in some disc around time zero in the complex plane. We show that such transforms are related to a system of analytic vectors for the generator of the process, and we state conditions which allow for holomorphic extension of these transforms into a strip which contains the positive real axis. Based on these extensions we develop a functional series expansion of these transforms in terms of the constituents of the generator. As application, we show that for multidimensional affine Itô-Lévy processes with state dependent jump part the Fourier transform is holomorphic in a time strip under some stationarity conditions, and give log-affine series representations for the transform.

Key words: Itô-Lévy processes, holomorphic transforms, affine processes

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1. Introduction

Transforms are an important tool in the theory of (ordinary and partial) differential equations and in stochastic analysis. In probability theory the Fourier

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transform of a random variable, which represents the characteristic function of the corresponding distribution, is widely used. Fourier (and Laplace) transforms have become increasingly popular in mathematical finance as well. On the one hand, for example, via a complex Laplace transform and the convolution theorem one may derive pricing formulas for European options (e.g., see [9],[10]). On the other hand, Laplace and Fourier transforms are known in closed form for many classes of processes. A famous example is the so-called Lévy-Khintchine formula which provides an explicit expression for the characteristic function of a Lévy process. More recent financial literature goes well beyond Lévy processes and attempts to establish explicit or semi-explicit formulas for derivatives where underlyings are modelled, for instance, by affine processes (see among others [6],[8] and [7]) or more general Itô-Lévy processes (e.g. see [14]). Theoretical analysis of affine processes is done in the seminal paper by Duffie, Filipović and Schachermayer [7] and has led to a unique characterization of affine processes. In particular, it is shown that the problem of determining the (conditional) Fourier transform of an affine process X_s corresponds to the problem of solving a system of generalized Riccati differential equations in the time variable s (see [7]). Although closed form solutions of this system can be found in important cases, there is no generic approach to solve such a system in the general multi-dimensional case. In this article we establish some kind of functional series representation for the Fourier transform, hence the characteristic function of the process under consideration and, in principle, for more general transforms. The most natural one is a Taylor expansion in time s around $s_0 = 0$. Unfortunately, it turns out that in many cases the resulting power series converges only in $s = s_0 = 0$. This problem corresponds to a difficulty which is well known in semi-group theory and in the theory of parabolic differential equations: small time expansions for the solutions of parabolic equations are usually possible in a neighborhood of some $s_0 > 0$, while an expansion around $s_0 = 0$ may have zero convergence radius. In this paper we prove that for a generator with affine coefficients the Fourier transform extends holomorphically into a disc around $s_0 = 0$ and a strip containing the positive real axes, under some mild regularity conditions. Then, for multi-dimensional affine processes we obtain convergent expansions for the Fourier transform and its logarithm on the whole time line. Hence, we have (affine) series representations for the exponent of the characteristic function of a general multi-dimensional affine process. More generally, we develop a framework based on a concept of analytic vectors which allows for functional series expansions for a class of holomorphic transforms which covers the standard Fourier transform.

The outline of the paper is as follows. The basic setup is described in Section 2. In Section 3 we introduce the notion of analytic vectors associated with a given generator and study functional series expansions for the corresponding transform. Section 4 is devoted to the Cauchy problem for affine generators. In Section 5 we derive series representations for the logarithm of the Fourier transform corresponding to a generator with affine coefficients. Section 6 gives an explicit representation for these expansions in a one-dimensional case. Finally, Section 7 contains results for affine Itô-Lévy processes which mainly follow from previous sections. More technical proofs are given in the Appendix.

2. Basic setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a standard filtered probability space where the filtration $(\mathcal{F}_t)$ satisfies 'the usual conditions'. On this space we consider for each $x \in \mathbb{R}^n$ a compensated Poisson random measure $\widetilde{N}(x, dt, dz, \omega) = N(x, dt, dz, \omega) - v(x, dz)dt$ on $\mathbb{R}_+ \times \mathbb{R}^n$, where N is a Poisson random measure with (deterministic) intensity kernel of the form $v(x, dz)dt = \mathbb{E}\, N(x, dt, dz)$ satisfying $v(x, B) < \infty$ for any $B \in \mathcal{B}(\mathbb{R}^n)$ such that $0 \notin \overline{B}$ (closure of B). Hence N is determined by

$$P\left[N(x, (0, t], B) = k\right] = \exp(-tv(x, B))\frac{t^k v^k(x, B)}{k!}, \quad k = 0, 1, 2, ...$$

In particular, for any $B \in \mathcal{B}(\mathbb{R})$ with $0 \notin \overline{B}$ and $x \in \mathbb{R}^n$, the process $M_t^{B,x} := \widetilde{N}(x, (0, t], B)$ is a (true) martingale. Further, we assume that the kernel v satisfies,

$$v(x, \{0\}) = 0, \quad \int_{\mathbb{R}^n} (|z|^2 \wedge |z|)v(x, dz) < \infty, \quad x \in \mathbb{R}^n.$$

Let us assume $W(t)$ to be a standard Brownian motion in $\mathbb{R}^m$ living on our basic probability space, and consider the Itô-Lévy SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dW(t) + \int_{\mathbb{R}^n} z\widetilde{N}(X_{t-}, dt, dz), \quad X_0 = x, \tag{2.1}$$

for deterministic functions $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, which satisfy sufficient regularity and/or mutual consistency conditions such that (2.1) has a unique strong solution X, called an Itô-Lévy process, which can be regarded as a strong Markov process (e.g., see [17], [14]).

As a well-known fact, the above process X can be connected to some kind of evolution equation in a natural way. In this context we consider a 'pseudo generator'

$$A^\sharp : \mathcal{D}(A^\sharp) \subset \mathcal{C}^{(2)} \subset \mathcal{C} \longrightarrow \mathcal{C}, \tag{2.2}$$

where $\mathcal{C} := C(\mathbb{R}^n)$ is the space of continuous functions $f : \mathbb{R}^n \to \mathbb{C}$, equipped with the topology of uniform convergence on compacta, and $\mathcal{C}^{(2)}$ is the space of functions $f \in \mathcal{C}$ which are two times continuously differentiable. Further, $f \in \mathcal{D}(A^\sharp)$ iff $f \in \mathcal{C}^{(2)}$, and

$$\begin{aligned} A^\sharp f(x) := \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i} \\ + \int_{\mathbb{R}^n}\left[f(x+z) - f(x) - \frac{\partial f}{\partial x}\cdot z\right] v(x,dz), \text{ with } a := \sigma\sigma^\top, \end{aligned} \tag{2.3}$$

exists and is such that $A^\sharp f \in \mathcal{C}$. In this respect we assume that the building blocks of the operator (2.3), $a(x)$, $b(x)$, and $v(x,B)$ for any B with $0 \notin \overline{B}$, have bounded derivatives of any order.

Clearly, $\mathcal{D}(A^\sharp)$ is dense in $\mathcal{C}$ and we henceforth require that the operator $A^\sharp$ thus defined is closable. In general, closability of an integro-differential operator of type (2.3) will in particular depend on the characteristics of the measure v and the chosen topology. In this respect, the following proposition provides sufficient conditions on the measure v for the above specified topology.

Proposition 2.1. *Let the measure v be of the form $v(x,dz) = \gamma(x,z)\nu(dz)$, where the function γ has bounded partial derivatives of any order, and ν is* *a Borel measure with bounded support satisfying* $\int |z|^2 \nu(dz) < \infty$*. Then the operator* $A^\sharp$ *in (2.3) is closable.*

The proof is given in the Appendix.

Remark 2.2. *Without going into further detail we note that the above closability restrictions on the measure* $\upsilon(x, dz)$ *may be relaxed, to unbounded support for example, when the domain of* $A^\sharp$ *is restricted to functions with certain growth constrains. However, the restrictions of Proposition 2.1 do not exclude (pure) jump-processes with infinite activity. Also, in practice, heavy tailed jump measures restricted to a large enough bounded domain may be considered.*

The closure of $A^\sharp$ is denoted by

$$A : \mathcal{D}(A) \subset \mathcal{C} \longrightarrow \mathcal{C}. \tag{2.4}$$

As such A can be seen as a relaxation of the notion of a generator of a strongly continuous Feller-Dynkin semigroup associated with the process X, for which the Hille-Yosida theorem applies. This semigroup is usually defined on the Banach space $C_0(\mathbb{R}^n)$, i.e. the set of continuous functions on $\mathbb{R}^n$ which vanish at infinity, equipped with the supremum norm, and its generator coincides literally with $A^\sharp$ on a dense subdomain of $C_0(\mathbb{R}^n)$. By slight abuse of terminology however, we will also refer to A as 'generator' when A is considered in connection with the process X given via (2.1). Let now $\mathfrak{F} := \{f_u,\, u \in I\} \subset \mathcal{C}$ be a dense subset of bounded continuous functions $f_u : \mathbb{R}^n \to \mathbb{C}$ which have bounded derivatives of any order. With respect to the (closed) generator (2.4) we consider for each $f_u \in \mathfrak{F}$ the (generalized) Cauchy problem

$$\begin{aligned} \frac{\partial \widehat{p}}{\partial s}(s, x, u) &= A\widehat{p}(s, x, u), \\ \widehat{p}(0, x, u) &= f_u(x), \quad s \geq 0, \quad x \in \mathfrak{X} \subset \mathbb{R}^n, \end{aligned} \tag{2.5}$$

where $\mathfrak{X}$ is some open (maximal) domain, and assume that problem (2.5) has a unique solution for $0 \leq s < \infty$. In this context existence and uniqueness results can be found in [11], (see also [2] for generalizations and results for unbounded initial data). Another proof can be obtained via purely stochastic methods using Malliavin calculus, see [3]. In particular, if some global ellipticity condition is satisfied we may have $\mathfrak{X} = \mathbb{R}^n$. For mixed type operators, i.e. for operators where the type of the second order differential part may vary in space (which may happen for differential operators with affine coefficients for example), existence, uniqueness, and the maximal domain $\mathfrak{X}$ has to be considered case by case. We underline, however, that in this article the main focus is on functional series representations for the solution of (2.5), and we therefore merely assume that sufficient regularity conditions for the coefficients in (2.3) (hence (2.1)) are fulfilled.

Remark 2.3. *In our analysis we often consider the pseudo generator (2.3) and its closure (2.4) on $\mathcal{C} := C(\mathfrak{X})$, for an open domain $\mathfrak{X} \subset \mathbb{R}^n$, rather than $C(\mathbb{R}^n)$. For notational convenience (while slightly abusing notation) we will denote these respective operators with $A^\sharp$ and A also.*

If A is the generator of the process (2.1), the solution $\widehat{p}(s,x,u)$ has the probabilistic representation

$$\widehat{p}(s,x,u) = \mathbb{E}\left[\, f_u(X_s^{0,x})\right],$$

where $X^{0,x}$ is the unique strong solution of (2.1) with $X_0^{0,x} = x$ a.s. We refer to $\widehat{p}(s,x,u)$ as *generalized transform* of the process $X_s^{0,x}$ associated with $\mathfrak{F}$. As a canonical example we may consider

$$f_u(x) := e^{\mathrm{i}u^\top x}, \quad u \in \mathbb{R}^n, \tag{2.6}$$

in which case (2.5) yields the characteristic function $\widehat{p}(s,x,u) = \mathbb{E}[e^{\mathrm{i}u^\top X_s^{0,x}}]$.

By using multi-index notation, the integral term in (2.3) may be formally expanded as

$$\int_{\mathbb{R}^n}\left[f(x+z) - f(x) - \frac{\partial f}{\partial x}\cdot z\right]v(x,dz) =$$
$$\sum_{|\alpha|\geq 2}\frac{1}{\alpha!}\partial_{x^\alpha}f(x)\int z^\alpha v(x,dz) =: \sum_{|\alpha|\geq 2}\frac{1}{\alpha!}m_\alpha(x)\partial_{x^\alpha}f(x).$$

Hence, we may write formally the generator as an infinite order differential operator

$$A = \sum_{|\alpha|>0} \mathfrak{a}_\alpha(x)\partial_{x^\alpha} \tag{2.7}$$

with obvious definitions of the coefficients $\mathfrak{a}_\alpha(x)$ for $|\alpha| > 0$.

3. Analytic vectors and transforms

First we introduce the notion of a set of analytic vectors associated with an operator A.

Definition 3.1. $\mathfrak{F} = \{f_u,\, u \in I\}$ *is a set of analytic vectors for an operator* A *in an open region* $\mathfrak{X}$, *if*

(i) $A^k f_u$ *exists for any* $u \in I$ *and* $k \in \mathbb{N}$,

(ii) *for every* $u \in I$ *there exists* $R_u > 0$ *such that for all* $x \in \mathfrak{X}$,

$$\lim_{k\to\infty} \sup_{r\geq k} \sqrt[r]{\frac{|A^r f_u(x)|}{r!}} \leq R_u^{-1}$$

where the limit is uniform over any compact subset of $\mathfrak{X}$ *(hence a locally uniform Cauchy-Hadamard criterion).*

Example 3.2. *For unbounded self-adjoint operators on a Hilbert space, analytic vectors can be constructed via their spectral resolution [15]. In [16] unbounded operators on sequence spaces are studied which are represented by exponentiable infinite matrices. For such operators the standard basis composes a system of analytic vectors in fact.*

If $\mathfrak{F}$ is a set of analytic vectors in the sense of Definition 3.1 then for all $x \in \mathfrak{X}$ the map

$$s \to P_s f_u(x) := \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k f_u(x), \quad |s| < R_u \tag{3.1}$$

is holomorphic in the complex disc $D_0 := \{s \in \mathbb{C} : |s| < R_u\}$ and the series converges uniformly in x over any compact subset of $\mathfrak{X}$. In fact, $P_s f_u(x)$ coincides with $\widehat{p}(s, x, u)$ for $0 \leq s < R_u$.

Proposition 3.3. *If $\mathfrak{F}$ is a set of analytic vectors, the map $(s,x) \to P_s f_u(x)$ defined in* (3.1) *satisfies* (2.5) *for all s, $|s| < R_u$ and $x \in \mathfrak{X}$. In particular we have $P_s f_u(x) = \widehat{p}(s,x,u)$, $0 \le s < R_u$.*

Proof. Obviously, $P_0 f_u(x) = f_u(x)$ for $x \in \mathfrak{X}$. Set (see Remark 2.3)

$$P_s^{(N)} f_u(x) := \sum_{k=0}^{N} \frac{s^k}{k!} A^k f_u(x),$$

then both $P_s^{(N)} f_u(x)$ and

$$AP_s^{(N)} f_u(x) := \sum_{k=0}^{N} \frac{s^k}{k!} A^{k+1} f_u(x)$$

converge uniformly for any x in a compact subset of $\mathfrak{X}$ and for any s satisfying $|s| < R_u - \varepsilon$ with arbitrary small ε. Hence, since A is closed,

$$AP_s f_u(x) = \sum_{k=0}^{\infty} \frac{s^k}{k!} A^{k+1} f_u(x) = \frac{\partial}{\partial s} \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k f_u(x) = \frac{\partial}{\partial s} P_s f_u(x),$$

and we are done. $\square$

In order to study generalized transforms associated with a set of analytical vectors $\mathfrak{F}$ in domains containing the non-negative real axis we introduce for $\eta > 0$ the sequence

$$q_k^{(\eta)}(x,u) := \frac{1}{k!} \prod_{r=0}^{k-1} (\eta^{-1} A + rI) f_u(x) \tag{3.2}$$

$$=: \frac{1}{k!} \sum_{r=0}^{k} c_{k,r} \eta^{-r} A^r f_u(x), \quad x \in \mathfrak{X}, \quad u \in I, \quad k = 0,1,2,... \tag{3.3}$$

In (3.3) the coefficients $c_{k,r}$, $0 \le r \le k$, are determined by the identity

$$\prod_{r=0}^{k-1} (z+r) = z(z+1) \cdot \ldots \cdot (z+k-1) \equiv \sum_{r=0}^{k} c_{k,r} z^r,$$

and are usually called unsigned Stirling numbers of the first kind. These numbers satisfy $c_{0,0} = 1$ and

$$c_{k,0} \equiv 0, \quad c_{k,k} \equiv 1,$$

$$c_{k+1,j} = kc_{k,j} + c_{k,j-1}, \quad 1 \le j \le k, \tag{3.4}$$

if $k \ge 1$. Obviously, the following recursion is equivalent to (3.2),

$$(k+1)q_{k+1}^{(\eta)}(x,u) = \eta^{-1} A q_k^{(\eta)}(x,u) + kq_k^{(\eta)}(x,u), \quad k \ge 0, \quad x \in \mathfrak{X}. \tag{3.5}$$

The next theorem provides a functional series representation for the solution of (2.5) for all $s \ge 0$, under certain conditions.

Theorem 3.4. *Let $\mathfrak{F}$ be a set of analytic vectors in the sense of Definition 3.1, $u \in I$ be fixed, and the sequence $(q_k^{(\eta)})$ be defined as in (3.3). Let $\widehat{p}$ be the solution of the Cauchy problem (2.5). Then the following statements are equivalent:*

(i) There exists a constant $R_u > 0$ such that for each $x \in \mathfrak{X}$, the map $s \to \widehat{p}(s,x,u)$ has a holomorphic extension to the domain

$$G_{R_u} := \{z : |z| < R_u\} \cup \{z : \operatorname{Re} z > 0 \ \wedge \ |\operatorname{Im} z| < R_u\},$$

see Figure 1.

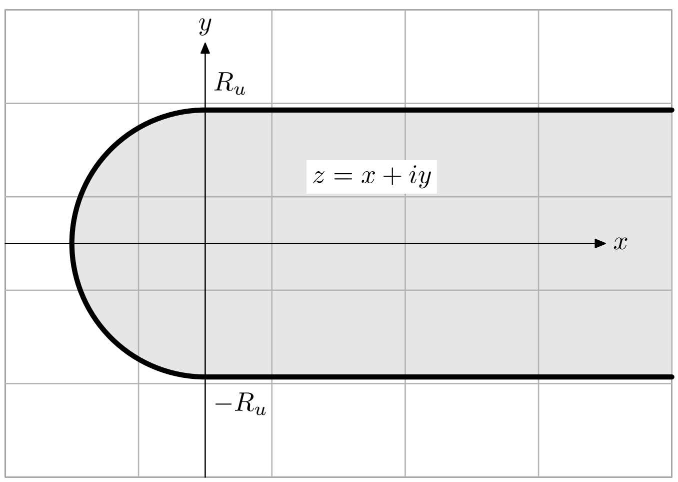

Figure 1: Domain G_{R_u} on the complex plane

(ii) There exists an $\eta_u > 0$ such that for each $x \in \mathfrak{X}$ the following series representation holds:

$$\widehat{p}(s,x,u) = \sum_{k=0}^{\infty} q_k^{(\eta_u)}(x,u) \left(1 - e^{-\eta_u s}\right)^k, \quad 0 \le s < \infty.$$

Moreover, the series converges uniformly for (x, s) *running through any compact subset of* $\mathfrak{X} \times \{s \in \mathbb{R} : 0 \leq s < R_u\}$.

(*iii*) *The solution* $\widehat{p}$ *of the Cauchy problem (2.5) is holomorphically extendable to* $[0, \infty)$, *there exists* $\eta_u > 0$ *such that*

$$\overline{\lim}_{k\to\infty} \sqrt[k]{\left|q_k^{(\eta_u)}(x, u)\right|} \leq 1, \quad x \in \mathfrak{X}, \tag{3.6}$$

and, there exists ε_u, $0 < \varepsilon_u < 1$, *such that the series*

$$\sum_{k=0}^{\infty} q_k^{(\eta_u)}(x, u) w^k \tag{3.7}$$

converges uniformly for (x, w) *running through any compact subset of* $\mathfrak{X} \times \{w \in \mathbb{C} : |w| < 1 - \varepsilon_u\}$.

Proof. See Appendix.

Remark 3.5. *From the proof of Theorem 3.4 it is clear that the implication* $(iii)' \Rightarrow (i)$, *where statement* $(iii)'$ *consists of (3.6), and (3.7) with* $\varepsilon_u = 0$ *holds as well. That is, loosely speaking, if in* (*iii*) *series (3.7) converges uniformly on all compact subsets of* $\mathfrak{X} \times \{w \in \mathbb{C} : |w| < 1\}$, *the holomorphy assumption on* $\widehat{p}$ *can be dropped.*

Remark 3.6. *In order to use the representation in* (*ii*) *one has to choose* η_u. *In fact,* η_u *can be related to* R_u *via* $\eta_u = \pi/R_u$ *and hence increases with decreasing* R_u.

It is important to note Theorem 3.4 concerns the solution of the Cauchy problem (2.5) connected with a general operator A. In particular, all criteria in this theorem are of pure analytic nature and via (3.2), respectively (3.3), exclusively formulated in terms of the $A^k f_u(x)$, i.e. coefficients in Definition 3.1. In the case where A is the generator of a Feller Dynkin process one can formulate a sufficient *probabilistic* criterion for Theorem 3.4-*(i)*:

Proposition 3.7. *Let* $\mathfrak{F}$ *be a set of analytic vectors in the sense of Definition 3.1 and let the Markov process* $\{X_t\}$ *be associated with the generator* A. *If* *in addition, for every $u \in I$ there exists a radius R_u such that for any $t \ge 0$*

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} \left| A^k \mathbb{E}[f_u(X_t^{0,x})] \right| < \infty, \quad 0 \le s < R_u, \tag{3.8}$$

uniformly in x over any compact subset of $\mathfrak{X}$, then Theorem 3.4-(i) *holds.*

The statement is a direct consequence of the following "quasi" semi-group property of the transition operator P_t.

Proposition 3.8. *Let $\mathfrak{F}$ be a set of analytic vectors satisfying (3.8). Then, for all $x \in \mathfrak{X}$ and all $t \ge 0$, the generalized transform $\widehat{p}(t+s, x, u)$ can be represented as*

$$\widehat{p}(t+s, x, u) = \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k \mathbb{E}[f_u(X_t^{0,x})], \quad 0 \le s < R_u, \tag{3.9}$$

where the series converges uniformly in x over any compact subset of $\mathfrak{X}$.

Proof. Denote the right-hand-side of (3.9) by $\widetilde{p}(t, s, x, u)$. Obviously, $\widetilde{p}(t, 0, x, u) = \mathbb{E}[f_u(X_t^{0,x})]$. Set

$$\widetilde{p}^{(N)}(t, s, x, u) := \sum_{k=0}^{N} \frac{s^k}{k!} A^k \mathbb{E}[f_u(X_t^{0,x})],$$

then both $\widetilde{p}^{(N)}(t, s, x, u)$ and

$$A\widetilde{p}^{(N)}(t, s, x, u) = \sum_{k=0}^{N} \frac{s^k}{k!} A^{k+1} \mathbb{E}[f_u(X_t^{0,x})]$$

converge for $N \to \infty$ uniformly over any compact subset of $\mathfrak{X}$, and s with $|s| < R_u - \varepsilon$, for an arbitrary small $\varepsilon > 0$. Hence, for $|s| < R_u - \varepsilon$, we have

$$\frac{\partial}{\partial s} \widetilde{p}^{(N)}(t, s, x, u) = \sum_{k=0}^{N-1} \frac{s^k}{k!} A^{k+1} \mathbb{E}[f_u(X_t^{0,x})] = A\widetilde{p}^{(N-1)}(t, s, x, u),$$
$$\widetilde{p}(t, 0, x, u) = \widehat{p}(t, x, u)$$

and thus, by closeness of the operator A and uniqueness of the Cauchy problem (2.5)-(2.7), we have $\widetilde{p}(t, s, x, u) = \widehat{p}(t+s, x, u)$.

The following proposition provides a situation in a semigroup context where a much stronger version of the condition (i) in Theorem 3.4 applies. It also sheds light on the connection between semi-group theory and holomorphic properties of generalized transforms.

Proposition 3.9. *Let $C_0(\mathbb{R}^n)$ be the Banach space of continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ which vanish at infinity, equipped with supremum norm: $||f|| := \sup_{x\in\mathbb{R}^n} |f(x)|$. Let $A : \mathfrak{D}(A) \subset C_0(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ be the generator of the Feller-Dynkin semi-group $(P_s)_{s\geq 0}$ associated with the process X, i.e. $P_s f(x) = \mathbb{E}\left[f(X_s^{0,x})\right]$, $f \in C_0(\mathbb{R}^n)$. Suppose that the family $\mathfrak{F}$ is such that $f_u \in \mathfrak{D}(A^k)$ for each $u \in I$ and all integer $k \geq 0$, and that for each $u \in I$,*

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} \left\|A^k f_u\right\| < \infty, \quad 0 \leq s < R_u.$$

Then for each $u \in I$,

$$P_s f_u = \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k f_u, \quad 0 \leq s < R_u, \tag{3.10}$$

with convergence in $C_0(\mathbb{R}^n)$. Thus, the map $s \to P_s f_u$ for $0 \leq s < R_u$ extends via (3.10) to the complex disc $D_0 := \{s \in \mathbb{C} : |s| < R_u\}$. In particular, for each $x \in \mathbb{R}^n$ the map $s \to P_s f_u(x)$ is holomorphic in D_0. Moreover, for each $t \geq 0$, we may extend the map $s \to P_{t+s} f_u$, $0 \leq s < R_u$ to the disc D_0 via,

$$P_{s+t} f_u = P_t P_s f_u = \sum_{k=0}^{\infty} \frac{s^k}{k!} P_t A^k f_u = \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k P_t f_u, \quad s \in D_0. \tag{3.11}$$

Proof. See Appendix.

Under the conditions of Proposition 3.9, $\mathfrak{F} = \{f_u,\ u \in I\}$ is a set of analytic vectors for the generator A in the sense of Definition 3.1 with $\mathfrak{X} = \mathbb{R}^n$. Moreover, due to Proposition 3.9 the map

$$s \to P_{s+t} f_u(x) = \sum_{k=0}^{\infty} \frac{s^k}{k!} P_t A^k f_u(x) = \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k E[f_u(X_t^{0,x})],$$

is holomorphic in D_0 for each $x \in \mathbb{R}^n$ and hence Theorem 3.4-(i) is fulfilled.

In this paper we do not stick to the semigroup framework because we want to avoid the narrow corset conditions of Proposition 3.9. We also want to consider operators A with unbounded (for instance, affine) coefficients and sets $\mathfrak{F}$ of functions that do not vanish at infinity (for example, (2.6)). Such situations may lead to the violation of condition $f_u \in \mathfrak{D}(A^k)$, $k \in \mathbb{N}$ in the sense of Proposition 3.9. In particular, in the next Sections 4-5 we will focus on general operators A with affine coefficients and in Section 7 on affine processes related to affine generators satisfying a kind of admissibility conditions.

4. Affine generators

Let us now consider generators of the form (2.3) with affine coefficients. In this section A may or may not be a generator of some Feller-Dynkin process. The next theorem and its corollaries say that the extended Fourier basis $\{\tilde{f}_u(x) := e^{\mathrm{i}u^\top x},\ u \in \mathbb{C}^n\}$ is a set of analytical vectors for an operator A of the form (2.7), where the coefficients $\mathfrak{a}_\alpha(x)$ are affine functions of x and satisfy certain growth conditions for $|\alpha| \to \infty$. Moreover, an explicit estimate for the radius of convergence is given.

Theorem 4.1. *Let A be a generator of the form (2.7) with affine coefficients $\mathfrak{a}_\alpha(x)$, i.e. for all multi-indexes α,*

$$\mathfrak{a}_\alpha(x) =: c_\alpha + x^\top d_\alpha, \quad x \in \mathfrak{X}, \tag{4.1}$$

where $\mathfrak{X} \subset \mathbb{R}^n$ is an open region, c_α is a scalar constant, and $d_\alpha \in \mathbb{R}^n$ is a constant vector. Assume that the series

$$\sum_{|\alpha|>0} \mathfrak{a}_\alpha(x) u^\alpha \tag{4.2}$$

converges absolutely for all $u \in \mathbb{C}^n$. This is in particular fulfilled under the conditions of Proposition 2.1. Then, for every $u \in \mathbb{C}^n$ and $x \in \mathfrak{X}$ it holds

$$\left| \frac{A^r \tilde{f}_u(x)}{\tilde{f}_u(x)} \right| \leq (r+1)!\, 2^{nr} (1 + \|x\|)^r \theta^r(\|u\|) \tag{4.3}$$

with $\|x\| = \max_{i=1,\ldots,n} |x_i|$,

$$\theta(v) := \sum_{k\geq 1} 2^k (1+v)^k \mathcal{D}_k^{\mathfrak{a}}, \quad v \in \mathbb{R}_+ \tag{4.4}$$

and

$$\mathcal{D}_k^{\mathfrak{a}} := \sup_{x\in\mathfrak{X}} \max_{|\alpha|=k,\,|\beta|\leq 1} \frac{|\partial_{x^\beta}\mathfrak{a}_\alpha(x)|}{1+\|x\|}.$$

The proof of Theorem 4.1 is given in the Appendix.

Corollary 4.2. *If in Theorem 4.1 the region* $\mathfrak{X}$ *is bounded, the generalized Fourier basis constitutes a set of analytic vectors for the affine operator* A *in* $\mathfrak{X}$.

Corollary 4.3. *If in Theorem 4.1 there exists for any* $\varsigma > 0$ *a constant* M *(which may depend on* $\varsigma > 0$*) such that*

$$\mathcal{D}_k^{\mathfrak{a}} \leq M\varsigma^k/k! \quad k \geq 1,$$

then

$$\theta(v) \leq M \exp\left(2\varsigma(1+v)\right).$$

Corollary 4.4. *If in Theorem 4.1 it holds that* $\mathfrak{a}_\alpha(x) \equiv 0$ *for* $|\alpha| > 2$ *(generator of diffusion type) then*

$$\theta(v) \leq C(1+v)^2, \quad C > 0.$$

For an affine operator A the sequence (3.2) can be explicitly constructed via the next proposition, which is proved in the Appendix.

Proposition 4.5. *Let* A *be an affine generator as in Theorem 4.1 and define*

$$\begin{aligned}\mathfrak{b}_\beta(x,u) &:= \partial_{u^\beta}\frac{Af_u(x)}{f_u(x)} = \sum_{\alpha\geq 0}\mathfrak{a}_{\alpha+\beta}(x)\frac{(\alpha+\beta)!}{\alpha!}(\mathfrak{i}u)^\alpha \\ &=: \mathfrak{b}_\beta^0(u) + \sum_{\kappa,\,|\kappa|=1}\mathfrak{b}_{\beta,\kappa}^1(u)\,x^\kappa,\end{aligned}$$

with $\mathfrak{a}_0 := 0$. *We set* $A^r f_u(x) =: g_r(x,u) f_u(x)$ *and, for fixed* $\eta > 0$, $q_r^{(\eta)}(x,u) =: h_r(x,u)f_u(x)$ *(the dependence on* η *is suppressed in* h *for notational convenience), where both* g_r *and* h_r *are polynomials in* x *of degree* r. *It holds*

$$g_r(x,u) =: \sum_{|\gamma|\leq r} g_{r,\gamma}(u)x^\gamma, \quad h_r(x,u) =: \sum_{|\gamma|\leq r} h_{r,\gamma}(u)x^\gamma, \tag{4.5}$$

where g_r *and* h_r *satisfy* $g_0 \equiv g_{0,0} \equiv h_0 \equiv h_{0,0} \equiv 1$, *and for* $r \geq 0$, *respectively,*

$$
\begin{aligned}
g_{r+1,\gamma} &= \sum_{|\beta|\leq r-|\gamma|} \binom{\gamma+\beta}{\beta} g_{r,\gamma+\beta}\mathfrak{b}^0_\beta \\
&+ \sum_{|\kappa|=1,\, \kappa\leq\gamma} \; \sum_{|\beta|\leq r+1-|\gamma|} \binom{\gamma-\kappa+\beta}{\beta} g_{r,\gamma-\kappa+\beta}\mathfrak{b}^1_{\beta,\kappa}, \text{ and,} \\
(r+1)h_{r+1,\gamma} &= \sum_{|\beta|\leq r-|\gamma|} \eta^{-1}\binom{\gamma+\beta}{\beta} h_{r,\gamma+\beta}\mathfrak{b}^0_\beta \\
&+ \sum_{|\kappa|=1,\, \kappa\leq\gamma} \; \sum_{|\beta|\leq r+1-|\gamma|} \eta^{-1}\binom{\gamma-\kappa+\beta}{\beta} h_{r,\gamma-\kappa+\beta}\mathfrak{b}^1_{\beta,\kappa} + rh_{r,\gamma}(u),
\end{aligned}
\tag{4.6}
$$

where $|\gamma| \leq r+1$, *and empty sums are defined to be zero.*

Remark 4.6. *Depending on the open set* $\mathfrak{X}$ *we may consider instead of (4.5) for an* $x_0 \in \mathfrak{X}$ *expansions in* $x - x_0$ *rather than in* x. *For simplicity we henceforth assume* $\{0\} \in \mathfrak{X}$ *which, if necessary, may be realized by a translation of the state space.*

A natural question is whether affine generators are the only ones for which the Fourier basis constitutes a set of analytic vectors. For this paper we leave this issue as an open problem but the following proposition shows that at any case the set of such generators is rather "thin".

Let us put $\mathfrak{X} = [-\pi, \pi]$ and

$$A = \frac{1}{2}a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}.$$

Proposition 4.7. *The set of coefficients* $(a(x), b(x))$ *such that for an arbitrary* $M > 0$

$$\|A^N f_u\|_{L^2(\mathfrak{X})} \gtrsim M^N N!, \quad N \to \infty,$$

is dense in $L^2(\mathfrak{X}) \times L^2(\mathfrak{X})$.

Proof. Without loss of generality let us assume that $b(x) \equiv 0$ and $u > 0$. The general case can be considered along the same ideas and is only formally more complicated. Any $a \in L^2(\mathfrak{X})$ may be approximated (in L^2-sense) by a finite Fourier series

$$a(x) \approx \sum_{l=1}^{n} a_l e^{ilx}.$$

Thus, for given $\varepsilon > 0$ we can find natural n and amplitudes a_l ($a_n \neq 0$) such that

$$\left\| a(x) - \sum_{l=1}^{n} a_l e^{ilx} \right\|_{L^2(\mathfrak{X})} \leq \varepsilon.$$

The corresponding approximative operator is given by

$$\widetilde{A} := \sum_{l=1}^{n} \widetilde{A}_l = \sum_{l=1}^{n} a_l e^{ilx} \frac{\partial^2}{\partial x^2}.$$

Using the fact that for any $s_1, \ldots, s_k \in \mathbb{N}$,

$$\widetilde{A}_{s_1} \cdots \widetilde{A}_{s_k} e^{iux} = (-1)^k a_{s_1} \cdots a_{s_k} \prod_{l=0}^{k-1} \left(u + \sum_{j=0}^{l} s_j \right)^2 e^{i(u+\sum_{j=1}^{k} s_j)x},$$

and setting

$$F_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f_u^{-1}(x) \widetilde{A}^N f_u(x) dx, \quad k \in \mathbb{N},$$

we have $F_k = 0$ for $k > nN$, and for $N \to \infty$

$$F_{nN} = (-1)^N a_n^N \prod_{l=0}^{N-1} (u + nl)^2 \sim (-1)^N a_n^N n^{2N} ((N-1)!)^2 (N-1)^{2u/n}.$$

Further, by Parseval's identity it holds

$$\|\widetilde{A}^N f_u\|_{L^2(\mathfrak{X})} = \left[2\pi \sum_{k=0}^{nN} |F_k|^2 \right]^{1/2},$$

and then we are done.

Obviously, Proposition 4.7 may be formulated with respect to any bounded interval.

5. Log-affine representations for the affine Cauchy problem

In [7] a Markov process X is called regular affine if for every s, $s \geq 0$, the characteristic function $\widehat{p}$ of $X_s^{0,x}$ has the form

$$\widehat{p}(s, x, u) = \mathbb{E}[f_u(X_t^{0,x})] = \exp\left(C(s, u) + x^\top D(s, u) \right), \tag{5.1}$$

with $f_u(x) = \exp(\mathrm{i}u^\top x)$. As a main result, it is shown in [7] that (under certain conditions) the vector valued function D satisfies for all $s > 0$ a generalized system of Riccati equations,

$$\partial_s D(s,u) = R(D(s,u)), \tag{5.2}$$

and the function C is then obtained by

$$C(s,u) = \int_0^s Q\left(D(\tau,u)d\tau\right). \tag{5.3}$$

The vector valued function R and real valued function Q in (5.2) and (5.3) respectively, are determined by the relation

$$\left(Q(\mathrm{i}u) + x^\top R(\mathrm{i}u)\right) f_u(x) = A f_u(x). \tag{5.4}$$

In general the equation for D in (5.2) is impossible to solve analytically. In this section we will derive (under certain conditions) general functional series expansions for (5.1), hence in particular for C and D, for which all ingredients can be obtained from the generator A in a direct algebraic way.

Consider the Cauchy problem (2.5) for affine generators A of the form (2.7), under the assumption (4.2). As in (4.1) we set $\mathfrak{a}(x) = c_\alpha + x^\top d_\alpha$. The ansatz

$$\widehat{p}(s,x,u) = \exp\left(C(s,u) + x^\top D(s,u)\right), \tag{5.5}$$

for scalar $C(s,u)$ and vector valued $D(s,u)$, where $C(0,u) = 0$ and $D(0,u) = \mathrm{i}u$, for the Cauchy problem (2.5) yields,

$$\partial_s C + x^\top \partial_s D = \sum_{|\alpha|>0} \mathfrak{a}_\alpha(x) D^\alpha = \sum_{|\alpha|>0} c_\alpha D^\alpha + \sum_{|\alpha|>0} x^\top d_\alpha D^\alpha,$$

and so

$$\partial_s C = \sum_{|\alpha|>0} c_\alpha D^\alpha, \quad \partial_s D = \sum_{|\alpha|>0} d_\alpha D^\alpha,$$
$$C(0,u) = 0, \quad D(0,u) = \mathrm{i}u.$$

We thus have a system of ordinary differential equations (ODEs), which reads component-wise

$$\partial_s C = \sum_{|\alpha|>0} c_\alpha D^\alpha, \quad \partial_s D_j = \sum_{|\alpha|>0} d_\alpha^{(j)} D^\alpha, \quad j = 1, ..., n,$$

$$C(0,u) = 0, \quad D_j(0,u) = \mathrm{i}u_j. \tag{5.6}$$

By assumption (4.2), the series

$$\sum_{|\alpha|>0} c_\alpha y^\alpha, \quad \sum_{|\alpha|>0} d_\alpha^{(j)} y^\alpha, \quad j = 1, ..., n, \tag{5.7}$$

are absolutely convergent for all $y \in \mathbb{R}^n$, and thus define terms-wise differentiable $C^{(\infty)}(\mathbb{R}^n)$ functions. In particular, they are locally Lipschitz and so according to standard ODE theory the system (5.6) has for fixed $u \in \mathbb{R}^n$ a unique solution $(C(s,u), D(s,u))$ for $0 \le s < s_u^\infty \le \infty$, where $(s, C(s,u), D(s,u))$ leaves any compact subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, when $s \uparrow s_u^\infty$.

Remark 5.1. *By a general theorem from analysis (e.g., see [4]), it follows that the solution of (5.6) extends component-wise holomorphically in s into a disc around $s = 0$, due to the analyticity of (5.7). This implies that (5.5) is holomorphic in s. So, besides Theorem 4.1, also along this line one may show that (5.5) can be represented as a power series of the form (3.1). I.e., in particular, the Fourier basis (2.6) constitutes a set of analytic vectors for the affine operator A. However, the direct approach in the proof of Theorem 4.1 (see Appendix) leads to an explicit estimate* (4.3) *and allows for investigating possible extensions of $\widehat{p}(s,x,u)$ into a strip containing the real axis in the complex plane (see Theorem 5.4). Moreover, it also suggests the line to follow in cases where A is not affine and/or the function base is not of the form (2.6).*

Let us suppose that for fixed $u \in \mathbb{R}^n$ the statements of Theorem 3.4 hold. Then we obtain for $0 \le s < s_u^\infty \le \infty$, $x \in \mathfrak{X}$,

$$\widehat{p}(s,x,u) = \exp\left(C(s,u) + x^\top D(s,u)\right) \tag{5.8}$$
$$= \sum_{k=0}^\infty q_k^{(\eta_u)}(x,u)(1 - e^{-\eta_u s})^k.$$

Since $q_0^{(\eta_u)}(x,u) = f_u(x) = \exp\left(\mathrm{i}u^\top x\right) \neq 0$ we have, taking into account the boundary conditions for C and D, at least for small enough $\varepsilon > 0$,

$$C(s,u) + x^\top D(s,u) = \sum_{k=0}^\infty \rho_k^{(\eta_u)}(x,u)(1 - e^{-\eta_u s})^k, \quad 0 \le s < \varepsilon, \tag{5.9}$$

where by a standard lemma on the power series expansion of the logarithm of a power series,

$$\rho_0^{(\eta_u)}(x,u) = \ln q_0^{(\eta_u)}(x,u) = \mathrm{i}u^\top x, \tag{5.10}$$
$$\rho_k^{(\eta_u)}(x,u) = \frac{1}{f_u(x)}\left[q_k^{(\eta_u)}(x,u) - \frac{1}{k}\sum_{j=1}^{k-1} j\rho_j^{(\eta_u)}(x,u)q_{k-j}^{(\eta_u)}(x,u)\right], \quad k \geq 1.$$

Thus by (5.9), the $\rho_k^{(\eta_u)}$ are necessarily affine in x!

Remark 5.2. *It is possible to prove directly that the functions* $\rho_k^{(\eta_u)}$ *defined above are affine in* x *using Proposition 4.5 via a (rather laborious) induction procedure, so without using a local solution of (5.6).*

Theorem 5.3. *Suppose that for fixed* $u \in I$ *the statement of Theorem 3.4-(i) holds for an open region* $\mathfrak{X}$ *and, in addition, for* $s \in G_{R_u}$ *and* $x \in \mathfrak{X}$ *it holds* $\widehat{p}(s,x,u) \neq 0$. *Then, with* $\rho_k^{(\eta_u)}(x,u) =: \rho_k^{(\eta_u,0)}(u) + x^\top \rho_k^{(\eta_u,1)}(u)$ *determined by (5.10), we have*

$$\widehat{p}(s,x,u) = \exp\left[\sum_{k=0}^{\infty}\left(\rho_k^{(\eta_u,0)} + x^\top\rho_k^{(\eta_u,1)}(u)\right)(1-e^{-\eta_u s})^k\right], \quad 0 \leq s < \infty.$$

Proof. Let $u \in I$ and $x \in \mathfrak{X}$ be fixed. Since G_{R_u} is simply connected and $s \to \widehat{p}(s,x,u)$ is holomorphic and non-zero in G_{R_u}, there exists a single-valued branch $s \to L(s,x,u)$ of the multi-valued complex logarithm such that $\widehat{p}(s,x,u) = \exp(L(s,x,u))$ for all $s \in G_{R_u}$. Along the same line as in Theorem 3.4 we then argue that there exists an $\eta_u > 0$ such that $w \to L(\Phi_{\eta_u}(w),x,u)$ (see the proof of Theorem 3.4) is holomorphic in the unit disc $\{w : |w| < 1\}$, hence, there exists $\widetilde{\rho}_k(x,u)$ such that

$$L(\Phi_{\eta_u}(w),x,u) = \sum_{k\geq 0}\widetilde{\rho}_k(x,u)w^k, \quad 0 \leq |w| < 1, \quad \text{and so}$$

$$L(s,x,u) = \sum_{k\geq 0}\widetilde{\rho}_k(x,u)(1-e^{-\eta_u s})^k, \quad 0 \leq s < \infty.$$

Since the later expansion must coincide with (5.9) for small s, it follows that necessarily $\widetilde{\rho}_k(x,u) = \rho_k^{(\eta_u)}(x,u)$ and the theorem is proved.

Let us now pass to another interesting log-affine representation for the characteristic function. From (4.5) and (4.6) we derive formally

$$\sum_{r\geq 0} q_r^{(\eta)}(x,u)(1-e^{-\eta s})^r = e^{\mathrm{i}u^\top x}\sum_{r\geq 0}\sum_{\gamma\geq 0} h_{r,\gamma}(u)x^\gamma(1-e^{-\eta s})^r 1_{|\gamma|\leq r}$$
$$= e^{\mathrm{i}u^\top x}\sum_{\gamma\geq 0} x^\gamma \sum_{r\geq 0} h_{|\gamma|+r,\gamma}(u)(1-e^{-\eta s})^{|\gamma|+r}.$$

Suppose that the requirements of Theorem 5.3 hold for fixed u and $\eta=\eta_u$. Then, using Theorem 4.1, it is easy to show that for small enough $\varepsilon>0$,

$$\sum_{\gamma\geq 0}\|x\|_\infty^{|\gamma|}\sum_{r\geq 0}|h_{|\gamma|+r,\gamma}(u)||w|^{|\gamma|+r}<\infty, \quad \text{if} \quad |w|<\varepsilon, \quad \|x\|_\infty<\varepsilon$$

(see Remark 4.6). Thus, for $|s|$ and $\|x\|_\infty$ small enough we obtain

$$\begin{aligned}\ln\widehat{p}(s,x,u) &= \mathrm{i}u^\top x + \ln\left(\sum_{\gamma\geq 0} x^\gamma\sum_{r\geq 0} h_{|\gamma|+r,\gamma}(u)(1-e^{-\eta_u s})^{|\gamma|+r}\right)\\ &= C(s,u)+x^\top D(s,u),\end{aligned}$$

with (in multi-index notation)

$$C(s,u)=\ln\left(\sum_{r\geq 0} h_{r,0}(u)(1-e^{-\eta_u s})^r\right), \tag{5.11}$$
$$D^\kappa(s,u)=\mathrm{i}u^\kappa+\frac{\sum_{r\geq 1} h_{r,\kappa}(u)(1-e^{-\eta_u s})^r}{\sum_{r\geq 0} h_{r,0}(u)(1-e^{-\eta_u s})^r}, \quad |\kappa|=1.$$

However, the left- and right-hand-sides of (5.11) are holomorphic for all $s\in G_{R_u}$ and we so arrive at the representation

$$\begin{aligned}\widehat{p}(s,x,u)=\exp\Bigg[&\ln\left(\sum_{r\geq 0} h_{r,0}(u)(1-e^{-\eta_u s})^r\right)+\mathrm{i}u^\top x\\ &+x^\top\frac{\sum_{r\geq 1} h_r(u)\,(1-e^{-\eta_u s})^r}{\sum_{r\geq 0} h_{r,0}(u)(1-e^{-\eta_u s})^r}\Bigg], \quad s\in G_{R_u},\ x\in\mathfrak{X},\end{aligned} \tag{5.12}$$

with

$$h_r(u):=[h_{r,i}(u)]_{i=1,\dots,n},$$

where for $1\leq i\leq n$, the multi-index $(\delta_{ij})_{j=1,\dots,n}$ is identified with i.

Particularly due to the explicit estimate (4.3) for affine generators in Theorem 4.1, we may proof the next theorem (which is a non-probabilistic version of Proposition 3.7 in the situation where A is affine).

Theorem 5.4. *Let $\mathfrak{X}$ be a bounded domain. Assume that the system* (5.6) *is non-exploding, i.e. $s_u^\infty = \infty$, and that for any fixed $u \in \mathbb{R}^n$ the solution $D(s,u)$ remains bounded as $s \to \infty$. Then, there exists a radius $R_u > 0$ such that for any $t \geq 0$ the map $s \to \widehat{p}(t+s,x,u)$, $0 \leq s < R_u$ has a holomorphic extension to the disc $\{s \in \mathbb{C} : |s| < R_u\}$. Moreover, it holds*

$$\widehat{p}(t+s,x,u) = \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k \widehat{p}(t,\cdot,u)\,(x), \quad |s| < R_u, \quad x \in \mathfrak{X}. \tag{5.13}$$

Remark 5.5. *The maximal extension radius R_u satisfies*

$$R_u \geq \frac{1}{2^n\,\theta(\|D^*(u)\|)} \inf_{x\in\mathfrak{X}} \frac{1}{1+\|x\|}, \tag{5.14}$$

where function θ is defined in (4.4) *and $\|D^*(u)\| = \sup_{s>0} \|D(s,u)\|$.*

Proof. Denote the right-hand-side of (5.13) by $\widetilde{p}(t,s,x,u)$. Obviously, $\widetilde{p}(t,0,x,u) = \widehat{p}(t,0,x,u)$. Let us define

$$\widetilde{p}^{(N)}(t,s,x,u) := \sum_{k=0}^{N} \frac{s^k}{k!} A^k \widehat{p}(t,\cdot,u)\,(x).$$

Since $\widehat{p}(t,x,u) = \exp(C(t,u) + x^\top D(t,u))$, Theorem 4.1 implies that the series

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} A^k \widehat{p}(t,\cdot,u)\,(x) \tag{5.15}$$

is absolutely and uniformly convergent on any compact subset of $\mathfrak{X} \times \{s \in \mathbb{C} : |s| < R_u\}$ if R_u satisfies (5.14). So, both $\widetilde{p}^{(N)}(t,s,x,u)$ and

$$A\widetilde{p}^{(N)}(t,s,x,u) = \sum_{k=0}^{N} \frac{s^k}{k!} A^{k+1} \widehat{p}(t,\cdot,u)\,(x)$$

converge for $N \to \infty$ uniformly over any compact subset of $\mathfrak{X}$ and s with $|s| < R_u - \varepsilon$ for any $\varepsilon > 0$. Hence, for $|s| < R_u - \varepsilon$

$$\frac{\partial}{\partial s}\widetilde{p}^{(N)}(t,s,x,u) = \sum_{k=0}^{N-1} \frac{s^k}{k!} A^{k+1} \widehat{p}(t,\cdot,u) = A\widetilde{p}^{(N-1)}(t,s,x,u),$$

$$\widetilde{p}(t,0,x,u)=\widehat{p}(t,x,u),$$

and thus, by closeness of the operator A and uniqueness of the Cauchy problem (2.5)-(2.7), we have $\widetilde{p}(t,s,x,u)=\widehat{p}(t+s,x,u)$.

6. Full expansion of a specially structured one dimensional affine system

Let us consider Cauchy problem (2.5) for $n=1$ with $f_u(x)=\exp(\mathfrak{i}ux)$, where the jump-kernel in the generator A (see (2.3)) has a special affine structure of the form

$$v(x,dz)=:(\lambda_0+\lambda_1 x)\nu(dz),$$

and where the diffusion coefficients have a similar structure,

$$b(x)=(\lambda_0+\lambda_1 x)\theta, \quad a(x)=(\lambda_0+\lambda_1 x)\vartheta,$$

for some constants $\lambda_0,\lambda_1,\theta,\vartheta\in\mathbb{R}$, and measure ν. So, in Proposition 4.5 the $\mathfrak{a}_\alpha$ have the form $\mathfrak{a}_l=:(\lambda_0+\lambda_1 x)\eta_l$ where

$$\mu_0:=0, \quad \mu_1=\theta, \quad \mu_2:=\frac{1}{2}\left(\vartheta+\int z^2\nu(dz)\right), \quad \mu_l:=\frac{1}{l!}\int z^l\nu(dz), \quad l>2.$$

Hence, in Proposition 4.5, the $\mathfrak{b}_\beta$ in (4.6) have the form

$$\begin{aligned}\mathfrak{b}_r(x,u)&=\mathfrak{b}_r^0(u)+x\mathfrak{b}_r^1(u)\\&=:(\lambda_0+\lambda_1 x)\sum_{l\geq 0}\mu_{l+r}\frac{(l+r)!}{l!}(\mathfrak{i}u)^l=:(\lambda_0+\lambda_1 x)\frac{d^r}{du^r}\mathfrak{h}(u) \quad r\geq 0,\end{aligned}$$

where $\mathfrak{h}(u):=\sum_{l\geq 0}\mu_l(\mathfrak{i}u)^l$. It is now possible to show via (4.6) that for $r\geq 1$,

$$\begin{aligned}g_r(x,u)=&\sum_{\substack{p>0,\,q\geq 0\\0<n_1<\ldots<n_q,\ m_1,\ldots,m_q\geq 0,\\r=p+n_1m_1+\cdots+n_qm_q}}\frac{1}{r!}\pi^{(p)}_{(n_1,m_1),\cdots,(n_q,m_q)}\lambda_1^{r-p}(\lambda_0+\lambda_1 x)^p\\&\times\ \mathfrak{h}^p(u)\prod_{j=1}^{q}\left(\mathfrak{h}^{n_j-1}(u)\frac{d^{n_j}}{du^{n_j}}\mathfrak{h}(u)\right)^{m_j}, \quad (6.1)\end{aligned}$$

with the following integer recursion procedure:

Initialization: $\pi^{(p)} \equiv 1, \quad \pi^{(0)}_{(n_1,m_1),\cdots,(n_q,m_q)} \equiv 0, \quad p,q \geq 1.$

For all $n_i > 0$, $m_i \geq 0$, with $1 \leq i \leq q$, $p, q \geq 1$:

Reduction rule I: If $m_j = 0$, for some j, $1 \leq j \leq q$, then

$$\pi^{(p)}_{(n_1,m_1),\cdots,(n_q,m_q)} = \pi^{(p)}_{(n_1,m_1),\cdots,(n_{j-1},m_{j-1}),(n_{j+1},m_{j+1}),\cdots,(n_q,m_q)}.$$

Reduction rule II:

$$\pi^{(p)}_{(n_1,m_1),\cdots,(n_{q-1},m_{q-1}),(n_q,m_q)} = \sum_{j=1}^{q} \binom{p+n_j-1}{n_j} \pi^{(p+n_j-1)}_{(n_1,m_1),\cdots,(n_j,m_j-1),\cdots,(n_q,m_q)} + \pi^{(p-1)}_{(n_1,m_1),\cdots,(n_q,m_q)}.$$

In fact, the above recursion procedure follows automatically after substituting (6.1) as ansatz into (4.6). Finally, we obtain the $q_k^{(\eta)}$ for the series expansion in Theorem 3.4 by (3.3), i.e.

$$q_k^{(\eta)}(x,u) =: \frac{1}{k!} \sum_{r=0}^{k} c_{k,r} \eta^{-r} g_r(x,u)\, f_u(x).$$

Remark 6.1. *If the measure ν is finite,*

$$\begin{aligned} \mathfrak{h}(u) &= \mathrm{i}u\theta - \frac{1}{2}\vartheta u^2 + \sum_{l \geq 2} \frac{(\mathrm{i}u)^l}{l!} \int z^l \nu(dz) \\ &= \phi(u) - 1 + (\mathrm{i}\theta - \phi'(0))\, u - \frac{1}{2}\vartheta u^2, \end{aligned}$$

where ϕ is the characteristic function of ν. Hence, in this case $\mathfrak{h}$ and all its derivatives may be computed from ϕ.

7. Application to affine processes

Affine processes have become very popular in recent years due to their analytical tractability in the context of option pricing, and their rather rich dynamics. Many well-known models such as Heston and Bates models fall into the class of affine jump diffusions. Option pricing in these models is usually done via the Fourier method which requires knowledge of the Fourier transform of the process in closed form (see e.g. [6]). The functional series representations for affine generators developed in this paper, in particular (5.12), can be directly applied to affine processes. Let us recall the characterization of a regular affine process as given in [7].

Definition 7.1. *We call a strong Markov process $\{X_t\}$ with generator A a regular affine process if A is of the form* (2.3) *and all functions*

$$a_{ij}(x),\ b_i(x),\ v(x,dz) \quad i,j=1,\dots,m$$

are affine in x (see (4.1)*), and satisfy the set of admissability conditions spelled out in [7, Definition 2.6]. These conditions guarantee that A is the generator of a Feller-Dynkin (strong) Markov process X in a subspace of the form $\mathbb{R}^l \times \mathbb{R}_+^{n-l} \subset \mathbb{R}^n$ for some $0 \le l \le n$.*

The next theorem provides a sufficient condition for convergence of the series representation in Theorem 3.4-(ii), hence representation (5.12), for regular affine processes.

Theorem 7.2. *Let $\{X_s\}$ be a regular affine process which has a non-degenerated limiting distribution for $s \to \infty$, and has a generator A which satisfies the moment condition (4.2). Then the (conditional) characteristic function $\widehat{p}(s,x,u) = \mathbb{E}[f_u(X_s^{0,x})]$, with $f_u(x) = e^{\mathrm{i}u^\top x}$, has a representation according to Theorem 3.4-(ii):*

$$\widehat{p}(s,x,u) = \sum_{k=0}^{\infty} q_k^{(\eta_u)}(x,u)\left(1-e^{-\eta_u s}\right)^k, \quad 0 \le s < \infty.$$

Moreover, the scaling factor η_u may be chosen according to the inequality:

$$\eta_u \ge C\,\theta(L(1+\|u\|_2^2)),$$

where the (monotonic) function θ is defined in (4.4)*, $L > 0$ is a constant independent of x, and C is a constant generally depending on x.*

Proof. Following [7], $\widehat{p}(s,x,u)$ has representation of the form (5.5) for $0 \le s < \infty$. The existence of a limiting distribution implies in particular that $D(s,u)$ in

(5.5) is bounded for all $s \geq 0$. Moreover, as shown in [7, Section 7], $\widehat{p}(s,x,u)$ is the characteristic function of some infinitely divisible distribution for all $s > 0$ (hence also in the limit $s \to \infty$). As a consequence (see [19]), there exists an $M > 0$ independent of s such that

$$\lim_{\|u\|_2 \to \infty} \|u\|_2^{-2} \left|\log \widehat{p}(s,x,u)\right| < M.$$

This implies that $\|D(s,u)\| \leq L(1 + \|u\|_2^2)$ for some constant $L > 0$ not depending on x and $s \geq 0$. Now we apply Theorem 5.4 and Remark 5.5.

Remark 7.3. *The existence of a limiting stationary distribution is a sufficient condition for the boundedness of $D(s,u)$. In fact, there are affine processes which have no limit distribution but bounded $D(s,u)$ (a trivial example is standard Brownian motion). The study of existence of limiting (and stationary) distributions for affine processes is currently under active research, e.g. see [13] or [12].*

8. Appendix

Proof of Proposition 2.1

Let us split the operator $A^\sharp$ in a diffusion and an integral component,

$$A_D^\sharp f(x) := \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i},$$
$$A_I^\sharp f(x) := A^\sharp f(x) - A_D^\sharp f(x)$$

Suppose that in the given topology,

$$f_n \to 0 \quad \text{and} \quad A^\sharp f_n \to g. \tag{8.1}$$

Let $\varphi \in C_\kappa^\infty(\mathbb{R}^n)$ be an arbitrary test function with compact support. Then

$$\int \varphi(x) A_D^\sharp f_n(x) dx = \int f_n(x) \left(A_D^\sharp\right)' \varphi(x) dx$$

where

$$\left(A_D^\sharp\right)' \varphi(x) = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)\varphi(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i(x)\varphi(x)).$$

Since f_n is uniformly bounded on the compact support of φ we have by dominated convergence

$$\int f_n(x)\left(A_D^\sharp\right)'\varphi(x)dx \to 0,$$

hence

$$\int \varphi(x)A_D^\sharp f_n(x)dx \to 0. \tag{8.2}$$

For the integral part we have by Fubini,

$$\int \varphi(x)A_I^\sharp f_n(x)dx = \int \nu(dz)\int \left(f_n(x+z) - f_n(x) - z\cdot\frac{\partial f_n}{\partial x}\right)\varphi(x)\gamma(x,z)dx$$
$$= \int\nu(dz)\int f_n(x)\left(\varphi(x-z)\gamma(x-z,z) - \varphi(x)\gamma(x,z) - z\cdot\frac{\partial}{\partial x}\left(\varphi(x)\gamma(x,z)\right)\right)dx.$$

Since by assumption both the measure ν and the function φ have compact support, the latter integral can be restricted to $(z,x)\in K_1\times K_2$ for compact subsets $K_{1,2}\subset\mathbb{R}^n$. Thus, since by assumption the derivatives of $x\to\gamma(x,z)$ are continuous in (x,z), we have

$$\varphi(x-z)\gamma(x-z,z)-\varphi(x)\gamma(x,z)-z\cdot\frac{\partial}{\partial x}\left(\varphi(x)\gamma(x,z)\right) = O(|z|^2),\ (z,x)\in K_1\times K_2.$$

Thus, since $\int |z|^2\nu(dz)<\infty$, it follows by (8.1) and dominated convergence that $\int \varphi(x)A_I^\sharp f_n(x)dx \to 0$, and then by (8.2),

$$\int \varphi(x)A^\sharp f_n(x)dx \to 0.$$

On the other hand also $A^\sharp f_n$ is uniformly bounded on the compact support of φ, so again by dominated convergence and (8.1),

$$\int \varphi(x)A^\sharp f_n(x)dx \to \int \varphi(x)g(x)dx = 0.$$

The function φ is arbitrary, hence $g=0$.

Proof of Theorem 3.4

$(i) \implies (ii)$: Let $\mathcal{U} := \{z\in\mathbb{C} : |z|<1\}$ be the unit disc in the complex plane. Consider for $\eta>0$ the map

$$\Phi_\eta : z \longrightarrow -\frac{1}{\eta}\mathrm{Ln}(1-z).$$

Obviously, there exists an $\eta_u > 0$,

$$(0-, \infty) \subset \Phi_{\eta_u}(\mathcal{U}) \subset G_{R_u}$$

(i.e., for $\epsilon_0 > 0$ small enough $(-\epsilon_0, \infty) \subset \Phi_{\eta_u}(\mathcal{U})$). Moreover, the map Φ_{η_u} is injective on $\mathcal{U}$. Thus, (denoting the extension in (i) with $\widehat{p}$ as well) for each $x \in \mathfrak{X}$, the function $\widehat{p}(\Phi_{\eta_u}(w), x, u)$ is holomorphic in $\mathcal{U}$ and has a series expansion

$$w \longrightarrow \widehat{p}(\Phi_{\eta_u}(w), x, u) =: \sum_{k=0}^{\infty} \widetilde{q}_k(x, u) w^k, \quad |w| < 1,$$

and as a consequence,

$$\widehat{p}(z, x, u) = \sum_{k=0}^{\infty} \widetilde{q}_k(x, u)(1 - e^{-\eta_u z})^k, \quad z \in \Phi_{\eta_u}(\mathcal{U}), \quad x \in \mathfrak{X}. \tag{8.3}$$

Since (8.3) holds in particular for $z \in (0-, \infty)$, we have in a (possibly small) ε-disk around $z = 0$,

$$\widehat{p}(z, x, u) = \sum_{k=0}^{\infty} \widetilde{q}_k(x, u)(1 - e^{-\eta_u z})^k = \sum_{k=0}^{\infty} \frac{z^k}{k!} A^k f_u(x), \quad 0 \leq |z| < \varepsilon, \tag{8.4}$$

due to Proposition 3.3. By taking $z = 0$ we have

$$\widehat{p}(0, x, u) = \widetilde{q}_0(x, u) = f_u(x).$$

We know from the exponential generating function of Stirling numbers of the second kind $S_{n,k}$ that

$$\frac{(\exp(z) - 1)^k}{k!} = \sum_{n=0}^{\infty} S_{n,k} \frac{z^n}{n!} \tag{8.5}$$

(cf. for example [5], Vol.II, Sec. 21.9, (9.1), p.1041). [1] Hence,

$$\begin{aligned} (-1)^k \eta_u^{-k} A^k f_u(x) &= \\ (-1)^k \eta_u^{-k} \frac{\partial^k}{\partial z^k} \sum_{l=0}^{\infty} \tilde{q}_l(x, u)(1 - \exp(-\eta_u z))^l \Bigg|_{z=0} &= \sum_{l=0}^{k} S_{k,l} (-1)^l l! \tilde{q}_l(x, u). \end{aligned}$$

[1] We thank an anonymous referee for his remarks and references concerning the theory of Stirling numbers, which have led to a substantial shortage of our original proof.

Using Stirling inversion, (see [1] for example), we have

$$u_k = \sum_{l=0}^{k} S_{k,l} v_l \Leftrightarrow v_l = \sum_{k=0}^{l} (-1)^{l-k} c_{l,k} u_k,$$

with $c_{l,k}$ defined in (3.4). Hence,

$$\tilde{q}_l(x,u) = \frac{1}{l!} \sum_{k=0}^{l} c_{l,k} \eta_u^{-k} A^k f_u(x).$$

Next we prove the uniform convergence as stated in (ii). Let us take an arbitrarily fixed *compact* subset $\mathcal{K} \subset \mathfrak{X}$, and an arbitrarily fixed κ with $0 < \kappa < R_u$. Let now $\epsilon > 0$. Take a fixed δ with $0 < \delta < 1$ such that $\kappa/(\delta R_u) < 1$. Due to Definition 3.1 there exists a number N_δ such that

$$|A^r f_u(x)| \leq \frac{r!}{(\delta R)^r}$$

for all $n > N_\delta$ and $x \in \mathcal{K}$. With $w := 1 - e^{-\eta_u s}$, we then have, using a well known property of Stirling numbers, for all $0 \leq s \leq \kappa < R_u$, $x \in \mathcal{K}$, and all $N > N_\delta$,

$$\begin{aligned}
\left| \sum_{k=N}^{\infty} q_k^{(\eta_u)}(x,u) w^k \right| &= \left| \sum_{k=N}^{\infty} \frac{w^k}{k!} \sum_{r=0}^{k} c_{k,r} \eta_u^{-r} A^r f_u(x) \right| \\
&= \left| \sum_{k=0}^{\infty} \frac{w^k}{k!} \sum_{r=0}^{\infty} c_{k,r} \eta_u^{-r} A^r f_u(x) 1_{r \leq k} 1_{N \leq k} \right| \\
&\leq \sum_{r=0}^{\infty} \eta_u^{-r} |A^r f_u(x)| \sum_{k=\max(N,r)}^{\infty} \frac{|w|^k}{k!} c_{k,r} \\
&\leq \sum_{r=0}^{N} \eta_u^{-r} |A^r f_u(x)| \sum_{k=N}^{\infty} \frac{|w|^k}{k!} c_{k,r} + \\
&+ \sum_{r=N+1}^{\infty} \eta_u^{-r} |A^r f_u(x)| \sum_{k=r}^{\infty} \frac{|w|^k}{k!} c_{k,r} =: (I) + (II).
\end{aligned}$$

For $N > N_\delta$, we have for the second term

$$(II) \leq \sum_{r=N+1}^{\infty} \frac{r!}{(\eta_u \delta R_u)^r} \sum_{k=r}^{\infty} \frac{|w|^k}{k!} c_{k,r} = \sum_{r=N+1}^{\infty} \frac{1}{(\delta R_u)^r} \left| \frac{\ln(1-|w|)}{\eta_u} \right|^r$$

$$\leq \sum_{r=N+1}^{\infty} \left(\frac{\kappa}{\delta R_u}\right)^r.$$

Thus, since $\kappa/(\delta R_u) < 1$, $(II) < \epsilon/2$ for $N > N_1 > N_\delta$. For the first term we may write

$$(I) = \sum_{r=0}^{N} \eta_u^{-r} |A^r f_u(x)| \sum_{k=N}^{\infty} \frac{|w|^k}{k!} c_{k,r} = \sum_{r=0}^{N} \frac{|A^r f_u(x)|}{r!} r! \eta_u^{-r} \sum_{k=N}^{\infty} \frac{|w|^k}{k!} c_{k,r}$$
$$= \sum_{r=0}^{N} \frac{|A^r f_u(x)|}{r!} \left|\frac{\ln(1-|w|)}{\eta_u}\right|^r \zeta_{r,N}$$

with

$$\zeta_{r,N} = 1 - \frac{r! \sum_{k=r}^{N-1} \frac{|w|^k}{k!} c_{k,r}}{|\ln(1-|w|)|^r}.$$

Note that $0 \leq \zeta_{r,N} \leq 1$, and that $\lim_{N\to\infty} \zeta_{r,N} = 0$ for all $r \geq 0$. Due to our assumptions we then have

$$(I) \leq C \sum_{r=0}^{N_\delta} \zeta_{r,N} 1_{r\leq N} + \sum_{r=0}^{\infty} \left(\frac{\kappa}{\delta R_u}\right)^r \zeta_{r,N} 1_{r\leq N} \tag{8.6}$$

for all $0 \leq s \leq \kappa < R_u$, $x \in \mathcal{K}$. Due to dominated convergence, the right-hand-side of (8.6) converges to zero as $N \to \infty$.

$(ii) \implies (iii)$: Is obvious, take $\varepsilon_u := 1 - e^{-\eta_u R_u}$.

$(iii) \implies (i)$ Let η_u and ε_u be such that (iii) holds. We may then define (see the proof of (ii))

$$\widetilde{p}(z, x, u) = \sum_{k=0}^{\infty} q_k^{(\eta_u)}(x, u)(1 - e^{-\eta_u z})^k, \quad x \in \mathfrak{X}, \tag{8.7}$$

which is holomorphic in $z \in \Phi_{\eta_u}(\mathcal{U})$. We first note that $\widetilde{p}(0, x, u) = f_u(x)$. Next we consider for $0 \leq s < -\eta_u^{-1} \ln \varepsilon_u$,

$$\widetilde{p}^{(N)}(s, x, u) = \sum_{k=0}^{N} q_k^{(\eta_u)}(x, u)(1 - e^{-\eta_u s})^k,$$

which satisfies

$$\begin{aligned}\frac{\partial}{\partial s}\widetilde{p}^{(N)}(s,x,u) &= \sum_{k=1}^{N} k q_k^{(\eta_u)}(x,u)(1-e^{-\eta_u s})^{k-1}\eta_u e^{-\eta_u s} \\ &= -\eta_u \sum_{k=1}^{N} k q_k^{(\eta_u)}(x,u)(1-e^{-\eta_u s})^{k} \\ &+ \eta_u \sum_{k=0}^{N-1}(k+1) q_{k+1}^{(\eta_u)}(x,u)(1-e^{-\eta_u s})^{k} \qquad (8.8) \\ &= -\eta_u N q_N^{(\eta_u)}(x,u)(1-e^{-\eta_u s})^{N} \\ &+ \sum_{k=0}^{N-1} A q_k^{(\eta_u)}(x,u)(1-e^{-\eta_u s})^{k} \qquad (8.9)\end{aligned}$$

by some rearranging and using (3.5). Since due to (iii) the first term in (8.9) vanishes for $N \to \infty$, we obtain

$$\frac{\partial}{\partial s}\widetilde{p}(s,x,u) = \lim_{N\to\infty}\frac{\partial}{\partial s}\widetilde{p}^{(N)}(s,x,u) = \lim_{N\to\infty} A\widetilde{p}^{(N-1)}(s,x,u),$$

together with

$$\lim_{N\to\infty}\widetilde{p}^{(N)}(s,x,u) = \widetilde{p}(s,x,u).$$

From the uniform convergence as stated in (iii) it follows easily that the two series in (8.8), the first term in (8.9), and so also the second term in (8.9) converge uniformly in the same sense. Thus, the above limits are uniform on compacta accordingly. Since the operator A is closed, we so obtain

$$\frac{\partial}{\partial s}\widetilde{p}(s,x,u) = A\widetilde{p}(s,x,u), \quad 0 \le s < -\eta_u^{-1}\ln\varepsilon_u,$$

and by uniqueness of the Cauchy problem associated with the operator A we thus have

$$\widehat{p}(s,x,u) = \widetilde{p}(s,x,u) = \sum_{k=0}^{N} q_k^{(\eta_u)}(x,u)(1-e^{-\eta_u s})^{k}, \quad 0 \le s < -\eta_u^{-1}\ln\varepsilon_u.$$

Because of the assumption that $\widehat{p}(s,x,u)$ is holomorphically extendable in each s, $0 \le s < \infty$, we then must have $\widehat{p}(s,x,u) = \widetilde{p}(s,x,u)$ for $0 \le s < \infty$. Finally, it is not difficult to see that there exists $R'_u > 0$ such that $G_{R'_u} \subset \Phi_{\eta_u}(\mathcal{U})$, hence (i) is proved. $\square$

Proof of Proposition 3.9

From the Taylor formula for semi-groups it follows that

$$P_s f_u = \sum_{k=0}^{r} \frac{s^k}{k!} A^k f_u + \frac{1}{r!} \int_0^s (s-\tau)^r P_\tau A^{r+1} f_u d\tau.$$

Due to (3.10), for $0 \le \tau \le s < R_u$ we have

$$\left\| P_\tau A^{r+1} f_u \right\| \le \sup_{0 \le \tau \le s} \|P_\tau\| \left\| A^{r+1} f_u \right\| \le \sup_{0 \le \tau \le s} \|P_\tau\| \left(\frac{1}{R_u} + \varepsilon \right)^{r+1} (r+1)!$$

for any $\varepsilon > 0$. It thus follows that

$$\left\| P_s f_u - \sum_{k=0}^{r} \frac{s^k}{k!} A^k f_u \right\| \le \left(\frac{1}{R_u} + \varepsilon \right)^{r+1} s^{r+1} \sup_{0 \le \tau \le s} \|P_\tau\|,$$

which converges to zero when $r \to \infty$, if $|s| < R_u/(1 + \varepsilon R_u)$. Since $\varepsilon > 0$ is arbitrary, the first statement is proved.

The commutation property $A^k P_t f_u = P_t A^k f_u$ and the boundedness of P_t for $t \ge 0$ imply that for $|s| < R_u$,

$$\sum_{k=0}^{\infty} \frac{|s|^k}{k!} \left\| A^k P_t f_u \right\| = \sum_{k=0}^{\infty} \frac{|s|^k}{k!} \left\| P_t A^k f_u \right\| \le \|P_t\| \sum_{k=0}^{\infty} \frac{|s|^k}{k!} \left\| A^k f_u \right\| < \infty.$$

Since $P_t f_u \in \mathfrak{D}(A^k)$ for all $k \ge 0$, (3.11) follows.

Proof of Theorem 4.1

For $r \ge 0$ define $A^r \tilde{f}_u =: g_r \tilde{f}_u$ with $\tilde{f}_u(x) = \exp\left[\mathrm{i} u^\top x \right]$, and write

$$A^{r+1} \tilde{f}_u = A\left(g_r \exp(\mathrm{i} u^\top x) \right) = \sum_{|\alpha| \ge 1} \mathfrak{a}_\alpha(x) \partial_{x^\alpha} \left(g_r \exp(\mathrm{i} u^\top x) \right).$$

Leibniz formula implies

$$\begin{aligned} A^{r+1} \tilde{f}_u &= \sum_{|\alpha| \ge 1} \mathfrak{a}_\alpha(x) \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial_{x^\beta} g_r \partial_{x^{\alpha-\beta}} \exp(\mathrm{i} u^\top x) \\ &= \left(\sum_{|\alpha| \ge 1} \mathfrak{a}_\alpha(x) \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (\mathrm{i} u)^{\alpha-\beta} \partial_{x^\beta} g_r \right) \exp(\mathrm{i} u^\top x). \end{aligned}$$

Hence, the following recurrent formula holds

$$g_{r+1} = \sum_{|\alpha| \ge 1} \mathfrak{a}_\alpha(x) \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (\mathrm{i} u)^{\alpha-\beta} \partial_{x^\beta} g_r. \tag{8.10}$$

Similar formulas for derivatives of g_{r+1} can be obtained:

$$\partial_{x^\rho} g_{r+1} = \sum_{|\alpha|\geq 1} \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (\mathrm{i}u)^{\alpha-\beta} \sum_{\eta\leq\rho} \frac{\rho!}{\eta!(\rho-\eta)!} \partial_{x^\eta} \mathfrak{a}_\alpha \ \partial_{x^{\rho-\eta+\beta}} g_r.$$

Since the underlying process is affine, all derivatives of $\mathfrak{a}$ of order higher than one are zero and thus, by induction, g_r is polynomial in x of degree at most equal r. We so get for $|\rho| \leq r+1$,

$$\partial_{x^\rho} g_{r+1} = \sum_{\eta\leq\rho,\, |\eta|\leq 1} \frac{\rho!}{\eta!(\rho-\eta)!} \sum_{|\alpha|\geq 1} \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (\mathrm{i}u)^{\alpha-\beta} \partial_{x^\eta} \mathfrak{a}_\alpha \ \partial_{x^{\rho-\eta+\beta}} g_r$$

By defining

$$\Gamma_r := \max_{|\beta|\leq r} |\partial_{x^\beta} g_r| ,$$

we obtain the following estimate for $x \in \mathfrak{X}$,

$$|\partial_{x^\rho} g_{r+1}| \leq \Gamma_r (1+\|x\|) \sum_{\substack{\eta\leq\rho,\\ |\eta|\leq 1}} \frac{\rho!}{\eta!(\rho-\eta)!} \sum_{|\alpha|\geq 1} \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} |u|^{\alpha-\beta} \mathcal{D}^{\mathfrak{a}}_{|\alpha|}$$

with $|u| := [|u_1| , ..., |u_n|]$. Hence, by the simple relations

$$\sum_{\{\eta:|\eta|\leq 1\}} \frac{\rho!}{\eta!(\rho-\eta)!} = 1+|\rho|, \qquad \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} |u|^{\alpha-\beta} \leq (1+\|u\|)^{|\alpha|},$$

$$\sum_{\alpha,|\alpha|=k} 1 = \frac{(k+n-1)!}{k!(n-1)!} \leq 2^{n+k},$$

with $\|u\| = \max_{i=1,\ldots,n} |u_i|$, we have

$$\begin{aligned} \Gamma_{r+1} &\leq 2^n \Gamma_r (r+2)(1+\|x\|) \sum_{k\geq 1} 2^k (1+\|u\|)^k \mathcal{D}^{\mathfrak{a}}_k \qquad (8.11) \\ &= 2^n \Gamma_r (r+2)(1+\|x\|)\theta(\|u\|), \end{aligned}$$

where the series in (8.11) is convergent due to assumption (4.2). As a consequence, (4.3) holds. □

Proof of Proposition 4.5

From (8.10) we have with $\mathfrak{a}_0 := 0$,

$$g_{r+1} = \sum_{\alpha,\beta,\gamma\geq 0} \mathfrak{a}_\alpha \frac{\alpha!}{\beta!(\alpha-\beta)!} (\mathrm{i}u)^{\alpha-\beta} g_{r,\gamma} \frac{\gamma!}{(\gamma-\beta)!} x^{\gamma-\beta} 1_{|\gamma|\leq r} 1_{\beta\leq\alpha} 1_{\beta\leq\gamma}$$

$$
\begin{aligned}
&= \sum_{\beta,\gamma\geq 0} g_{r,\gamma+\beta}\binom{\gamma+\beta}{\beta} x^{\gamma} 1_{|\gamma+\beta|\leq r}\mathfrak{b}_{\beta} \\
&= \sum_{|\gamma|\leq r} x^{\gamma} \sum_{|\beta|\leq r-|\gamma|} g_{r,\gamma+\beta}\binom{\gamma+\beta}{\beta}\mathfrak{b}^{0}_{\beta} \\
&+ \sum_{|\gamma|\leq r}\sum_{|\beta|\leq r-|\gamma|} g_{r,\gamma+\beta}\binom{\gamma+\beta}{\beta}\sum_{\kappa,\,|\kappa|=1}\mathfrak{b}^{1}_{\beta,\kappa}x^{\gamma+\kappa} \\
&= \sum_{|\gamma|\leq r+1} x^{\gamma} \sum_{|\beta|\leq r-|\gamma|} g_{r,\gamma+\beta}\binom{\gamma+\beta}{\beta}\mathfrak{b}^{0}_{\beta} \\
&+ \sum_{|\gamma|\leq r+1} x^{\gamma} \sum_{|\kappa|=1,\ \kappa\leq\gamma}\ \sum_{\beta,\,|\beta|\leq r+1-|\gamma|} g_{r,\gamma-\kappa+\beta}\binom{\gamma-\kappa+\beta}{\beta}\mathfrak{b}^{1}_{\beta,\kappa},
\end{aligned}
$$

where empty sums are to be interpret as zero. The second recursion follows from $(r+1)h_{r+1} = \eta^{-1}\widetilde{h}_{r+1} + rh_r$ with $A(h_r f_u) = \widetilde{h}_{r+1} f_u$ and $\widetilde{h}_{r+1}$ computed via (4.6) with g_r replaced by h_r. $\square$

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