Let \((X_n)_n\) be a discrete-time Markov chain on a finite space \(\mathcal{X}\), with positive transition probability \((P_{xy})_{x,y \in \mathcal{X} \times \mathcal{X}}\), and define:

\[
L^n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{X_i}, \quad \text{(empirical/occupation measure)}
\]

\[
L_{2,n} := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{(X_i, X_{i+1})}, \quad \text{(pair empirical measure)}
\]

\[
L_{2,n,\text{per}} := \frac{1}{n} \sum_{i=1}^{n-1} \mathbb{1}_{(X_i, X_{i+1})} + \frac{1}{n} \mathbb{1}_{(X_n, X_1)}. \quad \text{(periodic pair empirical measure)}
\]

1. Use Stirling’s formula to formally calculate, for \(\nu^n \to \nu\),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(L_{2,n,\text{per}} = \nu^n).
\]

2. Use the fact that the pair empirical measure \(L_{2,n}\) satisfies a large-deviation principle to show that the occupation measure \(L^n\) satisfies a large-deviation principle in \(\mathcal{M}_1(\mathcal{X})\), with some rate functional \(J\). Is \(J\) good?

3. In fact, both results are also true when the transition matrix is not necessarily positive, only the rate functional must be slightly changed. Use the fact that occupation measures \(L^n\) on a finite space satisfy a large-deviation principle with some good rate functional \(J\) to show that the pair empirical measure \(L_{2,n}\) satisfies a large-deviation principle.