1. Prove the following Laplace principle

$$\lim_{n \to \infty} \frac{1}{n} \log (e^{-an} + e^{-bn}) = -\min\{a, b\} \quad \text{for any } a, b \in [0, \infty).$$  \hspace{1cm} (1)

(It’s a very nice simple trick, but don’t spend too much time on it if you don’t see it ;-) )

Let $\mathcal{X}$ be a topological space (with its Borel $\sigma$-algebra).

2. Let $(\mu_n)_n$ satisfy an LDP with rate functional $I : \mathcal{X} \to [0, \infty]$. Show that $\inf_{\mathcal{X}} I = 0$, and argue that this may not hold when $(\mu_n)_n$ satisfies a weak LDP only.

3. Let $(\mu_n)_n$ satisfy a weak LDP with rate functional $I : \mathcal{X} \to [0, \infty]$, and assume that $(\mu_n)_n$ is exponentially tight. We are going to show that $(\mu_n)_n$ satisfies an LDP.

   (a) Take any closed set $G \subset \mathcal{X}$ and let $K_\eta$ be compact sets for which the sequence is exponentially tight. Write down the large-deviation upper bounds for the sets $G \cap K_\eta$ and $K_\eta^c$.

   (b) Derive upper bounds on $\mu_n(G \cap K_\eta)$ and $\mu_n(K_\eta)$ for $n$ sufficiently large (this will involve an arbitrary $\epsilon > 0$).

   (c) Use this together with the Laplace principle (1) to show that

   $$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(G) \leq -\min\{\inf_{G \cap K_\eta} I, \eta\}.$$  

   (d) Make a smart choice for $\eta$ to conclude the LDP upper bound for $G$.  