Theorem 0.1 (Varadhan's Lemma on arbitrary sets). *Assume:*

- $A \subset \mathcal{X}$ *is a measurable subset of a metric or Hausdorff topological vector space,*
- $(\mu_n)_{n\in\mathbb{N}}$ *satisfies a large-deviation principle in* $\mathcal{X}$ *with good rate functional* $I : \mathcal{X} \to [0, \infty]$,
- $f : \mathcal{X} \to \mathbb{R}$ *is continuous,*
- f *is either bounded from above, or the following tail condition holds:*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \int_{\{x\in\mathcal{X}: f(x)\geq M\}} e^{nf(x)}\, \mu_n(dx) = -\infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log \int_A e^{nf(x)}\, \mu_n(dx) = \sup_{x\in A}[f(x) - I(x)].$$

Proof. Exactly the same as Varadhan's Lemma on the full space $\mathcal{X}$, which does not use $\mu_n(\mathcal{X}) = 1$ at all! □

We can now use this to prove:

Proposition 0.2 (Exercise 1: Large deviations of a tilted measure). *Let* $(\mu_n)_n$ *satisfy a large-deviation principle in a metric space* $\mathcal{X}$ *with good rate functional* $I : \mathcal{X} \to [0, \infty]$. *Given* $f \in C_b(\mathcal{X})$, *define the tilted measure, for any measurable set* $A \subset \mathcal{X}$:

$$\nu_n(A) := \frac{1}{Z_n} \int_A e^{nf(x)}\, \mu_n(dx), \qquad\qquad Z_n := \int_{\mathcal{X}} e^{nf(x)}\, \mu_n(dx).$$

Then $(\nu_n)_n$ *satisfies a large-deviation principle with rate functional*

$$J(x) := I(x) - f(x) - \inf[I - f].$$

Proof. Take any open set $U \subset \mathcal{X}$. We don't need to check the tail condition in Varadhan's lemma since f is bounded. Hence by the above Varadhan Lemma, restricted to the set U:

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(U) &= \liminf_{n\to\infty} \frac{1}{n} \log \int_U e^{nf(x)}\, \mu_n(dx) - \frac{1}{n} \log \int_{\mathcal{X}} e^{nf(x)}\, \mu_n(dx) \\
&\geq \liminf_{n\to\infty} \Big[\frac{1}{n} \log \int_U e^{nf(x)}\, \mu_n(dx)\Big] + \liminf_{n\to\infty} \Big[-\frac{1}{n} \log \int_{\mathcal{X}} e^{nf(x)}\, \mu_n(dx)\Big] \\
&= \sup_{x\in U} \big[I(x) - F(x)\big] - \sup_{x\in\mathcal{X}} \big[I(x) - F(x)\big] = -\inf_{x\in U} J(x).
\end{aligned}$$

The same argument can be used for the upper bound. □

Remark 0.3. *Note that* Z_n *acts as a normalisation factor to make sure* ν_n *is a probability measure. This factor yields* $-\inf[I - f]$ *in the rate functional, which is also a normalisation to make sure that* $\inf J = 0$*...!*

Lemma 0.4 (Laplace principle on $\mathbb{R}$). *For any measurable set* $A \subset \mathbb{R}$ *and measurable function* $g : \mathbb{R} \to \mathbb{R}$ *with* $\int_{\mathbb{R}} e^{-g(x)}\, dx < \infty$,

$$\lim_{n\to\infty} \tfrac{1}{n} \log \int_A e^{-ng(x)}\, dx = -\operatorname*{essinf}_{x\in A} g(x).$$

Definition 0.5. *A sequence* $(\mu_n)_n$ *satisfies a large-deviation principle in a topological space* $\mathcal{X}$ with speed γ_n *and rate functional* $I : \mathcal{X} \to [0, \infty]$ *whenever:*

- $\lim_{n\to\infty} \gamma_n = \infty$,

- $\liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(U) \geq -\inf_U I$ *for all open* $U \subset \mathcal{X}$,
- $\liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(G) \leq -\inf_G I$ *for all closed* $G \subset \mathcal{X}$,
- I *is lower semicontinuous.*

Proposition 0.6. *Let* $\mu_n(dx) := \sqrt{\frac{\log n}{\pi}} n^{-x^2}\, dx$. *Then*

(a) μ_n *satisfies a large-deviation principle in* $\mathbb{R}$ *with speed* $\log n$ *and rate functional* $I(x) = x^2$,

(b) μ_n *satisfies a large-deviation principle in* $\mathbb{R}$ *with speed* n *and rate functional*

$$\tilde{I} \equiv 0. \tag{1}$$

(c) μ_n *satisfies a large-deviation principle in* $\mathbb{R}$ *with speed* $\log\log(n)$ *and rate functional*

$$\hat{I}(x) = \begin{cases} 0, & x = 0, \\ \infty, & x \neq 0. \end{cases} \tag{2}$$

Proof. *(a)* By the Laplace principle, for any open or closed $A \subset \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mu_n(A) = \lim_{n\to\infty} \frac{1}{2\log n} \log\left(\frac{\log n}{\pi}\right) + \frac{1}{\log n} \int_A e^{-(\log n)x^2}\, dx = -\inf_A x^2.$$

(b) and *(c)* are clear from *(a)*. □

Remark 0.7. *Both exercises show that Varadhan/Laplace becomes especially simple for probabilities with Lebesgue densities!*

Remark 0.8. *Trivial large-deviation rate functionals of the form* (2) *or* (1) *usually mean that you're not using the right scaling (=large-deviation speed).*