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ON CONVERGENCE AND STABILITY OF THE EXPLICIT DIFFERENCE METHOD FOR SOLUTION OF NONLINEAR SCHRÖDINGER EQUATIONS*

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Abstract. The first boundary value problem for a nonlinear Schrödinger equation is investigated. The conditional convergence and stability on the initial data of the explicit three-level difference scheme of DuFort–Frankel type in C and W_2^1 norms are proved. Grid analogues of energy conservation laws and grid multiplicative inequalities are used.

Key words. nonlinear Schrödinger equation, boundary problem, DuFort–Frankel, difference scheme

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1. Introduction.

We consider the first boundary value problem for a nonlinear Schrödinger-type equation. Such equations appear, for example, in many models of nonlinear optics [1] and in models of energy transfer in molecular systems [2]. They are also used in quantum mechanics, seismology, plasma physics, and other fields of science.

There are a lot of studies on the numerical solution of initial and initial-boundary problems for the nonlinear Schrödinger equations [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. We are interested in finite difference methods that have some grid analogues of conservation laws. The importance of these discrete laws is discussed in [3]. For example, the difference schemes of Crank–Nicolson type have such property. In [4, 5, 6] unconditional convergence and stability are proved for the schemes of this type. But, unfortunately, these schemes are implicit.

On the contrary, many explicit schemes are unstable—for example, Euler schemes [7]. Some Euler-type schemes are conditionally stable. A three-layered explicit difference scheme of DuFort–Frankel type is also conditionally stable. This scheme was introduced for the Schrödinger equation in [7, 8, 9]. The consistency of this scheme requires the condition $\tau/h \to 0$, where τ and h are time and space grid steps.

In [8, 9] the linear Schrödinger equations were investigated and stability of the schemes was proved. In [7] nonlinear equations were also discussed, and the grid analogue of the conservative law in the space L_2 was obtained. But there was no proof of the convergence and stability of the difference scheme. Thus, our paper sufficiently extends the results of [7, 8, 9].

As we know, the finite difference scheme of DuFort–Frankel type has not yet been investigated widely and fully for the nonlinear Schrödinger equation. It seems also that the schemes of this type could be implemented for parallel computations. This fact also increases the importance and actuality of our paper.

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We would like to draw attention to the novelty of the method, which was used during the investigation of the problem. This method was based on the new type of a priori estimates and was developed in [4, 5, 6]. In these papers the method was used for two-layered implicit difference schemes for the nonlinear Schrödinger equation. The method has allowed one to avoid restriction on time and space grid steps.

In the present case the restriction on these grid steps appears due to approximation and stability properties of the scheme. Unless it is not possible to avoid these restrictions, the method works well and helps to prove convergence and stability of the schemes even in the case of three-layered schemes.

In the case of cubic nonlinearity we have obtained the analogues of conservation laws in the spaces L_2 and W_2^1. In the more general case we have a new type of a priori estimate. Under the condition $\tau/h^2 \le \nu < 1/2|a|$, where a is the constant from the equation and ν is some arbitrary constant, the convergence and stability of the difference scheme in the norms of spaces C and W_2^1 were proved.

In section 2 we formulate the problem and prove the grid analogues of the embedding theorem and the multiplicative inequality. In section 3 we prove the grid analogues of the conservation laws for a solution of the cubic Schrödinger equation. In section 4 we prove the convergence and stability of the difference scheme for the cubic Schrödinger equation in the spaces L_2 and C. Finally, in section 5 we prove the convergence and stability on initial data of the difference scheme in the spaces C and W_2^1 for more general nonlinearity.

2. Statement of the problem: Auxiliary statements. We consider the first initial-boundary value problem for the cubic Schrödinger equation

$$\frac{\partial u}{\partial t} = ia\frac{\partial^2 u}{\partial x^2} - i\lambda|u|^2u, \qquad (x,t) \in Q, \tag{2.1}$$

with initial and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0;T], \qquad u(x,0) = u_0(x), \quad x \in [0;1]. \tag{2.2}$$

Here $i = \sqrt{-1}$, $Q = (0;1) \times (0;T)$, a, λ are real constants, $a \neq 0$; $u(x,t)$ is a complex-valued function.

We define the inner product between two functions $v(x)$ and $w(x)$ as

$$(v,w) = \int_0^1 v(x)w^*(x)dx.$$

Let L_p and W_2^1 denote the Sobolev spaces of complex-valued functions with the norms

$$\|v\|_{L_p} = \Big(\int_0^1 |v(x)|^p dx\Big)^{1/p}, \qquad \|v\|_{W_2^1} = \Big(\|v\|_{L_2}^2 + \Big\|\frac{\partial v}{\partial x}\Big\|_{L_2}^2\Big)^{1/2}.$$

Here $w^*(x)$ is the complex conjugate of $w(x)$. Also we define $\operatorname{Re} v$ and $\operatorname{Im} v$ as real and imaginary parts of the complex expression v.

Let $C(\bar{Q})$ be the space of continuous functions with the norm

$$\|v\|_{C(\bar{Q})} = \max_{(x,t)\in\bar{Q}} |v(x,t)|.$$

It is well known that the solution of the problem (2.1), (2.2) satisfies the following conservation laws for all $t \in [0,T]$:

$$\|u(t)\|_{L_2} = I_1(t) = I_1(0); \tag{2.3}$$

$$\left\|\frac{\partial u}{\partial x}(t)\right\|_{L_2}^2 + (\lambda/2a)\|u(t)\|_{L_4}^4 = I_2(t) = I_2(0). \tag{2.4}$$

We introduce a uniform grid with steps τ and h in the domain $\bar{Q}$: $\bar{Q}_h = \bar{\omega}_h \times \bar{\omega}_\tau$ and $Q_h = \omega_h \times \omega_\tau$. We consider that $\tau = T/M$, $t_j = j\tau$, $h = 1/N$, $x_l = lh$, $\bar{\omega}_\tau = \{t_j; j = 0, \dots, M\}$, $\omega_\tau = \{t_j; j = 1, \dots, M-1\}$, $\bar{\omega}_h = \{x_l; l = 0, \dots, N\}$, $\omega_h = \{x_l; l = 1, \dots, N-1\}$.

We shall use the grid analogues of the Sobolev spaces L_{ph} and W_{2h}^1. C_h denotes the analogue of the space $C(\bar{Q})$. Let us define inner products at the grid $\bar{\omega}_h$,

$$(u, v) = \sum_{l=1}^{N-1} u_l v_l^* h, \qquad (u, v] = \sum_{l=1}^{N} u_l v_l^* h.$$

The norms in this grid are defined as follows:

$$\|u\|_{L_{ph}}^p = \sum_{l=1}^{N-1} |u_l|^p h, \quad \|u]|^2 = (u, u], \quad \|u\|^2 = (u, u), \quad \|u\|_{W_{2h}^1}^2 = \|u\|^2 + \|u_{\bar{x}}]|^2.$$

We denote $p = p_l^j = p(x_l, t_j)$, $\hat{p} = p_l^{j+1}$, $\check{p} = p_l^{j-1}$, $\dot{p} = (\check{p} + \hat{p})/2$, $p_+ = p_{l+1}^j$, $p_- = p_{l-1}^j$, $\bar{p} = p_- + p_+$, $p_t = (\hat{p} - \check{p})/2\tau$, $p_{\bar{x}} = (p - p_-)/h$, $p_{x\nwarrow} = (\check{p} - p_-)/h$, $p_{x\swarrow} = (p - \check{p}_-)/h$.

We shall prove grid analogues of one embedding theorem and a multiplicative inequality.

LEMMA 2.1. *Let $v_0 = \hat{v}_0 = v_N = \hat{v}_N = 0$. Then*

$$\max\left\{\|\hat{v}\|_{C_h}, \|v\|_{C_h}\right\} \le 0.5\left(\|\hat{v}_{x\nwarrow}]| + \|\hat{v}_{x\swarrow}]|\right). \tag{2.5}$$

Proof. We denote by $v_{1;l}$ and $v_{-1;l}$ the grid functions $\hat{v}_l$ and v_l. Then, for all $l = 1, \dots, N-1$, we have

$$|v_{1;l}| = \left|\sum_{k=-l}^{-1} (v_{(-1)^{k+1};l+k+1} - v_{(-1)^k;l+k})\right| \le \sum_{k=-l}^{-1} \left|v_{(-1)^{k+1};l+k+1} - v_{(-1)^k;l+k}\right|.$$

Similarly we may write

$$|v_{1;l}| = \left|\sum_{k=0}^{N-l-1} (v_{(-1)^{k+1};l+k+1} - v_{(-1)^k;l+k})\right| \le \sum_{k=0}^{N-l-1} \left|v_{(-1)^{k+1};l+k+1} - v_{(-1)^k;l+k}\right|.$$

Summing these two inequalities, we obtain

$$2|v_{1;l}| \le \sum_{k=-l}^{N-l-1} \left|v_{(-1)^{k+1};l+k+1} - v_{(-1)^k;l+k}\right| = \sum_{k=1}^{N} \left|v_{(-1)^{k-l};k} - v_{(-1)^{k-l-1};k-1}\right|.$$

Similarly we have

$$2|v_{-1;l}| \le \sum_{k=1}^{N} \left|v_{(-1)^{k-l+1};k} - v_{(-1)^{k-l};k-1}\right|.$$

From the estimate $\max\{\|\hat{v}\|_{C_h}, \|v\|_{C_h}\} \leq \max_l (|v_l| + |\hat{v}_l|)$ and from the last two inequalities it follows that

$$2\max\{\|\hat{v}\|_{C_h}, \|v\|_{C_h}\} \leq \sum_{k=1}^{N} |\hat{v}_{k;x\nwarrow}|\, h + \sum_{k=1}^{N} |\hat{v}_{k;x\swarrow}|\, h.$$

The estimate (2.5) follows from Cauchy's inequality. The lemma is proved.

In an analogous manner we prove a grid multiplicative inequality.

LEMMA 2.2. *Let $v_0 = \hat{v}_0 = v_N = \hat{v}_N = 0$. Then*

$$\max\{\|\hat{v}\|^2_{C_h}, \|v\|^2_{C_h}\} \leq 0.5\big(\|\hat{v}\| + \|v\|\big)\big(\|\hat{v}_{x\nwarrow}]| + \|\hat{v}_{x\swarrow}]|\big). \tag{2.6}$$

Proof. For all $l = 1, \ldots, N-1$, we have

$$|v_{1;l}|^2 = \big||v_{(-1)^0;l}|^2 - |v_{(-1)^{-l};l-l}|^2\big| = \left|\sum_{k=-l}^{-1} \big(|v_{(-1)^{k+1};l+k+1}|^2 - |v_{(-1)^k;l+k}|^2\big)\right|$$
$$\leq \sum_{k=-l}^{-1} \big||v_{(-1)^{k+1};l+k+1}|^2 - |v_{(-1)^k;l+k}|^2\big|.$$

Similarly we may write

$$|v_{1;l}|^2 \leq \sum_{k=0}^{N-l-1} \big||v_{(-1)^{k+1};l+k+1}|^2 - |v_{(-1)^k;l+k}|^2\big|.$$

Summing these two inequalities we obtain

$$2|v_{1;l}|^2 \leq \sum_{k=1}^{N} \big||v_{(-1)^{k-l};k}|^2 - |v_{(-1)^{k-l-1};k-1}|^2\big|.$$

Similarly we have

$$2|v_{-1;l}|^2 \leq \sum_{k=1}^{N} \big||v_{(-1)^{k-l+1};k}|^2 - |v_{(-1)^{k-l};k-1}|^2\big|.$$

From the last two inequalities it follows that

$$2(|v_{-1;l}|^2 + |v_{1;l}|^2) \leq \sum_{k=1}^{N} \big||v_{-1;k}|^2 - |v_{1;k-1}|^2\big| + \sum_{k=1}^{N} \big||v_{1;k}|^2 - |v_{-1;k-1}|^2\big|.$$

Note that

$$\sum_{k=1}^{N} \big||v_{(-1)^j;k}|^2 - |v_{(-1)^{j+1};k-1}|^2\big| = \sum_{k=1}^{N} \left|\frac{|v_{(-1)^j;k}| - |v_{(-1)^{j+1};k-1}|}{h}\right|$$
$$\times \big(|v_{(-1)^j;k}| + |v_{(-1)^{j+1};k-1}|\big)\, h \leq \left(\sum_{k=1}^{N} \left|\frac{v_{(-1)^j;k} - v_{(-1)^{j+1};k-1}}{h}\right|^2 h\right)^{1/2} (\|v_{-1}\| + \|v_1\|).$$

The estimate (2.6) follows from here. The lemma is proved.

In the following we shall use some well-known inequalities.

The embedding theorems for the grid functions v, $v_0 = v_N = 0$ [14] are

$$\|v\|_{L_{ph}} \le \|v\|_{C_h} \le 0.5\|v_{\bar{x}}]| \le 0.5\|v\|_{W^1_{2h}}. \tag{2.7}$$

A grid analogue of the Gronwall inequality [15] is

$$Y_j \le \left(\bar{Y}_0 + 2et_j \max_{0\le l<j}\{b_l\}\right)\exp(4dt_j). \tag{2.8}$$

Here $Y \ge 0$ and $b \ge 0$ are defined on the grid $\bar{\omega}_\tau$, $Y_0 \le \bar{Y}_0$; $e \ge 0$, $0 < \tau d \le 1/2$ and for all $j = 1, \dots, M$ the following inequality holds:

$$Y_j \le \bar{Y}_0 + \tau d \sum_{l=0}^{j-1}(Y_l + Y_{l+1}) + \tau e \sum_{l=0}^{j-1} b_l.$$

3. The difference scheme. Grid conservation laws. We relate the problem (2.1), (2.2) with the following DuFort–Frankel-type difference scheme:

$$p_t = ia\frac{\bar{p} - 2\dot{p}}{h^2} - i\lambda|p|^2\dot{p}, \qquad (x,t) \in Q_h, \tag{3.1}$$

$$p(x_0,t) = p(x_N,t) = 0, \quad t \in \bar{\omega}_\tau, \qquad p(x,0) = u_0(x), \quad x \in \bar{\omega}_h. \tag{3.2}$$

The solution on the first layer t_1 can be found using some two-layered scheme.

In [7] one case of a grid analogue of (2.3) for the difference scheme (3.1), (3.2) was investigated. In this section we will prove grid analogues of (2.3) and (2.4).

LEMMA 3.1 (grid analogue of (2.3)). *The equality*

$$\|p(t_{j+1})\|^2 + \|p(t_j)\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}(t_j), p(t_{j+1})) = I_{1h}(t_j) = I_{1h}(t_0) \tag{3.3}$$

is valid for the solution of difference scheme (3.1), (3.2) *for all* $j = 0, \dots, M-1$. *Let the condition on the ratio of time and space grid steps*

$$0 < \frac{2|a|\tau}{h^2} \le \nu < 1, \tag{3.4}$$

where ν *is an arbitrary constant, be satisfied. Then the following estimate is valid:*

$$\|p(t_{j+1})\|^2 + \|p(t_j)\|^2 \le \mu\left(\|p(t_1)\|^2 + \|p(t_0)\|^2\right). \tag{3.5}$$

Here and later $\mu = \frac{1+\nu}{1-\nu}$.

Proof. We take the inner product on both sides of (3.1) with $4\tau\dot{p}$. The real part of the obtained equality is

$$\operatorname{Re}(\hat{p} - \check{p}, \hat{p} + \check{p}) = -\frac{4a\tau}{h^2}\operatorname{Im}(\bar{p}, \dot{p}) + \frac{8a\tau}{h^2}\operatorname{Im}\|\dot{p}\|^2 + 4\lambda\tau\operatorname{Im}\sum_{i=1}^{N-1}|p_i|^2|\dot{p}_i|^2 h.$$

Thus,

$$\|\hat{p}\|^2 - \|\check{p}\|^2 + \frac{4a\tau}{h^2}\operatorname{Im}(\bar{p}, \dot{p}) = 0$$

and

$$\|\hat{p}\|^2 + \|p\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{p},\hat{p}) = \|p\|^2 + \|\check{p}\|^2 - \frac{2a\tau}{h^2}\operatorname{Im}(\bar{p},\check{p}).$$

Summing these equalities for time layers, and noticing that $\operatorname{Im}(\bar{p},\check{p}) = -\operatorname{Im}(\check{p},\bar{p})$ and

$$(\check{p},\bar{p}) = \sum_{l=1}^{N-1}\check{p}_l(p^*_{l-1}+p^*_{l+1})h = \sum_{l=0}^{N-2}\check{p}_{l+1}p^*_l h + \sum_{l=2}^{N}\check{p}_{l-1}p^*_l h = (\bar{\check{p}},p)$$

we obtain (3.3).

Now we can easily evaluate

$$|\operatorname{Im}(\bar{p}(t_k),p(t_{k+1}))| \le \|p(t_k)\|^2 + \|p(t_{k+1})\|^2,$$

and, using (3.4), we can write

$$\left|\frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}(t_k),p(t_{k+1}))\right| \le \nu(\|p(t_k)\|^2 + \|p(t_{k+1})\|^2).$$

The estimate (3.5) immediately follows from here. The lemma is proved.

LEMMA 3.2 (grid analogue of (2.4)). *The equality*

$$\|p_{x^{\nwarrow}}(t_{j+1})]|^2 + \|p_{x_{\swarrow}}(t_{j+1})]|^2 + \frac{\lambda}{a}\|p(t_{j+1})p(t_j)\|^2 = I_{2h}(t_j) = I_{2h}(t_0) \tag{3.6}$$

is valid for the solution of the difference scheme (3.1), (3.2) *for all* $j = 0,\dots,M-1$.

Proof. We take the inner product on both sides of (3.1) with $\hat{p}-\check{p}$. The imaginary part of the obtained equality is

$$\frac{1}{2\tau}\operatorname{Im}\|\hat{p}-\check{p}\|^2 = \frac{a}{h^2}\operatorname{Re}\bigl(\bar{p}-2\dot{p},\hat{p}-\check{p}\bigr) - \frac{\lambda}{2}\operatorname{Re}\bigl(|p|^2(\hat{p}+\check{p}),\hat{p}-\check{p}\bigr).$$

Thus,

$$\frac{2}{h^2}\operatorname{Re}\bigl(2\dot{p}-\bar{p},\hat{p}-\check{p}\bigr) + \frac{\lambda}{a}\left(\|p\hat{p}\|^2 - \|\check{p}p\|^2\right) = 0.$$

Note that

$$\frac{2}{h^2}\operatorname{Re}(2\dot{p}-\bar{p},\hat{p}-\check{p}) = \frac{2}{h^2}\left(\|\hat{p}\|^2 - \|\check{p}\|^2 - \operatorname{Re}(\bar{p},\hat{p}) + \operatorname{Re}(\check{p},\bar{p})\right).$$

Since $(\check{p},\bar{p}) = (\bar{\check{p}},p)$, it follows that

$$\frac{2}{h^2}\operatorname{Re}(2\dot{p}-\bar{p},\hat{p}-\check{p}) = \frac{2}{h^2}\left(\left(\|\hat{p}\|^2 - \operatorname{Re}(\bar{p},\hat{p}) + \|p\|^2\right) - \left(\|p\|^2 - \operatorname{Re}(\bar{\check{p}},p) + \|\check{p}\|^2\right)\right).$$

Using the condition (3.2), we notice that

$$2\left(\|\hat{p}\|^2 - \operatorname{Re}(\bar{p},\hat{p}) + \|p\|^2\right) = \sum_{l=1}^{N}\left(|p_{l-1}|^2 - 2\operatorname{Re}(p_{l-1}\hat{p}^*_l) + |\hat{p}_l|^2\right)h$$
$$+ \sum_{l=0}^{N-1}\left(|p_{l+1}|^2 - 2\operatorname{Re}(p_{l+1}\hat{p}^*_l) + |\hat{p}_l|^2\right)h = \sum_{l=1}^{N}|p_{l-1}-\hat{p}_l|^2 h + \sum_{l=0}^{N-1}|p_{l+1}-\hat{p}_l|^2 h.$$

Therefore,

$$\frac{2}{h^2}\left(\|\hat{p}\|^2 - \mathrm{Re}(\bar{p},\hat{p}) + \|p\|^2\right) = \|\hat{p}_{x\nwarrow}]|^2 + \|\hat{p}_{x\swarrow}]|^2.$$

Now we write

$$\|\hat{p}_{x\nwarrow}]|^2 + \|\hat{p}_{x\swarrow}]|^2 + \frac{\lambda}{a}\|p\hat{p}\|^2 = \|p_{x\nwarrow}]|^2 + \|p_{x\swarrow}]|^2 + \frac{\lambda}{a}\|\check{p}p\|^2,$$

and by summing these equalities for time layers we obtain (3.6). The lemma is proved.

4. Convergence and stability of the difference scheme. Suppose that the solution of the problem (2.1), (2.2) is smooth enough to satisfy the approximation of the difference scheme. Let $\Phi(t_j)$ be a truncation error. It is easy to find that this error is of order $O(\tau^2 + h^2 + (\tau/h)^2)$. Thus, the consistency of the scheme requires the condition $\tau/h \to 0$ to be fulfilled.

Suppose that the solution of the problem (2.1), (2.2) is also smooth enough to satisfy the following conditions:

$$\max_{t_j\in\omega_\tau}\{\|\Phi(t_j)\|_{L_{2h}}\} \to 0, \qquad \tau, h \to 0, \tag{4.1}$$

and

$$M_1 = \max_{t\in[0,T]}\|u(t)\|_{W_2^1} < \infty, \quad M_2 = \max_{t\in[0,T]}\left\|\frac{\partial u}{\partial t}(t)\right\|_{L_2} < \infty. \tag{4.2}$$

From here and from the embedding theorem $\overset{\circ}{W}{}_2^1 \to C$ it follows that

$$\max_{t\in[0,T]}\|u(t)\|_{L_2} \le \|u\|_{C(\bar{Q})} \le 0.5M_1. \tag{4.3}$$

Let $\varepsilon = u - p$ be an error of the solution. Then we have the following difference scheme for this error:

$$\varepsilon_t = \frac{ia}{h^2}(\bar{\varepsilon} - 2\dot{\varepsilon}) + \Psi + \Phi, \qquad (x,t)\in Q_h, \tag{4.4}$$

$$\varepsilon(x,0) = 0, \quad x\in\bar{\omega}_h, \qquad \varepsilon(x_0,t) = \varepsilon(x_N,t) = 0, \quad t\in\bar{\omega}_\tau. \tag{4.5}$$

Here

$$\Psi = -i\lambda\left(|u|^2\dot{u} - |p|^2\dot{p}\right).$$

Suppose that the function ε on the first layer satisfies the condition

$$\|\varepsilon(t_1)\|^2_{W^1_{2h}} \to 0, \qquad \tau, h \to 0. \tag{4.6}$$

We shall prove one more auxiliary lemma.

LEMMA 4.1. *Suppose that the conditions* (3.4) *and* (4.6) *are satisfied. Then there exist constants* $\tau_0 > 0$ *and* $h_0 > 0$ *such that for all positive* $\tau \le \tau_0$ *and* $h \le h_0$ *the following estimates for the solution of the problem* (3.1), (3.2) *are valid:*

$$\max_{1\le j\le M}(\|p_{x\nwarrow}(t_j)]| + \|p_{x\swarrow}(t_j)]|) \le M_3, \tag{4.7}$$

$$\|p\|_{C(\bar{Q}_h)} = \max_{(x_l,t_j)\in\bar{Q}_h} |p(x_l,t_j)| \le 0.5M_3. \tag{4.8}$$

Here $M_3 = M_3(a,\lambda,M_1,\nu)$.

Proof. At first we consider the case $\lambda/a \ge 0$. It is not difficult to derive the equality

$$\|\hat{p}_{x\nwarrow}]|^2 + \|\hat{p}_{x\swarrow}]|^2 = \|\hat{p}_{\bar{x}}]|^2 + \|p_{\bar{x}}]|^2 + \frac{2\tau^2}{h^2}\,\mathrm{Re}\Big(\frac{\hat{p}-p}{\tau},\frac{\hat{p}_- - p_-}{\tau}\Big). \tag{4.9}$$

Using the conditions (4.6), (4.9), (4.2), (2.7), we evaluate $I_{2h}(t_0)$, when τ and h are positive and small enough:

$$\|p_{x\nwarrow}(t_1)]|^2 + \|p_{x\swarrow}(t_1)]|^2 + \frac{\lambda}{a}\|p(t_1)p(t_0)\|^2 \le 2M_1^2 + \frac{\nu\tau}{|a|}M_2^2 + \frac{\lambda}{16a}M_1^4 \le 0.5M_3^2;$$

here $\tau \le |a|M_1^2/\nu M_2^2$, $M_3^2 = 2M_1^2(3+\lambda M_1^2/16a)$. The inequality (4.7) follows directly from here. At last, we use (2.5) and obtain (4.8).

Let $\lambda/a < 0$. In this case from the equality (3.6) for all $j = 0,\dots,M-1$ we can obtain the following estimate:

$$\|p_{x\nwarrow}(t_{j+1})]|^2 + \|p_{x\swarrow}(t_{j+1})]|^2 \le \|p_{x\nwarrow}(t_1)]|^2 + \|p_{x\swarrow}(t_1)]|^2 + \Big|\frac{\lambda}{a}\Big|\|p(t_{j+1})p(t_j)\|^2. \tag{4.10}$$

For the last term in the right-hand side of this inequality we can write the following estimates:

$$\begin{aligned}
\Big|\frac{\lambda}{a}\Big|\|p(t_{j+1})p(t_j)\|^2 &\le \Big|\frac{\lambda}{a}\Big|\|p(t_{j+1})\|_{C_h}\|p(t_j)\|_{C_h}\|p(t_{j+1})\|\,\|p(t_j)\| \\
&\le \Big|\frac{\lambda}{2a}\Big|\max\big\{\|p(t_{j+1})\|_{C_h}^2,\|p(t_j)\|_{C_h}^2\big\}\big(\|p(t_{j+1})\|^2 + \|p(t_j)\|^2\big) \\
&\le \Big|\frac{\lambda}{2a}\Big|\big(\|p_{x\nwarrow}(t_{j+1})]|^2 + \|p_{x\swarrow}(t_{j+1})]|^2\big)^{1/2}\big(\|p(t_{j+1})\|^2 + \|p(t_j)\|^2\big)^{3/2} \\
&\le \frac{1}{2}\big(\|p_{x\nwarrow}(t_{j+1})]|^2 + \|p_{x\swarrow}(t_{j+1})]|^2\big) + \frac{\lambda^2}{8a^2}\big(\|p(t_{j+1})\|^2 + \|p(t_j)\|^2\big)^3 \\
&\le \frac{1}{2}\big(\|p_{x\nwarrow}(t_{j+1})]|^2 + \|p_{x\swarrow}(t_{j+1})]|^2\big) + \frac{\mu^3M_1^6\lambda^2}{2^6a^2}.
\end{aligned}$$

Here we consequently use (2.6), (3.5), (4.3) and suppose that τ and h are positive and small enough. From this estimate and from the inequality (4.10), we find, similar to the case $\lambda/a \ge 0$,

$$\|p_{x\nwarrow}(t_{j+1})]|^2 + \|p_{x\swarrow}(t_{j+1})]|^2 \le 4M_1^2 + \frac{2\nu\tau}{|a|}M_2^2 + \frac{\mu^3\lambda^2}{2^5a^2}M_1^6 \le \frac{1}{2}M_3;$$

here we can take $\tau \le |a|M_1^2/\nu M_2^2$, $M_3^2 = M_1^2(12+\lambda^2\mu^3M_1^4/16a^2)$.

In an analogous way, we obtain from here the estimates (4.7) and (4.8). The lemma is proved.

Now we can prove the convergence of the difference scheme in C norm.

THEOREM 4.1. *Let the conditions* (3.4), (4.1), (4.2), (4.6) *be satisfied. Then the solution of the problem* (3.1), (3.2) *converges to the solution of the problem* (2.1), (2.2) *in the space* $C(\bar{Q}_h)$. *There exist constants* $\tau_0 > 0$ *and* $h_0 > 0$ *such that for all positive* $\tau \le \tau_0$ *and* $h \le h_0$ *the following estimate is valid:*

$$\|\varepsilon\|_{C(\bar{Q}_h)}^2 \le c_1\|\varepsilon(t_1)\| + c_2\max_{1\le j\le M-1}\big\{\|\Phi(t_j)\|\big\}. \tag{4.11}$$

Here $c_l = c_l\,(a,\lambda,\nu,M_1,T)$, $l=1,2$.

Proof. We take the inner product on both sides of (4.4) with $4\tau\dot{\varepsilon}$ and, similar to Lemma 3.1, we obtain the equality

$$\|\hat{\varepsilon}\|^2+\|\varepsilon\|^2+\frac{2a\tau}{h^2}\operatorname{Im}(\bar{\varepsilon},\hat{\varepsilon})=\|\varepsilon\|^2+\|\check{\varepsilon}\|^2+\frac{2a\tau}{h^2}\operatorname{Im}(\bar{\check{\varepsilon}},\check{\varepsilon})+4\tau\operatorname{Re}(\Psi,\dot{\varepsilon})+4\tau\operatorname{Re}(\Phi,\dot{\varepsilon}).$$

Using the estimates (4.3) and (4.8), we can evaluate the last two summations of the right-hand side of this equality. The first one may be evaluated

$$\begin{aligned}&4\tau\operatorname{Re}(\Psi,\dot{\varepsilon})=4\lambda\tau\operatorname{Im}\left(|u|^2\dot{u}-|p|^2\dot{p},\dot{\varepsilon}\right)=4\lambda\tau\operatorname{Im}\big(\big((|u|^2-|p|^2)\dot{u},\dot{\varepsilon}\big)+\big(|p|^2\dot{\varepsilon},\dot{\varepsilon}\big)\big)\\&\le\tau|\lambda|\,\big|\big((|u|+|p|)(|\hat{u}|+|\check{u}|)|\varepsilon|,(|\hat{\varepsilon}|+|\check{\varepsilon}|)\big)\big|\le 0.5\tau|\lambda|(M_1+M_3)M_1\|\varepsilon\|(\|\hat{\varepsilon}\|+\|\check{\varepsilon}\|)\\&\le d_1\tau\big((\|\varepsilon\|^2+\|\hat{\varepsilon}\|^2)+(\|\check{\varepsilon}\|^2+\|\varepsilon\|^2)\big);\end{aligned}$$

here $d_1=0.25|\lambda|(M_1+M_3)M_1$. We evaluate the next one as follows:

$$4\tau\operatorname{Re}(\Phi,\dot{\varepsilon})\le 2\tau|(\Phi,(\hat{\varepsilon}+\check{\varepsilon}))|\le 2\tau\|\Phi\|^2+\tau\big(\|\hat{\varepsilon}\|^2+\|\check{\varepsilon}\|^2\big).$$

After this evaluation of summations we sum the obtained inequalities for time layers from t_1 up to t_j and obtain the estimate

$$\begin{aligned}&\|\varepsilon(t_{j+1})\|^2+\|\varepsilon(t_j)\|^2+\frac{2a\tau}{h^2}\operatorname{Im}(\bar{\varepsilon}(t_j),\varepsilon(t_{j+1}))\\&\le\|\varepsilon(t_1)\|^2+\|\varepsilon(t_0)\|^2+\frac{2a\tau}{h^2}\operatorname{Im}(\bar{\varepsilon}(t_0),\varepsilon(t_1))+2\tau\sum_{k=1}^{j}\|\Phi(t_k)\|^2\\&+(d_1+1)\tau\sum_{k=1}^{j}\big((\|\varepsilon(t_{k+1})\|^2+\|\varepsilon(t_k)\|^2)+(\|\varepsilon(t_k)\|^2+\|\varepsilon(t_{k-1})\|^2)\big).\end{aligned}$$

Similarly as in Lemma 3.1, it follows from (3.4) that

$$\begin{aligned}&\|\varepsilon(t_{j+1})\|^2+\|\varepsilon(t_j)\|^2\le\mu\left(\|\varepsilon(t_1)\|^2+\|\varepsilon(t_0)\|^2\right)+d_2\tau\sum_{k=1}^{j}\|\Phi(t_k)\|^2\\&+d_3\tau\sum_{k=1}^{j}\left((\|\varepsilon(t_{k+1})\|^2+\|\varepsilon(t_k)\|^2)+(\|\varepsilon(t_k)\|^2+\|\varepsilon(t_{k-1})\|^2)\right);\end{aligned}$$

here $d_2=2/(1-\nu)$, $d_3=(d_1+1)/(1-\nu)$.

We use the grid Gronwall inequality (2.8), where $d=d_3$, $e=d_2$,

$$b_i=\|\Phi(t_{i+1})\|^2,\quad \bar{Y}_0=\mu\left(\|\varepsilon(t_1)\|^2+\|\varepsilon(t_0)\|^2\right),\quad Y_j=\|\varepsilon(t_{j+1})\|^2+\|\varepsilon(t_j)\|^2.$$

Thus, for all positive $\tau\le 1/2d_3$ the following estimate holds:

$$\begin{aligned}\|\varepsilon(t_{j+1})\|^2+\|\varepsilon(t_j)\|^2&\le\mu\exp(4Td_3)(\|\varepsilon(t_1)\|^2+\|\varepsilon(t_0)\|^2)\\&\quad+2d_2T\exp(4Td_3)\max_{1\le k\le M-1}\big\{\|\Phi(t_k)\|^2\big\}.\end{aligned}$$

Since $\varepsilon(t_0)=0$, we have

$$\|\varepsilon(t_j)\|^2\le d_4\|\varepsilon(t_1)\|^2+d_5\max_{1\le k\le M-1}\big\{\|\Phi(t_k)\|^2\big\}$$

for all $j = 0, \dots, M$. Here $d_4 = \mu \exp(4d_3T)$, $d_5 = 2d_2T\exp(4d_3T)$.

Let $\tau, h \to 0$. Then from (4.1) and (4.6) the convergence of the right-hand side of this inequality to 0 follows. Thus, we have obtained the convergence of the difference scheme (3.1), (3.2) in L_2 norm.

From the multiplicative inequality (2.6) it follows that

$$\|\hat{\varepsilon}\|^2_{C(\bar{Q}_h)} \leq \max_{t_j \in \bar{\omega}_\tau} \{\|\varepsilon(t_j)\|\} \max_{1\leq j\leq M} \big\{\|u_{x^\nwarrow}(t_j)]| + \|u_{x\swarrow}(t_j)]|\big\} + \max_{t_j \in \bar{\omega}_\tau} \{\|\varepsilon(t_j)\|\} \max_{1\leq j\leq M} \big\{\|p_{x^\nwarrow}(t_j)]| + \|p_{x\swarrow}(t_j)]|\big\}.$$

For positive and small enough τ, h and $\tau \leq 5|a|M_1^2/2\nu M_2^2$, it follows from the equality (4.9) and the condition (4.2) that

$$\|\hat{u}_{x^\nwarrow}]| + \|\hat{u}_{x\swarrow}]| \leq \left(4M_1^2 + \frac{2\tau\nu M_2^2}{|a|}\right)^{1/2} \leq 3M_1.$$

Therefore, from (4.7) and from the last estimate we obtain

$$\|\hat{\varepsilon}\|^2_{C(\bar{Q}_h)} \leq (3M_1 + M_3) \max_{t_j \in \bar{\omega}_\tau} \big\{\|\varepsilon(t_j)\|\big\}.$$

Here the right-hand side converges to 0.

From here (4.11) follows with the constants

$$c_1 = (3M_1 + M_3)\sqrt{d_4}, \qquad c_2 = (3M_1 + M_3)\sqrt{d_5}.$$

Note that $M_3 = M_3(a, \lambda, M_1, \nu)$. Thus, $c_l = c_l\,(a, \lambda, \nu, M_1, T)$, $l = 1, 2$. The theorem is proved.

We shall prove the stability of the difference scheme on initial data in C norm.

Let $u_1(x,t)$, $u_2(x,t)$ and p_1, p_2 be the solutions of the problems (2.1), (2.2) and (3.1), (3.2) with the initial data $u_{10}(x)$ and $u_{20}(x)$, respectively.

THEOREM 4.2. *Let the conditions of Theorem* 4.1 *be satisfied. Then there exist constants* $\tau_0 > 0$ *and* $h_0 > 0$ *such that for all positive* $\tau \leq \tau_0$ *and* $h \leq h_0$ *the following estimate holds:*

$$\|p_1 - p_2\|^2_{C(\bar{Q}_h)} \leq c_3\|u_{10} - u_{20}\|. \tag{4.12}$$

Here $c_3 = c_3\left(a, \lambda, \nu, T, \max_{t\in[T;0]}\{\|u_1(t)\|_{W_2^1}, \|u_2(t)\|_{W_2^1}\}\right)$.

Proof. We denote $z = p_1 - p_2$. Then

$$z_t = \frac{ia}{h^2}(\bar{z} - 2\dot{z}) - i\lambda\left(|p_1|^2\dot{p}_1 - |p_2|^2\dot{p}_2\right), \qquad (x,t) \in Q_h,$$

$$z(x,0) = u_{10}(x) - u_{20}(x), \quad x \in \bar{\omega}_h, \qquad z(x_0,t) = z(x_N,t) = 0, \quad t \in \bar{\omega}_\tau.$$

We take the inner product on both sides of this equation with $4\tau\dot{z}$. Similarly as in Theorem 4.1, we can obtain the inequality

$$\|z(t_{j+1})\|^2 + \|z(t_j)\|^2 \leq \mu\left(\|z(t_1)\|^2 + \|z(t_0)\|^2\right) + d_1\tau\sum_{k=1}^{j}\left(\left(\|z(t_{k+1})\|^2 + \|z(t_k)\|^2\right) + \left(\|z(t_k)\|^2 + \|z(t_{k-1})\|^2\right)\right);$$

here $d_1 = (\lambda/(1-\nu))\|p_1\|_{C(\bar{Q}_h)}\big(\|p_1\|_{C(\bar{Q}_h)} + \|p_2\|_{C(\bar{Q}_h)}\big)$. Since the condition (4.8) holds, the constant d_1 is bounded when τ and h are positive and small enough.

From the grid Gronwall inequality (2.8) it follows that

$$\max_{t_j\in\bar{\omega}_\tau} \|z(t_j)\|^2 \le d_2(\|z(t_1)\|^2 + \|z(t_0)\|^2).$$

Here $d_2 = d_2(a, \lambda, T, \|p_1\|_{C(\bar{Q}_h)}, \|p_2\|_{C(\bar{Q}_h)}, \nu)$.

When $\tau > 0$ is small enough, the condition $\|z(t_1)\| \le 2\|z(t_0)\|$ is satisfied. Similarly as in Lemma 4.1, we can prove the dependence of the norms $\|p_1\|_{C(\bar{Q}_h)}$ and $\|p_2\|_{C(\bar{Q}_h)}$ on the constants $\max_{t\in[T;0]}\|u_1(t)\|_{W_2^1}$ and $\max_{t\in[T;0]}\|u_2(t)\|_{W_2^1}$. From here and from the last two estimates the stability in L_2 follows:

$$\max_{t_j\in\bar{\omega}_\tau} \|z(t_j)\| \le d_3\|z(t_0)\|;$$

here $d_3 = d_3(a, \lambda, T, \nu, \max_{t\in[T;0]}\{\|u_1(t)\|_{W_2^1}, \|u_2(t)\|_{W_2^1}\})$.

Similarly as in Theorem 4.1, from the multiplicative inequality (2.6) the estimate (4.12) follows. The theorem is proved.

5. A general case of the problem. We consider the nonlinear Schrödinger equation

$$\frac{\partial u}{\partial t} = ia\frac{\partial^2 u}{\partial x^2} + f(u,u^*)u. \tag{5.1}$$

Here $f(u,u^*)$ is a polynomial with arguments u and u^* and $f(u,u^*) = f(-u,-u^*)$. We can find a continuous nondecreasing function $\varphi(y)$ that satisfies the conditions

$$|f(u,u^*)| \le \varphi(|u|), \quad \big|D^{\mathbf{j}} f(u,u^*)u\big| \le \varphi(|u|), \qquad |\mathbf{j}| = 1,2; \tag{5.2}$$

here $\mathbf{j}$ is a two-dimensional vector, $|\mathbf{j}| = j_1 + j_2$, $D^{\mathbf{j}} = \partial^{|\mathbf{j}|}/\partial u^{j_1}\partial u^{*j_2}$.

We relate (5.1) with the following difference scheme:

$$p_t = ia\frac{\bar{p} - 2\dot{p}}{h^2} + f(p,p^*)\dot{p}, \qquad (x,t)\in Q_h. \tag{5.3}$$

It can be proved that the following estimates for the nonlinear grid function $f(v,v^*)\dot{v}$ are satisfied:

$$|(f(v,v^*)\dot{v},\dot{v})| \le 0.5\varphi\big(\|v\|_{C_h}\big)\big(\|\hat{v}\|^2 + \|\check{v}\|^2\big), \tag{5.4}$$

$$\begin{aligned}\big|\big(f(v,v^*)\dot{v} - f(w,w^*)\dot{w}, \dot{v} - \dot{w}\big)\big| &\le \varphi\big(\max\big\{\|\check{v}\|_{C_h}, \|v\|_{C_h}, \|\hat{v}\|_{C_h}, \|w\|_{C_h}\big\}\big)\\ &\quad\times \left(\|\check{v}-\check{w}\|^2 + \|v-w\|^2 + \|\hat{v}-\hat{w}\|^2\right).\end{aligned} \tag{5.5}$$

From (5.5) we can obtain

$$|((f(v,v^*)\dot{v})_{\bar{x}}, \dot{v}_{\bar{x}}]| \le \varphi\big(\max\big\{\|\check{v}\|_{C_h}, \|v\|_{C_h}, \|\hat{v}\|_{C_h}\big\}\big)\left(\|\check{v}_{\bar{x}}]|^2 + \|v_{\bar{x}}]|^2 + \|\hat{v}_{\bar{x}}]|^2\right). \tag{5.6}$$

Also we have

(5.7)

$$\begin{aligned}&|((f(v,v^*)\dot{v}(t_k) - f(w,w^*)\dot{w}(t_k))_{\bar{x}}, \dot{z}_{\bar{x}}(t_k)]| \le \Big(\max_{l=-1,0,1}\{1, \|v_{\bar{x}}(t_{k+l})]|, \|w_{\bar{x}}(t_{k+l})]|\}\Big)\\ &\times\ c\varphi\Big(\max_{l=-1,0,1}\{\|v(t_{k+l})\|_{C_h}, \|w(t_{k+l})\|_{C_h}\}\Big)\Big(\max_{l=-1,0,1}\{\|z_{\bar{x}}(t_{k+l})]|^2\}\Big),\end{aligned}$$

where c is some constant and $z = v - w$.

One can prove the estimates (5.4)–(5.7) in a manner analogous to the similar estimates in [6].

We shall prove the convergence and stability of this new scheme in a different manner than the proof in sections 3 and 4. We shall obtain a new type of a priori estimates [4, 5, 6], instead of equalities of the type (3.3) or (3.6).

Let p be the solution of the difference scheme (5.3), (3.2). Let the condition (3.4) be satisfied. We can obtain the equality similarly to (3.3):

$$\|\hat{p}\|^2 + \|p\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}, \hat{p}) = \|p\|^2 + \|\check{p}\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{\check{p}}, p) + 4\tau \operatorname{Re}\sum_{l=1}^{N-1} f(p_l, p_l^*)|\dot{p}_l|^2 h.$$

Two estimates follow from here.

First, we evaluate the nonlinear part and the summations of $\frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}, \hat{p})$ type. We obtain

$$\|\hat{p}\|^2 + \|p\|^2 \le \mu(\|p\|^2 + \|\check{p}\|^2) + \frac{2\tau}{1-\nu}\varphi\big(\|p\|_{C_h}\big)\big(\|\hat{p}\|^2 + \|\check{p}\|^2\big). \tag{5.8}$$

Second, we sum the previous equalities for layers t_k, $k = 1, \dots, j-1$, use the estimate (5.4) and obtain the inequality

$$\begin{aligned} &\|p(t_j)\|^2 + \|p(t_{j-1})\|^2 \le \mu(\|p(t_1)\|^2 + \|p(t_0)\|^2) \\ &+ \frac{2\tau}{1-\nu}\varphi(\|p\|_{C(\bar{Q}_{t_j h})})\sum_{k=1}^{j-1}\big((\|p(t_{k+1})\|^2 + \|p(t_k)\|^2) + (\|p(t_k)\|^2 + \|p(t_{k-1})\|^2)\big). \end{aligned} \tag{5.9}$$

Here $\|p\|_{C(\bar{Q}_{t_j h})} = \max_{0\le k\le j}\{\|p(t_k)\|_{C_h}\}$.

We denote the fictitious nodes of the grid $(-h, \tau j)$ and $(1+h, \tau j)$, where $j = 0, \dots, M$. Let v_{-1} and v_{N+1} be the values of grid function on these nodes. We define the solution of the difference scheme on these nodes as follows: $p_{-1} = -p_1$ and $p_{N+1} = -p_{N-1}$. This corresponds to the boundary conditions (2.2) and to the equality $\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0$. Here u is the solution of the extended differential problem (5.1), (2.2) on the frontier of the domain. The extension is valid due to the zero boundary conditions and since the nonlinear function is odd.

We have defined earlier $p = p_l^j$. Thus, from (5.3) it follows that

$$p_{t\bar{x}} = ia\frac{\bar{p}_{\bar{x}} - 2\dot{p}_{\bar{x}}}{h^2} + (f(p,p^*)\dot{p})_{\bar{x}}, \quad j = 1, \dots, M-1, \quad l = 1, \dots, N.$$

We take the inner product on both sides of this equality with $4\tau\dot{p}_{\bar{x}}$ and obtain

$$\begin{aligned} &\|\hat{p}_{\bar{x}}]|^2 + \|p_{\bar{x}}]|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}_{\bar{x}}, \hat{p}_{\bar{x}}] \\ &= \|p_{\bar{x}}]|^2 + \|\check{p}_{\bar{x}}]|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{\check{p}}_{\bar{x}}, p_{\bar{x}}] + 4\tau\operatorname{Re}\left((f(p,p^*)\dot{p})_{\bar{x}}, \dot{p}_{\bar{x}}\right]. \end{aligned}$$

Again, two estimates follow from here.

First, from (2.7) and from the estimate of the nonlinear part we obtain

$$|((f(p,p^*)\dot{p})_{\bar{x}}, \dot{p}_{\bar{x}}]| \le (2/h)\|f(p,p^*)\|_{C_h}\|\dot{p}\|\,\|\dot{p}_{\bar{x}}]| \le (1/h)\varphi\big(\|p\|_{C_h}\big)\|\dot{p}_{\bar{x}}]|^2.$$

Hence,

$$\|\hat{p}_{\bar{x}}]|^2 + \|p_{\bar{x}}]|^2 \le \mu(\|p_{\bar{x}}]|^2 + \|\check{p}_{\bar{x}}]|^2) + \frac{2\tau}{(1-\nu)h}\varphi\big(\|p\|_{C_h}\big)\big(\|\hat{p}_{\bar{x}}]|^2 + \|\check{p}_{\bar{x}}]|^2\big). \tag{5.10}$$

Second, from (5.6) it follows that

$$\|p_{\bar{x}}(t_j)]|^2 + \|p_{\bar{x}}(t_{j-1})]|^2 \le \mu(\|p_{\bar{x}}(t_1)]|^2 + \|p_{\bar{x}}(t_0)]|^2) + \frac{4\tau}{1-\nu}\varphi\big(\|p\|_{C(\bar{Q}_{t_j h})}\big) \times \sum_{k=1}^{j-1}\big((\|p_{\bar{x}}(t_{k+1})]|^2 + \|p_{\bar{x}}(t_k)]|^2) + (\|p_{\bar{x}}(t_k)]|^2 + \|p_{\bar{x}}(t_{k-1})]|^2)\big)\,. \tag{5.11}$$

We sum the inequalities (5.8) and (5.10), use the condition (3.4), and obtain the estimate

$$\|\hat{p}\|^2_{W^1_{2h}} \le \mu(\|p\|^2_{W^1_{2h}} + \|\check{p}\|^2_{W^1_{2h}}) + \frac{h\nu}{|a|(1-\nu)}\varphi\big(\|p\|_{C_h}\big)\big(\|\hat{p}\|^2_{W^1_{2h}} + \|\check{p}\|^2_{W^1_{2h}}\big).$$

When $0 < h \le |a|(1-\nu)/2\nu\varphi(\|p\|_{C_h})$, we have

$$\|\hat{p}\|^2_{W^1_{2h}} \le (4\mu + 1)\max\big\{\|p\|^2_{W^1_{2h}}, \|\check{p}\|^2_{W^1_{2h}}\big\}. \tag{5.12}$$

We sum the inequalities (5.9) and (5.11) and obtain

$$\|p(t_j)\|^2_{W^1_{2h}} + \|p(t_{j-1})\|^2_{W^1_{2h}} \le \mu\big(\|p(t_1)\|^2_{W^1_{2h}} + \|p(t_0)\|^2_{W^1_{2h}}\big) + \frac{4\tau}{1-\nu}\varphi\big(\|p\|_{C(\bar{Q}_{t_j h})}\big) \times \sum_{k=1}^{j-1}\Big((\|p(t_{k+1})\|^2_{W^1_{2h}} + \|p(t_k)\|^2_{W^1_{2h}}) + (\|p(t_k)\|^2_{W^1_{2h}} + \|p(t_{k-1})\|^2_{W^1_{2h}})\Big)\,. \tag{5.13}$$

Assume that the truncation error satisfies the condition

$$\max_{t_j \in \omega_\tau}\Big\{\|\Phi(t_j)\|_{W^1_{2h}}\Big\} \to 0, \qquad \tau, h \to 0. \tag{5.14}$$

This is a natural condition, since from (3.4) it follows that

$$\|\Phi_{\bar{x}}(t_j)]| \le c\Big(\sum_{i=1}^{N}\Big|\frac{\tau^2 + h^2 + (\tau/h)^2}{h}\Big|^2 h\Big)^{1/2} \le c(\nu^2 h^3 + h + \nu h) = O(h).$$

An error of the solution of the problem (5.1), (2.2) satisfies the equalities (4.4), (4.5), where

$$\Psi = (f(u, u^*)\dot{u} - f(p, p^*)\dot{p})\,.$$

From (5.5) and (5.7), similarly as in [6] and in the proof of the estimate (5.13), we can obtain the inequality for the error of the solution:

$$\begin{aligned}\|\varepsilon(t_j)\|^2_{W^1_{2h}} + \|\varepsilon(t_{j-1})\|^2_{W^1_{2h}} &\le \mu(\|\varepsilon(t_1)\|^2_{W^1_{2h}} + \|\varepsilon(t_0)\|^2_{W^1_{2h}}) + \frac{2\tau}{1-\nu}\sum_{k=1}^{j-1}\|\Phi(t_k)\|^2_{W^1_{2h}} \\ &+ \frac{\tau}{1-\nu}\Big(1 + 4c\varphi\big(\max\{\|u\|_{C(\bar{Q})}, \|p\|_{C(\bar{Q}_{t_j h})}\}\big)\max_{0\le k\le j}\big\{1, \|u_{\bar{x}}(t_k)]|, \|p_{\bar{x}}(t_k)]|\big\}\Big) \\ &\times \sum_{k=1}^{j-1}\Big((\|\varepsilon(t_{k+1})\|^2_{W^1_{2h}} + \|\varepsilon(t_k)\|^2_{W^1_{2h}}) + (\|\varepsilon(t_k)\|^2_{W^1_{2h}} + \|\varepsilon(t_{k-1})\|^2_{W^1_{2h}})\Big)\,.\end{aligned} \tag{5.15}$$

We shall now prove the convergence of the difference scheme (5.3), (3.2).

THEOREM 5.1. *Let the conditions* (3.4), (4.2), (4.3), (4.6), (5.14) *be satisfied. Then the solution of the difference scheme* (5.3), (3.2) *converges to the solution of the problem* (5.1), (2.2) *in spaces* W^1_{2h} *and* $C(\bar{Q}_h)$. *There exist constants* $\tau_0 > 0$ *and* $h_0 > 0$ *such that for all positive* $\tau \le \tau_0$ *and* $h \le h_0$ *the following estimates hold:*

$$\max_{t_j\in\bar{\omega}_\tau}\big\{\|\varepsilon(t_j)\|_{W^1_{2h}}\big\} \le c_4\|\varepsilon(t_1)\|_{W^1_{2h}} + c_5\max_{t_j\in\omega_\tau}\big\{\|\Phi(t_j)\|_{W^1_{2h}}\big\}; \tag{5.16}$$

$$\|\varepsilon\|_{C(\bar{Q}_h)} \le 0.5c_4\|\varepsilon(t_1)\|_{W^1_{2h}} + 0.5c_5\max_{t_j\in\omega_\tau}\big\{\|\Phi(t_j)\|_{W^1_{2h}}\big\}; \tag{5.17}$$

here $c_l = c_l\,(a, \varphi, \nu, M_1, T)$, $l = 4, 5$.

Proof. We can prove this theorem in a manner analogous to the similar theorems for two-layer difference schemes in [4, 5, 6].

We shall prove boundedness of the function $p(t_j)$, $t_j \in \bar{\omega}_\tau$: $\|p(t_j)\|_{W^1_{2h}} \le 2M_1$. We use the method of mathematical induction.

When $j = 0$, it follows from (3.2) that

$$\|p(t_0)\|_{W^1_{2h}} \le 2\|u(t_0)\|_{W^1_{2h}} \le 2M_1.$$

Let $j = 1$. Since the condition (4.6) holds, we have $\|\varepsilon(t_1)\|_{W^1_{2h}} \le M_1$ for τ and h small enough. Then we obtain $\|p(t_1)\|_{W^1_{2h}} \le 2M_1$.

Let the estimates $\|p(t_k)\|_{W^1_{2h}} \le 2M_1$ be valid for all $k = 0, \dots, j-1$. From (2.7) we find $\|p\|_{C(\bar{Q}_{t_{j-1}h})} \le M_1$. Then for all $h \le h_0$, $h_0 = |a|(1-\nu)/(2\nu\varphi(M_1))$, we can use (5.12) and obtain

$$\|p(t_j)\|^2_{W^1_{2h}} \le 4(4\mu+1)M_1^2.$$

From the condition (2.7) it follows that

$$\|p\|_{C(\bar{Q}_{t_jh})} \le \sqrt{4\mu+1}M_1 \quad \text{and} \quad \max_{0\le k\le j}\{\|p(t_k)\|_{W^1_{2h}}\} \le 2\sqrt{4\mu+1}M_1.$$

Since (4.2) is valid, we can use the grid Gronwall inequality (2.8) for the estimate (5.15), when τ and h are positive and small enough, $\tau \le 1/2d_1$. Here

$$d_1 = \big(1 + 4c\varphi(\sqrt{4\mu+1}M_1)\max\{1, 2\sqrt{4\mu+1}M_1\}\big)/(1-\nu).$$

Thus,

$$\|\varepsilon(t_j)\|^2_{W^1_{2h}} \le d_2\|\varepsilon(t_1)\|^2_{W^1_{2h}} + d_3\max_{1\le l\le M-1}\big\{\|\Phi(t_l)\|^2_{W^1_{2h}}\big\}, \tag{5.18}$$

where $d_2 = \mu\exp(4d_1T)$, $d_3 = 4T\exp(4d_1T)/(1-\nu)$.

The right-hand side of this estimate converges to 0 when $\tau, h \to 0$. Thus, there exist constants $\tau_0 > 0$ and $h_0 > 0$ such that for all positive $\tau \le \tau_0$ and $h \le h_0$ the estimate $\|\varepsilon(t_j)\|_{W^1_{2h}} \le M_1$ holds. Therefore, $\|p(t_j)\|_{W^1_{2h}} \le 2M_1$. The induction step is proved.

Thus, the statement

$$\max_{t_j\in\bar{\omega}_\tau}\{\|p(t_j)\|_{W^1_{2h}}\} \le 2M_1$$

is valid and the estimate (5.18) holds for all $t_j \in \bar{\omega}_\tau$. It follows that the estimate (5.16) is valid. Using (2.7), we also obtain (5.17), where $c_4 = \sqrt{d_2}$ and $c_5 = \sqrt{d_3}$. The theorem is proved.

Similarly as in [4, 5, 6], we prove the stability of the difference scheme on initial data. Let $u_1(x,t)$, $u_2(x,t)$ and p_1, p_2 be the solutions of the problems (5.1), (2.2) and (5.3), (3.2) with the initial data $u_{10}(x)$ and $u_{20}(x)$, respectively.

THEOREM 5.2. *Let the conditions of Theorem* 5.1 *be satisfied. Then there exist constants* $\tau_0 > 0$ *and* $h_0 > 0$ *such that for all positive* $\tau \le \tau_0$ *and* $h \le h_0$ *the following estimates hold:*

$$\max_{t_j \in \bar{\omega}_\tau} \big\{ \|p_1(t_j) - p_2(t_j)\|_{W^1_{2h}} \big\} \le c_6 \|u_{10} - u_{20}\|_{W^1_{2h}}, \tag{5.19}$$

$$\|p_1 - p_2\|_{C(\bar{Q}_h)} \le 0.5 c_6 \|u_{10} - u_{20}\|_{W^1_{2h}}. \tag{5.20}$$

Here $c_6 = c_6\big(a, \varphi, \nu, T, \max_{t \in [0;T]}\{\|u_1(t)\|_{W^1_2}, \|u_2(t)\|_{W^1_2}\}\big)$.

Proof. Using the estimates (5.5) and (5.7), similarly as in Theorems 4.2 and 5.1, we can obtain the inequality

$$\|z(t_j)\|^2_{W^1_{2h}} + \|z(t_{j-1})\|^2_{W^1_{2h}} \le \mu(\|z(t_1)\|^2_{W^1_{2h}} + \|z(t_0)\|^2_{W^1_{2h}})$$
$$+ \tau d_1 \sum_{k=1}^{j-1} \Big((\|z(t_{k+1})\|^2_{W^1_{2h}} + \|z(t_k)\|^2_{W^1_{2h}}) + (\|z(t_k)\|^2_{W^1_{2h}} + \|z(t_{k-1})\|^2_{W^1_{2h}}) \Big).$$

Here $z = p_1 - p_2$ and

$$d_1 = 4c\varphi\big(\max\{\|p_1\|_{C(\bar{Q}_h)}, \|p_2\|_{C(\bar{Q}_h)}\}\big) \max_{t_j \in \bar{\omega}_\tau} \big\{1, \|p_1(t_j)\|_{W^1_{2h}}, \|p_2(t_j)\|_{W^1_{2h}}\big\}.$$

The boundedness of norms

$$\|p\|_{C(\bar{Q}_h)} \le M_1 \quad \text{and} \quad \max_{t_j \in \bar{\omega}_\tau} \{\|p(t_j)\|_{W^1_{2h}}\} \le 2M_1$$

was proved in Theorem 5.1. Thus, when τ and h are positive and small enough, we can use the grid Gronwall inequality for the estimate obtained before. (5.19) and (5.20) follow from here. The theorem is proved.

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