1. If $X_n(t)$ is a sequence of random processes such that for any finite set of times, $\big(X_n(t_1),\dots,X_n(t_M)\big)$ satisfies the large-deviation principle in $\mathcal{Z}^M$ with good rate functional $I_{t_1,\dots,t_M}$, then by Dawson-Gärtner, the path X_n satisfies the large-deviation principle in $\mathcal{Z}^{[0,T]}$ (equipped with the product topology) with good rate fuctional $I(x) = \sup_{0<t_1<\dots<t_M<T} I_{t_1,\dots,t_M}(x_{t_1},\dots x_{t_M})$. This exercise shows how to argue 'in the other direction'.

   Let X_n be a sequence of random processes and fix a finite number of times $0 < t_1 < \dots < t_M < T$. *(In the following, the answer might be "no, because, ...".)*

   (a) If X_n satisfies the large-deviation principle in $\mathcal{Z}^{[0,T]}$ (with product topology) with good rate functional $\mathcal{I} : \mathcal{Z}^{[0,T]} \to [0,\infty]$, can you show that $\big(X_n(t_1),\dots,X_n(t_M)\big)$ satisfies the large-deviation principle in $\mathcal{Z}^M$?

   (b) If X_n satisfies the large-deviation principle in $C([0,T];\mathcal{Z})$ (with supremum norm L^∞) with good rate functional $\mathcal{I} : C([0,T];\mathcal{Z}) \to [0,\infty]$, can you show that $\big(X_n(t_1),\dots,X_n(t_M)\big)$ satisfies the large-deviation principle in $\mathcal{Z}^M$?

   (c) If X_n satisfies the large-deviation principle in $L^1([0,T];\mathcal{Z})$ (with L^1- norm) with good rate functional $\mathcal{I} : L^1([0,T];\mathcal{Z}) \to [0,\infty]$, can you show that $\big(X_n(t_1),\dots,X_n(t_M)\big)$ satisfies the large-deviation principle in $\mathcal{Z}^M$?

2. Let $B(t)$ be a standard Brownian motion and $X_n(t)$ be the solution of the SDE

$$dX_n(t) = b(X_n(t))\,dt + \frac{1}{\sqrt{n}} dB(t). \qquad \text{(SDE)}$$

   Let us assume that $\phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. In that case it is known that (SDE) has a 'strong solution' and that the mapping $F : n^{-1/2}B \mapsto X_n$ is continuous from $C_{0,\cdot}([0,T])$ to $C_{0,\cdot}([0,T])$. More precisely, $f = F(g)$ is the unique solution f to

$$f(t) = \int_0^t \phi(f(s))\,ds + g(t), \qquad t \in [0,T].$$

   Show that the sequences of processes X_n satisfies a large-deviation principle in $C_{0,\cdot}([0,T])$ with a good rate functional.
   Hint: start with Schilder's Theorem and exploit the continuity of F.
   Validity check: what is the limit of X_n as $n \to \infty$ and does it minimise the rate functional?

   Remark: the more general "Freidlin-Wentzell Theory" also holds for SDE's of the form $dX_n(t) = \phi(X_n(t))\,dt + \sqrt{\sigma(X_n(t))/n}\,dB(t)$ in d dimensions, but this requires more work as the mapping F may fail to be continuous.