AN L^2_T-ERROR BOUND FOR TIME-LIMITED BALANCED TRUNCATION

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Abstract. Model order reduction (MOR) is often applied to spatially-discretized partial differential equations to reduce their order and hence decrease computational complexity. A reduced system can be obtained, e.g., by time-limited balanced truncation, a method that aims to construct an accurate reduced order model on a given finite time interval $[0, T]$. This particular balancing related MOR technique is studied in this paper. An L^2_T-error bound based on the truncated time-limited singular values is proved and is the main result of this paper. The derived error bound converges (as $T \to \infty$) to the well-known $\mathcal{H}_\infty$-error bound of unrestricted balanced truncation, a scheme that is used to construct a good reduced system on the entire time line. The techniques within the proofs of this paper can also be applied to unrestricted balanced truncation so that a relatively short time domain proof of the $\mathcal{H}_\infty$-error bound is found here.

Key words. model order reduction, linear time-invariant systems, time-limited balanced truncation, error bound

AMS subject classifications. 93A15, 93B99, 93C05, 93C15.

1. Introduction. Many phenomena in real life can be described by partial differential equations. Famous examples are the motion of viscous fluids, the description of water or sound waves and the distribution of heat. In order to solve these equations numerically it is required to discretize in time and space. Discretizing in space usually leads to large scale systems of ordinary differential equation which usually cause large computational effort. To overcome this burden, model order reduction (MOR) can be used to replace a high dimensional system by one of smaller order aiming to capture the main information of the original system.

In this paper, we consider the following linear, time-invariant system:

$$\dot{x}_o(t) = A_o x_o(t) + B_o u(t), \quad x_o(0) = 0, \quad y(t) = C_o x_o(t), \tag{1.1}$$

where $A_o \in \mathbb{R}^{n\times n}$ is assumed to be Hurwitz implying asymptotic stability of the state equation in (1.1). $B_o \in \mathbb{R}^{n\times m}$ is the input and $C_o \in \mathbb{R}^{p\times n}$ the output matrix. Moreover, let the control u be square integrable with respect to time, i.e., $\|u\|^2_{L^2_T} := \int_0^T \|u(s)\|_2^2\, ds < \infty$ for $T < \infty$. We study a MOR technique that is called time-limited balanced truncation (BT). It was introduced in [5] with the goal of constructing an accurate reduced order model (ROM) on a finite time interval $[0, T]$. The idea of balancing MOR schemes is to simultaneously diagonalize so-called Gramians in order to create a system in which the dominant reachable and observable states are the same. Then the states that only contribute very little to the system dynamics are truncated to obtain a ROM. For time-limited BT time-limited reachability and observability Gramians are aimed to be diagonalized. These are defined as follows

$$P_T := \int_0^T \mathrm{e}^{A_o s} B_o B_o^\top \mathrm{e}^{A_o^\top s}\, ds, \quad Q_T := \int_0^T \mathrm{e}^{A_o^\top s} C_o^\top C_o \mathrm{e}^{A_o s}\, ds \tag{1.2}$$

and it can be shown that they are the unique solutions to

$$A_o P_T + P_T A_o^\top + B_o B_o^\top - \mathrm{e}^{A_o T} B_o B_o^\top \mathrm{e}^{A_o^\top T} = 0, \tag{1.3a}$$

$$A_o^\top Q_T + Q_T A_o + C_o^\top C_o - \mathrm{e}^{A_o^\top T} C_o^\top C_o \mathrm{e}^{A_o T} = 0. \tag{1.3b}$$

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Throughout this paper let us assume that system (1.1) is completely reachable and observable which is equivalent to P_T and Q_T being positive definite, see [1]. In order to diagonalize P_T and Q_T a state space transformation is used. This basically means that the original matrices (A_o, B_o, C_o) are replaced by $(A, B, C) := (SA_oS^{-1}, SB_o, C_oS^{-1})$, where S is an invertible matrix. This transformation does not change the quantity of interest y but it can be chosen such that the Gramians of the transformed system are equal and diagonal, i.e., $SP_TS^\top = S^{-\top}Q_TS^{-1} = \Sigma_T = \operatorname{diag}(\sigma_{T,1}, \dots, \sigma_{T,n})$ with $\sigma_{T,1} \geq \dots \geq \sigma_{T,n} > 0$. These diagonal entries are called time-limited singular values and are given as the square root of the eigenvalues of P_TQ_T. The balancing transformation can be derived through the Cholesky factorizations $P_T = L_PL_P^\top$, $Q_T = L_QL_Q^\top$, and the singular value decomposition $X\Sigma_TY^\top = L_Q^\top L_P$. The matrix S and its inverse are then given by $S = \Sigma_T^{-\frac{1}{2}}X^\top L_Q^\top$ and $S^{-1} = L_PY\Sigma_T^{-\frac{1}{2}}$, see, e.g., [1]. Now, the ROM with state space dimension r is obtained by selecting the left upper $r \times r$ block of A and choosing the the first r rows of B as the input matrix as well as the first r columns of C as the output matrix.

Unrestricted BT is a method that has already been widely studied [1, 9]. It relies on the infinite Gramians which are obtained by taking the limit $T \to \infty$ in (1.2). In [10], the preservation of asymptotic stability in the ROM has been shown and in [4, 6] an $\mathcal{H}_\infty$-error bound was proved, moreover [1] contains an $\mathcal{H}_2$-error bound for unrestricted BT.

Asymptotic stability is not preserved in the ROM for the time-limited case. However, error bounds exist such as $\mathcal{H}_2$-type error bounds that are quite recent. They can be found in [2, 13]. An $\mathcal{H}_\infty$-error bound does not exist for the method considered here. However, there is one for a modified version of time-limited BT [7]. Time-limited BT for unstable systems is furthermore discussed in [8]. The main result of this paper is an L^2_T-error bound for time-limited BT that leads to the $\mathcal{H}_\infty$-bound in [4, 6] for $T \to \infty$. As a side effect a relatively short time domain proof of the bound in [4, 6] is presented which can be seen as a special case of the time-limited scenario. We conclude the paper by a numerical experiment in which the new error bound is tested.

2. Reduced system and error bound for time-limited BT.

In this section, we work with the balanced realization (A, B, C) of (1.1) introduced in Section 1 through the balancing transformation S. Thus, (1.3a) and (1.3b) become

$$A\Sigma_T + \Sigma_TA^\top = -BB^\top + F_TF_T^\top, \tag{2.1}$$

$$A^\top\Sigma_T + \Sigma_TA = -C^\top C + G_T^\top G_T, \tag{2.2}$$

i.e., $SP_TS^\top = S^{-\top}Q_TS^{-1} = \Sigma_T = \operatorname{diag}(\sigma_{T,1}, \dots, \sigma_{T,n}) > 0$, where $G_T := C_o\,\mathrm{e}^{A_oT}\,S^{-1}$ and $F_T := S\,\mathrm{e}^{A_oT}\,B_o$. We partition the balanced coefficients of (1.1) as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{2.3}$$

where $A_{11} \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times m}$ and $C_1 \in \mathbb{R}^{p\times r}$ etc. Furthermore, we partition the state variable x of the balanced realization and the time-limited Gramian

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \text{ and } \Sigma_T = \begin{bmatrix} \Sigma_{T,1} & \\ & \Sigma_{T,2} \end{bmatrix}, \tag{2.4}$$

where x_1 takes values in $\mathbb{R}^r$ (x_2 accordingly), $\Sigma_{T,1}$ contains the large time-limited singular values and $\Sigma_{T,2}$ the small ones. The ROM by time-limited BT then is

$$\dot{x}_r(t) = A_{11}x_r(t) + B_1u(t), \tag{2.5a}$$

$$y_r(t) = C_1x_r(t), \tag{2.5b}$$

where $x_r(0) = 0$. In the following, an L^2_T-error bound is proved. To do so, we define the variables

$$x_-(t) = \begin{bmatrix} x_1(t)-x_r(t) \\ x_2(t) \end{bmatrix} \text{ and } x_+(t) = \begin{bmatrix} x_1(t)+x_r(t) \\ x_2(t) \end{bmatrix}. \tag{2.6}$$

The system for x_- is given by

$$\dot{x}_-(t) = Ax_-(t) + \begin{bmatrix} 0 \\ h(t) \end{bmatrix}, \tag{2.7a}$$

$$y_-(t) = Cx_-(t) = Cx(t) - C_1x_r(t) = y(t) - y_r(t), \tag{2.7b}$$

where $h(t) := A_{21}x_r(t)+B_2u(t)$. We derive (2.7) by comparing the balanced system (1.1) with the reduced system (2.5) using the partitions in (2.3) and (2.4). The equation for x_+ is obtained in a similar manner. In comparison to (2.7a), the sign for the compensation term h is different and an additional control term appears:

$$\dot{x}_+(t) = Ax_+(t) + 2Bu(t) - \begin{bmatrix} 0 \\ h(t) \end{bmatrix}. \tag{2.8}$$

The proof of the error bound is simply based on applying the product rule in order to find suitable representations for $x_-^\top(t)\Sigma_T x_-(t)$ and $x_+^\top(t)\Sigma_T^{-1}x_+(t)$. These representations are then used to compute the desired bound. Deriving error bounds through the variables x_- and x_+ has been done before in [11, 12]. We start with a special case before we focus on the general one.

LEMMA 2.1. *Let $\Sigma_{T,2} = \sigma_T I$, y be the output of the full model (1.1) and y_r be the output of the ROM (2.5). Then, for $T > 0$, we have*

$$\|y - y_r\|_{L^2_T} \leq 2\sigma_T c_T \|u\|_{L^2_T},$$

where $c_T = \mathrm{e}^{0.5\max\{\|G_T\Sigma_T^{-\frac{1}{2}}\|_2^2, \|F_T^\top\Sigma_T^{-\frac{1}{2}}\|_2^2\}T}$.

Proof. We observe that $x_-(0) = 0$ due to the zero initial conditions of x and x_r. Combining this fact with the product rule, we determine an estimate for $x_-^\top(t)\Sigma_T x_-(t)$. Hence, inserting (2.7a), we find

$$\begin{aligned} x_-^\top(t)\Sigma_T x_-(t) &= 2\int_0^t x_-^\top(s)\Sigma_T \dot{x}_-(s)ds \\ &= 2\int_0^t x_-^\top(s)\Sigma_T \left(Ax_-(s) + \begin{bmatrix} 0 \\ h(s) \end{bmatrix}\right) ds \\ &= \int_0^t x_-^\top(s)(A^\top\Sigma_T + \Sigma_T A)x_-(s)ds + c_-(t) \end{aligned}$$

for $t \in [0, T]$, where $c_-(t) := 2\int_0^t x_-^\top(s)\Sigma_T \begin{bmatrix} 0 \\ h(s) \end{bmatrix} ds = 2\int_0^t x_2^\top(s)\Sigma_{T,2}h(s)ds$. The identity for c_- is obtained by using the partitions of Σ_T and x_- in (2.4) and (2.6), respectively. We insert (2.2) into the above equation for $x_-^\top(t)\Sigma_T x_-(t)$ and take (2.7b) into account. This leads to

$$\begin{aligned} x_-^\top(t)\Sigma_T x_-(t) &= \int_0^t x_-^\top(s)(G_T^\top G_T - C^\top C)x_-(s)ds + c_-(t) \\ &= c_-(t) - \|y - y_r\|_{L^2_t}^2 + \int_0^t x_-^\top(s)G_T^\top G_T x_-(s)ds. \end{aligned}$$

Since $x_-^\top(s)G_T^\top G_T x_-(s) = \|G_T\Sigma_T^{-\frac{1}{2}}\Sigma_T^{\frac{1}{2}}x_-(s)\|_2^2 \leq \|G_T\Sigma_T^{-\frac{1}{2}}\|_2^2\ x_-^\top(s)\Sigma_T x_-(s) \leq k\ x_-^\top(s)\Sigma_T x_-(s)$, where $k := \max\{\|G_T\Sigma_T^{-\frac{1}{2}}\|_2^2, \|F_T^\top\Sigma_T^{-\frac{1}{2}}\|_2^2\}$, Lemma A.1 gives

$$\begin{aligned} x_-^\top(t)\Sigma_T x_-(t) &\leq c_-(t) - \|y - y_r\|_{L^2_t}^2 + \int_0^t (c_-(s) - \|y - y_r\|_{L^2_s}^2)k\,\mathrm{e}^{k(t-s)}\,ds \\ &\leq c_-(t) - \|y - y_r\|_{L^2_t}^2 + \int_0^t c_-(s)k\,\mathrm{e}^{k(t-s)}\,ds. \end{aligned}$$

This implies that

$$\begin{aligned}\|y-y_r\|_{L_t^2}^2 &\leq c_-(t)+\int_0^t c_-(s)k\,\mathrm{e}^{k(t-s)}\,ds\\ &=\sigma_T^2\left(c_+(t)+\int_0^t c_+(s)k\,\mathrm{e}^{k(t-s)}\,ds\right),\end{aligned}\tag{2.9}$$

where $c_+(t):=2\int_0^t x_2^\top(s)\Sigma_{T,2}^{-1}h(s)ds$, exploiting that $\Sigma_{T,2}=\sigma_T I$. We derive an upper bound for the expression $x_+^\top(t)\Sigma^{-1}x_+(t)$ to further analyze (2.9). Through (2.8), it holds that

$$\begin{aligned}x_+^\top(t)\Sigma_T^{-1}x_+(t) &= 2\int_0^t x_+^\top(s)\Sigma_T^{-1}\dot{x}_+(s)ds \qquad (2.10)\\ &= 2\int_0^t x_+^\top(s)\Sigma_T^{-1}(Ax_+(s)+2Bu(s)-\left[\begin{smallmatrix}0\\ h(s)\end{smallmatrix}\right])ds\\ &= \int_0^t x_+^\top(s)(A^\top\Sigma_T^{-1}+\Sigma_T^{-1}A)x_+(s)ds\\ &\quad+\int_0^t x_+^\top(s)\Sigma_T^{-1}(4Bu(s)-2\left[\begin{smallmatrix}0\\ h(s)\end{smallmatrix}\right])ds.\end{aligned}$$

We multiply (2.1) with Σ_T^{-1} from the left and from the right and obtain

$$A^\top\Sigma_T^{-1}+\Sigma_T^{-1}A-\Sigma_T^{-1}F_TF_T^\top\Sigma_T^{-1}=-\Sigma_T^{-1}BB^\top\Sigma_T^{-1},$$

which, by the Schur complement condition on definiteness, implies

$$\begin{bmatrix} A^\top\Sigma_T^{-1}+\Sigma_T^{-1}A-\Sigma_T^{-1}F_TF_T^\top\Sigma_T^{-1} & \Sigma_T^{-1}B\\ B^\top\Sigma_T^{-1} & -I\end{bmatrix}\leq 0.\tag{2.11}$$

We multiply (2.11) with $\left[\begin{smallmatrix}x_+\\ 2u\end{smallmatrix}\right]^\top$ from the left and with $\left[\begin{smallmatrix}x_+\\ 2u\end{smallmatrix}\right]$ from the right. This leads to

$$4\|u\|_2^2+x_+^\top\Sigma_T^{-1}F_TF_T^\top\Sigma_T^{-1}x_+\geq x_+^\top\left(A^\top\Sigma_T^{-1}+\Sigma_T^{-1}A\right)x_++4x_+^\top\Sigma_T^{-1}Bu.$$

Applying this result to inequality (2.10) gives

$$x_+^\top(t)\Sigma_T^{-1}x_+(t)\leq 4\|u\|_{L_t^2}^2+\int_0^t\left\|F_T^\top\Sigma_T^{-1}x_+(s)\right\|_2^2ds-c_+(t),$$

exploiting that $x_+^\top(s)\Sigma_T^{-1}\left[\begin{smallmatrix}0\\ h(s)\end{smallmatrix}\right]=x_2^\top(s)\Sigma_{T,2}^{-1}h(s)$ using the partitions of x_+ and Σ_T. Since $\left\|F_T^\top\Sigma_T^{-1}x_+(s)\right\|_2^2\leq\left\|F_T^\top\Sigma_T^{-\frac{1}{2}}\right\|_2^2 x_+^\top(s)\Sigma_T^{-1}x_+(s)\leq k\;x_+^\top(s)\Sigma_T^{-1}x_+(s)$, we obtain

$$x_+^\top(t)\Sigma_T^{-1}x_+(t)\leq 4\|u\|_{L_t^2}^2-c_+(t)+k\int_0^t x_+^\top(s)\Sigma_T^{-1}x_+(s)ds.\tag{2.12}$$

Applying the Lemma of Gronwall (Lemma A.1) to (2.12) yields

$$x_+^\top(t)\Sigma_T^{-1}x_+(t)\leq 4\|u\|_{L_t^2}^2-c_+(t)+\int_0^t(4\|u\|_{L_s^2}^2-c_+(s))k\,\mathrm{e}^{k(t-s)}\,ds.\tag{2.13}$$

We then have that

$$\int_0^t\|u\|_{L_s^2}^2\,k\,\mathrm{e}^{k(t-s)}\,ds\leq\|u\|_{L_t^2}^2\left[-\mathrm{e}^{k(t-s)}\right]_{s=0}^t=\|u\|_{L_t^2}^2\left(\mathrm{e}^{kt}-1\right).\tag{2.14}$$

We insert (2.14) into (2.13) and get

$$c_+(t) + \int_0^t c_+(s) k \, \mathrm{e}^{k(t-s)} \, ds \leq 4 \left\| u \right\|_{L_t^2}^2 \mathrm{e}^{kt} .$$

Comparing this result with (2.9) gives

$$\left\| y - y_r \right\|_{L_t^2} \leq 2\sigma_T \, \mathrm{e}^{0.5kt} \left\| u \right\|_{L_t^2} , \tag{2.15}$$

which concludes this proof. □

Remark 1. *Let S be the balancing transformation, then $\Sigma_T^{-1} = SQ_T^{-1}S^\top = S^{-\top}P_T^{-1}S^{-1}$. $\|G_T\Sigma_T^{-\frac{1}{2}}\|_2^2$ and $\|F_T^\top\Sigma_T^{-\frac{1}{2}}\|_2^2$ can be expressed with the help of the matrices corresponding to the original system, since*

$$G_T\Sigma_T^{-\frac{1}{2}}\left(G_T\Sigma_T^{-\frac{1}{2}}\right)^\top = C_o \, \mathrm{e}^{A_oT} S^{-1}(SQ_T^{-1}S^\top)S^{-\top} \mathrm{e}^{A_o^\top T} C_o^\top = C_o \, \mathrm{e}^{A_oT} Q_T^{-\frac{1}{2}}\left(C_o \, \mathrm{e}^{A_oT} Q_T^{-\frac{1}{2}}\right)^\top ,$$

$$F_T^\top\Sigma_T^{-\frac{1}{2}}\left(F_T^\top\Sigma_T^{-\frac{1}{2}}\right)^\top = B_o^\top \, \mathrm{e}^{A_o^\top T} S^\top(S^{-\top}P_T^{-1}S^{-1})S \, \mathrm{e}^{A_oT} B_o = B_o^\top \, \mathrm{e}^{A_o^\top T} P_T^{-\frac{1}{2}}\left(B_o^\top \, \mathrm{e}^{A_o^\top T} P_T^{-\frac{1}{2}}\right)^\top .$$

Hence, the constant in Lemma 2.1 is $c_T = \mathrm{e}^{0.5\max\{\|C_o \, \mathrm{e}^{A_oT} Q_T^{-\frac{1}{2}}\|_2^2, \|B_o^\top \, \mathrm{e}^{A_o^\top T} P_T^{-\frac{1}{2}}\|_2^2\}T}$.

Lemma 2.1 is now used to prove the main result of this paper. The idea is to remove the time-limited singular values step by step and apply the above lemma several times.

Theorem 2.2. *Let $\tilde{\sigma}_{T,1}, \tilde{\sigma}_{T,2}, \ldots, \tilde{\sigma}_{T,\kappa}$ be the distinct diagonal entries of $\Sigma_{T,2}$, i.e., $\Sigma_{T,2} = \operatorname{diag}(\sigma_{T,r+1}, \ldots, \sigma_{T,n}) = \operatorname{diag}(\tilde{\sigma}_{T,1}, \ldots, \tilde{\sigma}_{T,\kappa})$. Moreover, let y and y_r be the outputs of the full model (1.1) and the reduced system (2.5), respectively, with zero initial conditions. Then, for $T > 0$, it holds that*

$$\left\| y - y_r \right\|_{L_T^2} \leq 2\left(\tilde{\sigma}_{T,1}c_{T,1} + \tilde{\sigma}_{T,2}c_{T,2} + \ldots + \tilde{\sigma}_{T,\kappa}c_{T,\kappa}\right) \left\| u \right\|_{L_T^2} ,$$

where $c_{T,i} = \mathrm{e}^{0.5\max\{\|G_T\Sigma_T^{-\frac{1}{2}}\left[\begin{smallmatrix} I_{r_{i+1}} \\ 0 \end{smallmatrix}\right]\|_2^2, \|F_T^\top\Sigma_T^{-\frac{1}{2}}\left[\begin{smallmatrix} I_{r_{i+1}} \\ 0 \end{smallmatrix}\right]\|_2^2\}T}$. Here, $I_{r_{i+1}}$ is the identity matrix of dimension r_{i+1} that is computed through $r_{i+1} = r_i + m(\tilde{\sigma}_{T,i})$ for $i = 1, 2 \ldots, \kappa - 1$ setting $r_1 = r$, where $m(\tilde{\sigma}_i)$ is the multiplicity of $\tilde{\sigma}_i$. Further, we have $c_{T,\kappa} = \mathrm{e}^{0.5\max\{\|G_T\Sigma_T^{-\frac{1}{2}}\|_2^2, \|F_T^\top\Sigma_T^{-\frac{1}{2}}\|_2^2\}T}$.

Proof. We apply Lemma 2.1 several times in order to prove this result. We use the triangle inequality to find a bound between the error of y and y_r:

$$\left\| y - y_r \right\|_{L_T^2} \leq \left\| y - y_{r_\kappa} \right\|_{L_T^2} + \left\| y_{r_\kappa} - y_{r_{\kappa-1}} \right\|_{L_T^2} + \ldots + \left\| y_{r_2} - y_r \right\|_{L_T^2} ,$$

where y_{r_i} is the output of the ROM with dimension r_i. In the first error term, only $\tilde{\sigma}_{T,\kappa}$ is removed from the system. Hence, we can apply Lemma 2.1 which gives

$$\left\| y - y_{r_\kappa} \right\|_{L_T^2} \leq 2\tilde{\sigma}_{T,\kappa}c_{T,\kappa} \left\| u \right\|_{L_T^2} .$$

We can apply Lemma 2.1 again for the error between y_{r_κ} and $y_{r_{\kappa-1}}$. This is because only $\tilde{\sigma}_{r_{\kappa-1}}$ is removed. Moreover, the matrix equations for the ROM with dimension r_κ has the same form as (2.1) and (2.2). To see this, the left upper blocks of (2.1) and (2.2) need to be selected. This delivers the same kind of equations with respective submatrices of A, B, C and, in particular (F_T, G_T, Σ_T) are replaced by $(\tilde{F}_T, \tilde{G}_T, \tilde{\Sigma}_T) := (\left[\begin{smallmatrix} I_{r_\kappa} & 0 \end{smallmatrix}\right] F_T, G_T \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} I_{r_\kappa} & 0 \end{smallmatrix}\right] \Sigma_T \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right])$. Due to Lemma 2.1, it follows that

$$\begin{aligned} \left\| y_{r_\kappa} - y_{r_{\kappa-1}} \right\|_{L_T^2} &\leq 2\tilde{\sigma}_{T,r_{\kappa-1}} \, \mathrm{e}^{0.5\max\{\|\tilde{G}_T\tilde{\Sigma}_T^{-\frac{1}{2}}\|_2^2, \|\tilde{F}_T^\top\tilde{\Sigma}_T^{-\frac{1}{2}}\|_2^2\}T} \left\| u \right\|_{L_T^2} \\ &= 2\tilde{\sigma}_{T,r_{\kappa-1}}c_{T,\kappa-1} \left\| u \right\|_{L_T^2} , \end{aligned}$$

since $\tilde{G}_T\tilde{\Sigma}_T^{-\frac{1}{2}} = G_T \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} I_{r_\kappa} & 0 \end{smallmatrix}\right] \Sigma_T^{-\frac{1}{2}} \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right] = G_T\Sigma_T^{-\frac{1}{2}} \left[\begin{smallmatrix} I_{r_\kappa} & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right] = G_T\Sigma_T^{-\frac{1}{2}} \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right]$ and $\tilde{F}_T^\top\tilde{\Sigma}_T^{-\frac{1}{2}} = F_T^\top \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} I_{r_\kappa} & 0 \end{smallmatrix}\right] \Sigma_T^{-\frac{1}{2}} \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right] = F_T^\top\Sigma_T^{-\frac{1}{2}} \left[\begin{smallmatrix} I_{r_\kappa} & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right] = F_T^\top\Sigma_T^{-\frac{1}{2}} \left[\begin{smallmatrix} I_{r_\kappa} \\ 0 \end{smallmatrix}\right]$. Repeatedly applying the above arguments to the other error terms, the claim follows. ∎

For the case of unrestricted BT, it holds that $F_T = 0$ and $G_T = 0$ in (2.1) and (2.2). This leads to $c_{T,i} = 1$ for $i = 1, 2 \dots, \kappa$ in Theorem 2.2 which is the bound proved in [4, 6]. Consequently, the techniques in the proofs of Lemma 2.1 and Theorem 2.2 can also be used for a rather short time domain proof for the bound in [4, 6]. It is even shorter for $F_T = 0$ and $G_T = 0$ since Gronwall's lemma doesn't have to be applied.

Moreover, we observe that $\|G_T\|_2^2$ and $\|F_T^\top\|_2^2$ decay exponentially for A being Hurwitz. Hence, $c_{T,i} \to 1$ for $T \to \infty$ and for all $i = 1, 2 \dots, \kappa$. Consequently, we see that for sufficiently large T the error bound is mainly characterized by the truncated time-limited singular values. If they are small, the error is expected to be small. Therefore, it makes sense to choose the reduced order dimension r based on the truncated singular values for sufficiently large terminal times T. However, the bound in Theorem 2.2 requires to know the balancing transformation S which is practically not computed. Therefore, we provide an upper bound for the result in Theorem 2.2 in the next corollary.

COROLLARY 2.3. *Under the assumptions of Theorem 2.2 and for $T > 0$ we have*

$$\|y - y_r\|_{L_T^2} \leq 2c_T\left(\tilde{\sigma}_{T,1} + \tilde{\sigma}_{T,2} + \ldots + \tilde{\sigma}_{T,\kappa}\right)\|u\|_{L_T^2},$$

where $c_T = \mathrm{e}^{0.5\max\{\|C_o\,\mathrm{e}^{A_oT}\,Q_T^{-\frac{1}{2}}\|_2^2, \|B_o^\top\,\mathrm{e}^{A_o^\top T}\,P_T^{-\frac{1}{2}}\|_2^2\}T}$.

Proof. For $i = 1, 2, \dots, \kappa - 1$ we have

$$\max\{\|G_T\Sigma_T^{-\frac{1}{2}}\begin{bmatrix} I_{r_{i+1}} \\ 0 \end{bmatrix}\|_2^2, \|F_T^\top\Sigma_T^{-\frac{1}{2}}\begin{bmatrix} I_{r_{i+1}} \\ 0 \end{bmatrix}\|_2^2\} \leq \max\{\|G_T\Sigma_T^{-\frac{1}{2}}\|_2^2, \|F_T^\top\Sigma_T^{-\frac{1}{2}}\|_2^2\}$$

and hence $c_{T,i} \leq c_{T,\kappa} = c_T$ using Remark 1. ∎

The constant c_T can be determined practically. We will computed the above bound for an example in the next section.

3. Numerical experiments.

We test the derived L_T^2-error bound stated in Corollary 2.3 with an example that is taken from `http://slicot.org/20-site/126-benchmark-examples-for-model-reduction`. The particular example is a heat equation in a thin rod, where the corresponding data can be found in the file "heat-cont.mat". Here, the state space dimension is $n = 200$ and a system with a single input and a single output is considered. We apply time-limited BT to this example. Furthermore, we fix the final time to $T = 12$ and choose two different normalized controls. These are $u_1 = \tilde{u}_1/\|\tilde{u}_1\|_{L_T^2}$ and $u_1 = \tilde{u}_2/\|\tilde{u}_2\|_{L_T^2}$, where $\tilde{u}_1(t) = \sin(2/5\,\pi t)$ and $\tilde{u}_2(t) = \cos(2\pi t)\,\mathrm{e}^{-t}$. The outputs of the original model (1.1) and the ROM (2.5) corresponding to u_i are denoted by y^i and y_r^i, respectively ($i = 1, 2$).

One can see from Table 3.1 that the error bound is relatively tight for the heat equation example and choice of T.

4. Conclusions.

In this paper, we described the procedure of time-limited balanced truncation, a balancing related model order technique that is applied to find a good reduced system on a finite time interval $[0, T]$. $\mathcal{H}_2$-type error bounds for this scheme have already been studied. However, no bound for the output error in L_T^2 existed so far. We closed this gap in this paper which is the main contribution here. The obtained L_T^2-error bound showed that the reduced order dimension can be found based on the truncated time-limited singular values for a sufficiently large terminal time T. Moreover, the bound converges to the $\mathcal{H}_\infty$-error

r	$\left\|y^1 - y_r^1\right\|_{L_T^2}$	$\left\|y^2 - y_r^2\right\|_{L_T^2}$	$2c_T \sum_{i=r+1}^{200} \sigma_i$
2	2.91e−04	1.62e−04	4.68e−03
4	1.88e−05	1.90e−05	2.55e−04
6	2.07e−07	3.26e−07	4.13e−06
8	1.67e−08	1.93e−08	2.56e−07

Table 3.1: L_T^2-error time-limited BT and error bounds for different reduced order dimensions r; $u = u_1, u_2$ and $T = 12$.

bound of unrestricted (classical) balanced truncation such that the main result of this paper can be seen as an extension of this $\mathcal{H}_\infty$-error bound.

Appendix A. Gronwall lemma.

In this appendix, we state a version of Gronwall's lemma that we used throughout this paper.

LEMMA A.1 (Gronwall lemma). *Let $T > 0$, $z, \alpha : [0, T] \to \mathbb{R}$ be continuous functions and $\beta : [0, T] \to \mathbb{R}$ be a nonnegative continuous function. If*

$$z(t) \leq \alpha(t) + \int_0^t \beta(s) z(s) ds,$$

then for all $t \in [0, T]$, it holds that

$$z(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(w)dw\right) ds. \tag{A.1}$$

Proof. The result is shown as in [3, Proposition 2.1]. □

REFERENCES

[1] A. C. Antoulas. *Approximation of large-scale dynamical systems.* Advances in Design and Control 6. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2005.

[2] I. P. Duff and P. Kürschner. Numerical computation and new output bounds for time-limited balanced truncation of discrete-time systems. *arXiv preprint:1902.01652*, 2019.

[3] E. Emmrich. Discrete versions of Gronwall's lemma and their application to the numerical analysis of parabolic problems. Preprint No. 637, TU Berlin, 1999.

[4] D. F. Enns. Model reduction with balanced realizations: An error bound and a frequency weighted generalization. *Proceedings of the 23rd IEEE Conference on Decision and Control*, pages 127–132, 1984.

[5] W. Gawronski and J. Juang. Model reduction in limited time and frequency intervals. *Int. J. Syst. Sci.*, 21(2):349–376, 1990.

[6] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error norms. 39(6):1115–1193, 1984.

[7] S. Gugercin and A. C. Antoulas. A Survey of Model Reduction by Balanced Truncation and Some New Results. *International Journal of Control*, 77(8):748–766, 2004.

[8] P. Kürschner. Balanced truncation model order reduction in limited time intervals for large systems. *Advances in Computational Mathematics*, pages 1–24, 2018.

[9] B. C. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Trans. Autom. Control*, 26:17–32, 1981.

[10] L. Pernebo and L. M. Silverman. Model reduction via balanced state space representations. *IEEE Transactions on Automatic Control*, 27(2):382–387, 1982.

[11] M. Redmann. Energy estimates and model order reduction for stochastic bilinear systems. *International Journal of Control*, 2018.

[12] M. Redmann. A new type of singular perturbation approximation for stochastic bilinear systems. *arXiv preprint:1903.11600*, 2019.

[13] M. Redmann and P. Kürschner. An output error bound for time-limited balanced truncation. *Systems & Control Letters*, 121:1–6, 2018.