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Blow-up versus boundedness in a nonlocal and nonlinear Fokker–Planck equation

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Abstract

We consider a Fokker-Planck equation on a compact interval where, as a constraint, the first moment is a prescribed function of time. Eliminating the associated Lagrange multiplier one obtains nonlinear and nonlocal terms. After establishing suitable local existence results, we use the relative entropy as an energy functional. However, the time-dependent constraint leads to a source term such that a delicate analysis is needed to show that the dissipation terms are strong enough to control the work done by the constraint. We obtain global existence of solutions as long as the prescribed first moment stays in the interior of an interval. If the prescribed moment converges to a constant value inside the interior of the interval, then the solution stabilises to the unique steady state.

1 Introduction

In this paper we discuss a model that was developed for a many-particle system relevant for lithium-ion batteries, see [DJ*10, DGH06]. Here the variable $x \in \Omega =]0,1[$ relates to the relative loading state of particles and $u(x,t)$ is the time-dependent probability density, i.e. $\int_\Omega u(x,t)\,\mathrm{d}x = 1$ for all t. The model takes the form

$$\begin{cases} \tau u_t(x,t) = \Big(\nu^2 u_x(x,t) + \psi'(x)u(x,t) - \Lambda(t)u(x,t)\Big)_x & \text{for } x \in \Omega,\ t>0, \\ \nu^2 u_x(x,t) + \psi'(x)u(x,t) - \Lambda(t)u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t>0, \\ \mathcal{C}(u(t)) := \int_\Omega x\, u(x,t)\,\mathrm{d}x = \ell(t) & \text{for } t \geq 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{1.1}$$

The potential ψ can be taken general but has to satisfy certain smoothness, namely it is a general potential satisfying

$$\psi \in \mathrm{H}^1([0,1]). \tag{1.2}$$

The Lagrange multiplier $\Lambda(t)$ is associated with the constraint $\mathcal{C}(u(t)) = \ell(t)$, where $\ell : [0,\infty[\to]0,1[$ is a given datum. In fact, Λ can easily be determined as

$$\Lambda(t) = \int_\Omega \nu^2 u_x(x,t) + \psi'(x)u(x,t)\,\mathrm{d}x + \tau\dot{\ell}(t).$$

After inserting Λ into (1.1) we arrive at a nonlinear Fokker-Planck equation, where the nonlinearity is quadratic and arises only through the nonlocal term $\Lambda(t)$.

In Section 2.1 the origins of this model and its physical relevance are discussed in more detail. In Section 3 we provide a local existence theory for the above system. After some preparation we use the semilinear structure of the problem to derive existence on small time intervals. Positivity and parabolic regularity are obtained. The quadratic nature of the problem is nontrivial and may lead to blow-up. Note that (1.1) after elimination of Λ takes the form

$$\begin{cases} \tau u_t(x,t) = \Big(\nu^2 u_x(x,t) + u(x,t)\Big[\psi'(x) - \mathbb{L}(u(t)) - \tau p(t)\Big]\Big)_x & \text{for } x \in \Omega,\ t > 0, \\ \nu^2 u_x(x,t) + u(x,t)\Big[\psi'(x) - \mathbb{L}(u(t)) - \tau p(t)\Big] = 0 & \text{for } x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.3}$$

where $p(t)$ plays the role of $\dot{\ell}(t)$ and for any $v \in \mathrm{C}(\bar{\Omega})$, $\mathbb{L}(v)$ is defined as

$$\mathbb{L}(v) := \nu^2 \left(v(1) - v(0)\right) + \int_\Omega \psi'(x) v(x)\,\mathrm{d}x. \tag{1.4}$$

We show that for this system blow-up occurs for suitable choices of p and initial conditions u_0.

To obtain global existence, one needs to remember $p = \dot{\ell}$ and that $\ell(0)$ is given by the initial condition. Hence

$$\ell(t) = \int_\Omega x\, u(x,0)\,\mathrm{d}x + \int_0^t p(t)\,\mathrm{d}t.$$

Global existence will depend on the additional assumption $\ell(t) \in\,]0,1[$ for all $t \geq 0$. Obviously, there does not exist a smooth probability density on $]0,1[$ with $\ell = 0$ or 1. To use this information we introduce the energy functional

$$\mathcal{A}(u) = \int_\Omega \nu^2 u(x)\, \ln u(x) + \psi(x) u(x)\,\mathrm{d}x,$$

which is in fact the relative entropy with respect to the equilibrium solution $\widehat{u}(x) = c\mathrm{e}^{-\psi(x)/\nu^2}$. In Section 2.2, equation (1.1) is formally rewritten as the abstract constraint gradient flow

$$\tau u_t = -K(u)\Big(\mathrm{D}\mathcal{A}(u) - \Lambda(t)\mathrm{D}\mathcal{C}(u)\Big), \qquad \mathcal{C}(u(t)) = \ell(t),$$

where $K(u)$ is the semi-definite, selfadjoint linear operator defined via

$$K(u)\xi = -\big(\, u\,\xi_x\,\big)_x,$$

which is the inverse of the Wasserstein metric tensor, see [JKO98, Ott01].

The crucial consequence of this structure is the energy-dissipation relation

$$\frac{d}{dt}\mathcal{A}(u(t)) = -\mathcal{D}(u(t), \dot{\ell}(t)) \text{ with } \mathcal{D}(u,p) = \int_\Omega \frac{(\nu^2 u_x + \psi' u)^2}{u}\,\mathrm{d}x - \mathbb{L}(u)^2 - p\mathbb{L}(u).$$

While it is easy to show via the Cauchy-Schwarz estimate that the sum of the first two terms in $\mathcal{D}$ is nonnegative, the third term, which arises through the work of the constraint, may have an arbitrary sign. A major task is to find good lower bounds for $\mathcal{D}$, which will be done in Section 4.1 in several steps. The main point is that $\mathcal{D}(u,p)$ needs to be estimated from below on the set

$$\mathcal{M}(\ell) := \{\, u \in L^1(\Omega) \;:\; u \geq 0,\ \int_\Omega u(x)\,\mathrm{d}x = 1,\ \int_\Omega x u(x)\,\mathrm{d}x = \ell \,\}.$$

Theorem 4.3 shows that for each $\delta \in \,]0,1/2[$ and $\psi \in \mathrm{H}^1(\Omega)$ there is a constant C_δ^ψ such that $\ell \in [\delta, 1{-}\delta]$ implies

$$\mathcal{D}(u,p) \geq -C_\delta^\psi |p| \quad \text{for all } u \in \mathcal{M}(\ell) \text{ and } p \in [-\tfrac{1}{\delta}, \tfrac{1}{\delta}].$$

Thus we can conclude that $\mathcal{A}(u(t))$ cannot blow-up along a solution. Employing the L-log L variant of [BHN94] of the Gagliardo-Nirenberg interpolation for the embedding of $L^\infty(\Omega)$ in $\mathrm{H}^1(\Omega)$ (see Lemma A.2) it is then possible to find an apriori estimate for the L^2 norm and global existence can be obtained for all $\ell \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}([0,\infty[)$ with $\ell(t) \in \,]0,1[$ for all $t \geq 0$.

Finally, in Section 5 we show that the solutions converge to a steady state if $\ell(t) \to \ell_* \in \,]0,1[$ in such a way that $\dot{\ell} \in L^1(]0,\infty[) \cap L^\infty(]0,\infty[)$. For this we exploit that for each ℓ_* there is exactly one steady state U_{ℓ_*} that is characterised by the fact that it is the unique minimiser of $\mathcal{A}$ on the set $\mathcal{M}(\ell_*)$. As a final result we show that $u(t) \to U_{\ell_*}$ in $L^2(\Omega)$ for $t \to \infty$.

The theory in Sections 4 and 5 share many similarities with the global existence and convergence theory for electro-reaction-diffusion systems studied in [GlH97]. This includes the usage of the L-log L variant [BHN94] of the Gagliardo-Nirenberg interpolation, the energy estimate via the energy-dissipation relation, and the introduction of the auxiliary variable $v = \big(u/U_\ell\big)^{1/2}$, where U_ℓ is the relevant equilibrium, see the proofs of Theorem 4.3 and Proposition 4.4. Our analysis is simpler in the respect that we only deal with a single scalar equation, however we treat the case of a driven system, where the time-dependent constraint leads to several subtle difficulties.

Starting from Section 2.2 we will set the parameters τ and ν equal to 1. We do this without loss of generality as explained at the end of Section 2.1.

2 Modelling and mathematical structures

2.1 Motivation: Modelling of many-particle storage systems

Here we explain how the above model is capable to describe the behaviour of ensembles of interconnected storage particles. Modern many-particle electrodes of rechargeable lithium-ion batteries belong to that class of storage systems. The electrode consists of a powder of 10^{10} - 10^{17} nano-particles that serve to reversibly store and release lithium atoms during the process of charging and discharging respectively. For more details of the functionality of the battery, see [DJ*10] and [DGH06].

The probability density to find a particle of the ensemble at time t in the loading state x is represented by the function $u : [0,T] \times \Omega \to [0,\infty[$. Thus it satisfies

$$\int_\Omega u(x,t)\,\mathrm{d}x = 1 \quad \text{for all} \quad t \in [0,T]\ . \tag{2.1}$$

The voltage of the battery linearly depends on the expectation value $\int_\Omega \mu(x)u(x,t)\,\mathrm{d}x$, where the *chemical potential* $\mu(x)$ is non-monotone. Finally, the capacity of the battery, i.e. the total loading state of the ensemble, is proportional to $1-\ell(t)$ with

$$\ell(t) = \mathcal{C}(u(t)) := \int_\Omega x u(x,t)\,\mathrm{d}x\ . \tag{2.2}$$

In the charging experiment the function $\ell \in \mathrm{C}^1([0,T])$ is prescribed for all $t \in [0,T]$. Thus (2.2) introduces a constraint on the probability density. Due to $\Omega =]0,1[$ we have

$$0 < \ell(t) < 1 \quad \text{for all} \quad t \in [0,T]\ . \tag{2.3}$$

Figure 1 shows the typical behaviour of the battery. The voltage-capacity diagram reveals two crucial phenomena. We observe hysteretic behaviour and horizontal branches, indicating a phase transition in the many-particle system during charging and discharging respectively.

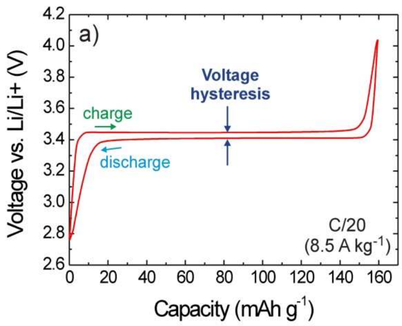

Figure 1: Voltage versus capacity of a battery with $FePO_4$ storage particles, see [DJ*10].

The time for full charging is 20 hours and hence very large with respect to the diffusional relaxation time τ of a single storage particle, which is about 1 second. Our mathematical model appropriately describes the charging-discharging process in that special case where the time to approach equilibrium of a single storage particle is much smaller than the time for full charging of the ensemble.

The evolution of the probability density $u(x,t)$ is described by the Fokker-Planck equation

$$\tau u_t = \big(u\Upsilon\big)_x \quad \text{with} \quad \Upsilon = -\Lambda(t) + \mu(x) + \nu^2\big(\log(u)\big)_x \quad \text{for} \quad x \in \Omega\ . \tag{2.4}$$

The equation contains a Lagrange multiplier Λ, which is associated with the constraint (2.2) and there appear two constant parameters $\tau > 0$ and $\nu^2 > 0$.

The evolution starts from smooth and non-negative initial data and we have homogenous no-flux boundary conditions, namely

$$u(x,0) = u_0(x) \quad \text{with} \quad \int_\Omega u(x,t)\,\mathrm{d}x = 1, \quad \Upsilon = 0 \quad \text{for} \quad x \in \partial\Omega = \{0,1\}\ . \tag{2.5}$$

By multiplying the first equation by x and integration over Ω, we see that Λ can be eliminated via

$$\Lambda(t) = \tau\dot{\ell}(t) + \nu^2(u(1){-}u(0)) + \int_\Omega \mu(x)u(x,t)\,\mathrm{d}x\ . \tag{2.6}$$

It is now easy to see that (2.4) gives exactly (1.1), and with the use of (2.6) we get (1.3).

We note that the observed hysteretic behaviour from Figure 1 is implied by the model in the parameter regime $\tau \ll 1$, $\nu^2 \ll 1$. Details of numerical simulations for various τ, ν^2 regimes are to be found in [DGH06], that also contains a careful description of the modelling. Another way of deriving macroscopic hysteresis in a many-particle system is discussed in [MiT10], where instead of the entropic diffusion term $\nu^2 u_{xx}$ spatial random fluctuations are used.

In particular, in [DGH06] it is shown that the Fokker-Planck equation (2.4) identically satisfies the 2^{nd} law of thermodynamics, which reads for the considered open system

$$\frac{d}{dt}\mathcal{A}(u(t)) - \Lambda(t)\ \dot{\ell}(t) \le 0\ , \tag{2.7}$$

where the total free energy of the system $\mathcal{A}(u)$ is given by

$$\mathcal{A}(u) = \int_\Omega (\nu^2 u(x)\ \log u(x) + \psi(x)u(x))\,\mathrm{d}x\ . \tag{2.8}$$

The newly introduced free energy of a single storage particle is related to the chemical potential by $\mu \equiv \psi'$.

From now on we set the constants τ and ν equal to 1. We can do so without loss of generality, since dividing (1.3) by ν^2 yields the equivalent PDE

$$\begin{aligned} \frac{\tau}{\nu^2}u_t(x,t) &= \Big(u_x(x,t) + \frac{\psi'}{\nu^2}(x)u(x,t) - \mathbb{L}_2(u(t)) - \frac{\tau}{\nu^2}p(t)\Big)_x, \\ \mathbb{L}_2(u(t)) &= \frac{\mathbb{L}(v)}{\nu^2} = (v(1)-v(0)) + \int_\Omega \frac{\psi'(x)}{\nu^2}v(x)\,\mathrm{d}x. \end{aligned}$$

This shows, that we can eliminate all occurrences of τ and ν on the right hand side by transforming the data to $\tilde{p}(t) = \tau p(t)/\nu^2$ and $\tilde{\psi}'(x) = \psi'(x)/\nu^2$. Another time transformation then easily lets the factor τ/ν^2 in front of the time derivative on the left hand side disappear.

2.2 Gradient systems driven by a constraint

As was observed in [JKO98], in the unconstraint case, i.e. without a condition as (2.2), the Fokker-Planck equation

$$u_t = \Big(u_x + \psi'(x)u\Big)_x \tag{2.9}$$

can be written as the gradient system

$$u_t = -K(u)\mathrm{D}\mathcal{A}(u), \quad \text{where } K(u)\psi = -\big(u\psi_x\big)_x. \tag{2.10}$$

Note that $K(u)$ is a selfadjoint, positive semidefinite operator, which can be inverted on (the tangent bundle of) function spaces satisfying the constraint (2.1) and being positive. Denoting the inverse by $G(u)$ equation (2.9) takes formally the form of a standard gradient system $G(u)u_t = -\mathrm{D}\mathcal{A}(u)$, where G denotes the metric tensor.

Moreover, (2.9) is also a transport equation (conservation law) of the form $u_t = \big\{u\tilde{v}\big\}_x$ with

$$\tilde{v} = \big(\mathrm{D}\mathcal{A}(u)\big)_x = \big(\psi + \log u\big)_x = \psi' + u_x/u$$

One of the main consequence of the gradient structure is a natural a priori estimate, called energy-dissipation estimate in terms of the functional $\mathcal{A}$ and the dissipation operator K. For the system (2.9) in the form $u_t = -K(u)\mathrm{D}\mathcal{A}(u)$ it reads

$$\frac{d}{dt}\Big(\mathcal{A}(u(t))\Big) = -\Big\langle \mathrm{D}\mathcal{A}(u), K(u)\mathrm{D}\mathcal{A}(u)\Big\rangle = -\int_\Omega \frac{(u_x + \psi' u)^2}{u}\,\mathrm{d}x \le 0.$$

This shows that $\mathcal{A}$ decreases along trajectories and that the only equilibria are those where $u_x + \psi' u \equiv 0$.

In the present case we have a *constraint gradient system*, $u_t = \big\{uv\big\}_x$, but now v is given by (2.4).

Finally, we return to the full problem (2.4) and (2.5), which we identify as a *constraint gradient system* in the form

$$u_t = -K(u)\Big(\mathrm{D}\mathcal{A}(u) - \Lambda\mathrm{D}\mathcal{C}(u)\Big), \qquad \mathcal{C}(u) = \ell(t), \tag{2.11}$$

where the operator K is given as in (2.10).

Testing with 1 and using the definition of K we immediately find that $\int_\Omega u\,\mathrm{d}x$ is constant along solutions. Moreover, taking the derivative of the constraint we immediately find the correct relation for $\dot{\ell}$, namely

$$\begin{aligned}\dot{\ell} &= \langle \mathrm{D}\mathcal{C}(u), u_t\rangle = \langle x, \Big(u\big((\log u{+}1) + \psi - \Lambda x\big)_x\Big)_x\rangle \\ &= -\langle 1, u\big(\tfrac{u_x}{u} + \psi' - \Lambda\big)\rangle = -\int_\Omega u_x{+}\psi' u\,\mathrm{d}x + \Lambda\int_\Omega u\,\mathrm{d}x.\end{aligned}$$

Using $\int_\Omega u\,\mathrm{d}x \equiv 1$ we find the adequate definition (2.6) for the Lagrange multiplier Λ.

Finally, we may take the derivative of $\mathcal{A}(u(t))$ to obtain the following crucial energy-dissipation estimate in terms of the data ℓ.

***Lemma* 2.1.** *Every sufficiently smooth solution u of* (1.1) *satisfies*

$$\begin{aligned}&\frac{d}{dt}\mathcal{A}(u(t)) = -\mathcal{D}(u,\dot{\ell}) \text{ where}\\ &\mathcal{D}(u,\dot{\ell}) = \int_\Omega \frac{(u_x{+}\psi' u)^2}{u}\,\mathrm{d}x - \Big(\int_\Omega u_x{+}\psi' u\,\mathrm{d}x\Big)^2 - \dot{\ell}\int_\Omega u_x{+}\psi' u\,\mathrm{d}x.\end{aligned} \tag{2.12}$$

Proof. Taking the derivative of $\mathcal{A}$ along a solution we find

$$\begin{aligned}\frac{d}{dt}\mathcal{A}(u(t)) &= \langle \mathrm{D}\mathcal{A}(u), u_t\rangle = \langle \big((\log u{+}1) + \psi, \Big(u\big((\log u{+}1) + \psi - \Lambda x\big)_x\Big)_x\rangle\\ &= -\int_\Omega \frac{(u_x + \psi' u)^2}{u}\,\mathrm{d}x + \Lambda\int_\Omega u_x + \psi' u\,\mathrm{d}x.\end{aligned}$$

Inserting formula (2.6) for Λ the assertion is established. ☐

A crucial step in our global existence result will be a suitable lower estimate for the dissipation functional $\mathcal{D}$, which is not automatically nonnegative for $\dot{\ell} \neq 0$, because of the work done by the changing constraint $\mathcal{C}(u(t)) = \ell(t)$.

3 Local existence of classical solutions

In this section we will inspect the solvability of the PDE (1.3). In this PDE the constraint $\mathcal{C}(u(t)) = \ell(t)$ is resolved and as a consequence the PDE is influenced only by the derivative $p := \dot{\ell}$. Also the datum function ψ which comes from the energy $\mathcal{A}$, see (2.6), has only influence through its derivative $\mu := \psi'$. Thus the results in this section are stated independently, only for Problem (1.3) with given data p and μ. The relation of solutions to ℓ and $\mathcal{A}$ are used in the later sections where we return to the investigation of the equivalent Problem (1.1).

In the sequel, $L^q(\Omega)$ denotes the usual complex Lebesgue space, with norm $\|\cdot\|_{L^q}$. For a function $u(x,t)$ depending on two variables, we write $u(t)$ for the function $\{x \mapsto u(x,t)\}$. This makes notation shorter, such that $\|u(t)\|_{L^q}$ is shorthand for $\|u(\cdot,t)\|_{L^q}$.

3.1 The semilinear equation: local existence and uniqueness

Our approach towards local existence of solutions basically follows a standard procedure for semilinear parabolic PDE's. We carry out the proofs, in order to incorporate two aims. We want that only some spatial $L^q(\Omega)$ norm of a solution with any $q > 1$ needs to be controlled near $t = T$ in order to extend the solution beyond time T. Furthermore we want our theory to hold for choices of μ which are only in some $L^q(\Omega)$ but not necessarily bounded, as the choice of μ in the model in [DGH06] has logarithmic singularities at the boundary.

We are looking for a solution of (1.3). By a solution we mean a function $u \in \mathrm{C}^1\big(]0,T_0[\,,L^q(\Omega)\big) \cap C\big(]0,T_0[\,,\mathrm{W}^{1,q}(\Omega)\big) \cap C\big([0,T_0[\,,L^q(\Omega)\big)$ such that for all $\varphi \in \mathrm{C}^\infty(\bar\Omega)$ and $t \in]0,T_0[$ there holds

$$\int_\Omega u_t(x,t)\varphi(x)\,\mathrm{d}x = -\int_\Omega \Big(u_x(x,t) + u(x,t)\Big[\mu(x) - \mathbb{L}(u(t)) - p(t)\Big]\Big)\varphi_x(x)\,\mathrm{d}x. \tag{3.1}$$

***Theorem* 3.1.** *Suppose that $p \in \mathrm{C}^\delta_{loc}([0,\infty[)$, $\mu \in \mathrm{L}^q(\Omega)$, and that $u_0 \in \mathrm{L}^q(\Omega)$ for some $q > 1$ and $\delta > 1/2$. Then there exists a maximal time $T_0 \in]0,\infty]$ and a uniquely determined solution of* (1.3) *(in the sense of* (3.1)*). Moreover we have the following alternative:*

$$\textit{Either } T_0 = \infty, \textit{ or } \|u(t)\|_{L^q} \to \infty \textit{ as } t \nearrow T_0 \quad \textit{for some } q > 1. \tag{3.2}$$

Proof. Existence and Uniqueness. We define $\mathbb{L}$ as in (1.4), $M := \int_0^1 u_0(x)\,\mathrm{d}x$, and $w_0(x) := \int_0^x u_0(z)\,\mathrm{d}z - Mx$. First we prove the existence of a solution to the problem

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) &= \Big[\mu(x) - p(t) - \mathbb{L}\left(w_x(t) + M\right)\Big](w_x(x,t) + M),\\ w(0,t) &= w(1,t) = 0,\\ w(x,0) &= w_0(x). \end{aligned} \tag{3.3}$$

The fact, that then the function $u := w_x + M$ is a solution to the original problem (1.3) follows as a regularity result. For $T \le 1$, $q > 1$ and $\beta \in]1 + 1/q, 2[$ we consider the space

$$\begin{aligned} X &:= \mathrm{C}([0,T],\mathrm{W}^{1,q}_0(\Omega)) \cap C_\gamma(]0,T]\,,\mathrm{W}^{\beta,q}_0(\Omega)),\\ \|v\|_X &:= \sup_{t\in[0,T]} \|v(t)\|_{\mathrm{W}^{1,q}_0} + \sup_{t\in]0,T]} t^\gamma \|v(t)\|_{\mathrm{W}^{\beta,q}_0}, \end{aligned}$$

where γ is specified below. The choice of β gives the compact injection $\mathrm{W}^{\beta,q}_0(\Omega) \hookrightarrow \mathrm{C}^1(\bar\Omega)$. Thus a constant $c_{\mathbb{L}}$ depending on $\|\mu\|_{L^q}$ exists such that for all $v \in \mathrm{W}^{1,q}_0(\Omega)$ there holds

$$|\mathbb{L}(v_x)| \le c_{\mathbb{L}} \|v\|_{\mathrm{W}^{\beta,1}_0} \qquad \text{and} \qquad |\mathbb{L}(v_x + M)| \le c_{\mathbb{L}}(\|v\|_{\mathrm{W}^{\beta,1}_0} + M)\,.$$

Let A denote the Dirichlet Laplacian on Ω. For a suitable $\gamma \in]0,1/2[$ and $\tilde R > 0$ we now show that F, defined as,

$$\begin{aligned} Fv(t) &:= \mathrm{e}^{-tA}w_0 + \int_0^t \mathrm{e}^{-(t-s)A} N(v(s))\,\mathrm{d}s, \qquad v \in S,\ t \in [0,T],\\ Nv(x,t) &:= -\Big(\mathbb{L}(v_x(t) + M) - \mu(x) + p(t)\Big) \cdot (v_x(x,t) + M), \end{aligned}$$

is a contractive selfmapping on the closed set

$$S := \Big\{v \in X : v(0) = w_0 \text{ and } \|x\|_X \le \tilde R + 2\|w_0\|_{\mathrm{W}^{1,q}_0} := R\Big\}\,,$$

and thus has a unique fixed point. A fixed point $\tilde v$ of F would be a mild solution to the above PDE. In the sequel we deduce restrictions for the choice of γ, depending on q and β.

First we show that for T small enough, F is a selfmapping on S. We define $R_M := R + M$ and examine

$$
\begin{aligned}
\|F(v)(t)\|_{\mathrm{W}_0^{1,q}} &\leq \|w_0\|_{\mathrm{W}_0^{1,q}} + \int_0^t (t-s)^{-\frac{1}{2}} \| \big(\mu(x) - p(s)\big)\big(v_x(s) + M\big) \|_{L^q} \,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{-\frac{1}{2}} \| \mathbb{L}\big(v_x(s) + M\big)\big(v_x(s) + M\big) \|_{L^q} \,\mathrm{d}s, \\
&\leq \|w_0\|_{\mathrm{W}_0^{1,q}} + \int_0^t (t-s)^{-\frac{1}{2}} \| \big(\mu(x) - p(s)\big) \|_{L^q} (\|v(s)\|_{\mathrm{W}_0^{\beta,q}} + M) \,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{-\frac{1}{2}} |\mathbb{L}\big(v_x(s) + M\big)| \, (\|v(s)\|_{\mathrm{W}_0^{1,q}} + M) \,\mathrm{d}s, \\
&\leq \|w_0\|_{\mathrm{W}_0^{1,q}} + \int_0^t (t-s)^{-\frac{1}{2}} s^{-\gamma} \left[\big(\|\mu(x)\|_{L^q} + \|p\|_{L^\infty}\big) R_M + c_{\mathbb{L}} R_M^2 \right] \,\mathrm{d}s, \\
&\leq \|w_0\|_{\mathrm{W}_0^{1,q}} + t^{\frac{1}{2}-\gamma} c_0 \int_0^1 (1-\sigma)^{-\frac{1}{2}} \sigma^{-\gamma} \,\mathrm{d}\sigma.
\end{aligned}
$$

Note that the integral exists by the choice of γ and the constant c_0 depends on upper bounds for M, $\tilde{R}$, $\|p\|_{L^\infty}$, $\|\mu\|_{L^q}$ and $\|w_0\|_{\mathrm{W}^{1,q}}$. Thus diminishing T if necessary, we have

$$
\sup_{t\in[0,T]} \|F(v)(t)\|_{\mathrm{W}_0^{1,q}} \leq \frac{R}{2}\,.
$$

As a second step we estimate

$$
\begin{aligned}
\|F(v)(t)\|_{\mathrm{W}_0^{\beta,q}} &\leq t^{-\frac{\beta-1}{2}} \|w_0\|_{\mathrm{W}_0^{1,q}} + \int_0^t (t-s)^{-\frac{\beta}{2}} \| \big(\mu(x) - p(s)\big)\big(v_x(s) + M\big) \|_{L^q} \,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{-\frac{\beta}{2}} \| \mathbb{L}\big(v_x(s) + M\big)\big(v_x(s) + M\big) \|_{L^q} \,\mathrm{d}s, \\
&\leq t^{-\frac{\beta-1}{2}} \|w_0\|_{\mathrm{W}_0^{1,q}} + \int_0^t (t-s)^{-\frac{\beta}{2}} s^{-\gamma} \left[\big(\|\mu(x)\|_{L^q} + \|p\|_{L^\infty}\big) R_M + c_{\mathbb{L}} R_M^2 \right] \,\mathrm{d}s, \\
&\leq t^{-\frac{\beta-1}{2}} \|w_0\|_{\mathrm{W}_0^{1,q}} + t^{1-\gamma-\frac{\beta}{2}} c_0 \int_0^1 (1-\sigma)^{-\frac{\beta}{2}} \sigma^{-\gamma} \,\mathrm{d}\sigma.
\end{aligned}
$$

Again the integral over σ exists by the choice of γ and β. Thus requiring $\beta < 1 + 2\gamma$, we get for sufficiently small T

$$
\sup_{t\in]0,T]} t^{\gamma} \|F(v)(t)\|_{\mathrm{W}_0^{\beta,q}} \leq \frac{R}{2}\,.
$$

which gives that F is a selfmapping on S.

Next we check, that F is a contraction on S. For $v_1, v_2 \in S$ we can estimate

$$\begin{aligned}
\|F(v_1)(t) - F(v_2)(t)\|_{\mathrm{W}_0^{1,q}} &\leq \int_0^t (t-s)^{-\frac{1}{2}} \|\big(\mu(x) - p(s)\big)\big(v_{1x}(s) - v_{2x}(s)\big)\|_{L^q}\, \mathrm{d}s \\
&\quad + \int_0^t (t-s)^{-\frac{1}{2}} \|\mathbb{L}\big(v_{1x}(s) - v_{2x}(s)\big)\big(v_{1x}(s) + M\big) \\
&\qquad\qquad - \mathbb{L}\big(v_{2x}(s) + M\big)\big(v_{2x}(s) - v_{1x}(s)\big)\|_{L^q}\, \mathrm{d}s, \\
&\leq \int_0^t (t-s)^{-\frac{1}{2}} \big(\|\mu(x)\|_{L^q} + \|p\|_{L^\infty}\big) \|v_1(s) - v_2(s)\|_{\mathrm{W}_0^{\beta,q}}\, \mathrm{d}s \\
&\quad + c_{\mathbb{L}} \int_0^t (t-s)^{-\frac{1}{2}} \|v_1(s) - v_2(s)\|_{\mathrm{W}_0^{\beta,q}} R_M\, \mathrm{d}s \\
&\quad + c_{\mathbb{L}} \int_0^t (t-s)^{-\frac{1}{2}} R_M s^{-\gamma} \|v_1(s) - v_2(s)\|_{\mathrm{W}_0^{1,q}}\, \mathrm{d}s, \\
&\leq t^{\frac{1}{2}-\gamma} c_1 \int_0^1 (1-\sigma)^{-\frac{1}{2}} \sigma^{-\gamma}\, \mathrm{d}\sigma \|v_1(s) - v_2(s)\|_X.
\end{aligned}$$

As before the constant c_1 depends on bounds for the given date. For small T this gives

$$\sup_{t\in[0,T]} \|F(v_1)(t) - F(v_2)(t)\|_{\mathrm{W}_0^{1,q}} \leq \frac{1}{4} \|v_1(s) - v_2(s)\|_X.$$

In a similar way we estimate

$$\begin{aligned}
\|F(v_1)(t) - F(v_2)(t)\|_{\mathrm{W}_0^{\beta,q}} &\leq \int_0^t (t-s)^{-\frac{\beta}{2}} \|\big(\mu(x) - p(s)\big)\big(v_{1x}(s) - v_{2x}(s)\big)\|_{L^q}\, \mathrm{d}s \\
&\quad + \int_0^t (t-s)^{-\frac{\beta}{2}} \|\mathbb{L}\big(v_{1x}(s) - v_{2x}(s)\big)\big(v_{1x}(s) + M\big) \\
&\qquad\qquad - \mathbb{L}\big(v_{2x}(s) + M\big)\big(v_{2x}(s) - v_{1x}(s)\big)\|_{L^q}\, \mathrm{d}s, \\
&\leq t^{1-\gamma-\frac{\beta}{2}} c_2 \int_0^1 (1-\sigma)^{-\frac{\beta}{2}} \sigma^{-\gamma}\, \mathrm{d}\sigma \|v_1(s) - v_2(s)\|_X.
\end{aligned}$$

The constant again depends on bounds for the given date. For small T this gives

$$\sup_{t\in]0,T]} t^\gamma \|F(v_1)(t) - F(v_2)(t)\|_{\mathrm{W}_0^{\beta,q}} \leq \frac{1}{4} \|v_1(s) - v_2(s)\|_X.$$

Thus F is a $1/2$ contraction on S provided, that T is sufficiently small. Hence it has a unique fixed point.

Let us discuss the interrelation of the occurring parameters. Remember that by assumption there holds $q > 1$. Then all our requirements for β, γ and q which we needed, namely

$$1 + \frac{1}{q} < \beta < 2, \qquad \beta < 1 + 2\gamma \qquad \text{and} \qquad 0 < \gamma < \frac{1}{2}.$$

are satisfied by choosing γ close to $1/2$.

The length of the existence time T depends on bounds for $\|\mu\|_{L^q}$, M, $\|p\|_{L^\infty}$, the choice of $\tilde{R}$ and the $\mathrm{W}_0^{1,q}(\Omega)$ norm of the initial value w_0. Thus by successively repeating the above reasoning we get an alternative provided that we have a uniform bound on $|p(t)|$. The maximal existence time T_0 is then either ∞ or if $T_0 < \infty$ then there must hold for every $q > 1$, that $\|w(t)\|_{\mathrm{W}_0^{1,q}} \to \infty$ as $t \to T_0$.

Regularity. By construction of the space X we know for every $\varepsilon > 0$, that $\|w(t)\|_{\mathrm{W}_0^{\beta,q}} \leq c_\varepsilon$, for all $t > \varepsilon$ Inserting this in the right hand side of the above PDE gives a linear parabolic equation with in $L^q(\Omega)$ bounded right hand side $N(w(t))$ on the time interval $[\varepsilon, T]$. According to known regularity theory for parabolic PDE's, see [Lun95, Prop.4.2.1], this results in the solution being even Hölder continuous in time. We have for all $\eta \in [0,1[$ that the solution w to the above PDE is of quality

$$w \in \mathrm{C}^{1-\frac{\eta}{2}}\big([\varepsilon, T], \mathrm{W}_0^{\eta,q}(\Omega)\big).$$

Now we use the assumption that p is Hölder continuous with the Hölder exponent $\delta > 0$. The right hand side $N(w(t))$ is then Hölder continuous in time on any interval $[\varepsilon, T]$. Thus again using known regularity theory, see [Lun95, Prop.4.3.4], we have that w is a classical solution to (3.3). Using this we can iteratively improve the Hölder continuity of w to get by the assumption $\delta > 1/2$, that there exists a small $\tilde{\delta} > 0$ such that

$$w \in \mathrm{C}^{1+\delta}\big([\varepsilon, T], L^q(\Omega)\big) \cap \mathrm{C}^{\delta}\big([\varepsilon, T], \mathrm{W}_0^{2,q}(\Omega)\big) \subset \mathrm{C}^{1+\tilde{\delta}}\big([\varepsilon, T], \mathrm{W}_0^{1,q}(\Omega)\big).$$

Finally we have that the time derivative w' is spatially weakly differentiable for positive times t and vanishes at the boundary. Thus we see that $u := w_x - M$ is a solution to (1.3) in the sense of (3.1), since for any $\varphi \in \mathrm{C}^\infty(\bar{\Omega})$ we have

$$\begin{aligned}
\int_\Omega u_t(x,t)\varphi(x)\,\mathrm{d}x &= -\int_\Omega w_t(x,t)\varphi_x(x)\,\mathrm{d}x,\\
&= -\int_\Omega \Big(w_{xx}(x,t) + (w_x(x,t)+M)\Big[\mu(x) - \mathbb{L}(w_x(t)+M) - p(t)\Big]\Big)\varphi_x(x)\,\mathrm{d}x,\\
&= -\int_\Omega \Big(u_x(x,t) + u\Big[\mu(x) - \mathbb{L}(u) - p(t)\Big]\Big)\varphi_x(x)\,\mathrm{d}x.
\end{aligned}$$

□

***Remark* 3.2.** *Better regularity of the data μ results in better spatial regularity of the solution.*

***Lemma* 3.3.** *Let the Assumptions of Theorem 3.1 with $q = 2$ hold. If the initial value $u_0 \in L^2(\Omega)$ is nonnegative, then the solution u to (1.3) is also nonnegative on $\Omega \times [0,T]$.*

Proof. We first show the nonnegativity if μ is a bounded function: since $q = 2$ we can apply the negative part $u^-(t)$ as a test function in (3.1) to get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^-(t)\|_{L^2}^2 &\leq -\|u_x^-(t)\|_{L^2}^2 + \int_\Omega u_x^-(x,t)u^-(x,t)\big(\mu - \mathbb{L}\big(u(t)\big) - p(t)\big)\,\mathrm{d}x,\\
&\leq \frac{1}{4}(\|\mu\|_{L^\infty} + R_M t^{-\gamma} + \|p\|_{L^\infty})^2\|u^-(t)\|_{L^2}^2.
\end{aligned}$$

Using Gronwall's lemma and $\|u_0^-\|_{L^2} = 0$, we deduce $\|u^-(t)\|_{L^2} = 0$ for all $t \in [0, T]$. If μ is unbounded, we define for any $k > 0$ the cut-off function $\mu_k := \min(k, \max(-k, \mu))$. By u_k we call the solution of (3.1), if μ is there replaced by μ_k. We consider the functions $w_k(x,t) := \int_0^x u_k(y,t)\,\mathrm{d}y - Mx$, being the solution to (3.3) when there μ is replaced by μ_k. Since $\|\mu_k\|_{L^2} \leq \|\mu\|_{L^2}$ one can find a common $T > 0$ such that each of the mappings

$$F_k v(t) := \mathrm{e}^{-tA} w_0 + \int_0^t \mathrm{e}^{-(t-s)A} \left[\mu_k - \mathbb{L}(v_x(s) + M) - p(s)\right](v_x(s) + M)\,\mathrm{d}s, \quad \text{for } v \in S. \tag{3.4}$$

is a contraction on X with contraction constant $\frac{1}{2}$. Let w_k denote the corresponding fixed point for F_k and w the fixed point for the mapping F from above. One easily calculates

$$\|w - w_k\|_X \leq \|Fw - F_k w_k\|_X \leq \|F_k w - F_k w_k\|_X + \|Fw - F_k w\|_X \leq \frac{1}{2}\|w - w_k\|_X + \|Fw - F_k w\|_X,$$

what leads to $\|w - w_k\|_X \leq 2\|Fw - F_k w\|_X$. Let us show that $\|Fw - F_k w\|_X$ approaches 0; one has

$$\begin{aligned} \|Fw(t) - F_k w(t)\|_{H^1} &\leq \Big\| \int_0^t \mathrm{e}^{-(t-s)A} (w(s)_x + M)\big(\mu - \mu_k\big)\,\mathrm{d}s \Big\|_{H^1}, \\ &\leq \|\mu - \mu_k\|_{L^2} t^{\frac{1}{2} - \gamma} \int_0^1 (1-\sigma)^{-\frac{1}{2}} \sigma^{-\gamma} R_M \,\mathrm{d}\sigma. \end{aligned}$$

Since $\mu_k \to \mu$ in $L^2(\Omega)$, we get, uniformly in t, $\|Fw(t) - F_k w(t)\|_{H^1} \to 0$. Similarly we get $t^\gamma \|Fw(t) - F_k w(t)\|_{H^\beta} \to 0$ uniformly for all $t \in [0, T]$. This gives for all $t \in [0, T]$ that $w_k(t) \to w(t)$ in $\mathrm{H}^1(\Omega)$, and thus $u_k(t) \to u(t)$ in $L^2(\Omega)$. Hence, $u(t)$ must also be a nonnegative function because each $u_k(t)$ – corresponding to the bounded coefficient function μ_k– is.

□

In the model derived in [DGH06], μ is not only in $L^2(\Omega)$, but inside of Ω it is a smooth function. This helps us to deduce strict positivity of the solution for positive times.

***Lemma* 3.4.** *Let the Assumptions of Theorem 3.1 hold. Furthermore assume that for all $\varepsilon > 0$ we have $\mu \in \mathrm{C}^1([\varepsilon, 1-\varepsilon])$ and $0 \neq u_0 \in L^2(\Omega)$ is nonnegative. Then the solution is strictly positive inside Ω for all positive times.*

Proof. Let u be the solution to problem (1.3). We define $\Omega_\varepsilon :=]\varepsilon, 1-\varepsilon[$ and

$$\psi(x,t) := \mu(x) - \mathbb{L}(u(t)) - p(t).$$

Consider the function $w_\varepsilon(x,t) := u(x,t)\mathrm{e}^{st}$ with $s_\varepsilon := -\sup_{x \in \Omega_\varepsilon} |\mu_x(x)| \geq 0$. This function then solves inside $\Omega_\varepsilon \times]\varepsilon, T]$

$$w_{\varepsilon t}(x,t) - w_{\varepsilon xx}(x,t) - w_{\varepsilon x}(x,t)\psi(x,t) = u(x,t)\mathrm{e}^{st}(x,t)(\mu_x(x) + s_\varepsilon). \tag{3.5}$$

The coefficients and initial as well as boundary values are spatially continuous and in time even Hölder continuous. This allows us to apply classical parabolic theory. We know from Theorem 3.1 that the initial and boundary values to this PDE are nonnegative. Due to conservation of mass, the initial function $w_{\varepsilon 0}(x) := u(x,\varepsilon)\mathrm{e}^{\varepsilon s}$ is positive inside Ω_ε for ε small enough. Even the right hand side of the PDE (3.5) is nonnegative. Hence using classical maximum principles, see for example [Eva98, Chapter 7.1 Theorem 9], we get $w > 0$ in $\Omega_\varepsilon \times]\varepsilon, T]$ and thus u is also strictly positive. This means by the arbitrariness of ε, that u is positive everywhere inside Ω for all positive times t. □

Lemma 3.5. *Assume that the solution u exists on a time interval $[t_0, T_*[$ and $\mu \in L^\infty(\Omega)$. Let c_0 denote the constant $\max\big(\|\mu\|_{L^\infty}, \|p\|_{L^\infty([t_0,T_*[)}\big)$, and let us put $a := \frac{36c_0^2}{5}$, $b := 4^4\left(\frac{\pi+1}{\pi}\right)^2$.*

i) Then the L^2-norm of u admits the following estimate:

$$\|u(t)\|_{L^2}^2 \le \frac{1}{\sqrt{\mathrm{e}^{2a(t_0-t)}\big[\frac{1}{\|u(t_0)\|_{L^2}^4} + \frac{b}{a}\big] - \frac{b}{a}}}, \tag{3.6}$$

as long as the expression under the square root is positive.

ii) Consequently, the L^2 norm does not explode on any interval $[t_0, T]$ as long as

$$T < t_0 + \frac{1}{2a}\log\Big(\frac{a}{b}\frac{1}{\|u(t_0)\|_{L^2}^4} + 1\Big).$$

Proof. We test the equation (1.3) with u and obtain for every $t \in]t_0, T_*[$

$$\begin{aligned}
\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_{t_0}^t \int_0^1 |u_x(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \le \\
\le \frac{1}{2}\|u(t_0)\|_{L^2}^2 + \int_{t_0}^t \|\mu\|_{L^\infty} \int_0^1 |u(x,s)||u_x(x,s)| \,\mathrm{d}x\,\mathrm{d}s + \\
+ \int_{t_0}^t |p(s) + \int_0^1 \mu(x)u(x,s)\,\mathrm{d}x + u(0,s) - u(1,s)| \int_0^1 |u(x,s)|\,|u_x(x,s)| \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.7}$$

Exploiting $\int_\Omega u \,\mathrm{d}x = 1$ and thus $|\int_\Omega \mu u \,\mathrm{d}x| \le c_0$, we obtain

$$|p(s) + \int_0^1 \mu(x)u(x,s)\,\mathrm{d}x| \le 2c_0,$$

while (A.4) from Lemma A.1 gives

$$|u(0,s) - u(1,s)| \le 2\sqrt{2}\sqrt{\frac{\pi+1}{\pi}}\|u(s)\|_{L^2}^{1/2}\|u_x(s)\|_{L^2}^{1/2}.$$

Additionally estimating $\int_0^1 |u(x,s)|\,|u_x(x,s)|\,\mathrm{d}x$ on the right hand side of (3.7) by $\|u(s)\|_{L^2}\|u_x(s)\|_{L^2}$, we derive from (3.7) the inequality

$$\frac{1}{2}\|u(t)\|_{L^2}^2 \le \frac{1}{2}\|u(t_0)\|_{L^2}^2 - \int_{t_0}^t \|u_x(s)\|_{L^2}^2\,\mathrm{d}s + 3c_0\int_{t_0}^t \|u(s)\|_{L^2}\|u_x(s)\|_{L^2}\,\mathrm{d}s+ \\ + \int_{t_0}^t 2\sqrt{2}\sqrt{\frac{\pi+1}{\pi}}\|u(s)\|_{L^2}^{3/2}\Big\|u_x(s)\Big\|_{L^2}^{3/2}\mathrm{d}s. \tag{3.8}$$

(3.8), equivalently written, reads as,

$$\|u(t)\|_{L^2}^2 \le -\int_{t_0}^t 2\|u_x(s)\|_{L^2}^2\,\mathrm{d}s + \|u(t_0)\|_{L^2}^2 + \tag{3.9}$$

$$\int_{t_0}^t 6c_0\|u(s)\|_{L^2}\|u_x(s)\|_{L^2} + 4\sqrt{2}\sqrt{\frac{\pi+1}{\pi}}\|u(s)\|_{L^2}^{3/2}\|u_x(s)\|_{L^2}^{3/2}\,\mathrm{d}s.$$

We estimate by Young's inequality

$$6c_0\|u(s)\|_{L^2}\|u_x(s)\|_{L^2} \le \frac{36c_0^2}{5}\|u(s)\|_{L^2}^2 + \frac{5}{4}\|u_x(s)\|_{L^2}^2 \tag{3.10}$$

and

$$4\sqrt{2}\sqrt{\frac{\pi+1}{\pi}}\|u(s)\|_{L^2}^{3/2}\|u_x(s)\|_{L^2}^{3/2} \le 4^4\Big(\frac{\pi+1}{\pi}\Big)^2\|u(s)\|_{L^2}^6 + \frac{3}{4}\|u_x(s)\|_{L^2}^2. \tag{3.11}$$

Applying this to (3.9), we get

$$\|u(t)\|_{L^2}^2 \le \|u(t_0)\|_{L^2}^2 + \int_{t_0}^t \frac{36c_0^2}{5}\|u(s)\|_{L^2}^2 + 4^4\Big(\frac{\pi+1}{\pi}\Big)^2\|u(s)\|_{L^2}^6\,\mathrm{d}s. \tag{3.12}$$

Putting $g : [0,\infty[\, \ni s \mapsto as + bs^3 = s(a+bs^2)$ and $y(s) := \|u(s)\|_{L^2}^2$, (3.12) can be read as the following integral inequality for y:

$$y(t) \le y(t_0) + \int_{t_0}^t g(y(s))\,\mathrm{d}s, \quad t \in]t_0, T_*[.$$

By straight forward computation one identifies the primitive of $\frac{1}{g}$ by

$$G : r \mapsto \frac{1}{a}\log\frac{r}{\sqrt{1+\frac{b}{a}r^2}} + \kappa, \tag{3.13}$$

for arbitrary choice of κ. Thus the inverse function G^{-1} is

$$G^{-1} : t \mapsto \frac{1}{\sqrt{\mathrm{e}^{-2a(t-\kappa)} - \frac{b}{a}}}.$$

According to the Bihari-Lemma ([Bih56], see also [BeB61, Ch. 4.5]) we get

$$y(t) \le G^{-1}(G(y(t_0)) + t - t_0) = \frac{1}{\sqrt{\mathrm{e}^{2a(t_0-t)}[\frac{1}{y(t_0)^2} + \frac{b}{a}] - \frac{b}{a}}}, \tag{3.14}$$

as long as the expression under the square root is positive.
ii) follows straightforward from i). □

3.2 Blow-up results

The identity (4.2b) for the time evolution of the first moments of solutions easily leads to the following blow-up criterion.

***Lemma* 3.6.** *Suppose that there exists $T > 0$ such that either*

$$\int_0^T p(t)\,\mathrm{d}t < -\int_\Omega x u_0(x)\,\mathrm{d}x \tag{3.15}$$

or

$$\int_0^T p(t)\,\mathrm{d}t > 1 - \int_\Omega x u_0(x)\,\mathrm{d}x. \tag{3.16}$$

Then the solution u of (1.3) *blows up before or at time T.*

Proof. Assuming that the maximal existence time T_0 of u exceeds T, we recall (4.2b) to see that

$$\int_0^T p(t)\,\mathrm{d}t = \int_\Omega x u(x,T)\,\mathrm{d}x - \int_\Omega x u_0(x)\,\mathrm{d}x.$$

This is compatible neither with (3.15) nor with (3.16), because the properties $u \geq 0$ and $\int_\Omega u\,\mathrm{d}x \equiv 1$ entail that

$$0 < \int_\Omega x u(x,t)\,\mathrm{d}x < 1 \qquad \text{for all } t \in \left]0, T_0\right[. \tag{3.17}$$

Thus, u must cease to exist before time T. □

As a particular consequence, we see that if p is sufficiently large then *all* solutions blow-up.

***Corollary* 3.7.** *Suppose that there exists $T > 0$ such that*

$$\left|\int_0^T p(t)\,\mathrm{d}t\right| \geq 1. \tag{3.18}$$

Then for all nonnegative u_0 fulfilling $\int_\Omega u_0\,\mathrm{d}x = 1$, the solution of (1.3) *blows up in finite time.*

Proof. In view of (3.17), the assumption (3.18) shows that any such u_0 satisfies either (3.15) or (3.16), so that the solution emanating from u_0 will blow-up before time T. □

Secondly, if merely $p \not\equiv 0$ then at least *some* initial data lead to non-global solutions.

***Corollary* 3.8.** *Suppose that $p \not\equiv 0$. Then there exists a nonnegative $u_0 \in \mathrm{C}_0^\infty(\Omega)$ with $\int_\Omega u_0\,\mathrm{d}x = 1$ such that the corresponding solution u of* (1.3) *blows up in finite time.*

Proof. Let $P(t) := \int_0^t p(s)\,\mathrm{d}s$ for $t \geq 0$. Since $P' = p$ on $]0,\infty[$, our assumption $p \not\equiv 0$ ensures that for some $T > 0$ we have $P(T) \neq 0$, which enables us to choose some $\varepsilon \in \left]0, \frac{1}{2}\right[$ such that $\varepsilon < |P(T)|$. We now fix any nonnegative $\zeta \in \mathrm{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\zeta \subset]1,2[$ and $\int_1^2 \zeta(\xi)\,\mathrm{d}\xi = 1$.
Assuming first that $P(T) < 0$, we then let

$$u_0(x) := \frac{1}{\varepsilon} \cdot \zeta\Big(\frac{x}{\varepsilon}\Big), \qquad x \in [0,1].$$

Then u_0 belongs to $\mathrm{C}^\infty(\bar{\Omega})$ and has its support contained in $]\varepsilon, 2\varepsilon[\subset \Omega$, and we easily compute $\int_\Omega u_0\,\mathrm{d}x = 1$. Moreover, its first moment satisfies

$$\int_\Omega x u_0(x)\,\mathrm{d}x = \int_\varepsilon^{2\varepsilon} x u_0(x)\,\mathrm{d}x \leq \varepsilon \cdot \int_\Omega u_0(x)\,\mathrm{d}x = \varepsilon < -P(T),$$

which entails that (3.15) is fulfilled, so that Lemma 3.6 asserts finite-time blow-up of the corresponding solution.
In the case $P(T) > 0$ we proceed similarly by defining

$$u_0(x) := \frac{1}{\varepsilon} \cdot \zeta\Big(\frac{1-x}{\varepsilon}\Big), \qquad x \in [0,1],$$

and showing that then (3.16) holds. ☐

4 Global existence for general ℓ

We now return to the situation where ℓ is a given datum such that $\ell(t) \in]0,1[$ for all time. For the local existence results in the previous section we only used $p(t) = \dot{\ell}(t)$ and hence the additional constraint $\ell(t) > 0$ and $\ell(t) < 1$ where only implicit.

We first make sure that the local solutions constructed above satisfies the constraint as expected and therefore turn out to be solutions to (1.1). For this we recall the general assumption

$$\ell \in \mathrm{C}^1([0,\infty[) \quad \text{and} \quad 0 < \ell(t) < 1 \text{ for all } t \geq 0. \tag{4.1}$$

The expected result is the following.

***Lemma* 4.1.** *Let (4.1) and (1.2) hold. Suppose that u is a classical solution of (1.3) in $\Omega \times]0,T[$ for some $T \in]0,\infty]$ satisfying $\int_\Omega u(x,0)\,\mathrm{d}x = 1$ and $\mathcal{C}(u(0)) = \ell(0)$. Then*

$$\int_\Omega u(x,t)\,\mathrm{d}x = 1 \qquad \textit{for all } t \in]0,T[\tag{4.2a}$$

and

$$\mathcal{C}(u(t)) = \int_\Omega x u(x,t)\,\mathrm{d}x = \ell(t) \qquad \textit{for all } t \in]0,T[\,. \tag{4.2b}$$

Proof. The first identity easily results by using $\varphi \equiv 1$ as a test function for (1.3), whereas (4.2b) follows upon choosing $\varphi(x,t) := x$ and using $\int_\Omega x u(x,0)\,\mathrm{d}x = \ell(0)$. ☐

***Remark* 4.2.** *As a simple consequence of Lemma 4.1 together with Lemma 3.6 we get a sufficient explosion condition. For $\ell \in \mathrm{W}^{1,\infty}(]0,\infty[)$, with $\ell(0) \in]0,1[$, let t_* be the first time such that $\ell(t_*) = 1$ or $\ell(t_*) = 0$. Then t_* must be an explosion time for the solution to* (1.1)*, if the solution does not cease to exist before time t_*. The rest of this section is devoted to the fact, that this condition is also necessary. Thus if ℓ stays inside of $]0,1[$, then the solution exists globally and does not explode. The solution then even stays bounded in $L^\infty(\Omega)$ on all bounded time intervals.*

4.1 Dissipation and energy control

The next result provides the fundamental estimate for the dissipation functional. We recall the energy dissipation function from Lemma 2.1, namely

$$\frac{d}{dt}\mathcal{A}(u(t)) = -\mathcal{D}(u(t),\dot{\ell}) \text{ with } \mathcal{D}(u,\dot{\ell}) = \int_\Omega \frac{W^2}{u}\,\mathrm{d}x - \Big(\int_\Omega W\,\mathrm{d}x\Big)^2 - \dot{\ell}\int_\Omega W\,\mathrm{d}x,$$

where $W = u_x + \psi' u$, as we have set $\nu = 1$. To obtain global existence we want to estimate $\mathcal{A}$ from above and hence $\mathcal{D}$ from below. As such our strategy is similar to those in [GlH97] for more complicated electro-reaction-diffusion systems. However, in our case the time-dependent constraint $\mathcal{C}(u(t)) = \ell(t)$ complicates the matter a lot. In particular, the lower estimates for $\mathcal{D}$ are much more difficult.

When estimating $\mathcal{D}$ from below we can of course take advantage of the constraints (4.2). Nevertheless, the difficulty is here that we cannot control $\int_\Omega W\,\mathrm{d}x$ easily. The first two terms in $\mathcal{D}$ form a nonnegative contribution, namely

$$\begin{aligned}\int_\Omega \frac{W^2}{u}\,\mathrm{d}x - \Big(\int_\Omega W\,\mathrm{d}x\Big)^2 &= \int_\Omega \frac{W^2}{u}\,\mathrm{d}x - \int_\Omega \sqrt{u}\frac{W}{\sqrt{u}}\,\mathrm{d}x\\ &\geq \int_\Omega \frac{W^2}{u}\,\mathrm{d}x - \int_\Omega u\,\mathrm{d}x\int_\Omega \frac{W^2}{u}\,\mathrm{d}x = 0,\end{aligned}$$

where we used the Cauchy-Schwarz estimate and $\int_\Omega u\,\mathrm{d}x = 1$. However, there is no hope to obtain a better lower estimate that allows to estimate the third term $\dot{\ell}\int_\Omega W\,\mathrm{d}x$. The reason is that the Cauchy-Schwarz estimate is an equality whenever $W = \beta u$ for some $\beta \in \mathbb{R}$. Thus, the functions $u = u_\beta : x \mapsto c\,\mathrm{e}^{\beta x - \psi(x)}$ lead to a vanishing contribution in the first two terms but may generate to an arbitrary large contribution in the third term. However, the additional constraint $\mathcal{C}(u) = \ell \in]0,1[$ selects a unique β, see Section 5.1. Hence, there one can expect to find a suitable lower bound when using both constraints.

The following result shows that these considerations can be made quantitative. We will estimate the deviation of a general u from a suitable chosen U_λ.

***Theorem* 4.3.** *Assume $\psi \in \mathrm{W}^{1,1}([0,1])$. Then, for each $\delta \in]0,1/2[$ there exists a constant $C_\delta^\psi \geq 0$ such that for all $\ell \in [\delta, 1-\delta]$ and all $\lambda \in [-1/\delta, 1/\delta]$ the following estimate holds:*

$$\mathcal{D}(u,\lambda) \geq -C_\delta^\psi|\lambda| \quad \text{for all } u \in \mathrm{H}^1(\Omega) \text{ with } \int_\Omega u(x)\,\mathrm{d}x = 1 \text{ and } \mathcal{C}(u) = \ell. \qquad (4.3)$$

Proof. There are two crucial steps in this proof. First we replace u by $v = \sqrt{u}$, which transforms the integral $\int_\Omega W^2/u \,\mathrm{d}x$ into the quadratic form $\int_\Omega (2v_x + \psi' v)^2 \,\mathrm{d}x$ and gives the new constraints for all t in the existence interval

$$\int_\Omega v(x,t)^2 \,\mathrm{d}x = 1 \quad \text{and} \quad \int_\Omega x v(x,t)^2 \,\mathrm{d}x = \ell(t). \tag{4.4}$$

Secondly we will decompose v into $V_\alpha + \eta$, where V_α is a function satisfying the first of the above constraints and making the first two terms of $\mathcal{D}$ vanish, i.e. the Cauchy-Schwarz estimate is sharp.

To be more precise we introduce the notations

$$\begin{aligned}
&\mathcal{V}(\ell) = \{\, v \in \mathrm{H}^1(\Omega) \,:\, v \geq 0, \text{ and } \quad (4.4) \text{ holds } \},\\
&D(v,\lambda) = \mathcal{D}(v^2,\lambda) = \int_\Omega w^2 \,\mathrm{d}x - \Big(\int vw \,\mathrm{d}x \Big)^2 - \lambda \int_\Omega vw \,\mathrm{d}x,\\
&w = 2v_x + \psi' v, \quad \gamma = \|w\|_{L^2}, \quad \text{and } \rho = \int_\Omega vw \,\mathrm{d}x.
\end{aligned}$$

Using $\|u\|_{L^2} = 1$ and the Cauchy-Schwarz estimate we have $\rho^2 \leq \gamma^2$.

The case $\rho = 0$ is trivial, because it gives $D(v,\lambda) \geq 0$. Hence, we assume $\rho > 0$ from now on. This implies $\gamma > 0$, and we first decompose v in the form

$$v = \frac{\rho}{\gamma^2} w + \xi \quad \text{with} \quad \int_\Omega \xi \, w \,\mathrm{d}x = 0,$$

which is a simple orthogonal projection. Hence, we find

$$1 = \|v\|_{L^2}^2 = \frac{\rho^2}{\gamma^2} + \|\xi\|_{L^2}^2 \quad \Rightarrow \quad \|\xi\|_{L^2}^2 = 1 - \frac{\rho^2}{\gamma^2}.$$

Recalling the definition of w in terms of v leads to $2v_x + \psi' v = w = \frac{\gamma^2}{\rho}\big(v - \xi\big)$. Solving this ODE with $\|v\|_{L^2} = 1$ gives the formula

$$v = \beta V_{\gamma^2/\rho} + \mathcal{K}_{\gamma^2/\rho}\xi \quad \text{where } \mathcal{K}_\alpha \xi(x) = \int_0^1 K_\alpha(x,y)\xi(y) \,\mathrm{d}y.$$

Here $V_\alpha(x) = c_\alpha \mathrm{e}^{(\alpha x - \psi(x))/2}$ with $c_\alpha > 0$ chosen such that $\|V_\alpha\|_{L^2} = 1$. The constant β is chosen such that $\|v\|_{L^2} = 1$. The kernel K_α is defined via

$$K_\alpha(x,y) = \begin{cases} \frac{\alpha V_\alpha(x)}{2V_\alpha(y)} & \text{for } \alpha > 0 \text{ and } 0 < x < y < 1,\\ -\frac{\alpha V_\alpha(x)}{2V_\alpha(y)} & \text{for } \alpha < 0 \text{ and } 0 < y < x < 1,\\ 0 & \text{otherwise.} \end{cases}$$

Using $\psi \in \mathrm{W}^{1,1}(\Omega)$, which implies $\psi \in \mathrm{C}(\bar{\Omega})$, the kernel can be estimated via

$$0 \leq K_\alpha(x,y) \leq \frac{C_K^\psi}{2} |\alpha| \mathrm{e}^{-|\alpha||x-y|/2} \text{ for } \alpha \neq 0 \text{ and } x,y \in [0,1],$$

where C^ψ_K depends only on ψ but not on α. Using this we can estimate $\widehat\xi := \mathcal{K}_{\gamma^2/\rho}\xi$ via $|\widehat\xi(x)| \le C^\psi_K \frac{\alpha}{2}\int_0^1 \mathrm{e}^{-|\alpha||x-y|/2}|\xi(y)|\,\mathrm{d}y$. Then using Young's inequality for convolutions $\widehat\xi = \phi * \xi$ in the form $\|\widehat\xi\|_{L^2(\mathbb{R})} = \|\phi\|_{L^1(\mathbb{R})}\|\xi\|_{L^2(\mathbb{R})}$, we have the uniform estimate

$$\|\mathcal{K}_\alpha\|_{\mathrm{Lin}(L^2(\Omega),L^2(\Omega))} \le C^\psi_K \text{ for all } \alpha \neq 0.$$

Now we write the final decomposition in the form

$$v = V_{\gamma^2/\rho} + \eta \quad \text{with } \eta = (\beta{-}1)V_{\gamma^2/\rho} + \mathcal{K}_{\gamma^2/\rho}\xi.$$

It is now essential to estimate η in terms of ρ/γ. We do this in terms of $\widehat\xi = K_{\gamma^2/\rho}\xi$, which satisfies $\|\widehat\xi\|_{L^2} \le C^\psi_K(1{-}\rho^2/\gamma^2)^{1/2}$. Recalling $\|V_{\gamma^2/\rho}\|_{L^2} = \|v\|_{L^2} = 1$, we always have $\|\eta\|_{L^2} \le 2$. For the case $\|\widehat\xi\|_{L^2} \le 1$ we improve this estimate with the relation

$$1 \ge \|\widehat\xi\|^2_{L^2} = \|v - \beta V_{\gamma^2/\rho}\|^2_{L_2} = 1 - 2\beta\int_\Omega vV_{\gamma^2/\rho}\,\mathrm{d}x + \beta^2.$$

Using $\int_\Omega vV_{\gamma^2/\rho}\,\mathrm{d}x > 0$ we conclude $\beta \ge 0$. Hence,

$$\begin{aligned}|1{-}\beta| \le |1{-}\beta^2| &= \big|\|v\|^2_{L_2} - \|\beta V_{\gamma^2/\rho}\|^2_{L_2}\big| = \Big|\int_\Omega (v{-}\beta V_{\gamma^2/\rho})(v{+}\beta V_{\gamma^2/\rho})\,\mathrm{d}x\Big| \\ &\le \Big|\int_\Omega \widehat\xi(2v{+}\widehat\xi)\,\mathrm{d}x\Big| \le (2{+}C^\psi_K)\|\widehat\xi\|_{L_2}.\end{aligned}$$

Combing this with the definition of η we find

$$\|\eta\|_{L^2} \le |\beta{-}1| + \|\widehat\xi\|_{L^2} \le (3{+}C^\psi_K)C^\psi_K\big(1{-}\rho^2/\gamma^2\big)^{1/2} \ \text{ if } \|\widehat\xi\|_{L^2} \le 1. \tag{4.5}$$

Now we are ready to estimate $D(v,\lambda)$ from below on the admissible set $\mathcal{V}(\ell)$. By our definitions of ρ and γ the functional D takes the form

$$D(v,\lambda) = \gamma^2 - \rho^2 - \lambda\rho,$$

where γ and ρ depend on $v \in \mathcal{V}(\ell)$. To estimate D we choose $\sigma_\delta \in \,]0,1[$ such that

$$(3{+}C^\psi_K)C^\psi_K\big(1{-}\sigma_\delta^2\big)^{1/2} \le \delta/2 < 1/4.$$

and distinguish two cases $\rho^2 \le \gamma^2\sigma_\delta^2$ and $\rho^2/\gamma^2 \in [\sigma_\delta^2, 1]$.

Case I, $|\rho| \le \gamma\sigma_\delta$: We easily find

$$D(v,\lambda) = \gamma^2 - \rho^2 - \lambda\rho \ge \gamma^2 - \gamma^2\sigma_\delta^2 - |\lambda|\gamma\sigma_\delta \ge -\frac{\lambda^2\sigma_\delta^2}{4(1{-}\sigma_\delta^2)} \ge -\frac{\sigma_\delta^2}{4\delta(1{-}\sigma_\delta^2)}\,|\lambda|\,,$$

where δ is from the statement of the theorem such that $|\lambda| \le 1/\delta$.

Case II, $\rho^2/\gamma^2 \in [\sigma_\delta^2, 1]$*:* Recalling $\|\widehat{\xi}\|_{L^2} \leq C_K^\psi(1{-}\rho^2/\gamma^2)^{1/2}$ we have $\|\widehat{\xi}\|_{L^2} \leq \delta/6 \leq 1$ and can use estimate (4.5) for η, namely $\|\eta\|_{L^2} \leq \delta/2$. Since $v = V_{\gamma^2/\rho} + \eta$ lies in $V(\ell)$ we obtain

$$\Big|\ell - \int_\Omega x V_{\rho^2/\gamma}(x)\,\mathrm{d}x\Big| = \Big|\int_\Omega x\big(v(x) - V_{\rho^2/\gamma}(x)\big)\,\mathrm{d}x\Big| \leq \int_\Omega \big|x\eta(x)\big|\,\mathrm{d}x \leq \|\eta\|_{L^2} \leq \delta/2.$$

We consider the function

$$m(\alpha) := \int_\Omega x V_\alpha(x)^2\,\mathrm{d}x.$$

It is easy to see that $m : \mathbb{R} \to]0,1[$ is differentiable, strictly increasing and satisfies $m(\alpha) \to 0$ for $\alpha \to -\infty$ and $m(\alpha) \to 1$ for $\alpha \to \infty$. Thus, for each $\delta \in]0,1/2[$ there is a constant a_δ such that $m(\alpha) \in [\delta/2, 1{-}\delta/2]$ implies $\alpha \in [-a_\delta, a_\delta]$.

Using the assumption $\ell \in [\delta, 1{-}\delta]$ we have shown that the decomposition $v = V_{\rho^2/\gamma} + \eta$ implies $m(\gamma^2/\rho) \in [\delta/2, 1{-}\delta/2]$. Thus, we conclude the estimate $a_\delta \geq |\gamma^2/\rho| \geq |\gamma|$, because $0 < |\rho| \leq \gamma$. Thus, we obtain the lower bound

$$D(v,\lambda) = \gamma^2 - \rho^2 - \lambda\rho \geq -a_\delta|\lambda|.$$

Combining the two cases we have established the desired estimate (4.3) with $C_\delta^\psi = \max\{a_\delta, \sigma_\delta^2/(4\delta(1{-}\sigma_\delta^2))\}$. □

Analysing the dependence of σ_δ and a_δ on δ in the above proof, it can be shown that C_δ^ψ can be estimated by $1/\delta^3$. However, it is possible that the estimates can be improved.

The above dissipation estimate is fundamental to control the growth of the energy $\mathcal{A}$. Under our main assumption (4.1) for ℓ we find for each $T > 0$ a constant $\delta > 0$ such that $\ell(t) \in [\delta, 1{-}\delta]$ and $|\dot{\ell}(t)| \leq 1/\delta$ for all $t \in [0,T]$. Hence we conclude the main energy estimate

$$\big|\mathcal{A}(u(t_2)) - \mathcal{A}(u(t_1))\big| \leq C_\delta \int_{t_1}^{t_2} |\dot{\ell}(s)|\,\mathrm{d}s \leq C_\delta^2(t_2{-}t_1) \quad \text{for } 0 \leq t_1 < t_2 \leq T. \tag{4.6}$$

In particular, $\mathcal{A}(u(t))$ cannot blow-up, if it is bounded initially.

For later use in the convergence theory in Section 5, we provide an improved energy-dissipation estimate, where the dissipation is not only bounded from below and even coercive but can also be bounded from below by an arbitrary positive multiple of the energy itself. The proof is a slight variant of the one above.

Proposition 4.4. *Assume $\psi \in \mathrm{H}^1(\Omega)$. Then, for each $\kappa > 0$ and each $\delta \in]0,1/2[$ there exists a constant $K_{\kappa,\delta}^\psi$ such that for all $\ell \in [\delta, 1{-}\delta]$ and all $\lambda \in [-1/\delta, 1/\delta]$ the following estimate holds:*

$$\mathcal{D}(u,\lambda) \geq \kappa\|u_x\|_{L^2} - K_{\kappa,\delta}^\psi \quad \text{and} \quad \mathcal{D}(u,\lambda) \geq \kappa\mathcal{A}(u) - K_{\kappa,\delta}^\psi \tag{4.7}$$

for all $u \in \mathrm{H}^1(\Omega)$ with $u \geq 0$, $\int_\Omega u\,\mathrm{d}x = 1$, and $\mathcal{C}(u) = \ell$.

Proof. We proceed exactly as in the proof of Theorem 4.3 and use the same notations.
Step 1: We first estimate

$$D_\kappa(v,\lambda) = D(v,\lambda) - \kappa\|v_x\|_{L^2}^{3/2}.$$

Because of $\gamma = \|2v_x{+}\psi' v\|_{L^2}$ and $\|v\|_{L^2} = 1$ we have $\|v_x\|_{L^2} \le \gamma + 1 + \frac{1}{2}\|\psi'\|_{L^2}^2$ and find

$$D_\kappa(v,\lambda) \ge \gamma^2 - \rho^2 - \lambda\rho - \kappa\gamma^{3/2} - C$$

where C depends on ψ and κ. This can be estimated from below via the two cases as before.

Case I, $|\rho| \le \gamma\sigma_\delta$: We obtain

$$D_\kappa(v,\lambda) \ge (1{-}\sigma_\delta^2)\gamma^2 - \tfrac{1}{\delta}\sigma_\delta\gamma - \kappa\gamma^{3/2} - C,$$

which is certainly bounded from below by a constant depending only on κ and σ_δ.
Case II, $\rho^2/\gamma^2 \in [\sigma_\delta^2, 1]$: As in the previous proof we find $|\rho| \le \gamma \le a_\delta$, giving

$$D_\kappa(v,\lambda) \ge \gamma^2 - \rho^2 - \tfrac{1}{\delta}|\rho| - \kappa\gamma^{3/2} - C$$

is trivially bounded from below.
Combining the two cases gives $D_\kappa(v,\lambda) \ge k^\psi_{\kappa,\delta}$ as desired.
Step 2: We now need to undo the substitution $u = v^2$ in $\mathcal{D}(u,\lambda) = D(\sqrt{u},\lambda)$. With $u_x = 2vv_x$ we find

$$\|u_x\|_{L^2}^2 = 4\|vv_x\|_{L^2}^2 = 4\|v\|_{L^\infty}^2\|v_x\|_{L^2}^2 \le C(1 + \|v_x\|_{L^2}^3),$$

where we have used $\|v\|_{L^\infty}^2 \le C\|v\|_{L^2}(\|v\|_{L^2}{+}\|v_x\|_{L^2}) = C(1{+}\|v_x\|_{L^2})$, see Lemma A.1. Using $v = \sqrt{u}$ we deduce

$$\mathcal{D}(u,\lambda) - \kappa\|u_x\|_{L^2} \ge D(v,\lambda) - c_1\kappa\|v_x\|_{L^2}^{3/2} - c_2 \;=\; D_{c_1\kappa}(v,\lambda) - c_2 \;\ge\; k^\psi_{c_1\kappa,\delta} - c_2 \;=:\; K^\psi_{\kappa,\delta}.$$

Thus, the first estimate in (4.7) is established.
Step 3: The second estimate in (4.7) is obtained by estimating $\mathcal{A}(u)$ from above. We have

$$\mathcal{A}(u) = \int_\Omega u\ln u{+}\psi u\,\mathrm{d}x \le \max\{\ln u{+}\psi\}\int_\Omega u\,\mathrm{d}x \le \ln\|u\|_{L^\infty} + \max\psi \le C(1{+}\|u_x\|).$$

Inserting this into the first estimate of (4.7), the second follows immediately. ☐

4.2 Improved a priori estimates

Based on the above energy bounds we derive new a priori estimates in $L^2(\Omega)$ as well as in $L^\infty(\Omega)$. To exploit the energy bound we can employ a variant of the "L log L" improved version of the classical Gagliardo-Nirenberg interpolation inequality:

$$\forall\,\varepsilon > 0\;\exists\,C_\varepsilon\;\forall\,w \in \mathrm{H}^1(\Omega):\;\; \|w\|_{L^\infty}^3 \le \varepsilon\|w_x\|_{L^2}^2\|w\ln|w|\|_{L^1} + C_\varepsilon\big(1{+}\|w\|_{L^1}^3\big). \tag{4.8}$$

The proof will be provided in Lemma A.2. We refer to [GlH97, GlM04] for similar uses of this inequality in reaction-diffusion systems.

From this, we are now able to derive an a-priori estimate for the L^2-norm, thus showing that blow-up is impossible under the assumption (4.1) for ℓ.

Proposition 4.5. *Assume that ψ and ℓ satisfy (1.2) and (4.1), respectively. Then for all $K > 0$ and T_0 there exists $C(K, T_0) > 0$ such that the following holds. If for some $t_0 \in [0, T_0[$ the solution u of (1.1) satisfies*

$$\mathcal{A}(u(t_0)) = \int_\Omega u(x, t_0)\big(\ln u(x, t_0) + \psi(x)\big)\,\mathrm{d}x \le K \quad \textit{and} \quad \|u(t_0)\|_{L^2} \le K, \tag{4.9}$$

then the following a priori estimate in $L^2(\Omega)$ holds:

$$\|u(t)\|_{L^2} \le C(K, T_0) \quad \textit{for all } t \in \,]t_0, T_0[\,. \tag{4.10}$$

Proof. First we test (1.3) with u to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 = \Big(\int_\Omega u_x\,\mathrm{d}x + \dot{\ell} + \int_\Omega \psi' u\,\mathrm{d}x\Big)\cdot \int_\Omega u u_x\,\mathrm{d}x - \int_\Omega \psi' u u_x\,\mathrm{d}x \tag{4.11}$$

for all $t \in \,]t_0, T[$. The integrals $\int_\Omega u^{k-1}u_x\,\mathrm{d}x = \frac{1}{k}(u(1)^k - u(0)^k)$ we estimate by $\frac{2}{k}\|u\|_{L^\infty}^k$, while the last term admits the estimate

$$-\int_\Omega \psi' u u_x\,\mathrm{d}x \le \frac{1}{4}\int_\Omega u_x^2\,\mathrm{d}x + \int_\Omega \psi'^2 u^2\,\mathrm{d}x \le \frac{1}{4}\|u_x\|_{L^2}^2 + \|\psi'\|_{L^2}^2\|u\|_{L^\infty}^2. \tag{4.12}$$

With $|\int_\Omega \psi' u\,\mathrm{d}x| \le \|\psi'\|_{L^1}\|u\|_{L^\infty}$, estimate (4.11) leads to

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \frac{3}{4}\|u_x\|_{L^2}^2 \le C_0\big(1 + \|u\|_{L^\infty}^3\big) \text{ with } C_0 = 3 + \|\psi'\|_{L^1} + \|\psi'\|_{L^2}^2 + \tfrac{1}{\delta}, \tag{4.13}$$

where $\delta > 0$ is such that $|\dot{\ell}(t)| \le 1/\delta$ for $t \in [0, T_0]$.

Next we employ the energy estimate (4.6) and the initial condition (4.9) giving

$$\mathcal{A}(u(t)) = \int_\Omega u(x, t)\big(\ln u(x, t) + \psi(x)\big)\,\mathrm{d}x \le C_1(K_\mathcal{A}) = K_\mathcal{A} := K + T_0/\delta. \tag{4.14}$$

Together with $\int_\Omega \psi u\,\mathrm{d}x \ge \int_\Omega \min(\psi) u\,\mathrm{d}x = \min\psi$ and the lower inequality $|\xi\ln\xi| \le \frac{2}{\mathrm{e}} + \xi\ln\xi$, valid for all $\xi > 0$, we find

$$\|u(t)\ln u(t)\|_{L^1(\Omega)} \le C_2(K_\mathcal{A}) = \frac{2}{\mathrm{e}} + C_1(K_\mathcal{A}) - \min\psi \quad \text{for all } t \in \,]t_0, T_0[\,.$$

An application of (4.8) with $\varepsilon(K_\mathcal{A}) := \frac{1}{4C_0C_2}$ shows that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \frac{1}{2}\|u_x(t)\|_{L^2}^2 \le C_3(K_\mathcal{A}) \quad \text{for all } t \in \,]t_0, T_0[\,. \tag{4.15}$$

Since $u(t)$ has mean value one there exist two positive constants C_4 and C_5 such that $C_4\|u(t)\|_{L^2}^2 - C_5 \le \|u_x(t)\|_{L^2}^2$. Using this in (4.15) results in the differential inequality

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \le -C_4\|u(t)\|_{L^2}^2 + C_5 + 2C_3(K_{\mathcal{A}}),$$

which gives for $C_5(K_{\mathcal{A}}) := (C_5 + 2C_3(K_{\mathcal{A}}))/C_4$,

$$\|u(t)\|_{L^2}^2 \le C(K, T_0) := \max\left\{\|u(t_0)\|_{L^2}^2, C_5(K_{\mathcal{A}})\right\}. \tag{4.16}$$

Where the dependence on T_0 in the constant $C(K, T_0)$ stems from $K_{\mathcal{A}}$ in (4.14). □

4.3 Global existence and boundedness properties

To obtain global existence for $t \in [0, \infty[$ we use a slightly weakened version of our basic assumption (4.1) on ℓ. We do no longer ask for continuous differentiability of ℓ, but use only $\ell \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}([0, \infty[)$. Additionally, we need to have $\ell(t) \in\,]0, 1[$. Thus, we impose that ℓ stays away from the boundary which implies that

$$\forall\, T > 0,\ \exists\, \delta \in\,]0, 1/2[\, :\ \ell(t) \in [\delta, 1-\delta] \text{ and } |\dot{\ell}(t)| \le 1/\delta \quad \text{for almost all } t \in [0, T]. \tag{4.17}$$

To obtain boundedness we have to impose this condition uniformly on $[0, \infty[$.

***Theorem* 4.6.** *Suppose that $\ell \in \mathrm{W}^{1,\infty}_{loc}([0, \infty[)$ satisfies (4.17). Then (1.1) admits a global classical solution u.*

Proof. We let $T_0 \in\,]0, \infty]$ denote the maximal existence time of the local-in-time solution u of (1.1). Assume $T_0 < \infty$, then on the one hand Theorem 3.1 implies $\|u(t)\|_{L^2} \to \infty$ as $t \nearrow T_0$ (use (3.2) with $q = 2$). On the other hand, Proposition 4.5 shows $\limsup_{t \nearrow T_0} \|u(t)\|_{L^2} < \infty$ (see (4.10)): From this contradiction we conclude $T_0 = \infty$. □

To obtain boundedness of the solution on the whole time interval $]0, \infty[$ we need to show that $\mathcal{A}$ remains bounded. For this we use the uniform version of (4.17) and the improved energy dissipation estimate (4.7) provided in Proposition 4.4.

***Theorem* 4.7.** *Assume that there exists $\delta \in\,]0, 1/2[$ such that $\ell \in \mathrm{W}^{1,\infty}([0, \infty[)$ satisfies $\ell(t) \in [\delta, 1-\delta]$ for all $t \in [0, \infty[$. Then the global solution u of (1.1) with $u_0 \in L^2(\Omega)$, which was obtained in Theorem 4.6, satisfies*

$$u \in L^\infty([0, \infty[\, , L^2(\Omega)).$$

Proof. For this we use the dissipation estimate (4.7) and obtain a differential inequality. For any positive κ we obtain

$$\frac{d}{dt}\mathcal{A}(u(t)) \le \mathcal{D}(u(t), \dot{\ell}(t)) \le -\kappa\mathcal{A}(u(t)) + K^{\psi}_{\kappa,\delta},$$

where, if necessary, the constant $\delta > 0$ from the assumption is made smaller to have $|\dot{\ell}(t)| \leq 1/\delta$ for a.a. $t \geq 0$ as well. From this estimate we easily obtain $\mathcal{A}(u(t)) \leq K_{\mathcal{A}} := \max\{\mathcal{A}(u(0)), K^{\psi}_{\kappa,\delta}\}$. We can use the estimates provided in the proof of Proposition 4.5. Note that (4.16) implies

$$\|u(t)\|_{L^2} \leq \max\{\|u_0\|_{L^2}, C_5(K_{\mathcal{A}})\},$$

which is independent of t, because here $K_{\mathcal{A}}$ is bounded independently of any time interval. □

***Remark* 4.8.** *The reasoning in Theorem 4.7 is still correct if the condition (4.17) holds only up to a finite time T_*. Then the assertion in Theorem 4.7 holds up to this time. Thus the condition that $\ell(t)$ touches the boundary of $]0,1[$ at time t_* is not only sufficient, as seen in Remark 4.2, but also necessary for t_* to be an explosion time. Hence for $\ell \in \mathrm{W}^{1,\infty}_{loc}([0,t_*[)$ the solution exists on the time interval $[0,t_*[$ if and only if $\ell(t) \in]0,1[$ for all $t \in [0,t_*[$.*

5 Convergence to the steady state if $\ell(t) \to \ell_* \in]0,1[$

In this section we show that the global solutions constructed in the previous section converge to the unique steady state if the constraint $\ell(t)$ converges in a suitable way. In Section 5.1 we first characterise the steady states as functions of the constraint ℓ. In particular, we show that they are the unique minimisers of $\mathcal{A}$ subject to the constraint $\mathcal{C}(u) = \ell$. In Section 5.2 we will then use properties of the dissipation functional $\mathcal{D}$ to show convergence of the solutions under the additional assumption that $\dot{\ell} \in L^1(]0,\infty[)$.

5.1 Characterisation of the steady states

The following lemma describes the structure of the set of equilibria of (1.1) satisfying (4.2a). In fact, all these steady states are explicitly known as setting $u_t \equiv 0$ leads to an ODE for u that can be solved explicitly. For $\beta \in \mathbb{R}$ we define the functions $u_\beta \in L^2(\Omega)$ via

$$u_\beta(x) = \frac{1}{c_\beta}\,\mathrm{e}^{\beta x - \psi(x)} \quad \text{with } c_\beta = \int_\Omega \mathrm{e}^{\beta x - \psi(x)}\,\mathrm{d}x. \tag{5.1}$$

By definition we have $u_\beta > 0$ and $\int_\Omega u_\beta\,\mathrm{d}x = 1$. It remains to study the first moment for which we set

$$M(\beta) = \int_\Omega x u_\beta(x)\,\mathrm{d}x.$$

The following result shows that M is strictly increasing with $M(\beta) \to 0$ and $u_\beta \overset{*}{\rightharpoonup} \delta_0$ (δ-distribution at $x = 0$) for $\beta \to -\infty$ and $M(\beta) \to 1$ and $u_\beta \overset{*}{\rightharpoonup} \delta_1$ (δ-distribution at $x = 1$) for $\beta \to \infty$.

***Lemma* 5.1.** *The functions u_β satisfy*

$$u_\beta \to 0 \qquad \begin{cases} \text{in } \mathrm{C}_{loc}(]0,1]) & \text{as } \beta \to -\infty, \\ \text{in } \mathrm{C}_{loc}([0,1[) & \text{as } \beta \to +\infty. \end{cases} \tag{5.2}$$

Moreover, M is strictly increasing with

$$M(\beta) = \int_\Omega x u_\beta(x)\,\mathrm{d}x \to \begin{cases} 0 & \text{as } \beta \to -\infty, \\ 1 & \text{as } \beta \to +\infty. \end{cases} \tag{5.3}$$

Consequently, for each $\ell \in\,]0,1[$ there exists a unique β with $\ell = M(\beta)$, which we denote by $\beta = B(\ell)$. Then,

$$U_\ell := u_{B(\ell)} \tag{5.4}$$

is the unique steady state u of (1.1) *with $\int_\Omega u\,\mathrm{d}x = 1$ and $\int_\Omega xu(x)\,\mathrm{d}x = \ell$.*

Proof. In order to derive (5.2), let us fix $x_0 \in\,]0,1]$ and assume that there exist $C_0 > 0$ and a sequence of numbers $\beta_k \to -\infty$ such that $u_{\beta_k}(x_0) \geq C_0$ for all k. Since $\psi \in \mathrm{H}^1(\Omega)$ we have $C_1 > 0$ such that $\|\psi\|_{\mathrm{C}} < C_1$ and so $\|\mathrm{e}^{\psi(\cdot)}\|_{\mathrm{C}} \leq C_2$. Then for $\beta < 0$ we can estimate u_β on $]0, x_0/2[$ by

$$\frac{|u_\beta(x_0)|}{|u_\beta(x)|} \leq |\mathrm{e}^{\beta(x_0-x)}|\,|\mathrm{e}^{\psi(x)-\psi(x_0)}| \leq |\mathrm{e}^{\frac{\beta}{2}x_0}|\,C_2^2 \;\to 0, \quad \text{as } \beta \to -\infty, \qquad \text{for all } x \in \left]0, \frac{x_0}{2}\right[.$$

Thus we can fix $\beta_0 < 0$ such that for all $x \in \left]0, \frac{x_0}{2}\right[$ we have $u_\beta(x) \geq \frac{4}{C_0 x_0} u_\beta(x_0) \geq \frac{4}{x_0}$ whenever $\beta < \beta_0$, which implies that for all sufficiently large k

$$1 = \int_\Omega u_{\beta_k} \geq \int_0^{\frac{x_0}{2}} u_{\beta_k} \geq 2\,.$$

Which is a contradiction to the construction (5.1). This proves the pointwise convergence to zero on $]0,1]$. By the same reasoning we can fix $\beta_1 < 0$ for any $x_0 \in\,]0,1[$, such that for all $x \in \left]\frac{1+x_0}{2}, 1\right]$ we have $u_\beta(x) \leq C_3 u_\beta(x_0)$ whenever $\beta < \beta_1$. This implies the uniform convergence on every subset $\left]\frac{1+x_0}{2}, 1\right]$ and thus the first claim in (5.2), whereas the second can be seen in a similar way.

Along with the property $\int_\Omega u_\beta\,\mathrm{d}x = 1$, this also entails (5.3): Indeed, given $\varepsilon > 0$, by (5.2) we can fix $\beta_\star < 0$ such that $u_\beta < \varepsilon$ in $\left]\frac{\varepsilon}{2}, 1\right[$ for all $\beta < \beta_\star$, whence

$$\begin{aligned} \textstyle\int_\Omega x u_\beta(x)\,\mathrm{d}x &\leq \tfrac{\varepsilon}{2} \textstyle\int_0^{\frac{\varepsilon}{2}} u_\beta(x)\,\mathrm{d}x + \int_{\frac{\varepsilon}{2}}^1 x \cdot \varepsilon\,\mathrm{d}x \\ &< \tfrac{\varepsilon}{2} \cdot 1 + \varepsilon \cdot \tfrac{1}{2} \quad \text{for all } \beta < \beta_\star, \end{aligned}$$

and the limit behaviour as $\beta \to +\infty$ can be proven similarly. Finally, to see that M is strictly increasing we use (5.1) to compute

$$\frac{d}{d\beta} \int_\Omega x u_\beta(x)\,\mathrm{d}x = \frac{\left(\int_\Omega x^2 \mathrm{e}^{\beta x - \psi(x)}\,\mathrm{d}x\right) \cdot \left(\int_\Omega \mathrm{e}^{\beta x - \psi(x)}\,\mathrm{d}x\right) - \left(\int_\Omega x\mathrm{e}^{\beta x - \psi(x)}\,\mathrm{d}x\right)^2}{\left(\int_\Omega \mathrm{e}^{\beta x - \psi(x)}\,\mathrm{d}x\right)^2}$$

for $\beta \in \mathbb{R}$. Since $\rho_1(x) := x\mathrm{e}^{\frac{1}{2}(\beta x-\psi(x))}$ and $\rho_2(x) := \mathrm{e}^{\frac{1}{2}(\beta x-\psi(x))}$, $x \in \bar{\Omega}$, are linearly independent, the Cauchy-Schwarz inequality says that

$$\Big(\int_\Omega \rho_1\rho_2 \,\mathrm{d}x\Big)^2 < \Big(\int_\Omega \rho_1^2 \,\mathrm{d}x\Big)\cdot\Big(\int_\Omega \rho_2^2 \,\mathrm{d}x\Big)$$

and thus ensures that $\frac{d}{d\beta}\int_\Omega x u_\beta(x)\,\mathrm{d}x > 0$ for each $\beta \in \mathbb{R}$. □

The next result characterises the above equilibria in terms of the energy functional $\mathcal{A}$ and the constraint $\mathcal{C}$.

***Proposition* 5.2.** *The functional $u \mapsto \mathcal{A}(u)$ attains its minimum on the set*

$$\mathcal{M}(\ell) := \{\, u \in L^1(\Omega) \;:\; u \geq 0,\; \int_\Omega u(x)\,\mathrm{d}x = 1,\; \int_\Omega x u(x)\,\mathrm{d}x = \ell \,\}$$

on exactly one point, namely U_ℓ defined in (5.4).

Proof. Note that $\mathcal{M}(\ell)$ is a strongly closed and convex subset of $L^1(\Omega)$. Moreover, the functional $\mathcal{A}$ is strictly convex. Hence, there is at most one minimiser.

We directly show that U_ℓ is the desired minimiser. The convexity of $u \mapsto u\ln u$ gives

$$\widetilde{u}\ln\widetilde{u} \geq u\ln u + (\ln u{+}1)(\widetilde{u}{-}u) \quad \text{for } u > 0 \text{ and } \widetilde{u} \geq 0.$$

Thus, for all $\widetilde{u} \in \mathcal{M}(\ell)$ we obtain

$$\begin{aligned}\mathcal{A}(\widetilde{u}) = \int_\Omega \widetilde{u}\ln\widetilde{u} + \psi\widetilde{u}\,\mathrm{d}x &\geq \int_\Omega U_\ell \ln U_\ell + (\ln U_\ell{+}1)(\widetilde{u}{-}U_\ell) + \psi\widetilde{u}\,\mathrm{d}x \\ &\overset{(i)}{=} \mathcal{A}(U_\ell) + \int_\Omega \big(B(\ell)x - \ln c_{B(\ell)}\big)(\widetilde{u}{-}U_\ell)\,\mathrm{d}x \overset{(ii)}{=} \mathcal{A}(U_\ell),\end{aligned}$$

where in (i) we used a cancellation of all terms involving ψ while in (ii) we use $\widetilde{u}, U_\ell \in \mathcal{M}(\ell)$. □

The following simple consequence will be useful to establish convergence to equilibria.

***Corollary* 5.3.** *Assume that the sequence $(u_k)_{k\in\mathbb{N}}$ satisfies*

$$u_k \rightharpoonup u_* \text{ in } L^2(\Omega), \quad \mathcal{C}(u_k) \to \ell_* \in\,]0,1[\,, \quad \mathcal{A}(u_k) \to \mathcal{A}(U_{\ell_*}).$$

Then, $u_* = U_{\ell_*}$ and $u_k \to U_{\ell_*}$ in $L^2(\Omega)$ strongly.

Proof. On the one hand, the strong continuity and convexity of $\mathcal{A}$ imply weak lower semicontinuity of $\mathcal{A}$. Hence, we have $\mathcal{A}(u_*) \leq \mathcal{A}(U_{\ell_*})$.

On the other hand $\mathcal{C}$ is weakly continuous, which implies $\mathcal{C}(u_*) = \ell_*$. Thus, Proposition 5.2 implies that u_* is equal to the unique minimiser U_{ℓ_*}.

Finally the strict convexity of $\mathcal{A}$ allows us to apply the Visintin's argument [Vis84]. The energy convergence $\mathcal{A}(u_k) \to \mathcal{A}(U_{\ell_*})$ turns the weak convergence $u_k \rightharpoonup U_{\ell_*}$ into the desired strong convergence. □

5.2 Vanishing dissipation and convergence

We now consider the case of global solutions for an $\ell \in \mathrm{W}^{1,\infty}(]0,\infty[)$ satisfying the following conditions

$$\dot{\ell} \in L^1(]0,\infty[) \cap L^\infty(]0,\infty[) \quad \text{and} \quad \exists\, \delta \in]0,1/2[\ \forall\, t \geq 0 : \ \ell(t) \in [\delta, 1-\delta]. \tag{5.5}$$

A simple consequence of this condition is that the limit

$$\ell_* := \lim_{t\to\infty} \ell(t)$$

exists. Moreover, Theorem 4.7 implies a classical solution $u \in L^\infty(]0,\infty[\,, L^2(\Omega))$. Our aim is now to show that $u(t) \to U_{\ell_*}$ in $L^2(\Omega)$ for $t \to \infty$. Our proof has two ingredients, both of which are related to the energy dissipation relations derived in Section 4.1. In the first step we will establish the convergence of $\mathcal{A}(u(t)) \to A_*$. In the second and final step we will exploit that the integral $\int_0^\infty \mathcal{D}(u(t), \dot{\ell}(t))\, \mathrm{d}t$ is finite.

***Lemma* 5.4.** *Assume that $\psi \in \mathrm{H}^1(\Omega)$ and that ℓ satisfies (5.5). Then, for every solution the following limit exists:*

$$A_* := \lim_{t\to\infty} \mathcal{A}(u(t)).$$

Proof. We recall the energy-dissipation (2.12) giving

$$\mathcal{A}(u(t_2)) + \int_{t_1}^{t_2} \mathcal{D}(u(t), \dot{\ell}(t))\, \mathrm{d}t = \mathcal{A}(u(t_1)) \quad \text{for } 0 \leq t_1 < t_2. \tag{5.6}$$

The dissipation estimate (4.3) gives $\mathcal{D}(u(t), \dot{\ell}(t)) \geq -C|\dot{\ell}(t)|$ for a fixed constant C. Thus, the function $\tau \mapsto a(\tau) := \mathcal{A}(u(\tau)) - C \int_0^\tau |\dot{\ell}(t)|\, \mathrm{d}t$ is nonincreasing. By the assumption $\dot{\ell} \in L^1(]0,\infty[)$ and the lower bound $\mathcal{A}(u) \geq -1/\mathrm{e} + \min \psi$ we know that a is bounded as well. Hence $a(t) \to a_*$ for $t \to \infty$. Thus, $\mathcal{A}(u(t)) \to a_* + C \int_0^\infty |\dot{\ell}(t)|\, \mathrm{d}t =: A_*$. □

We still have to show that A_* is related to $\ell_* = \lim_{t\to\infty} \ell(t)$. If we can show that $A_* = \mathcal{A}(U_{\ell_*})$, then Corollary 5.3 can be employed easily. To find the identity for A_* it will be enough to find one sequence $t_k \to \infty$ such that $\mathcal{D}(u(t_k), 0) \to 0$ and to employ the following result.

***Proposition* 5.5.** *Assume $\psi \in \mathrm{H}^1(\Omega)$ and consider a sequence $(u_k)_{k\in\mathbb{N}}$ with $u_k \in \mathcal{M}(\ell_k)$ such that*

$$u_k \rightharpoonup u_* \text{ in } L^2(\Omega), \quad \ell_k = \mathcal{C}(u_k) \to \ell_* \in]0,1[\,, \quad \mathcal{D}(u_k, 0) \to 0.$$

Then, $u_k \to U_{\ell_*}$ in $\mathrm{H}^1(\Omega)$ and $\mathcal{A}(u_k) \to \mathcal{A}(U_{\ell_*})$.

Proof. By the coercivity (4.7) of $\mathcal{D}$, we obtain that u_k is even bounded in $\mathrm{H}^1(\Omega)$. Thus, the weak convergence in $L^2(\Omega)$ implies $u_k \rightharpoonup u_*$ in $\mathrm{H}^1(\Omega)$. From this we obtain uniform convergence and conclude $\mathcal{A}(u_k) \to \mathcal{A}(u_*)$.

We already now $u_* \in \mathcal{M}(\ell_*)$, and it remains to identify u_* as U_{ℓ_*}. For this we use $\mathcal{D}(u_k,0) \to 0$. We introduce a new dependent variable z_k via the formula

$$u_k(x) = \mathrm{e}^{\rho_k(x)} z_k(x)^2 \quad \text{with } z_k(x) \geq 0 \text{ and } \rho_k(x) = \Lambda_k x - \psi(x), \tag{5.7}$$

where $\Lambda_k := \mathbb{L}u_k = u_k(1) - u_k(0) + \int_\Omega \psi' u_k \,\mathrm{d}x$. Doing some elementary calculations we find

$$\mathcal{D}(u_k,0) = \widehat{D}(z_k) := 4\int_\Omega \mathrm{e}^{\rho_k(x)}(z_k'(x))^2 \,\mathrm{d}x,$$

$$1 = \int_\Omega u_k(x)\,\mathrm{d}x = \int_\Omega \mathrm{e}^{\rho_k(x)}\big(z_k(x)\big)^2 \,\mathrm{d}x, \quad \ell_k = \int_\Omega x u_k(x)\,\mathrm{d}x = \int_\Omega x\mathrm{e}^{\rho_k(x)}\big(z_k(x)\big)^2 \,\mathrm{d}x.$$

As u_k converges to u_* we have $\Lambda_k \to \Lambda_* = \mathbb{L}u_*$ and $\rho_k \to \rho_* : x \mapsto \Lambda_* x - \psi(x)$.

Using $\widehat{D}(z_k) \to 0$ we conclude $z_k \to z_*$ in $\mathrm{H}^1(\Omega)$ strongly, where $z_*' \equiv 0$. From (5.7) we now see that $u_k \to \mathrm{e}^{\rho_*} z_*^2$, i.e. $u_* = \mathrm{e}^{\rho_*} z_*^2$. As z_* is constant, we see that u_* must be a multiple of u_{Λ_*}. However, due to Lemma 5.1, there is only one such multiple in $\mathcal{M}(\ell_*)$, namely U_{ℓ_*}. Thus, $u_* = U_{\ell_*}$ is established. Moreover $u_k \to U_{\ell_*}$ in $\mathrm{H}^1(\Omega)$ as $z_k \to z_*$ in $\mathrm{H}^1(\Omega)$. □

We are now ready to present our final convergent result.

***Theorem* 5.6.** *Assume that $\psi \in \mathrm{H}^1(\Omega)$ and that ℓ satisfies (5.5) with $\ell_* = \lim_{t\to\infty} \ell(t)$. Then, for every solution u we have $u(t) \to U_{\ell_*}$ in $L^2(\Omega)$ for $t \to \infty$.*

Proof. According to Lemma 5.4 we have $\mathcal{A}(u(t)) \to A_*$. Hence we can let $t_1 = 0$ and $t_2 \to \infty$ in the energy-dissipation relation (5.6) to obtain

$$\int_0^\infty \mathcal{D}(u(t),\dot{\ell}(t))\,\mathrm{d}t = \mathcal{A}(u(0)) - A_*.$$

As by Theorem 4.3 there holds $\mathcal{D}(u(t),\dot{\ell}(t)) \geq -C|\dot{\ell}(t)|$ we conclude that $t \mapsto \mathcal{D}(u(t),\dot{\ell}(t))$ lies in $L^1(]0,\infty[)$. Hence we can find a sequence $t_k \to \infty$ such that $\mathcal{D}(u(t_k),\dot{\ell}(t_k)) \to 0$, $\dot{\ell}(t_k) \to 0$. Thus Proposition 4.4 implies that $\|u(t_k)\|_{\mathrm{H}^1}$ is uniformly bounded for all k. This implies for a subsequence (not relabelled) that $u(t_k) \rightharpoonup u_*$ in $\mathrm{H}^1(\Omega)$ to some u_*. Since this even implies $\mathcal{D}(u(t_k),0) \to 0$, Proposition 5.5 is applicable, and we conclude $u(t_k) \to U_{\ell_*}$ and $A_* = \mathcal{A}(U_{\ell_*})$.

Now we consider a general sequence $\tau_k \to \infty$. Since $u(\tau_k)$ is bounded in $L^2(\Omega)$, see Theorem 4.7, we may assume $u(\tau_k) \rightharpoonup u_*$ in $L^2(\Omega)$ for some $u_* \in \mathcal{M}(\ell_*)$. From $u(\tau_k) \in \mathcal{M}(\ell(\tau_k))$ and $\tau_k \to \infty$, we obtain $u_* \in \mathcal{M}(\ell_*)$. Because of $\mathcal{A}(u(\tau_k)) \to A_* = \mathcal{A}(U_{\ell_*})$, Corollary 5.3 yields the desired result $u(\tau_k) \to U_{\ell_*}$ in $L^2(\Omega)$ strongly. As the possible limit of bounded sequences is unique, we have convergence of the whole family $u(t)$. □

We expect that the methods in [GlH97, Sect. 5.3] can be adapted to our case as well. Thus, if $\ell(t)$ converges exponentially to ℓ_*, i.e. $|\ell(t)-\ell_*| \leq C_0\mathrm{e}^{-\rho t}$, then there should exists $\lambda \in]0,\rho]$ and $C > 0$ such that the following exponential convergences hold:

$$|\mathcal{A}(u(t)) - \mathcal{A}(U_\ell)| \leq C\,\mathrm{e}^{-\lambda t} \quad \text{and} \quad \|u(t)-U_{\ell_*}\|_{L^2} \leq C\,\mathrm{e}^{-\lambda t/2}.$$

However, this is beyond of the aims of this paper.

A Appendix: Some embedding and inequalities

***Lemma* A.1.** *i) Let α, β be integers satisfying $0 \le \alpha < \beta$ and let $1 \le q, r \le \infty$, $0 \le p < \infty$. For the case q or r having the value ∞, we define formally $\frac{1}{\infty} = 0$. Then we define θ as*

$$\theta := \frac{\frac{1}{p} - \frac{1}{q} - \alpha}{\frac{1}{r} - \frac{1}{q} - \beta}.$$

If $\theta \in [\frac{\alpha}{\beta}, 1]$ then there exist constants $c_0, c_1 \ge 0$ such that for all $\varphi \in \mathrm{H}^{\beta,r}(\Omega) \cap L^q(\Omega)$ there holds

$$\|\frac{\partial^\alpha}{\partial x^\alpha}\varphi\|_{L^p} \le c_0 \|\frac{\partial^\beta}{\partial x^\beta}\varphi\|_{L^r}^\theta \|\varphi\|_{L^q}^{1-\theta} + c_1\|\varphi\|_{L^q}. \qquad (A.1)$$

ii) For all $0 < q < \infty$ and $0 < r \le \infty$ there exists $c > 0$ such that for all $\varphi \in \mathrm{H}^1(\Omega) \cap L^q(\Omega) \cap L^r(\Omega)$ there holds

$$\|\varphi\|_{\mathrm{C}([0,1])} \le (\frac{q}{2} + 1)^\theta \|\varphi_x\|_{L^2}^\theta \|\varphi\|_{L^q}^{1-\theta} + \|\varphi\|_{L^r}, \qquad (A.2)$$

$$\text{and} \qquad \|\varphi\|_{\mathrm{C}([0,1])} \le (\frac{q}{2} + 1)^\theta (\|\varphi\|_{L^2} + \|\varphi_x\|_{L^2})^\theta \|\varphi\|_{L^q}^{1-\theta}, \quad \text{with} \quad \theta = \frac{2}{q+2}. \qquad (A.3)$$

iii) For all $\psi \in \mathrm{H}^1(\Omega)$ it holds true

$$|\psi(1) - \psi(0)| \le 2\sqrt{2}\sqrt{\frac{\pi + 1}{\pi}} \|\psi\|_{L^2}^{1/2} \|\psi_x\|_{L^2}^{1/2}. \qquad (A.4)$$

Proof. i) This statement is taken from [Zhe04, Theorem 1.3.4].
ii) We know that $\mathrm{H}^1(\Omega) \subset \mathrm{C}(\bar{\Omega})$ such that we can define $x_*, x^* \in [0,1]$ as

$$|\psi(x_*)| \le |\psi(x)| \le |\psi(x^*)| \qquad \forall x \in [0,1].$$

Then for all $\beta > 1$ there holds

$$\begin{aligned}\|\psi\|_{L^\infty}^\beta = |\psi(x^*)|^\beta \le \Big| \int_{x_*}^{x^*} \left(|\psi|^\beta\right)_x \mathrm{d}x \Big| + |\psi(x_*)|^\beta \le \beta \int_0^1 |\psi|^{\beta-1}|\psi_x| + |\psi(x_*)|^\beta \,\mathrm{d}x \\ \le \beta \|\psi\|_{L^{2\beta-2}}^{\beta-1} \|\psi_x\|_{L^2} + |\psi(x_*)|^\beta. \qquad (A.5)\end{aligned}$$

Applying the bound $|\psi(x_*)| \le \|\psi\|_{L^r}$ and setting $\beta = \frac{q}{2} + 1 > 1$, this proves (A.2). *On the other hand keeping the choice of β we can proceed from* (A.5) *with*

$$\begin{aligned}\|\psi\|_{L^\infty}^\beta &\le \beta\|\psi\|_{L^{2\beta-2}}^{\beta-1}\|\psi_x\|_{L^2} + |\psi(x_*)|^{\beta-1}|\psi(x_*)| \le \beta\|\psi\|_{L^{2\beta-2}}^{\beta-1}\|\psi_x\|_{L^2} + \|\psi(x)\|_{L^{2\beta-2}}^{\beta-1}\|\psi(x)\|_{L^2} \\ &\le \beta\|\psi\|_{L^{2\beta-2}}^{\beta-1}(\|\psi\|_{L^2} + \|\psi_x\|_{L^2}).\end{aligned}$$

This then proves (A.3).
iii) We first observe that we have to estimate a linear form on $\mathrm{H}^1(\Omega)$ which vanishes on

constant functions. Therefore it suffices to give an estimate only for those functions which are orthogonal to the constants. (It is clear that a function ψ is orthogonal to the constants if and only if $\int_\Omega \psi \, \mathrm{d}x = 0$.) We estimate by means of (A.3), with $q = 2$,

$$|\psi(0) - \psi(1)| \le 2\|\psi\|_{\mathrm{C}([0,1])} \le 2\sqrt{2}\|\psi\|_{L^2}^{1/2}\sqrt{\|\psi\|_{L^2} + \|\psi_x\|_{L^2}}. \tag{A.6}$$

Using the estimating $\|\phi\|_{L^2} \le \frac{1}{\pi}\|\phi_x\|_{L^2}$ which holds for all for all $\phi \in \mathrm{H}^1(]0,1[)$ with $\int_0^1 \phi \, \mathrm{d}x = 0$, one obtains the assertion. □

We provide a Gagliardo-Nirenberg type estimate involving norms in $L \log L(\Omega)$. The proof consists of a modification of [BHN94, p. 1199].

***Lemma* A.2.** *Let $G \subset \mathbb{R}$ be a bounded interval. There exists $C > 0$ with the property that for all $\varepsilon > 0$ one can find $C_\varepsilon > 0$ such that*

$$\|w\|_{L^\infty}^3 \le \varepsilon\|w_x\|_{L^2}^2 \cdot \|w \ln|w|\|_{L^1} + C_\varepsilon + C\|w\|_{L^1}^3 \tag{A.7}$$

is valid for all $w \in \mathrm{H}^1(G)$.

Proof. Following the reasoning in [BHN94], we first invoke the Gagliardo-Nirenberg inequality (A.1) to find $c_1 > 0$ such that

$$\|z\|_{L^\infty}^3 \le c_1\|z_x\|_{L^2}^2 \cdot \|z\|_{L^1} + c_1\|z\|_{L^1}^3 \qquad \text{for all } z \in \mathrm{H}^1(G). \tag{A.8}$$

We now choose $N > 1$ large fulfilling $\frac{8c_1}{\ln N} \le \varepsilon$ and introduce $\chi \in \mathrm{W}_{loc}^{1,\infty}(\mathbb{R})$ by defining $\chi(s) := 0$ for $s \in [-N, N]$, $\chi(s) := |s|$ for $|s| \ge 2N$ and $\chi(s) := 2(|s| - N)$ for $N < |s| < 2N$. Then given $w \in \mathrm{H}^1(G)$, we evidently have

$$\|w - \chi(w)\|_{L^\infty} \le 2N$$

and furthermore

$$\|\chi(w)\|_{L^1} \le \int_{\{|w|>N\}} |w| \, \mathrm{d}x \le \frac{1}{\ln N} \cdot \|w \ln|w|\|_{L^1}.$$

Since $(1+\xi)^3 \le 2 \cdot (1+\xi^3)$ for $\xi \ge 0$, (A.8) furthermore yields

$$\begin{aligned} \|w\|_{L^\infty}^3 &\le 2\|\chi(w)\|_{L^\infty}^3 + 2\|w - \chi(w)\|_{L^\infty}^3 \\ &\le 2c_1\|(\chi(w))_x\|_{L^2}^2 \cdot \|\chi(w)\|_{L^1} + 2c_1\|\chi(w)\|_{L^1}^3 + 2^4N^3 \\ &\le \tfrac{8c_1}{\ln N} \cdot \|w_x\|_{L^2}^2 \cdot \|w \ln|w|\|_{L^1} + 2c_1\|w\|_{L^1}^3 + 3^4N^3, \end{aligned}$$

because $\|\chi'\|_{L^\infty(\mathbb{R})} = 2$ and $|\chi(s)| \le |s|$ for all $s \in \mathbb{R}$. In view of our definition of N, this proves (A.7) with $C := 2c_1$ and $C_\varepsilon := 2^4N^3$. □

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