1. Let $W_t$ be a $d$-dimensional Brownian motion with $W_0 = 0$ a.s.. Let $X^n_t := \frac{1}{\sqrt{n}} W_t$.

(a) Write down, for arbitrary times $0 < t_1 < t_2$, $x \in \mathbb{R}^d$, measurable set $A_2 \subset \mathbb{R}^d$, and fixed $n \in \mathbb{N}$,

$$
\nu^n_{t_1,t_2}(A_2|x) := P(X^n_{t_2} \in A_2 \mid X^n_{t_1} = x).
$$

(b) What is the large-deviation rate functional of $\nu^n_{t_1,t_2}(|x)$ as $n \to \infty$ (with speed $n$) and fixed $0 < t_1 < t_2, x \in \mathbb{R}^d$?

(c) Use the Markov property to write down, for $0 < t_1 < \ldots t_M, M \in \mathbb{N}$, and measurable sets $A_1, \ldots A_M \subset \mathbb{R}^d$, and fixed $n \in \mathbb{N}$:

$$
\mu^n_{t_1,\ldots t_M}(A_1 \times \ldots \times A_2) = P(X^n_{t_1} \in A_1, \ldots, X^n_{t_M} \in A_M).
$$

(\text{It could simplify notation to use integration variables } x_1, \ldots, x_M \text{ and to set } x_0 := 0 \text{ and } t_0 := 0).

(d) Use the Laplace principle to prove the large-deviation principle of $\mu^n_{t_1,\ldots t_M}$.

(\text{In this finite-dimensional setting it is easy to see that the rate function is indeed good...})

(e) Use the Dawson-Gärtner Theorem to prove the large-deviation principle of the path measure

$$
\mu^n(B) := P((X^n_t)_{t \in [0,\infty)} \in B),
$$

for $B \in \mathbb{R}^{d|0,\infty}$, equipped with the product topology.

(f) Validity check: what is the deterministic limit of $X^n$, and is the rate functional 0 in this limit?

In fact, it can be shown that the supremum over times is a limit where $M \to \infty$ and $\sup_{i \leq M} |t_i - t_{i-1}| \to 0$.

(g) For a smooth path $x : [0,\infty) \to \mathbb{R}^d$, use $\frac{x_{t_i} - x_{t_{i-1}}}{t_i - t_{i-1}} \approx \dot{x}(t_i)$ and the Riemann integral to formally simplify the rate functional.

Compare this rate functional to “Schilder’s Theorem”, see for example König, Satz 2.3.1, or Dembo & Zeitouni, Th. 5.2.3.

\footnote{Probabilistic literature usually takes Brownian motion with generator $f \mapsto \frac{1}{2} \Delta f$, whereas in analytic literature $f \mapsto \Delta f$ is more common. So if you are a factor 2 off in this exercise, it’s probably due to that.}