Numerical reconstruction of elastic obstacles from the far-field data of scattered acoustic waves

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Workshop on Inverse Problems for Waves: Methods and Applications

1 Direct Problem: Elastic Obstacle in Fluid.

Obstacle and wave.

due to excitation in $\Omega^c$: determine pressure and velocity of fluid in $\Omega^c$, get displacement and stress of elastic body in $\Omega$. 
Partial differential equations.

Navier equation, time-harmonic Lamé equ., reduced viscoelastodynamic equ.:

\[ \Delta u(x) + \rho \omega^2 u(x) = 0, \quad x \in \Omega, \]
\[ \Delta u(x) := \mu \Delta u(x) + (\lambda + \mu) \nabla \cdot u(x) \]

Helmholtz equation for scattered field \( p^s = p - p^{inc} \) (\( p^{inc}(x) := e^{ik_0 x} \)):

\[ \Delta p^s(x) + k_0^2 p^s(x) = 0, \quad x \in \Omega^c \]

Notation.

where traction \( t \):

\[ t := t(u) := 2\mu \frac{\partial u}{\partial n} \bigg|_x + \lambda \nabla \cdot u \bigg|_x \bigg|_\Gamma + \mu \left\{ \begin{array}{ll} n \times \left[ \nabla \times u \right] \bigg|_\Gamma & \text{if } d = 3 \\ n_2 \partial_{n_2} u_2 - n_1 \partial_{n_1} u_1 & \text{if } d = 2 \end{array} \right. \]

- \( \omega \) frequency \((\omega > 0)\)
- \( \rho \) density of body \((\rho > 0)\)
- \( \lambda, \mu \) Lamé constants \((\mu > 0, \lambda + \mu > 0)\)
- \( c \) speed of sound \((c > 0)\)
- \( k_0 \) wave number, \( k_0^2 = \omega^2/c^2 \)
- \( \rho_f \) density of fluid \((\rho_f > 0)\)
- \( n \) normal at points of \( \Gamma \) exterior w.r.t. \( \Omega \)

Boundary conditions.

Sommerfeld’s radiation condition at infinity for \( p^s \):

\[ \frac{x}{|x|} \cdot \nabla p^s(x) + ik_0 p^s(x) = o \left( |x|^{-(d-1)/2} \right), \quad |x| \to \infty \]

coupling via transmission condition:

\[ t(x) = -\left\{ p^s(x) + p^{inc}(x) \right\} n(x), \quad x \in \Gamma \]
\[ \rho_f \omega^2 u(x) \cdot n(x) = \left\{ \frac{\partial p^s(x)}{\partial n} + \frac{\partial p^{inc}(x)}{\partial n} \right\}, \quad x \in \Gamma \]

Indeed: \( \rho_f \partial_x \left\{ u(x)e^{-i\omega t} \right\} \cdot n(x) = ma = F = -\nabla \left\{ p(x)e^{-i\omega t} \right\} \cdot n(x) \)

Boundary value problem for partial differential equations. FEM.

Navier in \( \Omega \),
Helmholtz in \( \Omega_\infty \),
coupling on \( \Gamma \),

if truncated domain, then nonlocal condition on \( \Gamma_0 \): 
\[ \nabla^2 \left[ \partial_n p^s \right] + 0.5p^s - K_{0n}^2 |p^s| = 0 \]
(choose \( \Gamma_0 \) s.t. \( K_{0n}^2 \) is not an eigenvalue of \( -\Delta \) on interior of \( \Gamma_0 \))

variational formulation, FEM (cf., e.g., Márquez/Meddahi/Selgas)
Variational formulation. FEM.

$$
\mathcal{B} \left( \begin{pmatrix} u \\ p' \\ \sigma \\ v \\ q' \\ \chi \end{pmatrix} \right) = - \int_{\Gamma} p^{\text{inc}} n \cdot \mathbf{v} + \int_{\Gamma} \frac{\partial p^{\text{inc}}}{\partial n} \mathbf{q},
$$

$$\forall \mathbf{v} \in [\mathcal{H}^1(\Omega)]^d, \: q' \in \mathcal{H}^1(\Omega), \: \chi \in \mathcal{H}^{-1/2}(\Gamma_0).$$

Integral equation method. Method of fundamental solutions.

$$
\mathcal{B}(\cdots) = \int_{\Omega} \left\{ \lambda \nabla \cdot u \nabla \cdot v + \frac{\mu}{2} \sum_{i,j=1}^{d} \left[ \partial u_j \partial v_j + \partial u_i \partial v_i \right] - \rho \omega^2 u \cdot \mathbf{v} \right\} + \int_{\Gamma} p' n \cdot \mathbf{v} + \int_{\Omega} \left\{ \nabla p' \cdot \nabla q' - k_n^2 p' \mathbf{q} \right\} + \rho \omega^2 \int_{\Gamma} u \cdot n \mathbf{q} - \int_{\Gamma} \mathbf{q} \cdot \mathbf{q} + \int_{\Gamma} \left\{ V_{1\sigma}^{\text{ac}} + \frac{1}{2} \mathbf{K}_{1\sigma}^{\text{ac}} \right\} p' \mathbf{X}
$$

Acoustic potentials in nonlocal boundary value condition.

$$
K^{\text{ac}}_{1\sigma} p' (x) := \int_{\Gamma_0} \frac{\partial G^{\text{ac}}(x,y;k_w)}{\partial V(y)} p'(y) d\Gamma_0 y,
$$

$$
V^{\text{ac}}_{1\sigma} \sigma (x) := \int_{\Gamma_0} G^{\text{ac}}(x,y;k_w) \sigma(y) d\Gamma_0 y,
$$

$$
G^{\text{ac}}(x,y;k_w) := \frac{1}{4} H_0^{(1)} (k_w |x-y|)
$$

with $H_0^{(1)}$ the Hankel function of the first kind and order zero.

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Elastic potential in integral equation method.

$$
V_{1\sigma}^{\text{el}} u(x) := \int_{\Gamma} G^{\text{el}}(y,x) \Phi_{1\sigma}(y) d\Gamma_1 y,
$$

with fundamental Green’s tensor (Kupradze matrix)

$$
G^{\text{el}}(x,y) := \frac{1}{\mu} \left( G^{\text{ac}}(x,y;k_s) \delta_j + \frac{1}{k_f^2} \left[ \frac{\partial^2 (G^{\text{ac}}(x,y;k_s) - G^{\text{ac}}(x,y;k_p))}{\partial x_i \partial x_j} \right]_{i,j=1} \right)
$$

with the compressional wave number $k_s := \rho \omega^2 / (\lambda + 2\mu)$ and the shear wave number $k_p := \rho \omega^2 / \mu$

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Alternatively, representation by potentials:

$$
\begin{align*}
p &= V_{1\sigma}^{\text{el}} \phi_1 \\
u &= V_{1\sigma}^{\text{el}} \phi_0
\end{align*}
$$

Integral equations on $\Gamma$:

$$
\begin{align*}
\tau V_{1\sigma}^{\text{el}} \phi_1 + V_{1\sigma}^{\text{el}} \phi_1 n &= - p^{\text{inc}} n \\
\rho \omega^2 n \cdot V_{1\sigma}^{\text{el}} \phi_0 - \partial_n V_{1\sigma}^{\text{el}} \phi_0 &= \partial_n p^{\text{inc}}
\end{align*}
$$

(c.f., e.g., Barnett/Betoke for Helmholtz equation)

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Far-field pattern.

Far-field pattern $p^\infty$:

\[
p^f(x) = e^{ik_0|x|/|x|^{d-1/2}} \rho^\infty \left( \frac{x}{|x|} \right) + \mathcal{O} \left( \frac{1}{|x|^{(d+1)/2}} \right), \quad |x| \to \infty
\]

\[
p^\infty(x) = c_d \int_{\Gamma'} \left( [i k_0 n_x \hat{x} y + \partial_x p(y)] e^{-ik_0 y^2} dy \right.
\]

where $c_d$ is a constant ($c_2 = \frac{1}{\sqrt{2\pi}}$) and where the right-hand side of the last equation is an integral operator with smooth kernel.

Example: Non-convex curve.

non-convex curve $\Gamma'$ enclosed by circle $\Gamma''$:

For example:

\[
u_0(x_1, x_2) = 1 \frac{J_1 \left( \omega \sqrt{\frac{p}{\mu}} \sqrt{x_1^2 + x_2^2} \right)}{\sqrt{x_1^2 + x_2^2}} \left( -x_2 \right)
\]

over disc $\Omega = \{ x \in \mathbb{R}^2 : |x| < r_f \}$, where

$J_1$ Bessel function of first kind
$r_f := \frac{1}{20} \sqrt{\frac{p}{\mu}}$, $r_f := 5.135 622 301 840 682 556$,

not that $r_f$ is a root of the transcendental equ. $J'_1(x) = J_1(x)$

Far-field pattern.

Example: Non-convex curve.

real valued $x$- and $y$-components of displacement:

\[
u_0(x_1, x_2) = 1 \frac{J_1 \left( \omega \sqrt{\frac{p}{\mu}} \sqrt{x_1^2 + x_2^2} \right)}{\sqrt{x_1^2 + x_2^2}} \left( -x_2 \right)
\]

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for exceptional domains (cf. Hargé and Natroshvili/Sadunishvili/Sigua):

\[\exists \text{ eigensolutions} = \text{nontrivial solutions of homogeneous equations} \]

\[(u, p) = (u_0, 0) \text{ with Jones mode } u_0\]

\[
\Delta u_0(x) + \rho \omega^2 u_0(x) = 0, \quad x \in \Omega
\]

\[
t(u_0)(x) = 0, \quad x \in \Gamma
\]

\[
u_0(x) \cdot n = 0, \quad x \in \Gamma
\]

Jones modes.

Example of Jones mode.

Example of Jones mode.
Chosen constants.

- Frequency: \( \omega = 1.5707963267948966 \text{ kHz} \)
- Density of body: \( \rho = 6.75 \cdot 10^{-8} \text{ kg/m}^3 \)
- Lamé constant: \( \lambda = 1.287373095 \text{ Pa} \)
- Lamé constant: \( \mu = 0.66315 \text{ Pa} \)
- Speed of sound: \( c = 1500 \text{ m/s} \)
- Density of fluid: \( \rho_f = 2.5 \cdot 10^{-8} \text{ kg/m}^3 \)
- Direction of incoming plane wave: \( \nu = (1.0)^\top \)

Pressure.

Simulated pressure field:

Displacement field.

Simulated x-coordinate of displacement field:

Far field.

Far-field pattern:
Inverse Problem.

Goals of reconstruction.

Goals for a method based on:
— parametric representation of the boundary curve
— local optimization scheme

■ Suppose we know the topology of the obstacle:
Ω diffeomorphic to ball/disc
■ Suppose we know a "good" initial solution.
■ Seek a reconstruction with high precision.

■ Here, we do not consider alternative methods which determine
the topology and work well even if "good" initial solutions are not available:
cf. e.g. the sampling method for fluid-solid interaction by Monk/Selgas

Theoretical results.

■ Given the far-field pattern \(p^\infty\) for all possible directions \(v\) of incidence: Find
the shape of the obstacle.
  • D. Natroshvili, S. Kharibegashvili, and Z. Tediashvili
    uniqueness for domains:
    ∗ with simply connected complement
    ∗ with parametrizations in \(C^2, 0 < \alpha \leq 1\)
    uniqueness true even in the case of anisotropic elastic obstacle and of
    generalized Helmholtz equation in the fluid
  • P. Monk and V. Selgas
    uniqueness for domains:
    ∗ with simply connected complement
    ∗ with parametrizations in \(C^2\)
    ∗ for which \(\beta\) Jones modes
    uniqueness true even in the case where the Lamé constants depend on
the obstacle

■ Given the far-field pattern \(p^\infty\) for a single or a finite number of incidence
directions \(v\): Find the shape of the obstacle.
  • uniqueness problem open

Parametrization.

parametrization of star-shaped domain:
\[
\Gamma = \Gamma^r \ := \ \left\{ r(\hat{x}) \hat{x} : \hat{x} \in \mathbb{S}^{d-1} \right\},
\]
\[
r(\phi) = a_0 + \sum_j \left\{ a_j \cos(j \phi) + b_j \sin(j \phi) \right\}
\]

parametrization to avoid constraint \(r_i < r(\hat{x}) < r_e\):
\[
\Gamma^r := \left\{ \tilde{r}(\hat{x}) : \hat{x} \in \mathbb{S}^{d-1} \right\},
\]
\[
\tilde{r}(\hat{x}) := \frac{r_i + r_e}{2} + \frac{r_e - r_i}{\pi} \arctan \left( r_i \right)
\]

Look for unknown boundary of star-shaped domain: \(\Gamma \sim r \sim \{a_j, b_j\}\)
Mapping of inverse problem.

Fix \( p^{\text{inc}} \) and consider the mapping

\[
F : H^{2+\epsilon}(S^{d-1}) \rightarrow L^2(S^{d-1})
\]

\[ I^\gamma = r \mapsto p^\infty \]

where \( p^\infty \) is the far-field of \( p^s \) and \( p^s \) is the pressure part of the solution \((u, p^s)\) to the direct problem including the interface \( \Gamma = I^\gamma \)

given: \( p^\infty \)

find: \( r_{\text{sol}} \) s.t.

\[ F(r_{\text{sol}}) = p^\infty \]

**Lemma (Continuity)**

The “curve-to-far field” mapping \( F \) is continuous even at boundaries \( I^\gamma \) for which there exist Jones modes.

Proof.

- direct problem, boundary/transmission value problem:
  - solution \((u, p^s)\) exists
  - solution \((u, p^s)\) not unique
  - pressure component \( p^s \) is unique
- invariant subspace: subspace orthogonal to Jones modes for frequency \( \omega \):
  - unique solutions in this subspace
  - partial solution \((u(\omega'), p^s(\omega'))\) is analytic w.r.t. parameter \( \omega' \), i.e.,

\[
(u(\omega), p^s(\omega)) = \frac{1}{\pi} \int \frac{1}{\omega - \omega'} (u(\omega'), p^s(\omega')) d\omega'
\]

with \( \omega' \in \gamma \) not a Jones frequency
- pair \((u(\omega'), p^s(\omega'))\) depends continuously on curve \( I^\gamma \)

3

Reduction to Optimization Problems.

Equivalent optimization problem.

First equivalent optimization problem:

find least-squares solution \( r_{\text{min}} \) which is a minimizer of the following optimization problem:

\[
\inf_{r \in H^{2+\epsilon}(S^{d-1})} J(r)
\]

\[
J(r) := \| F(r) - p^\infty \|^2_{L^2(S^{d-1})}
\]
“Equivalent” optimization problem.

**First “equivalent” optimization problem:**

find approximate solution \( r_{\text{min}} \) which is a minimizer of the following optimization problem:

\[
\min_{r \in H^{2+\epsilon}((\mathbb{S}^2, 1))} J(r)
\]

\[
J(r) := \| F(r) - p_{\text{noisy}} \|_{L^2(\mathbb{S}^2, 1)}^2 + \gamma \| r \|_{H^2(\mathbb{S}^2, 1)}^2
\]

where \( \gamma \) is a small regularization parameter

Suppose

\[
\| p_{\text{noisy}} - p_{\text{noisy}} \|_{L^2(\mathbb{S}^2, 1)}^2 < \text{const.} \gamma
\]

**Next “equivalent” optimization problem.**

second “equivalent” optimization problem:

find approximate solution \( (r_{\text{min}}, u_{\text{min}}, p_{\text{min}}) \) which is a minimizer of the following optimization problem:

\[
\min_{r \in H^{2+\epsilon}((\mathbb{S}^2, 1)), u \in [H^2(\mathbb{S}^2, 1)], p \in \mathcal{H}} J_{\gamma}(r, u, p)
\]

\[
J_{\gamma}(r, u, p) := \| \text{far-field}(p) - p_{\text{noisy}} \|_{L^2(\mathbb{S}^2, 1)}^2 + \| \Delta u + \ldots - 0 \|_{L^2(\mathbb{S}^2, 1)}^2 + \| \Delta p + \ldots - 0 \|_{L^2(\mathbb{S}^2, 1)}^2
\]

\[
+ \| r(u) + pn \ldots n \|_{L^2(\mathbb{S}^2, 1)}^2 + \| \nabla (\partial_n p_{\text{inc}}) + \ldots \|_{L^2(\mathbb{S}^2, 1)}^2
\]

\[
+ \gamma \| r \|_{H^{2+\epsilon}(\mathbb{S}^2)}^2 + \gamma \| u \|_{H^2(\mathbb{S}^2)}^2 + \gamma \| p \|_{\mathcal{H}}^2
\]

where \( \gamma \) is a small regularization parameter

**Comparison of three approaches.**

- **First method:**
  - smallest number of optimization parameters
  - complicated objective functional
  - computation of objective functional requires solution of direct problem
  - we have implemented this FEM based Newton iteration using
    - grid generator “netgen” (cf. Schöberl)
    - solver “pardiso” (cf. Schenk/Gärtner/Fichtner)

- **Second method:**
  - huge set of optimization parameters
  - solution of direct problem not needed (good if \( \exists \) Jones mode)
  - not implemented

- **Third method:**
  - large but not huge set of optimization parameters
  - solution of direct problem not needed (good if \( \exists \) Jones mode)
  - additional difficulties due to ill-posed potential representation
  - possible: advanced algorithm with \( \Gamma_r \) and \( \Gamma_{\gamma} \) updated during iteration process (compare, e.g., You/Miao/Liu & Ivanyshyn/Kress/Serranho)
  - we have implemented this Kirsch-Kress algorithm

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Convergence of FEM based Newton iteration.

\[ \mathcal{F}(r) = \|F(r) - p_{noisy}^n\|^2_{L^2(S^d-1)} + \gamma \|r\|^2_{H^{2+\epsilon}(S^d-1)} \rightarrow \min \]

**Theorem (FEM based Newton iteration)**

Assume the noise satisfies \( p^n - p_{noisy}^n \) of optimization problem.

i) \( \gamma > 0 \): \( \exists \) unique solution \( r^*: \)
   - Subsequence \( r^i_{noisy} \) such that
     a) \( r^i_{noisy} \rightarrow r^* \) strongly in \( H^{2+\epsilon}(S^d-1) \) for \( n \rightarrow \infty \)
     b) \( r^i_{noisy} \rightarrow r^* \) weakly in \( H^{2+\epsilon}(S^d-1) \) for \( n \rightarrow \infty \)
     c) \( F(r^*) = p^\infty \)

ii) Suppose \( \exists \) solution \( r^* \):
   - Subsequence \( r^i_{noisy} \) such that
     a) \( r^i_{noisy} \rightarrow r^* \) strongly in \( H^{2+\epsilon}(S^d-1) \) for \( \gamma \rightarrow 0 \)
     b) \( r^i_{noisy} \rightarrow r^* \) weakly in \( H^{2+\epsilon}(S^d-1) \) for \( \gamma \rightarrow 0 \)

Convergence of Kirsch-Kress algorithm.

Proof.

\[ \mathcal{B} : \begin{pmatrix} \bar{\phi}_T \\ \phi \end{pmatrix} \rightarrow \begin{pmatrix} \gamma \nabla \cdot \bar{\psi} + \nabla \times \nabla \times \bar{\psi} \\ \rho_j \omega^2 n \cdot \nabla \bar{\phi} - \nabla \cdot \nabla \times \nabla \phi \end{pmatrix} \]

\[ \mathcal{B} : [L^2(\Gamma_\varepsilon)]^d \times L^2(\Gamma) \rightarrow [L^2(\Gamma)]^d \times L^2(\Gamma) \]

Essential lemma: The image \( \text{im} \mathcal{B} \) of operator \( \mathcal{B} \) is dense in the subspace

\[ \{ \langle \bar{\phi}, \phi \rangle \in [L^2(\Gamma)]^d \times L^2(\Gamma) : \langle \bar{\phi}, u_0 \rangle = 0, \forall u_0 \text{ Jones mode} \} \}

Derivatives and Quadrature.

Derivatives for gradient based optimization schemes

- derivatives for FEM based Newton method:
  - shape optimization techniques
  - get gradients by solving the FEM system of the direct problem with new right-hand side
  - 2D case: direct solver with LU factorization

- derivatives for Kirsch-Kress method:
  - reduces to simple differentiation of Green’s kernels
  - fourth order derivatives of Helmholtz kernel

Quadratures for Kirsch-Kress method

- no quadrature!
  - layer functions \( \phi_b \) and \( \phi_{b*} \) in \( H^{-1}(\Gamma) \) and \( [H^{-1}(\Gamma_\varepsilon)]^2 \), respectively
  - layer functions \( \phi_{b*} \) and \( \phi_{b*} \): linear combinations of Dirac \( \delta \) functions
Optimization algorithm.

Which optimization scheme?

- FEM based Newton method:
  - small number of parameters
  - Gauss-Newton method

- Kirsch-Kress method:
  - larger number of parameters
  - conjugate gradient method (nonlinear variant)
  - Gauss-Newton and Levenberg-Marquardt method: solve linear systems of dimension larger than those of the direct solver

Scaling.

Scaling of optimization scheme

- number of necessary iteration depends on conditioning of optimization problem
- natural scaling
  - scale far-field values in accordance with measurement uncertainties
  - scale parameters in accordance with the reconstruction requirements
- scaling for a fast iterative solution adapted in accordance with numerical tests
  - calibration constants before the several terms of the objective functional (e.g. constant $c$ and regularization parameter $\gamma$)
  - replace optimization parameters by multiple of parameters in order to get gradients with components of equal size

$$r = r/c_r, \quad \phi_i = \phi_i/c_i, \quad \phi_e = \phi_e/c_e$$

Reconstruction of egg domain.

initial solution and egg domain:

Fourier coefficients:

$$a_0 = 0$$
$$a_1 = -1, \quad a_2 = 0.1, \quad a_3 = 0.01, \quad a_4 = -0.001, \quad a_5 = 0.0001$$
$$b_1 = 1, \quad b_2 = 0.1, \quad b_3 = 0.01, \quad b_4 = 0.001, \quad b_5 = 0.0001$$
Reconstruction of non-convex domain.

initial solution and non-convex domain:

Fourier coefficients:

\[
\begin{align*}
a_0 &= 0 \\
a_1 &= 1 \\
a_2 &= 0.10 \\
a_3 &= 0.04 \\
a_4 &= 0.016 \\
a_5 &= 0.008 \\
b_1 &= -1 \\
b_2 &= 0.02 \\
b_3 &= -1.500 \\
b_4 &= -0.010 \\
b_5 &= 0.008
\end{align*}
\]

Data and Scaling.

far-field data:

- simulated far-field data
- computed in 80 uniformly distributed direction
- FEM computation over finer FEM triangulation (meshsize smaller at least by factor 0.25)

scaling:

- for Kirsch-Kress method with 44 discretization points on each curve: \(c = 4.000, c_r = 1, c_j = 0.1, c_e = 0.005\)

Reconstruction results.

Results:

- Good reconstruction with FEM based Newton method for both domains even without regularization terms \((\gamma = 0)\)
- Good reconstruction with Kirsch-Kress method for egg domain
- No reconstruction with Kirsch-Kress method for non-convex domain
  - Regularized solution of direct problem obtained with the Tikhonov term in our optimization scheme is bad
  - Regularization with truncated SVD representation and for a suitable very small range of regularization parameters: reasonable regularized solution of direct problem
  - Inverse crime (compute far-field data via integral equations): Good reconstruction with Kirsch-Kress method for non-convex domain
- Good reconstruction with Kirsch-Kress method for non-convex domain if curves \(\Gamma_i\) and \(\Gamma_e\) are close to unknown curve \(\Gamma\)

Reconstruction by FEM based Newton method.

Convergence of FEM based Newton

far-field data simulated by FEM computation on higher level:

reconstruction error \(err := \|r - r_{FEM}\|_{L^\infty}\) and number of iterations \(it\) depending on meshsize \(h\) of FEM discretization

\[
\begin{array}{ccc}
h & err & it \\
0.5 & 1.2596 & 0 \\
0.25 & 0.0759 & 6 \\
0.125 & 0.0247 & 8 \\
0.0625 & 0.00876 & 10 \\
0.03125 & 0.00156 & 10 \\
0.01562 & 0.000726 & 15 \\
0.0078125 & 0.000363 & 18 \\
0.00390625 & 0.000157 & 18 \\
\end{array}
\]

egg domain

non-convex domain
Reconstruction by Kirsch-Kress method for egg domain.

Different optimization schemes for Kirsch-Kress method

Method of conjugate gradients: too slow or different limit
Gauss-Newton method with regularization: GNw
Levenberg-Marquardt method with regularization: LMw
Levenberg-Marquardt method “without” regularization: LMo
(code and standard choice of parameters by M. Lourakis)

<table>
<thead>
<tr>
<th>points on ( \Gamma )</th>
<th>( \gamma )</th>
<th>GNw</th>
<th>LMw</th>
<th>LMo</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>( 4 \times 10^{-8} )</td>
<td>1.2596 (0)</td>
<td>1.2596 (0)</td>
<td>1.2596 (0)</td>
</tr>
<tr>
<td>44</td>
<td>( 2.5 \times 10^{-13} )</td>
<td>0.05427 (13)</td>
<td>0.05461 (30)</td>
<td>0.06793 (30)</td>
</tr>
<tr>
<td>88</td>
<td>( 4 \times 10^{-14} )</td>
<td>0.0002136 (13)</td>
<td>0.002007 (320)</td>
<td>0.002095 (320)</td>
</tr>
</tbody>
</table>

Error \( ||r - r_{KK}||_{L^\infty} \) and number of iterations for Kirsch-Kress method egg domain

Reconstruction by Kirsch-Kress method for non-convex domain.

- What shall we do with an initial solution like the disc?
  - curves \( \Gamma_i \) and \( \Gamma_e \) must be close to iterative solution:
    - update \( \Gamma_i \) and \( \Gamma_e \) during the iterative solution process
    - choose \( \Gamma_i \) and \( \Gamma_e \) by their radial functions:

\[
\begin{align*}
  r_i &= r - \frac{1}{2} \\
  r_e &= r + \frac{1}{2}
\end{align*}
\]

with \( r \) the radial function of the last iterative solution \( \Gamma = \Gamma^* \)

- Note that this is the setting for which the Kirsch-Kress method is convergent according to the previous test!
  - for fixed \( \Gamma_i \) and \( \Gamma_e \):
    - perform one step of Gauss-Newton method, but reduce the iteration step s.t. solution curve remains between \( \Gamma_i \) and \( \Gamma_e \)
  - if the iteration step remains small:
    - fix \( \Gamma_i \) and \( \Gamma_e \), and perform more Gauss-Newton steps

For \( \gamma = 10^{-8}, c = 10^6, c_\tau = 1, c_\gamma = 1, c_\nu = 0.2 \), and 352 discretization points on each curve:

- initial deviation of radial functions: 0.296
- number of iterations: 11
- reconstruction error: 0.000 279

1st step: initial solution and first iterate of Gauss-Newton method

Reconstruction by Kirsch-Kress method for non-convex domain.
Reconstruction by Kirsch-Kress method for non-convex domain.

2nd step: initial solution and first iterate of Gauss-Newton method

3rd step: initial solution and first iterate of Gauss-Newton method

4th step: initial solution and first iterate of Gauss-Newton method

5th step: initial solution and first iterate of Gauss-Newton method
Reconstruction by Kirsch-Kress method for non-convex domain.

6th step: initial solution and first iterate of Gauss-Newton method

7th step: initial solution and first iterate of Gauss-Newton method

8th step: initial solution and first iterate of Gauss-Newton method

9th step: initial solution and first iterate of Gauss-Newton method
Reconstruction by Kirsch-Kress method for non-convex domain.

Last step: 10 Gauss-Newton iterations

Noisy data.

Perturbation of far-field data

perturbed far-field data of egg domain: Add random number uniformly distributed in $[-\epsilon, \epsilon]$

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$|\gamma - \gamma_{FEM}|_\infty$</th>
<th>$|\gamma - \gamma_{KK}|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.</td>
<td>0.001568</td>
<td>0.002136</td>
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<tr>
<td>0.001</td>
<td>0.002637</td>
<td>0.003640</td>
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<tr>
<td>0.005</td>
<td>0.007156</td>
<td>0.02041</td>
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<tr>
<td>0.01</td>
<td>0.01368</td>
<td>0.05686</td>
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<tr>
<td>0.05</td>
<td>0.05433</td>
<td>0.09997</td>
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<tr>
<td>0.1</td>
<td>0.1087</td>
<td></td>
</tr>
</tbody>
</table>

Kirsch-Kress with 44 points on $\Gamma$

Reconstruction of curve with 14 Fourier coefficients.

Curve with 14 Fourier coefficients: Reconstruction with only 10 coefficients.

- additional non-zero coefficients for the non-convex domain: $a_6 = 0.004$, $a_7 = 0.001$, $b_6 = -0.004$, and $b_7 = 0.001$
- radial deviation of curve with 14 Fourier non-zero coefficients to curve with 10 is 0.0075
- initial solution $a_i^{ini} = 0.75 a_i$ (for $a_i^{ini} = 0$: convergence only for $h = 0.03125$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$err$</th>
<th>it</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.1147</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>0.03812</td>
<td>7</td>
</tr>
<tr>
<td>0.125</td>
<td>0.01878</td>
<td>7</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.01688</td>
<td>7</td>
</tr>
<tr>
<td>0.03125</td>
<td>0.01678</td>
<td>7</td>
</tr>
</tbody>
</table>

FEM based Newton iteration for non-convex domain with 14 coefficients

Kirsch-Kress method for $\Gamma_i$ and $\Gamma_e$ close to $\Gamma$: reduces radial deviation error to 0.000898 after 12 iteration

Reconstruction of domain with Jones modes.

initial solution and non-convex domain:

- Kirsch-Kress algorithm:
  - nmb-discr.pnts. 176, $\gamma = 4 \cdot 10^{-14}$, $c = 200$, $c_f = 1$, $c_i = 5$, $c_e = 0.05$
  - initial deviation 1.26, 8 iterations, reconstruction error 0.000814
- FEM based Newton iteration:
  - direct solver “pardiso” yields partial solution of variational system,
  - initial deviation 1.26, 13 iterations, reconstruction error 0.000492

Numerical reconstruction of elastic obstacles  WIAS, March 29, 2010 · Page 57 (64)
Conclusions.

Advantage of Kirsch-Kress method:
- high accuracy of reconstruction.
- fast computation.

Disadvantages of Kirsch-Kress method:
- sensitive to small perturbations of the far-field data.
- sensitive to scaling of optimization problem: a lot of parameters to be adapted.
- ill-posed integral equations: The Kirsch-Kress method does not require the solution of the direct problem. However, it works only if the direct problem is solvable by the ill-posed integral equations. To decrease the ill-posedness, the outer and inner curve should be chosen closer to the unknown curve.
- The conjugate gradient method is too slow for the Kirsch-Kress method. Advanced optimization schemes solve linear systems of equations which are larger than those of the direct solvers.

Algorithms work also for domains with Jones modes.

References.

Uniqueness results:


Numerical schemes:


