Truncated Gramians for Bilinear Systems and their Advantages in Model Order Reduction

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Abstract In this paper, we discuss truncated Gramians (TGrams) for bilinear control systems and their relations to Lyapunov equations. We show how TGrams relate to input and output energy functionals, and we also present interpretations of controllability and observability of the bilinear systems in terms of these TGrams. These studies allow us to determine those states that are less important for the system dynamics via an appropriate transformation based on the TGrams. Furthermore, we discuss advantages of the TGrams over the Gramians for bilinear systems as proposed in [1]. We illustrate the efficiency of the TGrams in the framework of model order reduction via a couple of examples, and compare to the approach based on the full Gramians for bilinear systems.

1 Introduction

Direct numerical simulations are one of the conventional methods to study physical phenomena of dynamical systems. However, extracting all the complex system dynamics generally leads to large state-space dynamical systems, whose direct simulations are inefficient and involve a huge computational burden. Hence, there is a need to consider *model order reduction* (MOR), aiming to replace these large-scale dynamical systems by systems of much smaller state dimension. MOR for linear systems has been investigated intensively in recent years and is widely used in numerous applications; see, e.g., [2, 8, 23]. In this work, we consider MOR for bilinear control systems, which can be considered as a bridge between linear and nonlinear systems and are of the form:

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$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum\nolimits_{k=1}^{m} N^{(k)} x(t) u_k(t) + Bu(t), \\ y(t) &= Cx(t), \quad x(0) = 0, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors of the system, respectively. The numbers m and p represent the quantity of inputs and outputs. All system matrices are of appropriate dimensions. The applications of bilinear systems can be seen in various fields [11, 18, 21]. Further, the applicability of the systems (1) in MOR for stochastic control problems is studied in [7, 17] and for MOR of a certain class of linear parametric systems in [4]. Our goal is to construct another low-dimensional bilinear system

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \sum\nolimits_{k=1}^{m} \hat{N}^{(k)} \hat{x}(t) u_k(t) + \hat{B}u(t), \\ \hat{y}(t) &= \hat{C}\hat{x}(t), \quad \hat{x}(0) = 0, \end{aligned} \tag{2}$$

where $\hat{A}, \hat{N}^{(k)} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times m}$ and $\hat{C} \in \mathbb{R}^{p\times r}$ with $r \ll n$, ensuring $y \approx \hat{y}$ for all admissible inputs $u \in L_2[0,\infty)$ with components u_k, $k = 1,\ldots,m$. Analogous to linear systems, we aim to obtain the reduced matrices via projection. For this, we construct two projection matrices $V, W \in \mathbb{R}^{n\times r}$ such that $W^T V$ is invertible, which allow us to determine the reduced matrices as:

$$\begin{aligned} &\hat{A} = (W^T V)^{-1} W^T AV, && \hat{N}^{(k)} = (W^T V)^{-1} W^T N^{(k)} V, \text{ for } k \in \{1,\ldots,m\}, \\ &\hat{B} = (W^T V)^{-1} W^T B \quad \text{and} && \hat{C} = CV. \end{aligned}$$

Clearly, it can be seen that the quality of the reduced system (2) depends on the choice of the projection matrices. Several methods for linear systems have been extended to bilinear systems such as balanced truncation [7] and interpolation-based MOR [3, 5, 10, 13]. In this work, we mainly focus on a balanced truncation based MOR technique for bilinear systems. Balanced truncation for linear systems, $N_k = 0$ in (1), has been studied in, e.g., [2, 19], and relies on controllability and observability Gramians of the system. Later on, the balancing concept for general nonlinear systems has been extended in a series of papers; see, e.g., [14, 16, 22], where a new notion of controllability and observability energy functionals has been introduced. Although theoretically the balancing concept for nonlinear systems is appealing, it is seldom applicable from the computational perspective. This is due to the fact that the energy functionals are solutions of nonlinear Hamilton-Jacobi equations, which are extremely expensive to solve for large-scale systems.

Subsequently, the generalized Gramians for bilinear systems have been developed in regards to MOR; see, e.g. [1], which are the solutions of generalized Lyapunov equations of the form

$$AP + PA^T + \sum\nolimits_{k=1}^{m} N^{(k)} P \left(N^{(k)}\right)^T = -BB^T, \tag{3a}$$

$$A^T Q + QA + \sum\nolimits_{k=1}^{m} \left(N^{(k)}\right)^T Q N^{(k)} = -C^T C, \tag{3b}$$

where A, $N^{(k)}$, B and C are as in (1). The connections between these Gramians and the energy functionals of bilinear systems have been studied in [7, 15]. Furthermore, the relations between the Gramians and the controllability/observability of bilinear systems have also been studied in [7]. However, the main bottleneck in using these Gramians in the MOR context is the computation of the Gramians, though recently there have been many advances in methods to determine the low-rank solutions of these generalized Lyapunov equations (3); see [6, 24].

This motivates us to investigate an alternative pair of Gramians for bilinear systems, which we call *Truncated* Gramians (TGrams). Regarding this, in Section 2 we recall balanced truncation for bilinear systems based on the Gramians (3). In Section 3, we propose TGrams for bilinear systems and investigate their connections with the controllability and observability of the bilinear systems. Moreover, we reveal the relation between these TGrams and energy functionals of the bilinear systems. Then, we discuss the advantages of considering these TGrams in the MOR context. Subsequently in Section 4, we provide a couple of numerical examples to illustrate the applicability of the TGrams for MOR of bilinear systems.

2 Background Work

In this section, we outline basic concepts of balanced truncation MOR. For this, let us consider a bilinear control system as in (1), then the controllability and observability of a state $x \in \mathbb{R}^n$ can be defined based on energy functionals as follows [22]:

$$E_c(x_0) = \inf_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad E_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \|y(t)\|^2 dt, \tag{4}$$

respectively. The functional E_c is measured in terms of the minimal input energy required to steer the system from $x(-\infty) = 0$ to a desired state x_0 at time $t = 0$. If the state x_0 is uncontrollable, then it requires infinite energy; that means $E_c(x_0) = \infty$. On the other hand, the functional E_o characterizes the output energy determined by a particular initial state x_0 using the uncontrolled system. If the state x_0 is unobservable, then it produces no output energy; $E_o(x_0) = 0$. In the linear case $(N_k = 0)$, these energy functionals can be represented exactly by the Gramians of the linear system:

$$E_c(x) = \tfrac{1}{2}\langle x, P_l^{\#} x\rangle \quad \text{and} \quad E_o(x) = \tfrac{1}{2}\langle x, Q_l x\rangle,$$

where $\langle \cdot,\cdot \rangle$ represents the Euclidean inner product, $P_l^{\#}$ denotes the Moore-Penrose pseudo inverse of P_l, and P_l and Q_l are the unique and positive semidefinite solutions of the following Lyapunov equations:

$$AP_l + P_l A^T + BB^T = 0 \quad \text{and} \quad A^T Q_l + Q_l A + C^T C = 0, \tag{5}$$

respectively. In case of a nonlinear setting, the functionals E_c and E_o can be determined by solving Hamilton-Jacobi nonlinear PDEs, which are quite expensive to solve for large-scale settings. For more details on these PDEs, we refer to [22]. However, for MOR of bilinear systems, the Gramians, namely the controllability (P) and the observability (Q) Gramians, are defined as

$$\begin{aligned} P &= \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \bar{P}_k(t_1,\dots,t_k)\bar{P}_k(t_1,\dots,t_k)^T dt_1 \cdots dt_k, \\ Q &= \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \bar{Q}_k(t_1,\dots,t_k)\bar{Q}_k(t_1,\dots,t_k)^T dt_1 \cdots dt_k, \end{aligned} \tag{6}$$

respectively, where

$$\begin{aligned} \bar{P}_1(t_1) &= e^{At_1}B, & \bar{P}_k(t_1,\dots,t_k) &= e^{At_k}\left[N^{(1)},\dots,N^{(k)}\right]\bar{P}_{k-1}, \\ \bar{Q}_1(t_1) &= e^{A^Tt_1}C^T, & \bar{Q}_k(t_1,\dots,t_k) &= e^{A^Tt_k}\left[\left(N^{(1)}\right)^T,\dots,\left(N^{(k)}\right)^T\right]\bar{Q}_{k-1}. \end{aligned} \tag{7}$$

Then, the connections between these Gramians and Lyapunov equations are derived in [1]. Therein, it is shown that these Gramians satisfy the generalized Lyapunov equations stated in (3). Though energy functionals for bilinear system cannot be determined exactly in terms of the Gramians of the latter system, the Gramians provide a lower bound (locally) for the input (controllability) energy functional and an upper bound (locally) for the output (observability) energy functional as follows:

$$E_c(x) \geq \tfrac{1}{2}\langle x, P^{-1}x\rangle, \qquad \text{and} \qquad E_o(x) \leq \tfrac{1}{2}\langle x, Qx\rangle, \tag{8}$$

in a small open neighborhood of the origin [7, 15], where in (8) we assume that P and Q are positive definite.

However, in the general case with $P, Q \geq 0$ it is shown in [7] that if the desired state $x_0 \notin \operatorname{Im} P$, then $E_c(x_0) = \infty$, and similarly if an initial state x_0 belongs to $\operatorname{Ker} Q$, then $E_o(x_0) = 0$. This shows that the states x_0 with $x_0 \in \operatorname{Ker} Q$ or $x_0 \notin \operatorname{Im} P$ do not play any role in the dynamics of the system; hence they can be removed. The main idea of balanced truncation lies in furthermore neglecting the almost uncontrollable and almost unobservable states (hard to control and hard to observe states).

In order to guarantee that hard to control and hard to observe states are truncated simultaneously, we need to find a transformation $x \mapsto T^{-1}x$, leading to a transformed bilinear system, whose controllability and observability Gramians are equal and diagonal, i.e.,

$$T^{-1}PT^{-T} = T^TQT = \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n). \tag{9}$$

Analogous to the linear case (see, e.g., [2]), having the factorizations of $P = LL^T$ and $L^TQL = U\Sigma^2U^T$, one finds the corresponding transformation matrix in $T = LU\Sigma^{-\frac{1}{2}}$. Now, w.l.o.g. we consider the following bilinear system being a balanced bilinear system:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \sum_{k=1}^{m}\begin{bmatrix}N_{11}^{(k)} & N_{12}^{(k)}\\ N_{21}^{(k)} & N_{22}^{(k)}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}u_k(t) + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u(t),$$
$$y(t) = \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1^T(t) & x_2^T(t)\end{bmatrix}^T,$$

with the controllability and observability Gramians equal to Σ :

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

$\sigma_i \geq \sigma_{i+1}$ and $x_1(t) \in \mathbb{R}^r$ and $x_2(t) \in \mathbb{R}^{n-r}$. Fixing r such that $\sigma_r > \sigma_{r+1}$, we determine a reduced-order system of order r by neglecting x_2 as follows:

$$\begin{aligned}\dot{x}_1(t) &= A_{11}x_1(t) + \textstyle\sum_{k=1}^{m} N_{11}^{(k)}x_1(t)u_k(t) + B_1u(t),\\ y_r(t) &= C_1x_1(t).\end{aligned} \tag{10}$$

This provides a good local reduced-order system, but unlike in the linear case, it is not clear how to quantify the error, occurring due to x_2 being removed.

3 Truncated Gramians for Bilinear Systems

As discussed in the preceding section, we need to solve two generalized Lyapunov equations in order to compute reduced-order systems via balanced truncation. Solving these generalized Lyapunov equations is a numerical challenge for large-scale settings, although there have been many advancements in this direction in recent times; see, e.g. [6, 24]. In this section, we seek to determine TGrams for bilinear systems and discuss their advantages in the balancing-type MOR.

We define TGrams for bilinear systems by considering only the first two terms in the series in (6), which are dependent on the first two *kernels* of the Volterra series of the bilinear system, as follows:

$$P_T = \int_0^\infty \bar{P}_1(t_1)\bar{P}_1^T(t_1)dt + \int_0^\infty\int_0^\infty \bar{P}_2(t_1,t_2)\bar{P}_2^T(t_1,t_2)dt_1dt_2, \tag{11a}$$
$$Q_T = \int_0^\infty \bar{Q}_1(t_1)\bar{Q}_1^T(t_1)dt_1 + \int_0^\infty\int_0^\infty \bar{Q}_2(t_1,t_2)\bar{Q}_2^T(t_1,t_2)dt_1dt_2, \tag{11b}$$

where $\bar{P}_i$ and $\bar{Q}_i$ are defined in (7). Next, we establish the relations between these truncated Gramians and the solutions of Lyapunov equations.

Lemma 1. *Consider the bilinear system* (1) *and let P_T and Q_T be the truncated controllability and observability Gramians of the system as defined in* (11). *Then, P_T and Q_T satisfy the following Lyapunov equations:*

$$AP_T + P_TA^T + \textstyle\sum_{k=1}^{m} N^{(k)}P_l\left(N^{(k)}\right)^T + BB^T = 0, \tag{12a}$$

$$A^TQ_T + Q_TA + \sum_{k=1}^{m} \left(N^{(k)}\right)^T Q_l N^{(k)} + C^TC = 0, \tag{12b}$$

respectively, where P_l and Q_l are the Gramians of the linear systems as shown in (5)*.*

The above lemma can be proven in a similar way as done in [1, Thm. 1]. Therefore, due to space limitations, we omit the proof. Next, we compare the energy functionals of the bilinear system and the quadratic forms given by the TGrams. Before we state the corresponding lemma, we introduce the *homogeneous* bilinear system, which is used to characterize the observability energy in the system, as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{k=1}^{m} N^{(k)}x(t)u_k(t), \\ y(t) &= Cx(t), \quad x(0) = x_0. \end{aligned} \tag{13}$$

Lemma 2. *Given the bilinear system* (1)*, with an asymptotically stable A, being asymptotically reachable from* 0 *to any state x. Let $P, Q > 0$ and $P_T, Q_T > 0$ be the Gramians and* TGrams *of the system, respectively. Then, there exists a small neighborhood W of* 0*, where the following relations hold:*

$$E_c(x) \geq \tfrac{1}{2}x^TP_T^{-1}x \geq \tfrac{1}{2}x^TP^{-1}x \quad \textit{and} \quad x \in W(0). \tag{14}$$

Furthermore, there also exists a neighborhood $\hat{W}$ of 0*, where the following holds:*

$$E_o(x) \leq \tfrac{1}{2}x^TQ_Tx \leq \tfrac{1}{2}x^TQx \quad \textit{and} \quad x \in \hat{W}(0), \tag{15}$$

where E_c and E_o are the energy functionals defined in (4)*.*

Proof. To prove (14), we follow the lines of reasoning in [7]. Let us assume that $x_0 \in \mathbb{R}^n$ is controlled by the input $u = u_{x_0} :]-\infty, 0] \to \mathbb{R}^m$, minimizing the input cost functional in the definition of $E_c(x_0)$. Using this input, we consider the homogeneous linear differential equation given by

$$\dot{\phi} = \left(A + \sum_{k=1}^{m} N^{(k)}u_k(t)\right)\phi =: A_u(t)\phi(t), \tag{16}$$

whose fundamental solution is denoted by Φ_u. Thus, if we consider a time-varying system as $\dot{x} = A_u(t)x + Bu$, then its controllability Gramian can be given by

$$P_u = \int_{-\infty}^{0} \Phi_u(0,\tau)BB^T\Phi_u(0,\tau)^T d\tau. \tag{17}$$

The input u also controls the time-varying system from 0 to x_0; therefore we have

$$\|u\|_{L^2}^2 \geq x_0^TP_u^{\#}x_0, \tag{18}$$

where $P_u^{\#}$ denotes the Moore-Penrose pseudo inverse of P_u. Alternatively, one can also determine P_u as an observability Gramian as

$$P_u = \int_{0}^{\infty} \Psi_u(t,0)BB^T\Psi_u(t,0)^T dt, \tag{19}$$

where Ψ_u is the fundamental solution of the dual system satisfying

$$\dot{\Psi}_u = \left(A^T + \sum_{k=1}^m \left(N^{(k)}\right)^T u_k(t)\right)\Psi_u, \quad \Psi_u(t,t) = I. \tag{20}$$

Note that we always choose x_0 in a small neighborhood W_0 of 0 such that only a small input is required to steer the systems from 0 to x_0, ensuring the asymptotic stability of $A_u(t)$. Hence, the matrix P_u is well-defined. Now, we define $\tilde{x}(t) = \Psi_u(t,0)x_0$, then we have

$$\begin{aligned}
x_0^T P_T x_0 &= -\int_0^\infty \frac{d}{dt}\left(\tilde{x}(t)^T P_T \tilde{x}(t)\right) dt \\
&= -\int_0^\infty \tilde{x}(t)^T \Big(AP_T + \sum_{k=1}^m N^{(k)} P_T u_k(-t) \\
&\qquad + P_T A^T + \sum_{k=1}^m P_T \left(N^{(k)}\right)^T u_k(-t)\Big)\tilde{x}(t)dt \\
&= -\int_0^\infty \tilde{x}(t)^T \left(AP_T + P_T A^T + \sum_{k=1}^m N^{(k)} P_l \left(N^{(k)}\right)^T\right)\tilde{x}(t)dt + \int_0^\infty \tilde{x}(t)^T \\
&\quad \times \sum_{k=1}^m \left(N^{(k)} P_l \left(N^{(k)}\right)^T - N^{(k)} P_T u_k(-t) - P_T \left(N^{(k)}\right)^T u_k(-t)\right)\tilde{x}(t)dt.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
-\int_0^\infty \tilde{x}(t)^T \left(AP_T + P_T A^T + \sum_{k=1}^m N^{(k)} P_l \left(N^{(k)}\right)^T\right)\tilde{x}(t)dt &= \int_0^\infty \tilde{x}(t)^T BB^T \tilde{x}(t)dt \\
&= x_0^T P_u x_0.
\end{aligned}$$

Moreover, if

$$\int_0^\infty \tilde{x}(t)^T \sum_{k=1}^m \left(N^{(k)} P_l \left(N^{(k)}\right)^T - N^{(k)} P_T u_k(-t) - P_T \left(N^{(k)}\right)^T u_k(-t)\right)\tilde{x}(t)dt \geq 0, \tag{21}$$

then we have $x_0^T P_T x_0 \geq x_0^T P_u x_0$. Furthermore, if we assume that the reachable state x_0 lies in a sufficiently small ball W in the neighborhood of 0 and $W(0) \subseteq W_0(0)$, then x_0 is reached with a sufficiently small input u, guaranteeing that the condition (21) is satisfied for all states $x_0 \in W(0)$. Hence, we obtain

$$x_0^T P_T^{-1} x_0 \leq x_0^T P_u^{-1} x_0, \quad \text{where} \quad x_0 \in W(0).$$

Furthermore, if the controllability Gramian P, which is the solution of (3a), is determined as a series [12], then it is easy to conclude that $P \geq P_T \geq 0$. That means, $x_0^T P^{-1} x_0 \leq x_0^T P_T^{-1} x_0$. Thus, we have $x_0^T P^{-1} x_0 \leq x_0^T P_T^{-1} x_0 \leq x_0^T P_u^{-1} x_0$, where $x_0 \in W(0)$.

Furthermore, along the lines of the proof [15, Thm 3.3], we can prove that

$$E_o(x_0,u) - \tfrac{1}{2}x_0^T Q_T x_0 = \int_0^\infty x(t)^T \mathscr{R}(t) x(t)dt,$$

where $\mathscr{R}(t) = \sum_{k=1}^{m} \left(Q_T N^{(k)} u_k(t) - \frac{1}{2}\left(N^{(k)}\right)^T Q_l N^{(k)} \right)$. For sufficiently small input u, it can be seen that $\mathscr{R}(t)$ is a negative semidefinite matrix. Hence, we get

$$E_o(x_0,u) - \tfrac{1}{2}x_0^T Q_T x_0 \leq 0 \quad \Rightarrow \quad E_o(x_0,u) \leq \tfrac{1}{2}x_0^T Q_T x_0.$$

Moreover, if the observability Gramian is determined as a series with positive semidefinite summands, then it can also be seen that $Q \geq Q_T$; hence

$$E_o(x_0,u) \leq \tfrac{1}{2}x_0^T Q_T x_0 \leq \tfrac{1}{2}x_0^T Q x_0.$$

This concludes the proof. □

To illustrate the relation between energy functionals, Gramians and TGrams of bilinear systems, we consider the same scalar example considered in [15].

Example 1. Consider a scalar example (a,b,c,η). We assume $a < 0$, $\eta^2 + 2a < 0$ and $bc \neq 0$ to ensure the existence of $P,Q > 0$. The energy functionals of the system can be determined by solving the corresponding nonlinear PDEs [15], which are:

$$E_c(x) = \frac{2a}{\eta^2}\left[\frac{\eta x}{\eta x + b} + \log\left(\frac{b}{\eta x + b}\right)\right] \quad \text{and} \quad E_o(x) = -\frac{1}{2}\left(\frac{c^2}{2a}\right)x^2.$$

The approximations of the energy functionals using the Gramians are:

$$E_c^{(G)}(x) = \frac{1}{2}\left(\frac{\eta^2 + 2a}{-b^2}\right)x^2 \quad \text{and} \quad E_o^{(G)}(x) = \frac{1}{2}\left(\frac{-c^2}{\eta^2 + 2a}\right)x^2.$$

The approximations of the energy functionals using TGrams are:

$$E_c^{(T)}(x) = a\left(-b^2 + \frac{\eta^2 b^2}{2a}\right)^{-1} x^2 \quad \text{and} \quad E_o^{(T)}(x) = \frac{1}{4a}\left(-c^2 + \frac{\eta^2 c^2}{2a}\right)x^2.$$

The comparison of these quantities by taking numerical values for $-a = b = c = \eta = 1$ is illustrated in Figure 1.

Next, we recall the discussion in [7] about definiteness of Gramians and controllability/observability of the bilinear systems. Following this discussion, we also show how controllability/observability of the bilinear systems are related to the TGrams.

Theorem 1.

(a) Consider the bilinear system (1) *and define its truncated controllability Gramian* P_T *as in* (11a). *If the final state* $x_0 \notin \operatorname{Im} P_T$, *then* $E_c(x_0) = \infty$.
(b) Consider the homogeneous bilinear system (13) *and assume that the truncated observability Gramian* Q_T *is defined as in* (11b). *If the initial condition* $x_0 \in \operatorname{Ker} Q_T$, *then the output* $y(t)$ *is zero for all* $t \geq 0$, *i.e.* $E_o(x_0) = 0$.

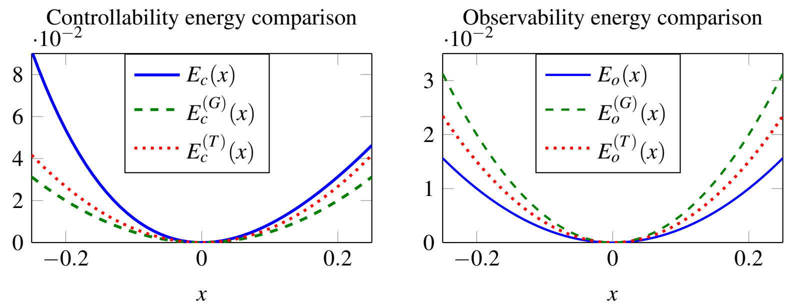


Fig. 1: The figure shows the comparison of the energy functionals of the system and their approximations via Gramians and TGrams as stated in Lemma 2.

Proof. The above theorem can be proven in a similar manner as [7, Thm. 3.1] using one of the important properties of positive semidefinite matrices. It is that the null space of the matrix $\mathscr{C}$, which is the sum of two positive semidefinite matrices $\mathscr{A}$ and $\mathscr{B}$, is the intersection of the null space of $\mathscr{A}$ and $\mathscr{B}$. In other words, if the vector v belongs to the null space of $\mathscr{C}$, then $\mathscr{A}v = 0$ and $\mathscr{B}v = 0$ as well. However, we skip a detailed proof due to the limited space. □

From Lemma 2 and Theorem 1, it is clear that the TGrams for bilinear systems can also be used to determine the states that absorb a lot of energy, and still produce very little output energy. However, there are several advantages of considering the TGrams over the Gramians for bilinear systems. Firstly, TGrams approximate the energy functionals of the bilinear systems more accurately (at least locally) as proven in Lemma 2 and illustrated in Example 1. Secondly, in order to compute TGrams, we require the solutions of four conventional Lyapunov equations, whereas the Gramians require the solutions of the generalized Lyapunov equations (3), which are indeed much more computationally cumbersome. Lastly, TGrams are of smaller rank as compared to Gramians; i.e. $P \geq P_T$ and $Q \geq Q_T$. This indicates that $\sigma_i(P \cdot Q) \geq \sigma_i(P_T \cdot Q_T)$, where $\sigma_i(\cdot)$ denotes the i-th largest eigenvalue of the matrix. This can be shown using Weyl's inequality [25]. Hence, if one chooses to truncate at machine precision, then the reduced system based on TGrams is probably to be of a small order; however, the relative decay of the Hankel singular values $\left(\sqrt{\sigma_i(P_T \cdot Q_T)}\right)$ so far lacks any analysis.

4 Numerical Results

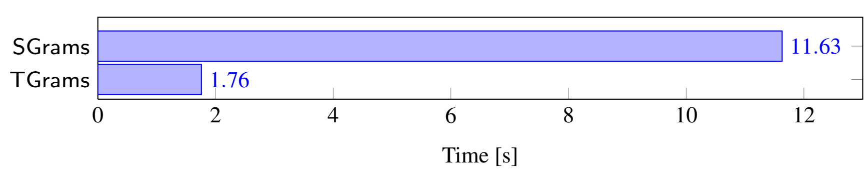
In this section, we illustrate the efficiency of the reduced-order systems obtained via the proposed TGrams for the bilinear system and compare it with that of the full Gramians [7]. We denote the Gramians for the bilinear system by SGrams (*standard* Gramians) from now on. In order to determine the low-rank factors of the Gramians for bilinear systems, we employ the most recently proposed algorithm in [24], which utilizes many of the properties of inexact solutions and uses the extended Krylov subspace method (EKSM) to solve the conventional Lyapunov equation up to a desired accuracy. To determine the low-rank factors of the linear Lyapunov equation, we also utilize EKSM to be in the same line. All the simulations were carried out in MATLAB® version 7.11.0(R2010b) on a board with 4 Intel® Xeon®E7-8837 CPUs with a 2.67-GHz clock speed, 8 Cores each and 1TB of total RAM.

4.1 Burgers' equation

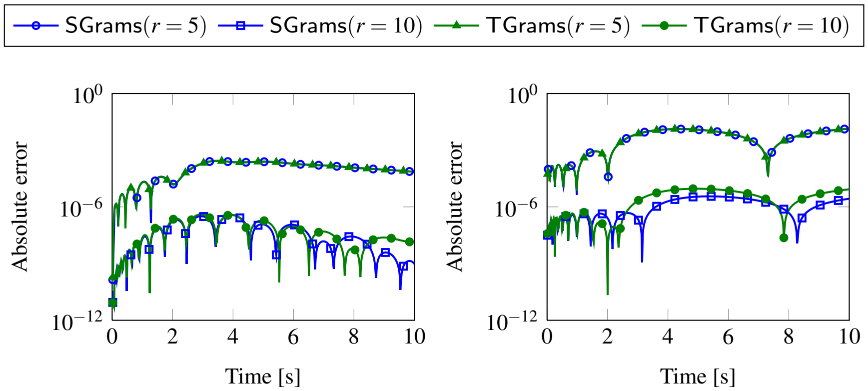
We consider a viscous Burgers' equation, which is one of the standard test examples for bilinear systems; see, e.g. [10]. Therein, one can also find the governing equation, boundary conditions and initial condition of the system. As shown in [10], a spatial semi-discretization of the governing equation using k equidistant nodes leads to an ODE system with quadratic nonlinearity. However, the quadratic nonlinear system can be approximated using Carleman bilinearization; see, e.g., [21]. The dimension of the approximated bilinearized system is $n = k + k^2$. We set the viscosity $\mu = 0.1$ and $k = 40$, and choose the observation vector C such that it yields an average value for the variable v in the spatial domain. The bilinearized system is not an $\mathscr{H}_2$ system, which can be checked by looking at the eigenvalues of the matrix $\mathscr{X} := (I \otimes A + A \otimes I + N \otimes N)$. If $\sigma(\mathscr{X}) \not\subset \mathbb{C}^-$, then the series determining its controllability Gramians diverges. To overcome this issue, we choose a scaling factor γ, which multiplies with the matrices B and N_k, and the input $u(t)$ is scaled by $\frac{1}{\gamma}$. For this example, we set $\gamma = 0.1$, ensuring $\sigma(\mathscr{X}) \subset \mathbb{C}^-$.

We determine reduced systems of orders $r = 5$ and $r = 10$ using SGrams and TGrams, and compare the quality of the reduced-order systems by using two arbitrary control inputs as shown in Figure 1. More importantly, we also show the CPU-time to determine the low-rank factors of SGrams and TGrams in the same figure.

Figure 1 shows that computing TGrams is much cheaper than SGrams. Moreover, we observe that the reduced systems based on TGrams are very much competitive to those of SGrams for both control inputs and both orders in Example 4.1.



(a) Comparison of CPU-time to compute SGrams and TGrams for Example 4.1.



(b) For an input $u(t) = t \cdot e^{-t} \cdot \sin(\pi t)$.

(c) For an input $u(t) = t \cdot e^{-t} + 1$.

Fig. 1: Comparisons of CPU-time and time-domain responses of the original and reduced-order systems for two different orders and for two inputs for the Burgers' equation example.

4.2 *Electricity cable impacted by wind*

Below, we discuss an example studied in [20]. Therein, a damped wave equation with Lévy noise is considered, which is transformed into a first order stochastic PDE (SPDE) and then discretized in space. The governing equation, which models the lateral displacement of an electricity cable impacted by wind, is

$$\frac{\partial^2}{\partial t^2}X(t,z) + 2\frac{\partial}{\partial t}X(t,z) = \frac{\partial^2}{\partial z^2}X(t,z) + \mathrm{e}^{-(z-\frac{\pi}{2})^2}u(t) + 2\,\mathrm{e}^{-(z-\frac{\pi}{2})^2}X(t-,z)\frac{\partial M(t)}{\partial t}$$

for $t, z \in [0,\pi]$, where M is a scalar, square integrable Lévy process with mean zero. The boundary and initial conditions are:

$$X(t,0) = X(t,\pi) = 0 \quad \text{and} \quad X(0,z) = 0, \; \frac{\partial}{\partial t}X(t,z)\bigg|_{t=0} \equiv 0.$$

An approximation for the position of the middle of the cable represents the output

$$Y(t) = \frac{1}{2\varepsilon}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} X(t,z)dz, \quad \varepsilon > 0.$$

Following [20], a semi-discretized version of the above SPDE has the following form with $x(0) = 0$ and $t \in [0,\pi]$:

$$dx(t) = [Ax(t) + Bu(t)]\,dt + Nx(s-)dM(s), \quad y(t) = Cx(t). \tag{22}$$

Here, A, $N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(t-) := \lim_{s\uparrow t} x(s)$ and y is the corresponding output. We, moreover, assume that the adapted control satisfies $\|u\|^2_{\mathscr{L}^2_T} := \mathbb{E}\int_0^T \|u(t)\|^2_{\mathbb{R}^m}\, dt < \infty$. For more details, we refer to [20].

In contrast to [20], we fix a different noise process, which allows the wind to come from two directions instead of just one. The noise term we choose is represented by a compound Poisson process $M(t) = \sum_{i=1}^{N(t)} Z_i$ with $(N(t))_{t\in[0,\pi]}$ being a Poisson process with parameter 1. Furthermore, $Z_1, Z_2, \ldots$ are independent uniformly distributed random variables with $Z_i \sim \mathscr{U}\left(-\sqrt{3}, \sqrt{3}\right)$, which are also independent of $(N(t))_{t\in[0,\pi]}$. This choice implies $\mathbb{E}[M(t)] = 0$ and $\mathbb{E}\left[M^2(1)\right] = 1$. BT for such an Ito type SDE (22) with the particular choice of M is also based on Gramians, which fulfill equations (3) with $m = 1$ and $N := N^{(1)}$. We fix the dimension of (22) to $n = 1000$ and set $u(t) = e^{w(t)} \sin(t)$, and then run several numerical experiments.

We apply BT based on SGrams as described in [9] and compute the reduced systems of order $r = 3$ and $r = 6$. Similarly, we determine the reduced systems of the same orders using TGrams. Next, we discuss the quality of these derived reduced systems and computational cost to determine the low-rank factors of SGrams and TGrams. In Figure 2, we see that the TGrams are computationally much cheaper as compared to the SGrams.

For the $r = 3$ case, clearly the reduced system based on TGrams outperforms the one based on the SGrams for all three trajectories (see Figure 3a). This is also true for the mean deviation as shown in the Figure 3c (left). For the $r = 6$ case, it is not that obvious anymore. The reduced system obtained by SGrams seems to be marginally more accurate, but still both methods result in very competitive reduced-order systems, see Figures 3b and 3c (right).

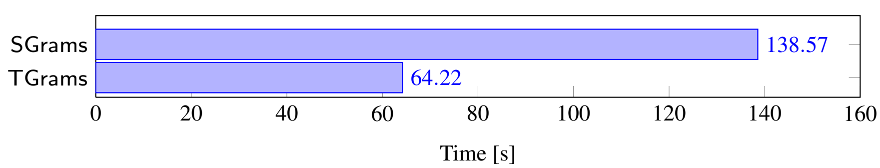
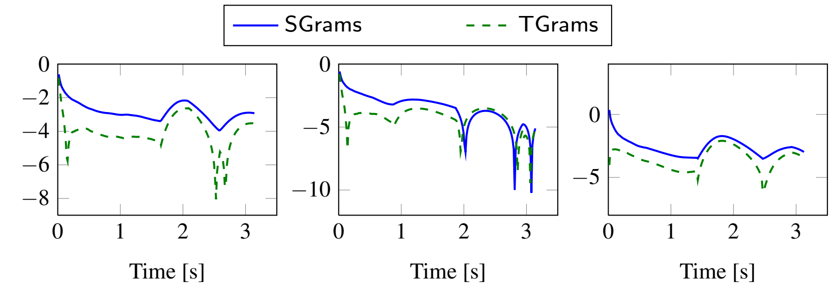


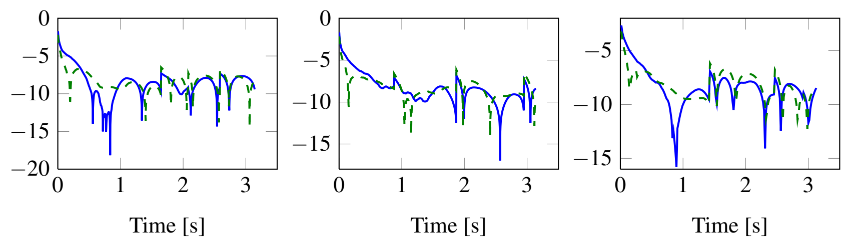
Fig. 2: Comparison of CPU-time to compute SGrams and TGrams for Example 4.2.

5 Conclusions

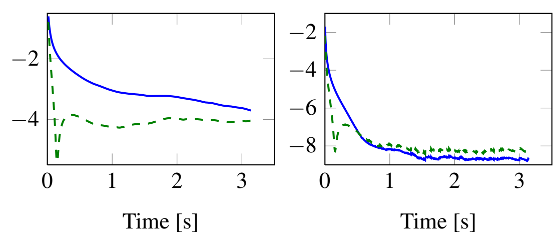
In this paper, we have proposed truncated Gramians for bilinear systems. These allow us to find the states, which are both hard to control and hard to observe, like the Gramians for bilinear systems. We have also shown that the truncated Gramians approximate the energy functionals of bilinear systems better (at least locally) as compared to the Gramians of the latter systems. We have presented how controlla-



(a) $\ln\left(\frac{|y(\omega,t)-y_r(\omega,t)|}{|y(\omega,t)|}\right)$ with reduced order dimension $r = 3$.



(b) $\ln\left(\frac{|y(\omega,t)-y_r(\omega,t)|}{|y(\omega,t)|}\right)$ with reduced order dimension $r = 6$.



(c) $\ln\left(\frac{\mathbb{E}|y(t)-y_r(t)|}{\mathbb{E}|y(t)|}\right)$, where $r = 3$ (left), 6 (right).

Fig. 3: Comparison of reduced-order systems for $u(t) = e^{w(t)}\sin(t)$.

bility and observability of bilinear systems are related to the truncated Gramians. Moreover, we have discussed advantages of the truncated Gramians in the model reduction context. In the end, we have demonstrated the efficiency of the proposed truncated Gramians in model reduction by means of two numerical examples.

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