An Iterative Method for Multiple Stopping: Convergence and Stability

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Abstract

We present a new iterative procedure for solving the multiple stopping problem in discrete time and discuss the stability of the algorithm. The algorithm produces monotonically increasing approximations of the Snell envelope, which coincide with the Snell envelope after finitely many steps. Contrary to backward dynamic programming, the algorithm allows to calculate approximative solutions with only a few nestings of conditional expectations and is, therefore, tailor-made for a plain Monte-Carlo implementation.

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1 Introduction

Financial derivatives with several early exercise rights play an important role in different markets, e.g. in electricity markets (swing options) and interest rate markets (chooser flexible caps). The pricing problem for such instruments is equivalent to a multiple stopping problem which is usually solved in practice by trinomial forests, see Jaillet et al. (2004) and the references therein. However, this pricing procedure is restricted to models for low-dimensional underlying processes, as trees tend to explode with increasing dimension of the underlying process, (so-called curse of dimensionality).

Obviously, multiple callable instruments with respect to a high dimensional interest rate model such as the popular Libor market model, and multiple callable options on a basket of several assets, do not meet this restriction. So new pricing methods for instruments with early exercise opportunities, based on high-dimensional underlying processes, are called for.

Only in recent years several approaches have been proposed to overcome the curse of dimensionality for American style derivatives, hence the case of a

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single exercise right. These methods basically rely on Monte-Carlo simulation and can be roughly divided into three groups. The first group directly employs a recursive scheme for solving the stopping problem, known as backward dynamic programming. Different techniques are applied to approximate the nested conditional expectations. The stochastic mesh method by Broadie and Glasserman (2004) and Broadie et al. (2000) and the least squares regression method of Longstaff and Schwartz (2001) are among the most popular approaches in this group. An alternative to backward dynamic programming is to approximate the exercise boundary by simulation, see e.g. Andersen (1999), Ibáñez and Zapatero (2004), and Milstein et al. (2004). The third group relies on a dual approach developed in Rogers (2002), Haugh and Kogan (2004), and in a multiplicative setting by Jamshidian (2003). By duality, tight upper bounds may be constructed from given approximative processes.

The methods in these three categories can be transferred from one to several exercise opportunities because the multiple stopping problem is equivalent to a system of nested single stopping problems. Meinshausen and Hambly (2004) suggest an extension of the Longstaff and Schwartz (2001)-algorithm to several exercise rights along these lines. Their main contribution is a derivation of the dual formulation under several exercise rights. Ibáñez (2004) presents a generalization of Ibáñez and Zapatero (2004) for multiple exercise opportunities.

The aim of the present paper is twofold: Firstly, we suggest an algorithm for the multiple stopping problem, which generalizes a procedure recently introduced by Kolodko and Schoenmakers (2006) for the single stopping problem. Secondly, we analyze stability of the algorithm under one as well as under several exercise rights.

Policy-improvement algorithms, such as the one proposed in Kolodko and Schoenmakers (2006), are mending one of the main drawbacks of the backward dynamic programming scheme: Suppose exercise can take place at one out of k time instances. Then, in order to obtain the value of the optimal stopping problem via backward dynamic programming, one has to calculate nested conditional expectations of order k. No approximation of the time 0 value is available prior to the evaluation of the kth nested conditional expectations. This prevents the use of plain Monte-Carlo simulations for approximating the conditional expectations and requires more complicated approximation procedures for these quantities. Contrary, the algorithm of Kolodko and Schoenmakers (2006) yields approximations of the time 0 value of the value function at every iteration step, which monotonically increase to the Snell envelope. This allows to calculate approximations of the Snell envelope via a plain Monte-Carlo simulation, if the underlying process is Markovian. Indeed, the it is shown in Kolodko and Schoenmakers (2006) that good approximations can be obtained even for very high ($d = 40!$)-dimensional problems.

In fact, the main advantage of the algorithm in Kolodko and Schoenmakers (2006) were lost, if a multi-exercise version would be straightforwardly defined as a nesting of one-exercise versions. This would cause nested conditional expectations in each iteration step and, thus, again prevent the use of a plain Monte Carlo implementation. Instead we present a multiple exercise version of the policy-improvement algorithm in a way that the order of nestings does not depend on the number of exercise rights. It is therefore tailored for plain Monte-Carlo simulation of the conditional expectations. We also prove that the algorithm coincides with the Snell envelope under L exercise rights after the same number of iterations as needed for the nested dynamic programming algorithm proposed in Carmona and Touzi (2006). This shows that our algorithm is theoretically as good as backward dynamic programming, but may be superior from a practical point of view.

The second contribution of our paper is a stability analysis for the policy-improvement algorithm of Kolodko and Schoenmakers (2006) and its multi-exercise extension. In the case of a single exercise right the stability result can be put in words as follows (recall, one can think of the stopping problem as an investor trying to maximize his expected gain): The shortfall of the investor's expected gain corresponding to m steps of the perturbed algorithm below the expected gain corresponding to m steps of the theoretical algorithm converges to zero. Surprisingly, it can happen that the perturbed algorithm performs better than the theoretical one (as is shown in example 4.1). A similar result is obtained in the multi-exercise case. This stability analysis provides the rigorous basis for a Monte-Carlo implementation of the algorithms.

Finally we note that standard policy iterations for dynamic programming (see e.g. Puterman, 1994; Kushner and Dupuis, 2001; Bertsekas, 2000) are based on lower approximations of the continuation value due to a (suboptimal) strategy. In contrast, the multiple stopping problem involves a lower approximation of the continuation value *and* the cashflow, hence a lower approximation on both sides of the inequality in the exercise criterion. The approximation of the cashflow is crucial to avoid additional nestings of conditional expectations and, therefore, reduces the computational complexity. In fact, in our algorithm both lower approximations are calculated by means of the same (suboptimal) strategy and nonetheless the monotone improvement property is established.

The paper is organized as follows: In Section 2 we pose the multiple stopping problem and explain its connection to the single stopping problem. Then in Section 3 we state the multiple exercise algorithm and prove its convergence. In particular, in Section 3.2 and 3.3 we put a main emphasis on the analysis of the building blocks of the algorithm, called one-step improvements. The results of Sections 3.2-3.3 are crucial for the discussion of stability in Section 4.

2 On the Multiple Stopping Problem

Suppose $(Z(i)\colon\ i = 0, 1, \ldots, k)$ is a nonnegative stochastic process in discrete time on a probability space $(\Omega, \mathcal{F}, P)$ adapted to some filtration $(\mathcal{F}_i :\ 0 \le i \le k)$ which satisfies

$$\sum_{i=1}^{k} E|Z(i)| < \infty.$$

We may think of the process Z as a cash-flow, which an investor may exercise L times. The investors' problem is to maximize his expected gain by exercising optimally. He is subjected to the additional constraint that he has to wait a minimal time $\delta \in \mathbb{N}$ between exercising two rights. The introduction of δ avoids mathematical trivialities, as otherwise the investor would exercise all rights at the same time. To emphasize that the introduction of δ is not a mathematical oddity, we will refer to δ as the *refracting period* following the terminology from swing options.

We now formalize the multiple stopping problem. For notational convenience we trivially extend the cash-flow process by $Z(i) = 0$ and $\mathcal{F}_i = \mathcal{F}_k$ for $i > k$. Let us define $\mathcal{S}_i(L, \delta)$ as the set of $\mathcal{F}_i$ stopping vectors $(\tau_1(i), \ldots, \tau_L(i))$ such that $i \le \tau_1(i)$ and, for all $2 \le j \le L$, $\tau_{j-1}(i) + \delta \le \tau_j(i)$. The multiple stopping problem may then be stated as follows: Find a family of stopping vectors $\tau^*(i) \in \mathcal{S}_i(L, \delta)$ such that for $0 \le i \le k$

$$E^{\mathcal{F}_i}\left[\sum_{j=1}^{L} Z(\tau_j^*(i))\right] = \operatorname{esssup}_{\tau \in \mathcal{S}_i(L,\delta)} E^{\mathcal{F}_i}\left[\sum_{j=1}^{L} Z(\tau_j)\right].$$

The process on the right hand side is called the *Snell envelope* of Z under L exercise rights and we denote it by $Y_L^*(i)$. We sometimes write $Y^*(i) = Y_1^*(i)$.

The case of one exercise right $L = 1$ is very well studied. We collect some facts, which can be found in Neveu (1975).

1. The Snell envelope Y^* of Z under one exercise rights is the smallest supermartingale that dominates Z.

2. A family of optimal stopping times for the stopping problem with one exercise rights is given by

   $$\tau^*(i) = \inf\{i \le j : \; Z(j) \ge E^{\mathcal{F}_j} Y^*(j+1)\}, \quad 0 \le i \le k.$$

   If several optimal stopping families exist, then the above family is the family of smallest optimal stopping times.

The multiple stopping problem can be reduced to L nested stopping problems with one exercise right. We briefly explain the reduction.

Define a sequence of processes $(X_0, \ldots, X_L, \ldots)$ as follows. $X_0 := 0$, $X_1 := Y_1^*$ is the Snell envelope of Z. X_L, $L \ge 2$, is the Snell envelope of the cash-flow $Z(i) + E^{\mathcal{F}_i} X_{L-1}(i+\delta)$ under one exercise right. We also define for $L = 1, 2, \ldots$,

$$\sigma_L^*(i) = \inf\{i \le j : \; Z(j) + E^{\mathcal{F}_j} X_{L-1}(j+\delta) \ge E^{\mathcal{F}_j} X_L(j+1)\}, \qquad i \ge 0, \quad (1)$$

i.e. the smallest optimal stopping families for the sequence of single stopping problems. It is straightforward to show by induction over L, that

$$Y_L^*(i) = X_L(i), \qquad 1 \le i \le k, \quad (2)$$

and a family of optimal stopping vectors for the multiple stopping problem with L exercise rights and cash-flow Z is given by

$$\begin{aligned} \tau_{1,L}^*(i) &= \sigma_L^*(i) \\ \tau_{d+1,L}^*(i) &= \tau_{d,L-1}^*(\sigma_L^*(i) + \delta) \quad 1 \le d \le L-1. \end{aligned} \quad (3)$$

Note that, due to the convention $Z(i) = 0$ for $i > k$, we have $\tau_{1,L}^*(i) = \sigma_L^*(i) = i$ for $i \ge k$.

By the above reduction, any algorithm for single optimal stopping problems can, in principle, be applied iteratively to the multiple stopping problem. For example, Carmona and Touzi (2006) suggested to apply backward dynamic programming iteratively to the L stopping problems. However, this approach leads to high nestings of conditional expectations, and, as a consequence, to tremendous simulation costs in a plain Monte Carlo approach.

3 An Algorithm for Multiple Stopping

3.1 The Algorithm

We are now going to present an algorithm which simultaneously improves the Snell envelope under $L = 1, \ldots, D$ exercise rights with the order of nested conditional expectations for a given number of iterations independent of L. In the case of a single exercise right it coincides with the procedure in Kolodko and Schoenmakers (2006). The building block of the algorithm is, as in the case of one exercise right, a policy improvement. More precisely, suppose we are given the families of stopping times

$$\sigma_L(i), \quad 0 \le i \le k, \quad 1 \le L \le D,$$

trivially extended with $\sigma_L(i) = i$ for $i > k$. Recall that k is the time horizon of the real cash-flow process. We are interested in the Snell envelope with L exercise rights for all $1 \le L \le D$ and refracting period δ. We interpret $\sigma_L(i)$ as the time, when the investor exercises (possibly in a suboptimal way) the first of his L rights, given that he has not exercised prior to time i. This interpretation requires that the stopping families σ_L under consideration are consistent in the sense of the following definition:

Definition 3.1. A family of integer-valued stopping times $(\tau(i):\ 0 \le i \le k)$ is said to be *consistent*, if for $0 \le i < k$,

$$\begin{aligned} &i \le \tau(i) \le k,\ \tau(k) \equiv k, \\ &\tau(i) > i \Rightarrow \tau(i) = \tau(i+1). \end{aligned} \tag{4}$$

Given consistent stopping families σ_L, $L = 1, 2, \ldots$, we define associated stopping families $\tau_{d+1,L}$ via,

$$\begin{aligned} \tau_{1,L}(i) &= \sigma_L(i) \\ \tau_{d+1,L}(i) &= \tau_{d,L-1}\left(\sigma_L(i) + \delta\right) \quad 1 \le d \le L-1. \end{aligned} \tag{5}$$

$\tau_{d,L}(i)$ can be interpreted as the time, when the investor exercises the dth of his L exercise rights, provided he has not exercised his first right prior to time i. An approximation of the Snell envelope with L exercise rights is now given by

$$Y_L(i; \sigma_1, \ldots \sigma_L) := E^{\mathcal{F}_i}\left[\sum_{d=1}^{L} Z(\tau_{d,L}(i))\right]. \tag{6}$$

Note, $Y_L(i; \sigma_1, \ldots \sigma_L)$ has a simple interpretation as the expected gain (conditional on $\mathcal{F}_i$) the investor obtains when he employs the stopping families $\sigma_1, \ldots, \sigma_L$ for exercising the cash-flows. We then introduce intermediate processes

$$\widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L) := \max_{i+1 \le p \le k} E^{\mathcal{F}_i}\left[\sum_{d=1}^{L} Z(\tau_{d,L}(p))\right] \tag{7}$$

on which a next exercise criterion is built,

$$\widetilde{\sigma}_L(i) := \inf\left\{ j \ge i;\ \ Z(j) + E^{\mathcal{F}_j} Y_{L-1}(j+\delta; \sigma_1, \ldots, \sigma_{L-1}) \ge \widehat{Y}_L(j; \sigma_1, \ldots, \sigma_L)\right\}, \tag{8}$$

with $Y_0(i) := 0$. Note that $\widetilde{\sigma}_L(k) = k$ since $\max \emptyset = -\infty$, and, obviously, the stopping families $\tilde{\sigma}_L$ are consistent for $1 \leq L \leq D$.

Given consistent starting families of stopping times $\sigma_L^{(0)}$, $1 \leq L \leq D$, we define iteratively,

$$\begin{aligned} \sigma_L^{(m)}(i) &:= \widetilde{\sigma}_L^{(m-1)}(i), \\ Y_L^{(m)}(i) &:= Y_L(i; \sigma_1^{(m)}, \ldots, \sigma_L^{(m)}). \end{aligned} \tag{9}$$

Canonical consistent starting families are given, for instance, by $\sigma_L^{(0)}(i) = i$, $L = 1, 2, \ldots$

Theorem 3.2. *Suppose the stopping families $\sigma_L^{(0)}(i)$ are consistent for all $1 \leq L \leq D$. Then, for all $m \in \mathbb{N}$, $1 \leq L \leq D$, and $0 \leq i \leq k$,*

$$Y_L^{(m+1)}(i) \geq Y_L^{(m)}(i).$$

Moreover, for $m \geq k - i$,

$$Y_L^{(m)}(i) = Y_L^*(i),$$

where Y_L^* denotes the Snell envelope of Z under L exercise rights.

The dynamic programming scheme suggests to define another approximation of the Snell envelope, namely

$$\begin{aligned} y_L(i; \sigma_1, \ldots, \sigma_L) \;:=\; & \max\{Z(i) + E^{\mathcal{F}_j} Y_{L-1}(i + \delta; \sigma_1, \ldots, \sigma_{L-1}), \\ & E^{\mathcal{F}_i}[Y_L(i + 1; \sigma_1, \ldots, \sigma_L)]\}, \end{aligned} \tag{10}$$

$1 \leq L \leq D$, $0 \leq i \leq k$, given consistent stopping families $\sigma_1, \ldots, \sigma_D$. Based on y_L we will also consider a modified algorithm, namely

$$\begin{aligned} \sigma_L^{(m)}(i) &:= \widetilde{\sigma}_L^{(m-1)}(i), \\ y_L^{(m)}(i) &:= y_L(i; \sigma_1^{(m)}, \ldots, \sigma_L^{(m)}), \end{aligned} \tag{11}$$

with the same stopping families $\sigma_L^{(m)}$ as in (9). y_L does not admit such an intuitive interpretation as Y_L, but the modified algorithm yields better approximations than the one based on Y_L.

Theorem 3.3. *All assertion of theorem 3.2 hold with $Y_L^{(m)}$ replaced by $y_L^{(m)}$. Moreover, for all $1 \leq L \leq D$, $0 \leq i \leq k$, and $m \in \mathbb{N}$,*

$$y_L^{(m)}(i) \geq Y_L^{(m)}(i)$$

(provided both algorithms are initiated with the same stopping families.)

We prove Theorems 3.2 and 3.3 in Section 3.4. Before we scrutinize the building blocks of the algorithm, which we will refer to as *one-step improvements* in the following subsections, let us briefly discuss the implementation of the algorithm.

On the Implementation of the Algorithm

Although Theorems 3.2 and 3.3 state convergence to the Snell envelope in finitely many steps, in practice only a few steps (not the whole algorithm) may be calculated. In this respect the main benefit of the theorems is that every step improves upon the previous one. Specifically, we recommend the following procedure, if the underlying process has a Markovian structure:

1. Choose appropriate starting families. If no better choice is known a priori, one can start with the canonical families $\sigma_L^{(0)}(i) = i$.
2. Apply the plain Monte-Carlo estimator to approximate all (conditional) expectations in the iteration steps based on (6) and (8), which are not known analytically.
3. The obtained approximations are (up to a simulation error) lower bounds of the Snell envelope. Upper bounds (up to simulation error) can be calculated from the lower bounds by the dual method of Meinshausen and Hambly (2004).

More detailed information on the implementation, including a generalization of the pseudo code from Schoenmakers (2005) to the case of two exercise rights, can be found in an addendum provided online (Bender and Schoenmakers, 2006). Simulation results in the case of a single exercise right are presented in Kolodko and Schoenmakers (2006) for a Bermudan Swaption on a 40 dimensional underlying.

3.2 A Generalization of the One-Step Improvement in the Case of One Exercise Right

We now investigate a single improvement step the case of one exercise right and generalize results of Kolodko and Schoenmakers (2006). These generalizations will be of crucial importance for investigating the stability of the proposed algorithm in Section 4.

Suppose a consistent stopping family $(\tau(i) : 1 \le i \le k)$ is given. We then define the process

$$Y(i;\tau) := E^{\mathcal{F}_i}\left[Z(\tau(i))\right]. \tag{12}$$

Based on the sequence $(\tau(i) : 1 \le i \le k)$ Kolodko and Schoenmakers (2006) construct a new family $(\widetilde{\tau}(i) : 1 \le i \le k)$ in the following way: Introduce an intermediate process

$$\widetilde{Y}(i;\tau) := \max_{p:\, i \le p \le k} E^{\mathcal{F}_i}\left[Z(\tau(p))\right], \tag{13}$$

which serves as a new exercise criterion, i.e.

$$\begin{aligned} \widetilde{\tau}(i) &:= \inf\{j : i \le j \le k,\ \widetilde{Y}(j;\tau) \le Z(j)\} \qquad (14)\\ &= \inf\{j : i \le j \le k,\ \max_{p:\, j \le p \le k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right] \le Z(j)\}, \quad 0 \le i \le k. \end{aligned}$$

Kolodko and Schoenmakers (2006), Theorem 3.1, show that $\widetilde{\tau}$ is an improvement of τ in the sense that the new strategy promises a higher expected gain for the investor than the old one, i.e.

$$Y(i;\widetilde{\tau}) \ge \widetilde{Y}(i;\tau) \ge Y(i;\tau).$$

Our first aim is to extend this chain of inequalities to a wider class of stopping families than $\tilde{\tau}$. To this end we first compare the intermediate processes $\widehat{Y}(i;\tau)$ and

$$\widehat{Y}(i;\tau) := \max_{p:\, i+1\le p\le k} E^{\mathcal{F}_i}\left[Z(\tau(p))\right]. \tag{15}$$

Lemma 3.4. *Suppose the stopping family τ is consistent. Then, for $0 \le i \le k$,*

$$\widetilde{Y}(i;\tau) = \mathbf{1}_{\{\tau(i)>i\}}\widehat{Y}(i;\tau) + \mathbf{1}_{\{\tau(i)=i\}} \max\left\{\widehat{Y}(i;\tau),\, Z(i)\right\}. \tag{16}$$

In particular,

$$Z(i) \ge \widetilde{Y}(i;\tau) \iff Z(i) \ge \widehat{Y}(i;\tau), \tag{17}$$

and

$$\widetilde{\tau}(i) = \inf\{j : i \le j \le k,\ \widehat{Y}(j) \le Z(j)\}. \tag{18}$$

Proof. By property (4), we have,

$$\begin{aligned} E^{\mathcal{F}_i}\left[Z(\tau(i))\right] &= E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tau(i)=i\}}Z(i)\right] + E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tau(i)>i\}}Z(\tau(i+1))\right] \\ &= \mathbf{1}_{\{\tau(i)=i\}}Z(i) + \mathbf{1}_{\{\tau(i)>i\}}E^{\mathcal{F}_i}\left[Z(\tau(i+1))\right]. \end{aligned}$$

Since

$$\widetilde{Y}(i;\tau) = \max\left\{\widehat{Y}(i;\tau),\, E^{\mathcal{F}_i}\left[Z(\tau(i))\right]\right\},$$

(16) follows with (17) and (18) as immediate consequences. □

We next define another stopping family, namely,

$$\widehat{\tau}(i) := \inf\{j : i \le j \le k,\ \widehat{Y}(j) < Z(j)\}. \tag{19}$$

Clearly by (18),

$$\widehat{\tau}(i) \ge \widetilde{\tau}(i). \tag{20}$$

Theorem 3.5. *Let $(\tau(i),\, 1 \le i \le k)$ be a consistent stopping family. Suppose $(\bar{\tau}(i),\, 1 \le i \le k)$ is also consistent and satisfies*

$$\widetilde{\tau}(i) \le \bar{\tau}(i) \le \widehat{\tau}(i) \quad 0 \le i \le k. \tag{21}$$

Then,

$$Y(i;\bar{\tau}) \ge \widetilde{Y}(i;\tau) \ge Y(i;\tau), \quad 0 \le i \le k.$$

Remark 3.1. Obviously, the choices $\bar{\tau} = \widetilde{\tau}$ and $\bar{\tau} = \widehat{\tau}$ are examples of a family $\bar{\tau}$ satisfying (4) and (21).

Proof. The second inequality is trivial. We prove the first inequality by backward induction over i. For $i = k$, note that $Y(k;\bar{\tau}) = Z(k) = \widetilde{Y}(k;\tau)$. Now suppose $0 \le i \le k-1$, and that the assertion is already proved for $i+1$. It holds $\{\bar{\tau}(i) = i\} \subset \{\widetilde{\tau}(i) = i\}$ by (21). Hence, we obtain on the set $\{\bar{\tau}(i) = i\}$,

$$Y(i;\bar{\tau}) = Z(i) \ge \widetilde{Y}(i;\tau).$$

However, on $\{\bar{\tau}(i) > i\}$ we have $\bar{\tau}(i) = \bar{\tau}(i+1)$ and hence the induction hypothesis yields,

$$\begin{aligned} Y(i;\bar{\tau}) &= E^{\mathcal{F}_i}\left[Z(\bar{\tau}(i+1))\right] = E^{\mathcal{F}_i}\left[Y(i+1;\bar{\tau})\right] \ge E^{\mathcal{F}_i}\left[\widetilde{Y}(i+1;\tau)\right] \\ &= E^{\mathcal{F}_i}\left[\max_{i+1\le p\le k} E^{\mathcal{F}_{i+1}}\left[Z(\tau(p))\right]\right] \ge \max_{i+1\le p\le k} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] = \widehat{Y}(i;\tau). \end{aligned}$$

Property (21) implies $\{\bar{\tau}(i) > i\} \subset \{\widehat{\tau}(i) > i\}$. Thus, on $\{\bar{\tau}(i) > i\}$, $\widehat{Y}(i;\tau) \geq Z(i)$ and, by (16), $\widehat{Y}(i;\tau) = \widetilde{Y}(i;\tau) \quad \text{on } \{\bar{\tau}(i) > i\}$. □

Motivated by the previous theorem we introduce the notion of an *improver*:

Definition 3.6. Suppose τ is a consistent stopping family. A stopping family $\bar{\tau}$ is called an *improver* of τ, if it is consistent and satisfies (21) for $0 \leq i \leq k$.

The next theorem provides another justification for the name 'improver'.

Theorem 3.7. *Suppose τ is a consistent stopping family and $\bar{\tau}$ is an improver of τ. Then*

$$Y(i;\tau) = Y^*(i) \quad \text{for } i \geq j+1$$

implies

$$Y(i;\bar{\tau}) = Y^*(i) \quad \text{for } i \geq j.$$

Proof. We will exploit the fact that the Snell envelope is the smallest supermartingale dominating Z. By Theorem 3.5 we have, for $0 \leq i \leq k-1$,

$$Y(i;\bar{\tau}) \geq \widetilde{Y}(i;\tau) \geq E^{\mathcal{F}_i}\left[Z(\tau(i+1))\right] = E^{\mathcal{F}_i}\left[Y(i+1;\tau)\right].$$

Therefore, for $j \leq i \leq k-1$,

$$Y(i;\bar{\tau}) \geq E^{\mathcal{F}_i}\left[Y^*(i+1)\right] \geq E^{\mathcal{F}_i}\left[Y(i+1;\bar{\tau})\right].$$

This means $(Y(i;\bar{\tau}),\ j \leq i \leq k)$ is a supermartingale. We may also deduce from Theorem 3.5 that for $0 \leq i \leq k$,

$$Y(i;\bar{\tau}) \geq \mathbf{1}_{\{\bar{\tau}(i)=i\}} Z(i) + \mathbf{1}_{\{\bar{\tau}(i)>i\}} \widetilde{Y}(i;\tau).$$

However, as in the proof of Theorem 3.5, we obtain

$$\mathbf{1}_{\{\bar{\tau}(i)>i\}} \widetilde{Y}(i;\tau) \geq \mathbf{1}_{\{\bar{\tau}(i)>i\}} \widehat{Y}(i;\tau) \geq \mathbf{1}_{\{\bar{\tau}(i)>i\}} Z(i).$$

Thus, $Y(\cdot,\bar{\tau})$ dominates Z. We thus have shown that $(Y(i;\bar{\tau}),\ j \leq i \leq k)$ is a supermartingale dominating Z. Therefore,

$$Y(i;\bar{\tau}) \geq Y^*(i) \quad \text{for } i \geq j.$$

The reverse inequality is trivial. □

Remark 3.2. The proof of the previous theorem shows, that for any improver $\bar{\tau}$,

$$Y(i;\bar{\tau}) \geq Z(i), \quad 0 \leq i \leq k. \tag{22}$$

We end this section with a comparison between different improvers.

Proposition 3.8. *Suppose τ is consistent and $\bar{\tau}$ is an improver of τ. Then, for all $0 \leq i \leq k$,*

$$Y(i;\widehat{\tau}) \geq Y(i;\bar{\tau}) \geq Y(i;\widetilde{\tau}).$$

Proof. We prove the second inequality. The proof of the first one is similar. For $i = k$ equality holds. Suppose $0 \leq i \leq k-1$ and the inequality is proved for $i+1$. Then, on $\{\bar{\tau}(i) > i\} \cap \{\widetilde{\tau}(i) > i\}$,

$$Y(i;\bar{\tau}) = E^{\mathcal{F}_i}\left[Y(i+1;\bar{\tau})\right] \geq E^{\mathcal{F}_i}\left[Y(i+1;\widetilde{\tau})\right] = Y(i;\widetilde{\tau})$$

by the induction hypothesis. On $\{\bar{\tau}(i) > i\} \cap \{\widetilde{\tau}(i) = i\}$ we have

$$Y(i;\bar{\tau}) \geq Z(i) = Y(i;\widetilde{\tau})$$

by (22). Finally, the set $\{\bar{\tau}(i) = i\} \cap \{\widetilde{\tau}(i) > i\}$ is evanescent by the definition of an improver. □

3.3 The One-Step Improvement in the Case of Several Exercise Rights

We now investigate the one-step improvement under several exercise rights defined in (5)–(8). To this end, suppose consistent stopping families $\sigma_1, \ldots, \sigma_D$ are given. The following obvious representations of $Y_L(i; \sigma_1, \ldots, \sigma_L)$ and $\widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L)$ allow to extend Theorem 3.5 to the case of several exercise rights.

Lemma 3.9. *Define for $2 \le L \le D$ and $0 \le i \le k$,*

$$Z_L(i; \sigma_1, \ldots \sigma_{L-1}) = Z(i) + E^{\mathcal{F}_i}\left[Y_{L-1}(i+\delta; \sigma_1, \ldots, \sigma_{L-1})\right]. \tag{23}$$

Then,

$$\begin{aligned} Y_L(i; \sigma_1, \ldots, \sigma_L) &= E^{\mathcal{F}_i}\left[Z_L(\sigma_L(i); \sigma_1, \ldots, \sigma_{L-1})\right], \\ \widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L) &= \max_{i+1 \le p \le k} E^{\mathcal{F}_i}\left[Z_L(\sigma_L(p); \sigma_1, \ldots, \sigma_{L-1})\right]. \end{aligned}$$

By the previous lemma we may rewrite $\widetilde{\sigma}_L$ defined in (8) as

$$\widetilde{\sigma}_L(i) = \inf\left\{ j \ge i;\;\; Z_L(j; \sigma_1, \ldots, \sigma_{L-1}) \ge \max_{p \ge j+1} E^{\mathcal{F}_j} Z_L(\sigma_L(p); \sigma_1, \ldots, \sigma_{L-1}) \right\}. \tag{24}$$

Consequently, the step from σ_L to $\widetilde{\sigma}_L$ is a one-step improvement with one exercise right and cash-flow $Z_L(\cdot; \sigma_1, \ldots, \sigma_{L-1})$.

As in the case of one exercise right we also consider the stopping family

$$\widehat{\sigma}_L(i) = \inf\left\{ j \ge i;\;\; Z_L(j; \sigma_1, \ldots, \sigma_{L-1}) > \max_{p \ge j+1} E^{\mathcal{F}_j} Z_L(\sigma_L(p); \sigma_1, \ldots, \sigma_{L-1}) \right\}. \tag{25}$$

Definition 3.10. A stopping family $\bar{\sigma}_L$ is said to be an *L-improver of σ_L with respect to* $(\sigma_1, \ldots, \sigma_{L-1})$, if $\bar{\sigma}_L$ is consistent and

$$\widetilde{\sigma}_L(i) \le \bar{\sigma}_L(i) \le \widehat{\sigma}_L(i). \tag{26}$$

In abuse of terminology we will simply speak of an *improver*, when L and $(\sigma_1, \ldots, \sigma_{L-1})$ are evident from the context.

We now state a generalization of Theorem 3.5, which justifies the name 'improver'.

Theorem 3.11. *Suppose consistent stopping families $\sigma_1, \ldots, \sigma_D$ are given with respective improvers $\bar{\sigma}_1, \ldots, \bar{\sigma}_D$. Then, for $1 \le L \le D$ the following chain of inequalities holds,*

$$\begin{aligned} & Y_L(i; \bar{\sigma}_1, \ldots, \bar{\sigma}_L) \ge Y_L(i; \sigma_1, \ldots, \sigma_{L-1}, \bar{\sigma}_L) \ge Y_L(i; \sigma_1, \ldots, \sigma_{L-1}, \widetilde{\sigma}_L) \\ \ge\;\; & \max\left\{ Y_L(i; \sigma_1, \ldots, \sigma_L),\, \widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L) \right\}. \end{aligned}$$

Proof. By the previous considerations $\bar{\sigma}_L$ is also a 1-improver of σ_L with respect to the cash-flow $Z_L(\cdot; \sigma_1, \cdots, \sigma_{L-1})$ (with the convention $Z_1 = Z$). In view of Lemma 3.9 the second inequality follows from Proposition 3.8 and the third one from Theorem 3.5. We will prove the first inequality by induction over L. Note that the inequality is trivial for $L=1$. The step from $L-1$ to L can be shown as follows. By Lemma 3.9,

$$\begin{aligned} & Y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_L) - Y_L(i;\sigma_1,\ldots,\sigma_{L-1},\bar{\sigma}_L) \\ = \; & E^{\mathcal{F}_i}\left[Z(\bar{\sigma}_L(i)) + Y_{L-1}(\bar{\sigma}_L(i)+\delta;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1})\right] \\ & -E^{\mathcal{F}_i}\left[Z(\bar{\sigma}_L(i)) + Y_{L-1}(\bar{\sigma}_L(i)+\delta;\sigma_1,\ldots,\sigma_{L-1})\right] \\ = \; & E^{\mathcal{F}_i}\left[Y_{L-1}(\bar{\sigma}_L(i)+\delta;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1}) - Y_{L-1}(\bar{\sigma}_L(i)+\delta;\sigma_1,\ldots,\sigma_{L-1})\right]. \end{aligned}$$

As the second and the third inequality are already proved, the induction hypothesis (for the first inequality) implies,

$$Y_{L-1}(\bar{\sigma}_L(i)+\delta;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1}) \geq Y_{L-1}(\bar{\sigma}_L(i)+\delta;\sigma_1,\ldots,\sigma_{L-1}).$$

Thus,

$$Y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_L) - Y_L(i;\sigma_1,\ldots,\sigma_{L-1},\bar{\sigma}_L) \geq 0.$$

□

By the very definition of y_L in (10) we get the following corollary of the previous theorem.

Corollary 3.12. *Suppose consistent stopping families $\sigma_1,\ldots,\sigma_D$ are given with respective improvers $\bar{\sigma}_1,\ldots,\bar{\sigma}_D$. Then the following chain of inequalities holds for $1 \leq L \leq D$,*

$$\begin{aligned} & y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_L) \geq y_L(i;\sigma_1,\ldots,\sigma_{L-1},\bar{\sigma}_L) \geq y_L(i;\sigma_1,\ldots,\sigma_{L-1},\widetilde{\sigma}_L) \\ \geq \; & y_L(i;\sigma_1,\ldots,\sigma_L). \end{aligned}$$

We are now ready to give the proofs of Theorems 3.2 and 3.3.

3.4 Proofs of Theorems 3.2 and 3.3

Proof of Theorem 3.2. The monotonicity assertion is a direct consequence of Theorem 3.11 since, by definition,

$$\begin{aligned} Y_L^{(m)}(i) &= Y(i;\sigma_1^{(m)},\ldots\sigma_L^{(m)}) \\ \sigma_d^{(m+1)} &= \widetilde{\sigma}_d^{(m)}, \quad 1 \leq d \leq L. \end{aligned}$$

Recall that the $\bar{\ }$ in Theorem 3.11, can always be replaced with $\widetilde{\ }$ by the definition of an improver.

Concerning the second assertion we will show by backward induction over i that for $m+i \geq k$ and all $1 \leq L \leq D$

$$\sigma_L^{(m)}(i) = \sigma_L^*(i), \tag{27}$$

where the family of stopping families $(\sigma_1^*,\ldots,\sigma_D^*)$ was defined in (1). For $i=k$ this claim is obvious. Suppose now it is proven for all $j \geq i+1$. Recall, in view of (3), $(\sigma_1^*,\ldots,\sigma_D^*)$ induces an optimal strategy, i.e.

$$Y_L(i;\sigma_1^*,\ldots,\sigma_L^*) = Y_L^*(i).$$

Thus, for $j \geq i+1$, $m+i \geq k$

$$Y_L^{(m-1)}(j) = Y_L(j; \sigma_1^{(m-1)}, \ldots, \sigma_L^{(m-1)}) = Y_L(j; \sigma_1^*, \ldots, \sigma_L^*) = Y_L^*(j)$$

and, consequently, by the supermartingale property of the Snell envelope,

$$\widehat{Y}_L(j-1; \sigma_1^{(m-1)}, \ldots, \sigma_L^{(m-1)}) = \max_{p \geq j} E^{\mathcal{F}_{j-1}} \left[Y_L^{(m-1)}(p) \right] = E^{\mathcal{F}_{j-1}} \left[Y_L^*(j) \right].$$

Hence, by (1), (2), (8), and (9), for $i+m \geq k$,

$$\begin{aligned} \sigma_L^{(m)}(i) &= \inf \Big\{ j \geq i; \;\; Z(j) + E^{\mathcal{F}_j} Y_{L-1}(j+\delta; \sigma_1^{(m-1)}, \ldots, \sigma_{L-1}^{(m-1)}) \\ &\quad \geq \widehat{Y}_L(j; \sigma_1^{(m-1)}, \ldots, \sigma_L^{(m-1)}) \Big\} \\ &= \inf \Big\{ j \geq i; \;\; Z(j) + E^{\mathcal{F}_j} Y_{L-1}^*(j+\delta) \geq E^{\mathcal{F}_j} \left[Y_L^*(j+1) \right] \Big\} \\ &= \sigma_L^*(i). \end{aligned}$$

Now (27) yields $Y_L^{(m)}(i) = Y_L^*(i)$ for $m+i \geq k$ and all $1 \leq L \leq D$. □

Proof of Theorem 3.3. We first show

$$y_L(i; \sigma_1, \ldots, \sigma_L) \geq Y_L(i; \sigma_1, \ldots, \sigma_L) \tag{28}$$

for all $1 \leq L \leq D$, $0 \leq i \leq k$, given consistent stopping families $\sigma_1, \ldots, \sigma_D$. By Lemma 3.9 and due to the consistency of σ_L,

$$\begin{aligned} Y_L(i; \sigma_1, \ldots, \sigma_L) &= \mathbf{1}_{\{\sigma_L(i)=i\}} \left(Z(i) + E^{\mathcal{F}_i} \left[Y_{L-1}(i+\delta; \sigma_1, \ldots, \sigma_{L-1}) \right] \right) \\ &\quad + \mathbf{1}_{\{\sigma_L(i)>i\}} E^{\mathcal{F}_i} [Y_L(i+1; \sigma_1, \ldots, \sigma_L)]. \end{aligned}$$

This proves (28) and, with the choice $(\sigma_1, \ldots, \sigma_D) = (\sigma_1^{(m)}, \ldots, \sigma_D^{(m)})$,

$$y_L^{(m)}(i) \geq Y_L^{(m)}(i).$$

In particular, for $m \geq k-i$ and applying theorem 3.2,

$$Y_L^*(i) \geq y_L^{(m)}(i) \geq Y_L^{(m)}(i) = Y_L^*(i).$$

The monotonicity in m follows directly from corollary 3.12. □

Remark 3.3. The proofs show that after any $m \geq k-i$ improvements, not only the $\widetilde{\ }$-improvement, the corresponding approximations coincides with the Snell envelope under L exercise rights up from time i on.

4 Stability

In this section we discuss the stability of the algorithms. We start with the analysis of the one-step improvement under one exercise right. Then we will prove stability of the Y-algorithm under one exercise right. Finally, stability of the y_L-algorithm for the general case is established under an additional assumption.

4.1 Stability of the One-Step Improvement ($L = 1$)

Suppose a consistent stopping family τ is given. As we cannot expect to know the conditional expectations analytically in general, but, may only calculate approximations, we consider instead of $\widetilde{\tau}(i)$ a sequence of stopping families

$$\widetilde{\tau}^{(N)}(i) := \inf\{j : i \leq j \leq k,\ \widehat{Y}(j;\tau) + \epsilon^{(N)}(j) \leq Z(j)\},$$

where $N \in \mathbb{N}$, and $\epsilon^{(N)}(i)$ is a sequence of $\mathcal{F}_i$-adapted processes.

We will first show by some simple examples that we must neither expect

$$\widetilde{\tau}^{(N)}(i) \to \widetilde{\tau}(i) \quad \text{in probability,}$$

nor

$$Y(0;\widetilde{\tau}^{(N)}) \to Y(0;\widetilde{\tau}), \tag{29}$$

when

$$\lim_{N\to\infty} \epsilon^{(N)}(i) = 0, \quad P - a.s.$$

Example 4.1. (i) Suppose $(\xi_N)_{N\in\mathbb{N}}$ is a sequence of independent binary trials with $P(\xi_N = 1) = P(\xi_N = 0) = 1/2$. We define the process $(Z(i) :\ i = 0, 1)$ by $Z(0) = Z(1) \equiv 1$. The σ-field $\mathcal{F}_0 = \mathcal{F}_1$ is the one generated by the sequence of trials. Moreover, the sequence of perturbations is defined by $\epsilon^{(N)}(0) = \xi_N/N$ and $\epsilon^{(N)}(1) = 0$. Then, starting with any consisting stopping family τ, we get

$$\widetilde{\tau}^{(N)}(0) = \xi_N.$$

In particular, no subsequence of $\widetilde{\tau}^{(N)}(0)$ converges in probability.

(ii) Let $\Omega = \{\omega_0, \omega_1\}$, $\mathcal{F}$ the power set of Ω, and $P(\{\omega_1\}) = 1/4 = 1 - P(\{\omega_0\})$. We define the process $(Z(i) :\ i = 0, 1, 2)$ by $Z(0) = Z(2) = 2$, and $Z(1, \omega_0) = 1$, $Z(1, \omega_1) = 3$. $\mathcal{F}_i$ is the filtration generated by Z. We start with the stopping family $\tau(i) = i$. As $E[Z(1)] = 3/2$, we have

$$Z(0) = 2 \geq \max\{3/2,\ 2\} = \max\{E[Z(1)],\ E[Z(2)]\} = \widehat{Y}(0, \tau).$$

Therefore, $\widetilde{\tau}(0) = 0$ and $Y(0;\widetilde{\tau}) = 2$. The perturbation sequence $\epsilon^{(N)}$ is defined to be $\epsilon^{(N)}(1) = \epsilon^{(N)}(2) \equiv 0$ and $\epsilon^{(N)}(0) = 1/N$. A straightforward calculation shows that for $N \geq 2$,

$$\widetilde{\tau}^{(N)}(0, \omega_0) = 2, \quad \widetilde{\tau}^{(N)}(0, \omega_1) = 1.$$

Thus,

$$Y(0;\widetilde{\tau}^{(N)}) = 9/4 > 2 = Y(0;\widetilde{\tau}),$$

which violates (29).

At first glance, Example 4.1 paints a rather sceptical picture of the stability properties of the one-step-improvement. Indeed, the best we can now hope for, is

(ia) there is a sequence $\bar{\tau}^{(N)}$ of improvers of τ such that

$$|\widetilde{\tau}^{(N)}(i) - \bar{\tau}^{(N)}(i)| \to 0 \quad P - a.s.$$

(iia) The shortfall of $Y(i;\widetilde{\tau}^{(N)})$ below $Y(i;\widetilde{\tau})$ converges to zero P-a.s.

Note, however, that convergence of the shortfall as in (iia) is the relevant question, not convergence of the distance as in (29), since the shortfall corresponds to a change for the worse of $\widetilde{\tau}^{(N)}$ compared to $\widetilde{\tau}$. As we are interested in an improvement it suffices to guarantee that such a change for the worse converges to zero. An additional improvement of $\widetilde{\tau}^{(N)}$ compared to $\widetilde{\tau}$ due to the error processes $\epsilon^{(N)}$ may be seen as a welcome side effect!

We now prove assertions (ia) and (iia). We first introduce a new sequence of stopping families which turns out to consist of improvers. Let us define $\bar{\tau}^{(N)}(k) = k$, and, for $i < k$,

$$\begin{aligned} \bar{\tau}^{(N)}(i) = i \quad &\Longleftrightarrow \quad (\widetilde{\tau}^{(M)}(i) > i \text{ for only finitely many } M) \\ &\qquad \vee (\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M \text{ and } \widetilde{\tau}^{(N)}(i) = i), \\ \bar{\tau}^{(N)}(i) \neq i \quad &\Longrightarrow \quad \bar{\tau}^{(N)}(i) = \bar{\tau}^{(N)}(i+1). \end{aligned}$$

We then have the following result:

Theorem 4.2. *Suppose*

$$\lim_{N\to\infty} \epsilon^{(N)}(i) = 0 \quad P-a.s.,$$

for all $0 \leq i \leq k$. *Then* $\bar{\tau}^{(N)}$ *is an improver of* τ *for every* $N \in \mathbb{N}$.

Proof. The consistency property (4) is satisfied by definition. We show (21) by backward induction over i. The case $i = k$ is immediate. Suppose now $0 \leq i \leq k-1$ and (21) is already shown for $i+1$. On $\{\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M\}$ we have, for infinitely many M (depending on ω),

$$Z(i) \geq \widehat{Y}(i;\tau) + \epsilon^{(M)}(i).$$

This means,

$$Z(i) \geq \widehat{Y}(i;\tau) \quad \text{on } \{\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M\},$$

as $\epsilon^{(M)}(i)$ tends to zero almost surely. However,

$$\{\bar{\tau}^{(N)}(i) = i\} \subset \{\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M\}.$$

Thus,

$$Z(i) \geq \widehat{Y}(i;\tau) \quad \text{on } \{\bar{\tau}^{(N)}(i) = i\}.$$

But this implies $\widetilde{\tau}(i) = i$ on $\{\bar{\tau}^{(N)}(i) = i\}$. Consequently, (21) holds on $\{\bar{\tau}^{(N)}(i) = i\}$.

On the other hand, $\{\bar{\tau}^{(N)}(i) > i\} \subset \{\widetilde{\tau}^{(M)}(i) > i \text{ for infinitely many } M\}$, and an analogous argument yields

$$Z(i) \leq \widehat{Y}(i;\tau) \quad \text{on } \{\bar{\tau}^{(N)}(i) > i\}.$$

Consequently, $\widehat{\tau}(i) > i$ and thus, by the induction hypothesis,

$$\bar{\tau}^{(N)}(i) = \bar{\tau}^{(N)}(i+1) \leq \widehat{\tau}(i+1) = \widehat{\tau}(i) \quad \text{on } \{\bar{\tau}^{(N)}(i) > i\}.$$

The induction hypothesis can be applied in the same way to show

$$\bar{\tau}^{(N)}(i) \geq \widetilde{\tau}(i) \quad \text{on } \{\bar{\tau}^{(N)}(i) > i\} \cap \{\widetilde{\tau}(i) > i\},$$

whereas this inequality is trivially satisfied on $\{\bar{\tau}^{(N)}(i) > i\} \cap \{\widetilde{\tau}(i) = i\}$. This completes the proof of (21). □

The next theorem completes the proof of assertion (ia).

Theorem 4.3.
$$|\widetilde{\tau}^{(N)}(i) - \overline{\tau}^{(N)}(i)| \to 0 \quad P-a.s.,$$

or equivalently,

$$P\left(\bigcap_{N\in\mathbb{N}}\bigcup_{M=N}^{\infty}\left\{\widetilde{\tau}^{(M)}(i) \neq \overline{\tau}^{(M)}(i)\right\}\right) = 0.$$

Proof. The statement is obvious for $i = k$. Suppose now $0 \le i \le k-1$ and that the statement is proved for $i+1$. Define,

$$A(N,i) = \bigcup_{M=N}^{\infty}\left\{\widetilde{\tau}^{(M)}(i) \neq \overline{\tau}^{(M)}(i)\right\}. \tag{30}$$

Clearly,
$$A(N,i) = B(N,i) \cup C(N,i) \cup D(N,i),$$

where

$$\begin{aligned}
B(N,i) &= \bigcup_{M=N}^{\infty}\left\{\widetilde{\tau}^{(M)}(i) = i\right\} \cap \left\{\overline{\tau}^{(M)}(i) > i\right\},\\
C(N,i) &= \bigcup_{M=N}^{\infty}\left\{\widetilde{\tau}^{(M)}(i) > i\right\} \cap \left\{\overline{\tau}^{(M)}(i) = i\right\},\\
D(N,i) &= \bigcup_{M=N}^{\infty}\left\{\widetilde{\tau}^{(M)}(i) > i\right\} \cap \left\{\overline{\tau}^{(M)}(i) > i\right\} \cap \left\{\widetilde{\tau}^{(M)}(i) \neq \overline{\tau}^{(M)}(i)\right\}.
\end{aligned}$$

Since the sets $B(N,i)$, $C(N,i)$, and $D(N,i)$ are decreasing in N, we have

$$\bigcap_{N\in\mathbb{N}} A(N,i) = \left(\bigcap_{N\in\mathbb{N}} B(N,i)\right) \cup \left(\bigcap_{N\in\mathbb{N}} C(N,i)\right) \cup \left(\bigcap_{N\in\mathbb{N}} D(N,i)\right).$$

We show, that the three sets on the right hand side are evanescent. Firstly, as $\overline{\tau}^{(M)}$ and $\widetilde{\tau}^{(M)}$ are consistent, it holds $D(N,i) \subset A(N,i+1)$. Hence, the intersection of the $D(N,i)$'s is a null set by the induction hypothesis. By the definition of $\overline{\tau}^{(M)}$ we have,

$$C(N,i) \subset \bigcup_{M=N}^{\infty}\left\{\widetilde{\tau}^{(M)}(i) > i\right\} \cap \left\{\widetilde{\tau}^{(K)}(i) > i \text{ for only finitely many } K\right\}.$$

Thus, the intersection of the $C(N,i)$'s is a null set. A similar argument applies for the intersection of the $B(N,i)$'s. ☐

Assertion (iia) follows from the next theorem.

Theorem 4.4. *Suppose that for all i, $0 \le i \le k$,*

$$\lim_{N\to\infty} \epsilon^{(N)}(i) = 0, \quad P-a.s.$$

Then, for all $0 \le i \le k$,

$$\lim_{N\to\infty} \left| Y(i;\widetilde{\tau}^{(N)}) - Y(i;\overline{\tau}^{(N)}) \right| = 0 \quad P-a.s.$$

and

$$\lim_{N\to\infty} \Big(Y(i;\widetilde{\tau}^{(N)}) - Y(i;\widetilde{\tau}) \Big)_{-} = 0, \quad P-a.s.$$

Remark 4.1. By the dominated convergence theorem the above convergences also hold in $L^1(P)$.

Proof. The first claim is easily derived from Theorem 4.3 and dominated convergence. The second one then follows from Proposition 3.8 and Theorem 4.2. □

Remark 4.2. Applying the first inequality in Proposition 3.8 we obtain,

$$\lim_{N\to\infty} \Big(Y(i;\widetilde{\tau}^{(N)}) - Y(i;\widehat{\tau}) \Big)_{+} = 0, \quad P-a.s.$$

Thus, convergence of $Y(i;\widetilde{\tau}^{(N)})$ to $Y(i;\widetilde{\tau})$ holds, whenever $\widetilde{\tau}(i) = \widehat{\tau}(i)$ for all $0 \le i \le k$.

4.2 Stability of the Algorithm: The Case $L = 1$

We are now going to explain how the stability result for the one-step improvement carries over to the algorithm in the case of one exercise right. We will make use of the following perturbed monotonicity result.

Proposition 4.5. *Suppose* (τ_N) *is a sequence of consistent stopping families and, for all* $0 \le i \le k$,

$$\lim_{N\to\infty} \left(Y(i;\tau_N) - Y(i;\tau) \right)_{-} = 0 \quad P-a.s.$$

Then, for all $0 \le i \le k$,

$$\lim_{N\to\infty} \left(Y(i;\widetilde{\tau}_N) - Y(i;\widetilde{\tau}) \right)_{-} = 0 \quad P-a.s.,$$

where

$$\widetilde{\tau}_N(i) := \inf\{ j : i \le j \le k,\ \widehat{Y}(j;\tau_N) \le Z(j) \}.$$

Remark 4.3. For a constant sequence $\tau_N = \sigma$ for all N, with σ being consistent, Proposition 4.5 states:

$$Y(i;\sigma) \ge Y(i;\tau) \quad \Longrightarrow \quad Y(i;\widetilde{\sigma}) \ge Y(i;\widetilde{\tau}).$$

Hence, the better the input stopping family, the better the improvement.

Proof. The statement will be proved by backward induction over i. The induction base $i = k$ is obvious. Suppose the statement is proved for some $1 \le i+1 \le k$.
We first note that by Remark 3.2,

$$\mathbf{1}_{\{\widetilde{\tau}(i)=i\}} \left(Y(i;\widetilde{\tau}_N) - Y(i;\widetilde{\tau}) \right)_{-} \le \left(Y(i;\widetilde{\tau}_N) - Z(i) \right)_{-} = 0. \tag{31}$$

We next show that the statement is true on the set $\{\widetilde{\tau}_M(i) = i$ for infinitely many $M\}$. For this we need the following preliminary consideration. By Jensen's inequality and the dominated convergence theorem, for all $p \geq i$ it holds,

$$\left(E^{\mathcal{F}_i}[Y(p;\tau_N)] - E^{\mathcal{F}_i}[Y(p;\tau)]\right)_- \leq E^{\mathcal{F}_i}\left[(Y(p;\tau_N) - Y(p;\tau))_-\right] \to 0.$$

Thus,

$$\lim_{N\to\infty}\left(\widehat{Y}(i;\tau_N) - \widehat{Y}(i;\tau)\right)_- = 0 \quad P-a.s., \tag{32}$$

since the max-operator is continuous with respect to the metric generated by the negative part. On $\{\widetilde{\tau}_M(i) = i$ for infinitely many $M\}$ we have for infinitely many M,

$$\widehat{Y}(i;\tau_M) \leq Z(i).$$

Since

$$\left(Z(i) - \widehat{Y}(i;\tau)\right)_- \leq \left(Z(i) - \widehat{Y}(i;\tau_M)\right)_- + \left(\widehat{Y}(i;\tau_M) - \widehat{Y}(i;\tau)\right)_-,$$

we may conclude from (32), that

$$Z(i) \geq \widehat{Y}(i;\tau) \quad \text{on } \{\widetilde{\tau}_M(i) = i \text{ for infinitely many } M\}.$$

Hence,

$$\{\widetilde{\tau}_M(i) = i \text{ for infinitely many } M\} \subset \{\widetilde{\tau}(i) = i\}.$$

On the latter set the statement was proved in (31).
It remains to verify the statement on

$$E(i) = \{\widetilde{\tau}_M(i) = i \text{ for only finitely many } M\} \cap \{\widetilde{\tau}(i) > i\}.$$

Define

$$N_0(i) = \mathbf{1}_{E(i)} \max\{N;\ \widetilde{\tau}_N(i) = i\} + 1,$$

and note that the process $N_0(i)$ is $\mathcal{F}_i$-adapted. Since

$$\widetilde{\tau}_N(i) > i \quad \text{on } \{N \geq N_0(i)\} \cap E(i),$$

it follows from the consistency of $\widetilde{\tau}_N$, the induction hypothesis, Jensen's inequality, and the dominated convergence theorem, that

$$\begin{aligned}
& \mathbf{1}_{\{N\geq N_0(i)\}\cap E(i)}\,(Y(i;\widetilde{\tau}_N) - Y(i;\widetilde{\tau}))_- \\
= \;& \mathbf{1}_{\{N\geq N_0(i)\}\cap E(i)}\left(E^{\mathcal{F}_i}[Y(i+1;\widetilde{\tau}_N)] - E^{\mathcal{F}_i}[Y(i+1;\widetilde{\tau})]\right)_- \\
\leq \;& E^{\mathcal{F}_i}\left[(Y(i+1;\widetilde{\tau}_N) - Y(i+1;\widetilde{\tau}))_-\right] \to 0.
\end{aligned}$$

□

For notational convenience we state the stability result of the algorithm for two improvement steps ($m = 2$) only. It is immediate, how this extends to higher iterations. We will also skip all subscripts, which are superfluous in the case of one exercise right. For instance, we write $\tau^{(1)}$ instead of $\tau^{(1)}_{1,1}$. First note that with $\tau = \tau^{(0)}$,

$$\begin{aligned}
\tau^{(1)}(i) &= \widetilde{\tau}(i), \\
\tau^{(2)}(i) &= \widetilde{\widetilde{\tau}}(i) = \inf\{j : i \leq j \leq k,\ \widehat{Y}(j;\widetilde{\tau}) \leq Z(j)\}.
\end{aligned}$$

Let us suppose that for $(N_1, N_2) \in \mathbb{N} \times \mathbb{N}$, sequences $\epsilon^{(N_1)}(i)$ and $\epsilon^{(N_1,N_2)}(i)$ are given such that for $0 \le i \le k$,

$$\lim_{N_1 \to \infty} \epsilon^{(N_1)}(i) = 0 \quad P - a.s.,$$

and, for $0 \le i \le k$ and $N_1 \in \mathbb{N}$,

$$\lim_{N_2 \to \infty} \epsilon^{(N_1,N_2)}(i) = 0 \quad P - a.s.$$

We then define

$$\begin{aligned} \widetilde{\tau}^{(N_1)}(i) &:= \inf\{j : i \le j \le k,\ \widehat{Y}(j;\tau) + \epsilon^{(N_1)}(j) \le Z(j)\}, \\ \widetilde{\widetilde{\tau}}^{(N_1)}(i) &:= \inf\{j : i \le j \le k,\ \widehat{Y}(j;\widetilde{\tau}^{(N_1)}) \le Z(j)\}, \\ \widetilde{\widetilde{\tau}}^{(N_1,N_2)}(i) &:= \inf\{j : i \le j \le k,\ \widehat{Y}(j;\widetilde{\tau}^{(N_1)}) + \epsilon^{(N_1,N_2)}(j) \le Z(j)\}. \end{aligned}$$

Theorem 4.4 now yields

$$\lim_{N_1 \to \infty} \left(Y(i;\widetilde{\tau}^{(N_1)}) - Y(i;\widetilde{\tau}) \right)_- = 0 \quad P - a.s., \tag{33}$$

$$\lim_{N_2 \to \infty} \left(Y(i;\widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y(i;\widetilde{\widetilde{\tau}}^{(N_1)}) \right)_- = 0 \quad P - a.s. \tag{34}$$

In view of (33) we obtain by Proposition 4.5,

$$\lim_{N_1 \to \infty} \left(Y(i;\widetilde{\widetilde{\tau}}^{(N_1)}) - Y(i;\widetilde{\widetilde{\tau}}) \right)_- = 0 \quad P - a.s. \tag{35}$$

From

$$\begin{aligned} & \left(Y(i;\widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y^{(2)}(i) \right)_- \\ \le\ & \left(Y(i;\widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y(i;\widetilde{\widetilde{\tau}}^{(N_1)}) \right)_- + \left(Y(i;\widetilde{\widetilde{\tau}}^{(N_1)}) - Y(i;\widetilde{\widetilde{\tau}}) \right)_- , \end{aligned}$$

we then obtain,

Theorem 4.6. *For all $0 \le i \le k$,*

$$\lim_{N_1 \to \infty} \lim_{N_2 \to \infty} \left(Y(i;\widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y^{(2)}(i) \right)_- = 0$$

P-almost surely and in $L^1(P)$.

The generalization of this result to m iteration steps may be put into words as follows:

The shortfall of the investor's expected gain corresponding to m perturbed steps of the algorithm below the expected gain corresponding to m theoretical steps converges to zero.

4.3 Stability under Several Exercise Rights

The stability issue becomes more involved under several exercise rights. One reason is that we cannot expect to have the inequality

$$Y_L(i; \bar{\sigma}_1, \ldots, \bar{\sigma}_L) \geq Y_L(i; \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_L),$$

where $\bar{\sigma}_1, \ldots, \bar{\sigma}_L$ are arbitrary improvers of $\sigma_1, \ldots, \sigma_L$, but only the inequalities stated in Theorem 3.11. The latter theorem suggest that we must confine ourselves with the following stability result for the one-step improvement under several rights.

Theorem 4.7. *Suppose $\sigma_1, \ldots, \sigma_D$ are consistent stopping families. Define for $1 \leq L \leq D$,*

$$\begin{aligned}\widetilde{\sigma}_L^{(N)}(i) &= \inf\{j \geq i;\ \ Z(j) + E^{\mathcal{F}_j} Y_{L-1}(j+\delta; \sigma_1, \ldots, \sigma_{L-1}) \\ &\quad \geq \widehat{Y}_L(j; \sigma_1, \ldots, \sigma_L) + \epsilon_L^{(N)}(j)\},\end{aligned}$$

where for all $1 \leq L \leq D$, $0 \leq i \leq k$,

$$\lim_{N\to\infty} \epsilon_L^{(N)}(i) = 0 \quad P-a.s.$$

Then, there are sequences of improver $\bar{\sigma}_1^{(N)}, \ldots, \bar{\sigma}_D^{(N)}$ of $\sigma_1, \ldots, \sigma_D$ such that, for all $1 \leq L \leq D$,

$$\lim_{N\to\infty} |\widetilde{\sigma}_L^{(N)}(i) - \bar{\sigma}_L^{(N)}(i)| = 0.$$

Moreover,

$$\lim_{N\to\infty} \left| Y_L(i; \widetilde{\sigma}_1^{(N)}, \ldots, \widetilde{\sigma}_L^{(N)}) - Y_L(i; \bar{\sigma}_1^{(N)}, \ldots, \bar{\sigma}_L^{(N)}) \right| = 0 \quad P-a.s.,$$

and

$$\lim_{N\to\infty} \Big(Y_L(i; \widetilde{\sigma}_1^{(N)}, \ldots, \widetilde{\sigma}_L^{(N)}) - Y_L(i; \sigma_1, \ldots, \sigma_{L-1}, \widetilde{\sigma}_L) \Big)_- = 0 \quad P-a.s.$$

Proof. In view of Lemma 3.9 and Theorem 3.11, the theorem follows by straightforward reduction to the case of one exercise right. □

A more satisfactory result can be derived for the y_L-algorithm under an additional assumption.

Theorem 4.8. *Under the assumptions and with the notations of the previous Theorem, suppose additionally, for $1 \leq i \leq k-1$, $1 \leq L \leq D$,*

$$P\left(Z_L(i; \sigma_1, \ldots, \sigma_{L-1}) = \max_{p \geq i+1} E^{\mathcal{F}_i}\left[Z_L(\sigma_L(p); \sigma_1, \ldots, \sigma_{L-1}) \right] \right) = 0. \tag{36}$$

Then, for $0 \leq i \leq k$, $1 \leq L \leq D$,

$$\lim_{N\to\infty} \left| y_L(i; \widetilde{\sigma}_1^{(N)}, \ldots, \widetilde{\sigma}_L^{(N)}) - y_L(i; \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_L) \right| = 0 \quad P-a.s.$$

Proof. Assumption (36) guarantees $\widetilde{\sigma}_L(i) = \widehat{\sigma}_L(i)$ for $1 \le i \le k$, $1 \le L \le D$. Thus, $\bar{\sigma}_L^{(N)}(i) = \widetilde{\sigma}_L(i)$ for $1 \le i \le k$, $1 \le L \le D$, and $N \in \mathbb{N}$ by the definition of an improver. (Here $\bar{\sigma}_L^{(N)}(i)$ denotes the sequence from the previous Theorem). Theorem 4.7 now yields for $1 \le i \le k$, $1 \le L \le D$,

$$\lim_{N\to\infty} \widetilde{\sigma}_L^{(N)}(i) = \widetilde{\sigma}_L(i) \quad P - a.s.$$

Hence, the assertion follows from the definition of y_L and an application of the dominated convergence theorem. (Recall, the definition of y_L does not involve the 0-value of the stopping families.) $\square$

We now discuss stability of the y_L-algorithm. Again we demonstrate the stability of the multiple stopping algorithm only for two steps ($m = 2$). Suppose we are given consistent starting families $\sigma_1, \ldots, \sigma_D$ (with suppressed superscript 0 in the notation of the algorithm). Recall that

$$\begin{aligned} \sigma_L^{(1)}(i) &:= \widetilde{\sigma}_L(i), \\ \sigma_L^{(2)}(i) &:= \widetilde{\sigma}_L^{(1)}(i) = \widetilde{\widetilde{\sigma}}_L(i). \end{aligned}$$

We shall suppose that (36) holds as well as its analogue for $\widetilde{\sigma}_L$, i.e. for $1 \le i \le k-1$, $1 \le L \le D$,

$$P\left(Z_L(i; \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{L-1}) = \max_{p \ge i+1} E^{\mathcal{F}_i}\left[Z_L(\widetilde{\sigma}_L(p); \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{L-1}) \right] \right) = 0. \tag{37}$$

We next consider the perturbed versions,

$$\begin{aligned} \widetilde{\sigma}_L^{(N_1)}(i) &= \inf\{ j \ge i;\ \ Z(j) + E^{\mathcal{F}_j} Y_{L-1}(j+\delta; \sigma_1, \ldots, \sigma_{L-1}) \\ &\qquad \ge \widehat{Y}_L(j; \sigma_1, \ldots, \sigma_L) + \epsilon_L^{(N_1)}(j) \}, \\ \widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}(i) &= \inf\{ j \ge i;\ \ Z(j) + E^{\mathcal{F}_j} Y_{L-1}(j+\delta; \widetilde{\sigma}_1^{(N_1)}, \ldots, \widetilde{\sigma}_{L-1}^{(N_1)}) \\ &\qquad \ge \widehat{Y}_L(j; \widetilde{\sigma}_1^{(N_1)}, \ldots, \widetilde{\sigma}_L^{(N_1)}) + \epsilon_L^{(N_1,N_2)}(j) \}, \end{aligned}$$

with

$$\begin{aligned} \lim_{N_1\to\infty} \epsilon_L^{(N_1)}(i) &= 0 \quad P - a.s., \\ \lim_{N_2\to\infty} \epsilon_L^{(N_1,N_2)}(i) &= 0 \quad P - a.s. \end{aligned}$$

We denote by $\widetilde{\widetilde{\sigma}}_L^{(N_1)}$ the theoretical $^\sim$-improvement of $\widetilde{\sigma}_L^{(N_1)}$. The additional assumption (36) now ensures that for $1 \le i \le k$

$$\lim_{N\to\infty} \widetilde{\sigma}_L^{(N)}(i) = \widetilde{\sigma}_L(i) \quad P - a.s.$$

(See the proof of Theorem 4.8.) Thus, we can write (applying Lemma 3.9),

$$\begin{aligned} \widetilde{\widetilde{\sigma}}_L^{(N_1)}(i) &= \inf\{ j \ge i;\ \ Z_L(j; \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{L-1}) \\ &\qquad \ge \max_{p \ge j+1} E^{\mathcal{F}_j} Z_L(\widetilde{\sigma}_L(p); \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{L-1}) + \widetilde{\epsilon}_L^{(N_1)}(i) \}, \end{aligned}$$

where

$$\lim_{N_1\to\infty} \widetilde{\epsilon}_L^{(N_1)}(i) = 0 \quad P-a.s.$$

We now define $\bar{\widetilde{\sigma}}_L^{(N_1)}(k) = k$, and

$$\begin{aligned}
\bar{\widetilde{\sigma}}_L^{(N_1)}(i) = i \quad &\iff \quad (\widetilde{\widetilde{\sigma}}_L^{(M)}(i) > i \text{ for only finitely many } M) \\
&\qquad \vee (\widetilde{\widetilde{\sigma}}_L^{(M)}(i) = i \text{ for infinitely many } M \text{ and } \widetilde{\widetilde{\sigma}}_L^{(N_1)}(i) = i), \\
\bar{\widetilde{\sigma}}_L^{(N_1)}(i) \neq i \quad &\implies \quad \bar{\widetilde{\sigma}}_L^{(N_1)}(i) = \bar{\widetilde{\sigma}}_L^{(N_1)}(i+1).
\end{aligned}$$

By Theorem 4.3, we have for all $1 \le i \le k$, $1 \le L \le D$,

$$\lim_{N_1\to\infty} |\widetilde{\widetilde{\sigma}}_L^{(N_1)}(i) - \bar{\widetilde{\sigma}}_L^{(N_1)}(i)| = 0.$$

However, assumption (37) implies that the improvers $\bar{\widetilde{\sigma}}_L^{(N_1)}(i)$ coincide with $\widetilde{\widetilde{\sigma}}_L(i)$ for $1 \le i \le k$. Hence, the dominated convergence theorem yields

$$\lim_{N_1\to\infty} \left| y_L(i; \widetilde{\widetilde{\sigma}}_1^{(N_1)}, \dots, \widetilde{\widetilde{\sigma}}_L^{(N_1)}) - y_L(i; \widetilde{\widetilde{\sigma}}_1, \dots \widetilde{\widetilde{\sigma}}_L) \right| = 0; \quad P-a.s. \tag{38}$$

On the other hand, a direct application of Theorem 4.8 gives

$$\lim_{N_2\to\infty} \left| y_L(i; \widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)}, \dots, \widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - y_L(i; \widetilde{\widetilde{\sigma}}_1^{(N_1)}, \dots \widetilde{\widetilde{\sigma}}_L^{(N_1)}) \right| = 0; \quad P-a.s. \tag{39}$$

The discussion is summarized in the following theorem.

Theorem 4.9. *Suppose (36)–(37). Then, for all* $0 \le i \le k$, $1 \le L \le D$,

$$\lim_{N_1\to\infty} \lim_{N_2\to\infty} \left| y_L(i; \widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)}, \dots, \widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - y_L^{(2)}(i) \right| = 0$$

P-almost surely and in $L^1(P)$.

The straightforward generalization to higher order iterations is left to the reader.

Remark 4.4. (i) Stability of the Y_L-algorithm can be proven along the same lines, if (36)–(37) also hold for $i = 0$. We emphasize however, that this additional assumption can always be violated by a bad choice of the constant $Z(0)$. This is, why we refrained from this additional assumption and presented the stability for the y_L-algorithm.

(ii) Assumptions (36)–(37) can be replaced by the weaker condition that the limits

$$\lim_{N_1\to\infty} \widetilde{\sigma}_L^{(N_1)}(i), \qquad \lim_{N_1\to\infty} \lim_{N_2\to\infty} \widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}(i)$$

exist for $1 \le i \le k-1$, $1 \le L \le D$. Then $y_L^{(2)}(i)$ in Theorem 4.9 must be replaced by some theoretical y_L-two-step improvement of $(\sigma_1, \dots, \sigma_l)$. The analogous result holds for the Y_L-algorithm, too, when the limits also exist for $i = 0$.

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References

Andersen, L. (1999) A Simple Approach to the Pricing of Bermudan Swaptions in a Multifactor LIBOR Market Model. *Journal of Computational Finance*, **3**, 5-32.

Bender, C., Schoenmakers, J. (2006) Addendum to 'An Iterative Method for Multiple Stopping: Convergence and Stability'. `http://www.wias-berlin.de/people/schoenma/`

Bertsekas, D. P. (2000) *Dynamic Programming and Optimal Control*. 2nd ed. Athena Scientific: Belmont, Mass.

Broadie, M., Glasserman, P. (2004) A stochastic mesh method for pricing high-dimensional American options. *Journal of Computational Finance,* **7**, 4, 35-72.

Broadie, M., Glasserman, P., Ha, Z. (2000) Pricing American Options by Simulations Using a Stochastic Mesh with Optimized Weights. *Probabilistic Constrained Optimization: Methodology and Applications*, S. Uryasev (ed.), Kluwer: Norwell, Mass., 32-50.

Carmona, R. and Dayanik, S. (2004) Optimal Multiple-Stopping of Linear Diffusions and Swing Options. Preprint.

Carmona, R., Touzi, N. (2006) Optimal Multiple-Stopping and Valuation of Swing Options. *Math. Finance*, fortcoming.

Haugh, M. B., Kogan, L. (2004) Pricing American Options: A Duality Approach. *Operations Research*, **52**, 258-270.

Ibáñez, A. (2004) Valuation by Simulation of Contingent Claims with Multiple Early Exercise Opportunities. *Math. Finance*, **14**, 223-248.

Ibáñez, A., Zapatero, F. (2004) Valuation of American Options through Computation of the Optimal Exercise Frontier. *J. Financial Quant. Anal.*, **39**, 2, 253-276.

Jaillet, P., Ronn, E. I., Tompaidis, S. (2004) Valuation of Commodity Based Swing Options. *Management Science*, **50**, 909-921.

Jamshidian, F. (2003) Minimax Optimality of American and Bermudan Claims and their Monte Carlo Valuation. Working Paper.

Kolodko, A., Schoenmakers, J. (2006) Iterative Construction of the Optimal Bermudan Stopping Time. *Finance Stoch.*, **10**, 27–49.

Kushner, H. J., Dupuis, P. (2001) *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd ed., New York: Springer.

Longstaff, F. A., Schwartz, R. S. (2001) Valuing American Options by Simulation: A Simple Least-Square Approach. *Review of Financial Studies*, **14**, 113-147.

Meinshausen, N., Hambly, B. M. (2004) Monte Carlo Methods for the Valuation of Multiple-Exercise Options. *Math. Finance*, **14**, 557-583.

Milstein, G.N., Reiß, O. and Schoenmakers, J. (2004) A New Monte Carlo Method for American Options. *Int. J. of Theoretical and Applied Finance*, **7**, 5, 591-614.

Neveu, J. (1975) *Discrete Parameter Martingales*. North-Holland: Amsterdam.

Puterman, M. L. (1994) *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons: New York.

Rogers, L. C. G. (2002) Monte Carlo Valuation of American Options. *Math. Finance*, **12**, 271-286.

Schoenmakers, J. (2005) *Robust Libor Modelling and Pricing of Derivative Products*. Chapman & Hall - CRC Press: Boca Raton, FL.