A functional $I : \mathcal{X} \to [0, \infty]$ is called *(sequentially) lower semicontinuous* if for all convergent sequences $x_n \to x$:

$$\liminf_{n\to\infty} I(x_n) \geq I(x).$$

A functional $I : \mathcal{X} \to [0, \infty]$ is called *(topologically) lower semicontinuous* if the sublevel sets $\{x \in \mathcal{X} : I(x) \leq C\}$ are closed. We shall mostly work with metric spaces, in which case the two concepts coincide.

1. Given $\theta > 0$, let

$$I(x) := \begin{cases} x \log \frac{x}{\theta} - x + \theta, & x > 0, \\ b, & x = 0, \\ \infty, & x < 0. \end{cases}$$

   For which values of b is I lower semicontinuous?

2. Assume that the random variables X_n satisfy a large deviation principle with rate functional $\mathcal{I} : \mathcal{X} \to [0, \infty]$. Define $Y_n := \phi(X_n)$ where $\phi : \mathcal{X} \to \mathcal{Y}$ is a continuous mapping between the two topological spaces $\mathcal{X}$ and $\mathcal{Y}$. Show the large deviation lower and upper bounds for Y_n for some candidate rate functional $\hat{I} : \mathcal{Y} \to [0, \infty]$.
   ($\hat{\mathcal{I}}$ is only a candidate for a large deviation rate functional, since it may fail to be lower semicontinuous).

3. Let $X_1, X_2, \dots$ be independent, identically distributed random variables, and set $S_n := \sum_{i=1}^n X_i$. Can you derive $I(x) := -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{1}{n} S_n \geq x)$ in the case where:

   a.) $X_1 \sim \text{Poisson}(\theta)$;

   b.) $X_1 \sim \text{Exponential}(\theta)$;

   c.) $X_1 \sim \text{Normal}(0, \sigma^2)$.