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A Mathematical Framework for Standard Generalized Materials in the Rate-Independent Case

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Abstract

Standard generalized materials are described by an elastic energy density and a dissipation potential. The latter gives rise to the evolution equation (flow law) for the internal variables. The energetic formulation provides a very weak, derivative-free form of this flow law. It is based on a global stability condition and an energy balance. Using time-incremental minimization problems, which allow for the usage of the rich theory in the direct method of the calculus of variations, it is possible to establish general, abstract existence results as well as convergence for numerical approximations. Applications to shape-memory materials and to magnetostrictive or piezoelectric materials are surveyed.

1 Introduction

On the mechanical side the theory of standard generalized materials was developed in the early 1970s, see [28, 64, 69, 81]. The mathematics for these models was studied in parallel, but was mostly restricted to the case of convex potentials with applications in small-strain elastoplasticity, cf. [31, 65].

The theory of rate-independent hysteresis operators advanced much further, see [11, 35, 36, 80], mainly in the field of scalar-valued hysteresis operators. In parallel, the mathematical theory of solid mechanics had major breakthroughs in the treatment of finite-strain elastostatics [6, 13] and in the study of microstructures in modern materials [7, 67].

The theory presented here is located in a triangle that has its corners in the rich area of existing engineering models, in the theory of hysteresis models, and in the methods of calculus of variations that were derived for nonconvex material models. The major fact is that rate independence is still so close to statics that very similar methods can be employed. Nevertheless it allows us to study evolutionary effects on slow timescales.

In Sect. 2 we will present the theory of standard generalized materials and will show how these models are linked to the so-called energetic formulation. In Sect. 3 we summarize the existence theory for energetic solutions developed in a quite abstract setting, see [22, 43, 49].

In Sect. 4 we discuss the question of approximation of the energetic formulation. Based on abstract Γ-convergence ideas it is possible to derive convergent results for numerical approximations via finite-element methods, see [37, 55]. Moreover, homogenization results are established, see [63]. Finally, a relaxation result is presented that is due to [37, 57].

The final section is devoted to a list of several applications. The whole work was initiated through the need for a better understanding of the hysteretic evolution of microstructure in shape-memory alloys [58, 60]. In the Sects. 5.1 to 5.4 we report on the development of the analysis of different models since then. Further applications occur in damage [20, 56], in delamination [34] and in brittle fracture [12, 15, 21]. The modeling of ferroelectric and magnetostricitive materials also fits into this framework, see Sects. 5.5 and 5.6. Moreover, the theory of elastoplasticity should be mentioned, since it is one of the major driving forces of the theory of rate-independent processes. The recent advances in this topic will be surveyed in another article of this volume, see [46].

2 Modeling Materials with Internal Variables

2.1 Standard Generalized Materials

This theory was developed in [28, 81] and has established a central rôle in the area of material modeling on the phenomenological level, see [23, 26, 45] for some recent references.

We consider an elastic body with reference domain $\Omega \subset \mathbb{R}^d$. The deformation $\varphi : \Omega \to \mathbb{R}^d$ gives rise to the strain tensor $F = \nabla\varphi$. We assume that the state in a material point $x \in \Omega$ is described by $F \in \mathbb{R}^{d\times d}$ and a further variable $z \in Z$ which is often called internal variable. Here z may denote plastic variables, damage, magnetization, polarization or some phase indicator. The admissible set Z is in general a submanifold (with boundary) of $\mathbb{R}^m$ for some $m \in \mathbb{N}$.

The material behavior is described by two constitutive functions, the stored-energy density $W = \widehat{W}(x, F, z)$ (also called elastic potential) and the dissipation potential $R = \widehat{R}(x, z, \dot z)$. While W is the potential for the stress-strain relation, R is the potential for the dissipational forces versus the rate $\dot z$, viz.,

$$T = \tfrac{\partial}{\partial F}\widehat{W}(x, F, z) \text{ and } f_{\text{diss}} = -\tfrac{\partial}{\partial \dot z}\widehat{R}(x, z, \dot z) \ .$$

The time evolution of the material is now described by the quasistatic elastic equilibrium

$$-\text{div}\left(\tfrac{\partial}{\partial F}\widehat{W}(x, \nabla\varphi, z)\right) = f_{\text{ext}} \text{ plus bound. cond.}$$

and by the flow law for the internal variable which involves the thermodynamically conjugated driving force $X_Z = -\frac{\partial}{\partial z}\widehat{W}(x, F, z)$, viz.,

$$-(f_{\text{diss}} + X_z) = 0 = \tfrac{\partial}{\partial \dot z}\widehat{R}(x, z, \dot z) + \tfrac{\partial}{\partial z}\widehat{W}(x, \nabla\varphi, z) \ .$$

Rate independence means that $\widehat{R}(x, z, \cdot)$ is homogeneous of degree 1. Then, $\frac{\partial}{\partial \dot z}\widehat{R}$ has to be understood as the multi-valued subdifferential of convex analysis

$$\partial_{\dot z}\widehat{R}(x, z, v) = \{\, \eta \in \mathrm{T}_z^* Z \mid \forall\, w \in \mathrm{T}_z Z\colon \widehat{R}(x, z, w) \geq \widehat{R}(x, z, v) + \langle \eta, w - v\rangle \,\} \ .$$

To provide a mathematical framework we introduce $\mathcal{F}$ as the set of admissible deformations, which is typically an affine subspace of some Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)$ due to the Dirichlet boundary conditions. Moreover, we let $\mathcal{Z}=\mathrm{L}^1(\Omega,Z)$ for the function space of admissible internal states. For the state space $\mathcal{Q}=\mathcal{F}\times\mathcal{Z}$ we set $q=(\varphi,z)$ and

$$\begin{aligned}\mathcal{E}(t,q) &= \textstyle\int_\Omega \widehat{W}(x,\nabla\varphi,z)\,\mathbb{D}z - \int_\Omega f_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathbb{D}x\,,\\ \mathcal{R}(z,\dot z) &= \textstyle\int_\Omega \widehat{R}(t,z(x),\dot z(x))\,\mathbb{D}x\,.\end{aligned}$$

Hence, the evolutionary problem takes the form

$$\begin{aligned}&\mathrm{D}_\varphi\mathcal{E}(t,\varphi(t),z(t))=0,\\ &0\in\partial_{\dot z}\mathcal{R}(z(t),\dot z(t))+\mathrm{D}_z\mathcal{E}(t,\varphi(t),z(t))\,.\end{aligned}\tag{1}$$

2.2 The Energetic Formulation

In general the manifold $Z\subset\mathbb{R}^m$ might be complicated and the definition of $\dot z$ might be nontrivial. Moreover, in rate-independent systems it is to be expected that solutions develop jumps. Hence, it is desirable to find a weaker formulation avoiding derivatives. For this we introduce the *dissipation distance* $D(x,\cdot,\cdot):Z\times Z\to[0,\infty]$ which is associated with the Finslerian dissipation metric $\widehat{R}(x,\cdot,\cdot):\mathrm{T}Z\to[0,\infty]$, viz.,

$$D(x,z_0,z_1)=\inf\{\,\textstyle\int_0^1\widehat{R}(x,\widetilde z(s),\dot{\widetilde z}(s))\mathbb{D}s\mid \widetilde z\in\mathrm{C}^1([0,1],Z),\widetilde z(0)=z_0,\widetilde z(1)=z_1\,\}\,.$$

On $\mathcal{Z}$ this induces the distance $\mathcal{D}$ with $\mathcal{D}(z_0,z_1)=\int_\Omega D(x,z_0(x),z_1(x))\,\mathbb{D}x$, and we are able to define the dissipation along an arbitrary path $z:[0,T]\to\mathcal{Z}$ via

$$\mathrm{Diss}_{\mathcal{D}}(z,[s,t])=\sup\{\,\sum_{j=1}^N\mathcal{D}(z(t_{j-1}),z(t_j))\mid N\in\mathbb{N},s\le t_0<t_1<\cdots t_N\le t\,\}\,.$$

For smooth paths is compatible with the classical dissipation

$$\mathrm{Diss}_{\mathcal{D}}(z,[s,t])=\textstyle\int_s^t\mathcal{R}(z(\tau),\dot z(\tau))\,\mathbb{D}\tau=\int_s^t\int_\Omega\widehat{R}(x,z(\tau,x),\dot z(\tau,x))\,\mathbb{D}x\,\mathbb{D}\tau\,.$$

Our weak form of (1) is the *energetic formulation* involving the *stability condition* (S) and the *energy balance* (E). A process $q=(\varphi,z):[0,T]\to\mathcal{F}\times\mathcal{Z}=\mathcal{Q}$ is called *energetic solution* for $(\mathcal{E},\mathcal{D})$, if for all $t\in[0,T]$ we have

$$\boxed{\begin{aligned}&\textbf{(S)}\quad q(t)\in\mathcal{S}(t)\stackrel{\mathrm{def}}{=}\{\,q\in\mathcal{Q}\mid\mathcal{E}(t,q)<\infty,\ \forall\,\widetilde q\in\mathcal{Q}\colon\mathcal{E}(t,q)\le\mathcal{E}(t,\widetilde q)+\mathcal{D}(q,\widetilde q)\,\}\\ &\textbf{(E)}\quad\mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(q,[0,t])=\mathcal{E}(0,q(0))+\textstyle\int_0^t\partial_s\mathcal{E}(s,q(s))\,\mathbb{D}s\,.\end{aligned}}$$

Here $\partial_s\mathcal{E}(s,q(s))=\frac{\partial}{\partial s}\mathcal{E}(s,q(s))$ is called the power of the external forces and we implicitly assume that $t\mapsto\partial_t\mathcal{E}(t,q(t))$ lies in $\mathrm{L}^1((0,T))$.

In the case that $\mathcal{Q}$ is a Banach space, that $\mathcal{E}$ and $\mathcal{R}$ are Gateaux differentiable and that the energetic solution q lies in $\mathrm{W}^{1,1}([0,T],\mathcal{Q})$ it is easy to see that (S)

implies $\mathrm{D}_\varphi \mathcal{E}(t,\varphi(t),z(t)) = 0$ and $O \in \partial_{\dot z}\mathcal{R}(z(t),0) + \mathrm{D}_z\mathcal{E}(t,\varphi(t),z(t))$. Moreover, differentiating (E) with respect to time yields $\mathrm{D}_z\mathcal{E}(t,\varphi(t),z(t))[\dot z(t)]+\mathcal{R}(z(t),\dot z(t)) = 0$. This is exactly (1). In the case that $\mathcal{E}(t,\cdot)$ is strictly convex on the Banach space $\mathcal{Q}$ and that $\mathcal{R}$ does not depend on z, it is shown in [59] that (1) is in fact equivalent to (S) & (E). See also [19, 52] for more general results on this equivalence.

However, as we are mostly interested in nonconvex models we will mainly focus on the energetic formulation (S) & (E). Note that a significant simplification occurs due to the fact that (S) is a purely static condition.

2.3 Formulations that Minimize Locally

A major drawback of the energetic formulation is that (S) involves a *global* stability condition, while *local* stability would be more physical. However, the word "local" means that we need to specify a topology in which neighborhoods will be defined. One physical way of doing this is to consider systems with small viscosity and to study the limit of vanishing viscosity,

$$\begin{aligned} 0 &= \varepsilon A_1\dot\varphi + \mathrm{D}_\varphi\mathcal{E}(t,\varphi,z)\ , \\ 0 &\in \partial\mathcal{R}(z,\dot z) + \varepsilon A_2\dot z + \mathrm{D}_z\mathcal{E}(t,\varphi,z)\ . \end{aligned}$$

A mathematical way of approaching the same problem is that of doing local minimization in the associated *time-incremental problem*

$$(\mathrm{IP})^\delta_{\mathrm{loc}} \quad q_k \in \mathrm{Argmin}\{\, \mathcal{E}(t_k,\widetilde q) + \mathcal{D}(q_{k-1},\widetilde q) \mid \widetilde q \in \mathcal{Q}, \|q_{k-1}-\widetilde q\| \le \delta \,\}\ ,$$

where $\|\cdot\|$ denotes a suitable norm.

It is shown in [18] that for the smooth finite-dimensional situation the associated solutions converge, after an arclength parameterization, to solutions of the following limit problem

$$0 \in \partial R_{\|\cdot\|}(z'(s)) + \mathrm{D}_z\mathcal{E}(t(s),z(s)) \quad \text{and} \quad 1 = t'(s) + \|z'(s)\|\ ,$$

where $R_{\|\cdot\|}(v) = R(v)$ for $\|v\| \le 1$ and ∞ else. Generalizations of this idea to the infinite dimensional setting will be discussed in [53].

3 Analysis of the Energetic Formulation

3.1 The Basic Abstract Assumptions

Our state space $\mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ is considered to be the product of two topological spaces $\mathcal{F}$ and $\mathcal{Z}$, both of which are assumed to be Hausdorffsch. Throughout all topological notions like compactness, closedness and (semi-)continuity are meant in the sequential sense. For convergence we write $\overset{\mathcal{Q}}{\to}$, $\overset{\mathcal{F}}{\to}$ and $\overset{\mathcal{Z}}{\to}$, respectively.

We start with the assumptions on $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$:

$$\forall\, z_1, z_2, z_3 \in \mathcal{Z} :\ \mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3)\ . \tag{2}$$

$$\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is lower semi-continuous}\ . \tag{3}$$

$$\begin{aligned} &\text{For compact } \mathcal{K} \subset \mathcal{Z} \text{ and } (z_k)_{k\in\mathbb{N}} \subset \mathcal{K} \text{ we have:} \\ &\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0 \quad \Longrightarrow \quad z_k \overset{\mathcal{Z}}{\to} z\ . \end{aligned} \tag{4}$$

For applications in continuum mechanics it is essential to allow $\mathcal{D}$ to attain the value $+\infty$ and to be unsymmetric, i.e., in general $\mathcal{D}(z_1, z_2) \neq \mathcal{D}(z_2, z_1)$.

An important abstract tool is a suitable generalization of Helly's selection principle, cf. [43]. If the functions $z_k : [0, T] \to \mathcal{K} \subset \mathcal{Z}$ with $\mathcal{K}$ compact satisfy $\mathrm{Diss}_{\mathcal{D}}(z_k, [0, T]) \le C < \infty$, then there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ and a limit function $z : [0, T] \to \mathcal{K} \subset \mathcal{Z}$, such that for all $t \in [0, T]$ we have $z_{k_j}(t) \overset{\mathcal{Z}}{\to} z(t)$ and $\mathrm{Diss}_{\mathcal{D}}(z, [0, T]) \le \liminf_{k\to\infty} \mathrm{Diss}_{\mathcal{D}}(z_k, [0, T])$.

For the energy functional $\mathcal{E}$ the following assumptions proved to be useful:

$$\forall\, t \in [0, T]\ \forall\, E \in \mathbb{R} :\quad \{\, q \in \mathcal{Q} \mid \mathcal{E}(t, q) \le E \,\} \text{ is compact}\ ; \tag{5}$$

$$\begin{aligned} &\exists\, c_0^E \in \mathbb{R}\ \exists\, c_1^E > 0\ \forall\, (t, q) \in [0, T] \times \mathcal{Q} \text{ with } \mathcal{E}(t, q) < \infty : \\ &\mathcal{E}(\cdot, q) \in \mathrm{C}^1([0, T], \mathbb{R}) \text{ and } |\partial_t \mathcal{E}(s, q)| \le c_1^E(\mathcal{E}(s, q) + c_0^E) \text{ on } [0, T]\ ; \end{aligned} \tag{6}$$

$$\begin{aligned} &\forall\, \varepsilon > 0\ \forall\, E \in \mathbb{R}\ \exists\, \delta > 0\ \forall\, q \text{ with } \mathcal{E}(0, q) \le E : \\ &|t_1 - t_2| \le \delta \quad \Longrightarrow \quad |\partial_t \mathcal{E}(t_1, q) - \partial_t \mathcal{E}(t_1, q)| \le \varepsilon\ ; \end{aligned} \tag{7}$$

$$\Big(q_k \in \mathcal{S}(t),\ \sup_{k\in\mathbb{N}} \mathcal{E}(t, q_k) < \infty,\ q_k \overset{\mathcal{Q}}{\to} q\Big) \implies \partial_t \mathcal{E}(t, q_k) \to \partial_t \mathcal{E}(t, q)\ . \tag{8}$$

The standard condition (5) implies lower semi-continuity and relative compactness of infimizing sequences. The other conditions concern the power of external forces $\partial_t \mathcal{E}$. Assumption (6) says that we are able to control the work of the external forces via the energy itself. The assumptions (7) and (8) concern continuity in t and q. They are easily checked in the Banach space setting if $\mathcal{E}$ has the form $\mathcal{E}(t, q) = \mathcal{E}_0(q) - \langle \ell(t), q\rangle$ with $\ell \in \mathrm{C}^1([0, T], \mathcal{Q}^*)$.

The final and crucial assumption controls the interplay of $\mathcal{E}$ and $\mathcal{D}$:

$$\forall\, t \in [0, T] :\ \mathcal{S}(t) \text{ is closed in } \mathcal{Q}\ . \tag{9}$$

In most applications of the present theory, the major work goes into establishing (9). There are a few abstract results that establish (9). For instance, if $\mathcal{D}$ is continuous on $\mathcal{Z}$, then (9) can be easily derived using (5).

The following lemma provides a more general condition. We refer to [43, 55, 57] for more discussion on ways to establish closedness of the stable set.

Lemma 3.1. *If for each sequence* $(q_k)_{k\in\mathbb{N}}$ *in* $\mathcal{S}(t)$ *with* $q_k \overset{\mathcal{Q}}{\to} q$ *and each* $\widetilde{q} \in \widetilde{Q}$ *there exists a* recovery sequence $(\widetilde{q}_k)_{k\in\mathbb{N}}$ *with* $\widetilde{q}_k \overset{\mathcal{Q}}{\to} \widetilde{q}$ *such that*

$$\limsup_{k\to\infty} \Big(\mathcal{E}(t,\widetilde{q}_k) + \mathcal{D}(q_k,\widetilde{q}_k) - \mathcal{E}(t,q_k) \Big) \le \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(q,\widetilde{q}) - \mathcal{E}(t,q)$$

holds, then $\mathcal{S}(t)$ *is closed.*

Proof: We start from $q_k \in \mathcal{S}(t)$ with $q_k \to q$ and have to show $q \in \mathcal{S}(t)$. Let $\widetilde{q}$ be an arbitrary test function. Then, by the assumption of the lemma there exist $\widetilde{q}_k, k \in \mathbb{N}$, with $\widetilde{q}_k \overset{\mathcal{Q}}{\to} \widetilde{q}$. From $q_k \in \mathcal{S}(t)$ we know $0 \le \mathcal{E}(t,\widetilde{q}_k) + \mathcal{D}(q_k,\widetilde{q}_k) - \mathcal{E}(t,q_k)$ and hence the $\limsup_{k\to\infty}$ is nonnegative. We conclude $\mathcal{E}(t,\widetilde{q}) + \mathcal{D}(q,\widetilde{q}) - \mathcal{E}(t,q) \ge 0$ and obtain $q \in \mathcal{S}(t)$. ∎

3.2 The Existence Result

We approach the time-continuous formulation (S) & (E) by the following time-incremental problem (IP). For a partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ and a given initial value $q_0 \in Q$ we let

$$\boxed{\begin{array}{ll} (\mathrm{IP})_\Pi & \text{Find } q_1, q_2, \dots, q_n \text{ such that} \\ & q_k \in \mathrm{Argmin}\{\, \mathcal{E}(t_k,\widetilde{q}) + \mathcal{E}(q_{k-1},\widetilde{q}) \mid \widetilde{q} \in \mathcal{Q} \,\}\,. \end{array}}$$

By assumption (3) and (5) it is immediate that $(\mathrm{IP})_\Pi$ is solvable and we are able to define the piecewise constant interpolant

$$q^\Pi : [0,T] \to \mathcal{Q} \text{ with } q^\Pi(t) = \begin{cases} q_{j-1} & \text{for } \ t \in [t_{j-1}, t_j)\,, \\ q_N & \text{for } \ t = T\,. \end{cases}$$

It is not difficult to see that the incremental solution satisfies $q^\Pi(t_j) \in \mathcal{S}(t_j)$ for $j = 1, \dots, N$ and

$$\mathcal{E}(t_j, q^\Pi(t_j)) + \mathrm{Diss}_{\mathcal{D}}(q^\Pi, [0,t_j]) \le \mathcal{E}(0, q^\Pi(0)) + \textstyle\int_0^{t_j} \partial_s \mathcal{E}(s, q^\Pi(s))\, \mathbb{D}s\,.$$

From this it is then possible to derive a priori estimates independent of Π for $\mathcal{E}(t, q^\Pi(t))$ and $\mathrm{Diss}_{\mathcal{D}}(q^\Pi, [0,T])$. Helly's selection principle for the z-component and the compactness of the sublevels of $\mathcal{E}$ allow us then to construct a converging subsequence and to pass to the limit. The final result reads as follows. We refer to [22, 43, 49] for the proof.

Theorem 3.2. *Let* $\Pi_k = \{0 = t_0^k < t_1^k < \cdots < t_{N_k}^k = T\}, k \in \mathbb{N}$, *be a sequence of partitions such that* $\phi(\Pi_k) = \max\{t_j^k - t_{j-1}^k \mid j = 1, \dots, N_k\}$ *tends to* 0. *Let* $q_0 \in \mathcal{S}(0)$ *be an initial condition and* $q^{\Pi_k} : [0,T] \to \mathcal{Q}$ *be piecewise constant interpolants of the solution of* $(IP)_{\Pi_k}$. *Then there exists a subsequence* $\overline{q}_n = q^{\Pi_{k_n}}$ *and an energetic solution* $q : [0,T] \to \mathcal{Q}$ *of (S) & (E) with* $q(0) = q_0$ *such that for all* $t \in [0,T]$ *the following holds*

(i) $\overline{z}_n(t) \xrightarrow{\mathcal{Z}} z(t)$,
(ii) $\mathcal{E}(t, \overline{q}_n(t)) \to \mathcal{E}(t, q(t))$,
(iii) $\mathrm{Diss}_{\mathcal{D}}(\overline{q}_n, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(q, [0,T])$,
(iv) $\exists$ *subsequence* $(N_l^t)_{l\in\mathbb{N}}$*:* $\overline{\varphi}_{N_l^t}(t) \xrightarrow{\mathcal{F}} \varphi(t)$ *for* $l \to \infty$.

Moreover, $\partial_t \mathcal{E}(\cdot, \overline{q}_n(\cdot)) \overset{*}{\rightharpoonup} \partial_t \mathcal{E}(\cdot, q(\cdot))$ *in* $\mathrm{L}^\infty((0,T))$.

The convergence of the φ-component occurs only on t-dependent subsequences $(N_l^t)_{l\in\mathbb{N}}$. Hence, in general, we cannot guarantee the measurability of the mapping $\varphi : [0,T] \to \mathcal{F}$. However, in [41, 42] it is shown that measurability can also be obtained by applying suitable results for measurable selections of multi-valued mappings.

3.3 Results Based on Convexity

The abstract result of the previous section can be improved if additional properties are available. We now assume that $\mathcal{Q}$ is a Banach space, such that convexity methods can be used. In general, one should distinguish three different spaces X, Y and Z. The space Z is the one that provides coercivity of the dissipation distance, i.e.,

$$\forall\, q_0, q_1 \in \mathcal{Q} : \ \mathcal{D}(q_0, q_1) \geq \|q_1 - q_0\|_Z \ . \tag{10}$$

The space Y measures the uniform convexity of $\mathcal{J}_{t,q} : \widetilde{q} \mapsto \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(q,\widetilde{q})$:

$$\forall\, q_0, q_1 \in \mathcal{Q} : \ \mathcal{J}_{t,q}\big(\tfrac{1}{2}(q_0+q_1)\big) \leq \tfrac{1}{2}(\mathcal{J}_{t,q}(q_0)+\mathcal{J}_{t,q}(q_1)) - \tfrac{\alpha}{2}\,\|q_0-q_1\|_Y^2 \tag{11}$$

for some $\alpha > 0$. Finally, X relates to the coercivity of $\mathcal{E}$, i.e.,

$$\forall\, q \in \mathcal{Q} : \ \mathcal{E}(t,q) \geq g(\|q\|_X) \tag{12}$$

for some $g \in \mathrm{C}^0([0,\infty), \mathbb{R})$ with $g(t) \to \infty$ for $t \to \infty$.

The abstract results of Sect. 3.2 immediately imply that any solution of (S) & (E) satisfies

$$q = (\varphi, z) \in \mathrm{L}^\infty([0,T], X) \quad \text{and} \quad z \in \mathrm{BV}([0,T], Z) \ .$$

For a proof of the following result we refer to Theorem 3.4 in [49].

Proposition 3.3. *Assume that* $\mathcal{E}$ *and* $\mathcal{D}$ *satisfy the joint convexity condition* (11) *for some* $\alpha > 0$ *and that there exists* $C_Y > 0$ *such that*

$$\forall\, t \in [0,T]\ \forall\, q_0, q_1 \in \mathcal{Q} : \ |\partial_t \mathcal{E}(t,q_0) - \partial_t \mathcal{E}(t,q_1)| \leq C_Y \|q_0 - q_1\|_Y \ .$$

Then, every solution q *of (S) & (E) satisfies*

$$\forall\, t_1, t_2 \in [0,T] : \ \|q(t_1) - q(t_2)\|_Y \leq \frac{C_Y}{\alpha} |t_1 - t_2| \ .$$

As a typical example we consider the case $\mathcal{Q} = \mathcal{Z} = X$ with

$$X = \mathrm{H}^1(\Omega) \ , \quad \mathcal{D}(z_0, z_1) = \int_\Omega |z_0(x) - z_1(x)| \, \mathbb{D}x \ ,$$
$$\mathcal{E}(t, z) = \int_\Omega W(\nabla z(x)) + \tfrac{\alpha}{2}|z(x)|^2 - f_{\mathrm{ext}}(t, x) z(x) \, \mathbb{D}x \ ,$$

with $\alpha > 0$, $f_{\mathrm{ext}} \in \mathrm{C}^1([0,T], \mathrm{L}^2(\Omega))$ and $W : \mathbb{R}^d \to [0, \infty)$, where W is convex and coercive, i.e., $W(A) \geq c|A|^2 - C$ for some $C, c > 0$ and all $A \in \mathbb{R}^d$. Then, we may choose $Z = \mathrm{L}^1(\Omega)$ and $Y = \mathrm{L}^2(\Omega)$.

In such situations it is possible to define $\dot{q}(t)$ almost everywhere, since jumps, which are allowed in the energetic formulation, can no longer occur. Hence, it is possible to study the local subdifferential formulation (1) instead. Using $q = (\varphi, z) \in X = Q$ and $\mathcal{R}(z, v) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathcal{D}(z, z + \varepsilon v)$ we write (1) in the compact form

$$X^* \ni 0 \in \partial_v \mathcal{R}(q(t), \dot{q}(t)) + \partial_q \mathcal{E}(t, q(t)) \text{ a.e. on } [0, T] \ . \tag{13}$$

This equation is called a doubly nonlinear equation and it relates to evolutionary quasi-variational inequalities (cf. [10]). We refer to [52, 59] for exact conditions which guarantee the equivalence between (S) & (E) and (13).

The latter work contains also a general existence result for Lipschitz continuous solutions to (13). Under quite severe additional assumptions it is even possible to prove uniqueness, see [10, 52, 59]. However, these assumptions are rarely met in material models except for very simple cases like linearized elastoplasticity with quadratic hardening, see [29, 31, 65]. Other uniqueness results are discussed in [61, 62] for piezoelectricity and in [4] for an isotropic model for shape-memory alloys, see also Sect. 5.

4 Approximation, Γ-Limits and Relaxation

In several circumstances it is desirable to consider sequences of functionals $(\mathcal{E}_k)_{k \in \mathbb{N}}$ and $(\mathcal{D}_k)_{k \in \mathbb{N}}$ which converge to limit functionals $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$, respectively, in a suitable sense. The main question is which type of convergence guarantees that limits $q : [0,T] \to \mathcal{Q}$ of solutions $q_k : [0,T] \to \mathcal{Q}$ for $(\mathcal{E}_k, \mathcal{D}_k)$ are solutions for $(\mathcal{E}_\infty, \mathcal{D}_\infty)$.

Typical applications of this idea occur for

- numerical approximations with $\mathcal{E}_k(t, q) = \mathcal{E}_\infty(t, q)$ for $q \in \mathcal{Q}_k \subset \mathcal{Q}$ and ∞ otherwise, where each Q_k is a finite-dimensional subspace of $\mathcal{Q}$ such that $Q_k \subset \mathcal{Q}_{k+1}$ and $\bigcup_{k \in \mathbb{N}} Q_k$ is dense in $\mathcal{Q}$.
- problems with singular perturbations (like sharp interface models) or with penalization terms
- constant sequences $\mathcal{E}_k = \mathcal{E}_1, \mathcal{D}_k = \mathcal{D}_1$, where $\mathcal{E}_1(t, \cdot)$ and $\mathcal{D}_1(\cdot, \cdot)$ are not lower semi-continuous and differ from their Γ-limits $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$.

The latter point relates to relaxations of rate-independent evolution which is an important topic in material modeling. It is a tool for deriving evolution equations for microstructures. We refer to [14, 48, 51, 60, 79] for discussions of this topic.

Here we present the theory originating from [37]. In [57] the abstract version was developed and in [55] it is applied to numerical approximation in several material models. The following version is a simplified version of the one developed in [57].

4.1 Γ-Convergence of Rate-Independent Systems

We let $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ and state first the conditions on the dissipation distances $(\mathcal{D}_k)_{k\in\mathbb{N}_\infty}$. Each $\mathcal{D}_k, k \in \mathbb{N}_\infty$, is a pseudo distance on $\mathcal{Z}$, i.e.,

$$\forall\, z_j \in \mathcal{Z} :\ \mathcal{D}_k(z_1, z_1) = 0 \text{ and } \mathcal{D}_k(z_1, z_3) \le \mathcal{D}_k(z_1, z_2) + \mathcal{D}_k(z_2, z_3)\ . \tag{14}$$

To obtain solutions of incremental problems we impose that

$$\forall\, k \in \mathbb{N}_\infty :\ \mathcal{D}_k : \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is lower semi-continuous}\,. \tag{15}$$

The limit distance $\mathcal{D}_\infty$ must be positive in the following sense

$$\begin{aligned} &\text{For compact } \mathcal{K} \subset \mathcal{Z} \text{ and } (z_k)_{k\in\mathbb{N}} \subset \mathcal{K} \text{ we have:} \\ &\min\{\mathcal{D}_\infty(z_k, z), \mathcal{D}_\infty(z, z_k)\} \to 0 \quad \Longrightarrow \quad z_k \overset{\mathcal{Z}}{\to} z\ . \end{aligned} \tag{16}$$

Finally, $\mathcal{D}_\infty$ must be bounded from above by the Γ-liminf of $(\mathcal{D}_k)_{k\in\mathbb{N}}$, i.e.,

$$\Big(z_k \overset{\mathcal{Z}}{\to} z \text{ and } \widetilde{z}_k \overset{\mathcal{Z}}{\to} \widetilde{z}\Big) \quad \Longrightarrow \quad \mathcal{D}_\infty(z, \widetilde{z}) \le \liminf_{k\to\infty} \mathcal{D}_k(z_k, \widetilde{z}_k)\ . \tag{17}$$

Next we state the conditions on the energy functionals. We start with the compactness of the sublevels:

$$\begin{aligned} &\forall\, t \in [0,T]\ \forall\, E \in \mathbb{R} : \\ &\text{(i) } \forall\, k \in \mathbb{N}_\infty :\ \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t,q) \le E \,\} \text{ is compact}\,, \\ &\text{(ii) } \textstyle\bigcup_{k\in\mathbb{N}} \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t,q) \le E \,\} \text{ is relatively compact}\,. \end{aligned} \tag{18}$$

The next three conditions provide suitable continuity properties of the powers $\partial_t\mathcal{E}_k(\cdot,\cdot)$ of the external forces.

$$\begin{aligned} &\exists\, c_0, c_1 > 0\, \forall\, k \in \mathbb{N}_\infty\ \forall\,(t,q) \in [0,T] \times \mathcal{Q} \text{ with } \mathcal{E}_k(t,q) < \infty : \\ &\mathcal{E}_k(\cdot,q) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t\mathcal{E}_k(s,q)| \le c_1(\mathcal{E}_k(s,q)+c_0) \text{ on } [0,T]\ ; \end{aligned} \tag{19}$$

$$\begin{aligned} &\forall\, \varepsilon > 0\ \forall\, E > 0\ \exists\, \delta > 0\ \forall\, k \in \mathbb{N}_\infty\ \forall\, q \in \mathcal{Q} \text{ with } \mathcal{E}_k(0,q) \le E : \\ &|t_1 - t_2| \le \delta \quad \Longrightarrow \quad |\partial_t\mathcal{E}_k(t_1,q) - \partial_t\mathcal{E}_k(t_2,q)| \le \varepsilon\ ; \end{aligned} \tag{20}$$

$$\Big(q_k \overset{\mathcal{Q}}{\to} q \text{ and } \sup_{k\in\mathbb{N}} \mathcal{E}_k(t,q_k) < \infty\Big) \quad \Longrightarrow \quad \partial_t\mathcal{E}_k(t,q_k) \to \partial_t\mathcal{E}(t,q)\ . \tag{21}$$

The final condition on $(\mathcal{E}_k)_{k\in\mathbb{N}_\infty}$ concerns the Γ-liminf, namely

$$q_k \overset{\mathcal{Q}}{\to} q \quad \Longrightarrow \quad \mathcal{E}_\infty(t,q) \le \liminf_{k\to\infty} \mathcal{E}_k(t,q_k)\ . \tag{22}$$

The crucial condition that connects the convergences of $\mathcal{D}_k$ to $\mathcal{D}_\infty$ and $\mathcal{E}_k$ to $\mathcal{E}_\infty$ involves the sets of stable states. For $k \in \mathbb{N}_\infty$ we have

$$\mathcal{S}_k(t) \overset{\text{def}}{=} \{\, q\in\mathcal{Q} \mid \mathcal{E}_k(t,q) < \infty \text{ and } \forall\, \widetilde{q}\in\mathcal{Q} : \mathcal{E}_k(t,q) \le \mathcal{E}_k(t,\widetilde{q}) + \mathcal{D}_k(q,\widetilde{q}) \,\}$$

and ask for the upper semi-continuity $\mathrm{Limsup}_{k\to\infty}\mathcal{S}_k(t) \subset \mathcal{S}_\infty(t)$, i.e.,

$$\Big(q_{k_\ell}\in\mathcal{S}_{k_\ell}(t) \text{ and } q_{k_\ell}\overset{\mathcal{Q}}{\to} q \text{ for } k_\ell\to\infty\Big) \quad\Longrightarrow\quad q\in\mathcal{S}_\infty(t)\ . \tag{23}$$

In typical applications in continuums mechanics it is hard to establish this condition. On the abstract level it is possible to provide sufficient conditions. For instance, we say that $\mathcal{E}_\infty$ is the Γ-*limit* of $(\mathcal{E}_k)_{k\in\mathbb{N}}$ if (22) holds and if for all $\widetilde{q}\in\mathcal{Q}$ there exists a recovery sequence $(\widetilde{q}_k)_{k\in\mathbb{N}}$ such that

$$\widetilde{q}_k \overset{\mathcal{Q}}{\to} \widetilde{q} \text{ and } \mathcal{E}_\infty(t,\widetilde{q}) \ge \limsup_{k\to\infty} \mathcal{E}_k(t,\widetilde{q}_k)\ . \tag{24}$$

A similar notion of Γ-limit holds for $(\mathcal{D}_k)_{k\in\mathbb{N}}$.

It is shown in [57] that in general (23) does not hold if $\mathcal{E}_k$ Γ-converges to $\mathcal{E}_\infty$ and $\mathcal{D}_k$ Γ-converges to $\mathcal{D}_\infty$. Even more, the following theorem may be false. The next lemma gives a positive result.

Lemma 4.1. *If $\mathcal{E}_\infty = \Gamma\text{-}\lim_{k\to\infty}\mathcal{E}_k$, i.e., (22) and (24) hold, and if $\mathcal{D}_k$ converges continuously to $\mathcal{D}_\infty$, i.e.,*

$$\big(\, z_k \overset{Z}{\to} z \text{ and } \widetilde{z}_k \overset{Z}{\to} \widetilde{z}\,\big) \quad\Longrightarrow\quad \mathcal{D}_k(z_k,\widetilde{z}_k) \to \mathcal{D}_\infty(z,\widetilde{z})\ , \tag{25}$$

then (23) holds.

Proof: Let $q_k = (\varphi_k, z_k) \in \mathcal{S}_k(t)$ be given such that $q_{k_\ell} \overset{\mathcal{Q}}{\to} q$. Moreover, let $\widetilde{q}$ be arbitrary. Then there exists a recovery sequence $\widetilde{q}_k = (\widetilde{\varphi}_k, \widetilde{z}_k)$ satisfying (24). Using (25) we conclude

$$\begin{aligned}
&\mathcal{E}_\infty(t,q) \le \liminf_{\ell\to\infty} \mathcal{E}_{k_\ell}(t,q_{k_\ell}) \\
&\le \liminf_{\ell\to\infty} \big(\mathcal{E}_{k_\ell}(t,\widetilde{q}_{k_\ell}) + \mathcal{D}_{k_\ell}(q_{k_\ell},\widetilde{q}_{k_\ell})\big) \ = \ \mathcal{E}_\infty(t,\widetilde{q}) + \mathcal{D}_\infty(q,\widetilde{q})\ .
\end{aligned}$$

Here we use first (22), next $q_k \in \mathcal{S}_k(t)$ and last (24) and (25). Since $\widetilde{q}\in\mathcal{Q}$ was arbitrary, we have $q\in\mathcal{S}_\infty(t)$. ∎

The following result is concerned with the so-called *incremental problem* $(\mathrm{IP})_k$. For this choose a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions with $\Pi_k = \{0 = t_0^k < t_1^k < \ldots < t_{N_k}^k = T\}$ and fineness $\phi(\Pi_k) = \max\{\, t_j^k - t_{j-1}^k \mid j = 1,\ldots,N_k \,\}$:

$$(\mathrm{IP})_k \qquad \begin{array}{l} \text{Given } q_0^k \in \mathcal{Q}, \text{ find iteratively} \\ q_j^k \in \mathrm{Argmin}\{\, \mathcal{E}_k(t_j^k, \widetilde{q}) + \mathcal{D}_k(q_{j-1}^k, \widetilde{q}) \mid \widetilde{q} \in \mathcal{Q} \,\}. \end{array}$$

Existence of solutions follows easily from (15) and (18). We define the constant interpolants $\overline{q}_k : [0,T] \to \mathcal{Q}$ via

$$\overline{q}_k(t) = q_{j-1}^k \text{ for } t \in [t_{j-1}^k, t_j^k) \text{ and } \overline{q}_k(T) = q_{N_k}^k \ .$$

Theorem 4.2. *Let the conditions* (14) *to* (23) *hold and let the partitions* Π_k, $k \in \mathbb{N}$, *satisfy* $\phi(\Pi_k) \to 0$ *for* $k \to \infty$. *Moreover, assume*

$$q_0^k \in \mathcal{S}_k(0),\ q_0^k \overset{\mathcal{Q}}{\to} q_0, \ \textit{and}\ \mathcal{E}_k(0, q_0^k) \to \mathcal{E}_\infty(0, q_0) \ .$$

Choose any sequence $(\overline{q}_k)_{k\in\mathbb{N}}$ *of constant interpolants of solutions to (IP)*$_k$. *Then, there exists a solution* $q : [0,T] \to \mathcal{Q}$ *of (S) & (E) associated with* $(\mathcal{E}_\infty, \mathcal{D}_\infty)$ *and* $q(0) = q_0$ *and a subsequence* $(\overline{q}_{k_\ell})_{\ell\in\mathbb{N}}$ *such that for all* $t \in [0,T]$ *the following holds:*

(i) $\mathcal{E}_{k_\ell}(t, \overline{q}_{k_\ell}(t)) \to \mathcal{E}_\infty(t, q(t))$,
(ii) $\mathrm{Diss}_{\mathcal{D}_{k_\ell}}(\overline{q}_{k_\ell}, [0,t]) \to \mathrm{Diss}_{\mathcal{D}_\infty}(q, [0,t])$,
(iii) $z_{k_\ell}(t) \overset{\mathcal{Z}}{\to} z(t)$,
(iv) $\exists$ *subsequences* $(K_n^t)_{n\in\mathbb{N}}$ *of* $(k_\ell)_{\ell\in\mathbb{N}}$: $\varphi_{K_n^t}(t) \overset{\mathcal{F}}{\to} \varphi(t)$ *for* $n \to \infty$.

Moreover, we have $\partial_t \mathcal{E}_{k_\ell}(\cdot, \overline{q}_{k_\ell}(\cdot)) \overset{*}{\rightharpoonup} \partial_t \mathcal{E}_\infty(\cdot, q(\cdot))$ *in* $\mathrm{L}^\infty([0,T])$.

4.2 Relaxation in Case of Missing Lower Semi-Continuity

In applications it may occur that for mechanically given functionals $\mathcal{E}$ and $\mathcal{D}$ it is not possible to choose a space $\mathcal{Q}$, such that the sublevels of $\mathcal{E}$ are compact. In particular, the time-incremental problems $(\mathrm{IP})_\Pi$ may not be solvable because of the missing lower semi-continuity, which has its mechanical counterpart in the formation of microstructure. In such situations it is desirable to find suitable *relaxations*, which allow for the calculation of suitable effective quantities associated with these microstructures. For rate-independent systems this question was first addressed in [60], where the separate relaxation $\mathcal{E}_\infty = \Gamma\text{-}\lim \mathcal{E}$ and $\mathcal{D}_\infty = \Gamma\text{-}\lim \mathcal{D}$, and further developed in [14, 20, 48, 57, 79]. Of course, in the case of a constant sequence the Γ-limit is simply the lower semi-continuous hull.

In [48, 57, 60] it is suggested to study the *approximate incremental problem*

$$(\mathrm{AIP})_{\Pi,\alpha} \qquad \begin{array}{l} \text{Given } q_0 \in \mathcal{Q}, \text{ find iteratively } q_1, q_2, \dots, q_N \text{ such that} \\ \mathcal{E}(t_j, q_j) + \mathcal{D}(q_{j-1}, q_j) \\ \qquad \le (t_j - t_{j-1})\alpha + \inf_{\widetilde{q}\in\mathcal{Q}} \mathcal{E}(t_j, \widetilde{q}) + \mathcal{D}(q_{j-1}, \widetilde{q}). \end{array}$$

For $\alpha > 0$ this problem always has solutions and the question arises as to how the solutions behave for $\alpha \to 0$ and for smaller and smaller time steps.

Choose sequence $(\Pi_k)_{k\in\mathbb{N}}$ and $(\alpha_k)_{k\in\mathbb{N}}$ with $0 < \alpha_k \to 0$ and $\phi(\Pi_k) \to 0$. Then, wor each $k \in \mathbb{N}$ a solution of $(\mathrm{AIP})_{\Pi_k,\alpha_k}$ exists and defines a piecewise constant interpolant $\overline{q}_k : [0,T] \to \mathcal{Q}$. In [57] it is shown under general abstract conditions that the interpolants contain a convergent subsequence in the sense above and that the limit $q : [0,T] \to \mathcal{Q}$ is an energetic solution for the Γ-limit potentials $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$. One simple sufficient condition is that $\mathcal{D}$ is already continuous, which implies $\mathcal{D}_\infty = \mathcal{D}$.

Another abstract relaxation result is derived in [51]. It uses a kind of elliptic regularization of the subdifferential inclusion (13). We consider a sequence of functionals

$$\mathcal{I}_{k,\delta}(q) = \int_0^T \mathbb{E}^{-t/\delta}(\mathcal{R}_k(\dot{q}(t)) + \tfrac{1}{\delta}\mathcal{E}_k(t,q(t)))\,\mathbb{D}t \ ,$$

where again each $\mathcal{R}_k : Q \to [0,\infty]$ is convex, lower semi-continuous and 1- homogeneous. If $\mathcal{R}_k$ and $\mathcal{E}_k$ were smooth, the Euler-Lagrange equation reads

$$\delta \mathrm{D}^2\mathcal{R}_k(\dot{q})[\ddot{q}] = \mathrm{D}\mathcal{R}_k(\dot{q}) + \mathrm{D}_q\mathcal{E}_k(t,q) \ ,$$

which in the formal limit $\delta \to 0$ converges to (13).

Using the 1-homogeneity of $\mathcal{R}_k$ it is proved in [51] that minimizers $q : [0,T] \to \mathcal{Q}$ of $\mathcal{I}_{k,\delta}$ satisfy the δ-independent energy balance

$$\mathcal{E}_k(t,q(t)) + \int_0^t \mathcal{R}_k(\mathbb{D}q) = \mathcal{E}_k(0,q(0)) + \int_0^t \partial_s\mathcal{E}_k(s,q(s))\,\mathbb{D}s \ .$$

As in Sect. 3.2 this implies a priori bounds independent of $\delta > 0$ and of k, if (19) is used. Fixing $\delta > 0$ and letting $k \to \infty$, we obtain a Γ-limit $\mathcal{I}_{\infty,\delta}$ in the form

$$\mathcal{I}_{\infty,\delta}(q) = \int_0^T \mathbb{E}^{-t/\delta}\left(\mathcal{R}_\infty(\dot{q}(t)) + \tfrac{1}{\delta}\mathcal{E}_\infty(t,q(t))\right)\,\mathbb{D}t \ ,$$

if $\mathcal{E}_\infty = \Gamma\text{-}\lim_{k\to\infty}\mathcal{E}_k$ and $\mathcal{R}_k$ converges continuously to $\mathcal{R}_\infty$. Finally, under these assumptions it is shown that for minimizers $q_{k,\delta} : [0,T] \to \mathcal{Q}$ the accumulation points for $k \to \infty$ and $\delta \to 0$ are in fact solutions of the energetic formulation (S) and (E) associated with $\mathcal{E}_\infty$ and $\mathcal{D}_\infty : (q,\widetilde{q}) \mapsto \mathcal{R}_\infty(\widetilde{q} - q)$.

4.3 Numerical Space Discretization

We indicate one of the main applications of the Γ-convergence results. Consider a reflexive Banach space $\mathcal{Q}$ equipped with its weak topology. This space is approximated by a nested sequence $(Q_k)_{k\in\mathbb{N}}$ of finite-dimensional subspaces such that their union is dense, viz., $Q_k \subset Q_{k+1} \subset \mathcal{Q}$ and $\overline{\cup_{k\in\mathbb{N}}Q_k} = \mathcal{Q}$. Finally, assume that the functionals $\mathcal{E}_\infty = [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ is strongly continuous and $\mathcal{D}_\infty : \mathcal{Q}\times\mathcal{Q} \to [0,\infty]$ is weakly continuous in addition to the assumption (2) to (9). Now define the finite-dimensional (space) approximations via

$$\mathcal{E}_k(t,q) = \begin{cases} \mathcal{E}_\infty(t,q) & \text{for } q \in Q_k \ , \\ \infty & \text{otherwise} \ , \end{cases} \qquad \mathcal{D}_k(q,\widetilde{q}) = \begin{cases} \mathcal{D}_\infty(q,\widetilde{q}) & \text{for } q,\widetilde{q} \in Q_k \ , \\ \infty & \text{otherwise} \ . \end{cases}$$

Then it is easy to see that the assumptions (14) to (22) are satisfied. To establish the upper semi-continuity of the stable sets we proceed as follows. Starting from $q_k \in \mathcal{S}_k(t)$ with $q_k \rightharpoonup q$ we need to show $q \in \mathcal{S}_\infty(t)$. For $\widetilde{q} \in \mathcal{Q}$ we choose $\widetilde{q}_k \in Q_k$ with $\widetilde{q}_k \to q$ (strongly). Then we have

$$0 \le \mathcal{E}_k(t, \widetilde{q}_k) + \mathcal{D}_k(q_k, \widetilde{q}_k) - \mathcal{E}_k(t, q_k) = \mathcal{E}(t, \widetilde{q}_k) + \mathcal{D}(q_k, \widetilde{q}_k) - \mathcal{E}(t, q_k) \ .$$

Using strong continuity for $\mathcal{E}$, weak continuity for $\mathcal{D}$ and weak lower semi-continuity for $\mathcal{E}$ we take the limsup of the last expression and find

$$0 \le \limsup_{k\to\infty} \big(\mathcal{E}(t, \widetilde{q}_k) + \mathcal{D}(q_k, \widetilde{q}_k) - \mathcal{E}(t, q_k)\big) \le \mathcal{E}(t, \widetilde{q}) + \mathcal{D}(q, \widetilde{q}) - \mathcal{E}(t, q) \ ,$$

which is the desired stability result, since $\widetilde{q}$ is arbitrary.

This theory is the basis for treating spatial discretizations of energetic formulations. In particular, Theorem 4.2 guarantees that each limit point of the *joint space-time discretization* provides a true solution of (S) and (E).

In [37] this numerical theory was developed for a model involving gradient Young measures to describe microstructures in shape-memory alloys. A more systematic treatment of different aspects of numerical space discretizations as well as penalizations or relaxations is given in [55]. Using more regularity and convexity assumptions full convergence results, without choosing subsequences, are established in [4]. This is closer to the highly developed theory in linearized elastoplasticity, see [1, 29, 31, 78].

5 Applications to Material Models

The theory of rate-independent processes finds applications in many areas. This includes the theory of superconductivity [68, 76], dry friction on surfaces [2, 44, 75], delamination [34], damage [56] and brittle fracture [12, 15, 21]. The latter three areas involve applications where the internal variable is active only on submanifolds of the elastic body. Here we restrict ourselves to those situations where the internal variable z is distributed throughout the body.

The original driving force of this theory was the dynamic problem of linearized elastoplasticity, however, nowadays many other applications occur in phase transformations in shape-memory materials, in magnetostriction, in piezo- or ferroelectricity, and in damage. Finite-strain elastoplasticity is another very active area for rate-independent modeling. This will be surveyed in [46] of this volume.

5.1 Shape-Memory Materials

We consider an elastic body $\Omega \subset \mathbb{R}^d$ in its reference configuration, which we assume to have a Lipschitz boundary. The deformation $\varphi : \Omega \to \mathbb{R}^d$ describes the elastic

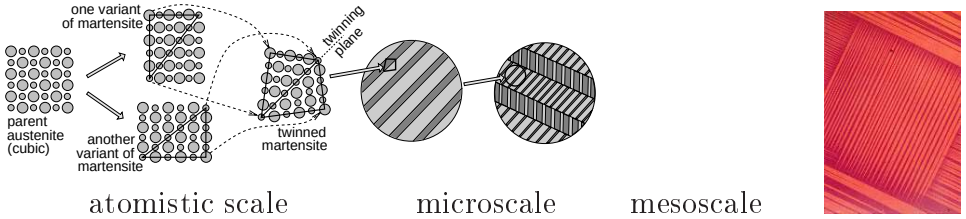

Figure 1: Sketch of the multiscale structure of a sequential laminate in a shape-memory alloy (left to middle). Microscopic view of laminates by Chu and James (right)

behavior in the case of large strains (also called "finite strain"), whereas the displacement $u : \Omega \to \mathbb{R}^d$, $x \mapsto \varphi(x) - x$, is used in the case of small strains (also called "infinitesimal strain"). The elastic properties depend on the strain tensors $\nabla\varphi^\top\nabla\varphi$ and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$, respectively.

Since the shape-memory effect relies on the fact that these materials have several phases, the corresponding stored-energy density (also called stress potential) has a multi-well structure, which is usually given in the form

$$W(x, \nabla\varphi) = \min\{\, W_j(x, \nabla\varphi) \mid j = 1, \dots, N \,\} .$$

Here N is the number of (variants of) phases including the austenite and the martensites. Each $W_j(x, \cdot) : \mathbb{R}^{d\times d} \to [0, \infty]$ is assumed to behave nicely in terms of lower semi-continuity and coercivity.

However, $W(x, \cdot)$ is in general not rank-one convex, and hence formation of microstructures is to be expected. This is compatible with the physics, since the shape-memory effect relies heavily on formation of martensitic laminates (also called "twinning"), see Fig. 1. We refer to [9, 71] for surveys on the mathematical modeling of *microsctructures* in shape-memory alloys.

To describe this mathematically it is advantageous to use gradient Young measures, see [7, 9, 38, 39, 66] for the static case and see [3, 27, 37, 48, 54, 71] for the evolution of microstructures. We will survey this work next. After that we will discuss several models which do not resolve the microstructure but keep certain volume fractions or effective properties, see [23, 24, 25, 42, 58, 60, 77]. Finally, we will indicate how these models may be generalized to include the temperature as an additional external parameter.

5.2 Models Using Gradient Young Measures

A gradient Young measure is a function over the physical domain Ω which takes values in the set of probability measures on the set $\mathbb{R}^{d\times d}$ of deformation gradients, namely

$$\mathrm{Prob}(\mathbb{R}^{d\times d}) := \{\, \mu \in \mathcal{M}(\mathbb{R}^{d\times d}) \mid \mu \geq 0,\ \textstyle\int_{\mathbb{R}^{d\times d}} 1\, \mu(\mathbb{D}A) = 1 \,\} .$$

However, the addition "gradient" means that only those measures are considered, that can be generated via a sequence of gradients of deformations.

We say that a bounded sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)$ generates the gradient Young measure $\mu:\Omega\to\mathrm{Prob}(\mathbb{R}^{d\times d})$ and write $\nabla\varphi_k \overset{\mathrm{YM}}{\to} \mu$, if for all $\psi\in \mathrm{C}^0_{\mathrm{c}}(\Omega\times\mathbb{R}^{d\times d})$ we have, for $k\to\infty$,

$$\int_\Omega \psi(x,\nabla\varphi_k(x))\,\mathbb{D}x \to \int_\Omega\int_{\mathbb{R}^{d\times d}} \psi(x,A)\mu(x,\mathbb{D}A)\,\mathbb{D}x\ .$$

Such μ have the additional property $\int_\Omega\int_{\mathbb{R}^{d\times d}}(1+|A|)^p\mu(x,\mathbb{D}A)\,\mathbb{D}x<\infty$, and we denote the set of all these measures by

$$\mathcal{G}_p(\Omega) = p\text{-integrable gradient Young measures}\ .$$

To model the hysteretic behavior in shape-memory materials with the energetic formulation discussed in Sect. 3 we need to introduce a phase indicator $z:\Omega\to Z_N$ where Z_N is usually taken as the Gibbs simplex

$$Z_N = \{\, z\in\mathbb{R}^N \mid z_i\geq 0,\ \textstyle\sum_{i=1}^N z_i = 1\,\}\ .$$

The components z_i of $z\in Z_N$ measure the volume fraction of phase i in a representative volume element. For gradient Young measures we extract the phase fractions via a continuous mapping

$$\zeta:\Omega\times\mathbb{R}^{d\times d}\to Z_N$$

such that $\zeta(x,A)=e_j$ (unit vector in $\mathbb{R}^N$), if $W(x,A)=W_j(x,A)\leq W_k(x,A)-\delta$ for $k\neq j$. Here $\delta>0$ is a suitable constant which is assumed to be much smaller than the depth of the wells.

Finally we introduce a dissipation distance $D: Z_N\times Z_N\to[0,\infty)$. It suffices to prescribe the values $\kappa_{j\to k}=D(e_j,e_k)>0$, such that the triangle inequality holds, i.e., $\kappa_{j\to\ell}\leq\kappa_{j\to k}+\kappa_{k\to\ell}$. Here $\kappa_{j\to k}$ denotes the energetic loss when the material jumps from a phase e_j into another phase e_k. Then, $D: Z_N\times Z_N\to[0,\infty)$ is defined via the optimal transport problem

$$D(z,\widetilde{z}) = \min\Big\{\ \sum_{j,k=1}^N m_{jk}\kappa_{j\to k}\ \Big|\ m_{jk}\geq 0,\ \sum_{k=1}^N m_{jk}e_j=\widetilde{z},\ \sum_{j=1}^N m_{jk}e_k = z\ \Big\}\ .$$

It is shown in Proposition 4.7 in [60], that there exists a convex, 1-homogeneous $R:\mathbb{R}^N\to[0,\infty)$ such that $D(z,\widetilde{z})=R(\widetilde{z}-z)$.

With these notations we now formulate the function spaces and the functionals. We assume that $\Omega\subset\mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and that $\Gamma_{\mathrm{Dir}}\subset\partial\Omega$ is a set of positive surface measure on which we describe Dirichlet boundary data. We let

$$\mathcal{F} = \{\,\varphi\in\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)\mid \varphi|_{\Gamma_{\mathrm{Dir}}}=\Phi_{\mathrm{Dir}}\,\}\times\mathcal{G}_p(\Omega) \text{ and } \mathcal{Z}=\mathrm{L}^1(\Omega,Z_N)\ .$$

The state space is $\mathcal{Q}=\mathcal{F}\times\mathcal{Z}$ and a state consists of a triple $q=(\varphi,\mu,z)$. We further let $\mathcal{Q}_0=\{\,(\varphi,\mu,z)\in\mathcal{Q}\mid \nabla\varphi=\mathrm{id}\bullet\mu,\ z=\zeta\bullet\mu\,\}$, where "$\bullet$" denotes the

contraction over $A \in \mathbb{R}^{d\times d}$ but not over $x \in \Omega$, i.e., $(\mathrm{id}\bullet\mu)(x) = \int_{\mathbb{R}^{d\times d}} A\,\mu(x,\mathbb{D}A)$ and $(\zeta\bullet\mu)(x) = \int_{\mathrm{R}^{d\times d}} \zeta(x,A)\mu(x,\mathbb{D}A)$. With the prescribed external volume and surface loadings

$$\langle \ell(t),\varphi\rangle = \int_\Omega f_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathbb{D}x + \int_{\Gamma_{\mathrm{Neu}}} g_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathbb{D}a(x)$$

we define the energy-storage functional

$$\mathcal{E}(t,q) = \int\limits_\Omega \int\limits_{\mathbb{R}^{d\times d}} W(x,A)\mu(x,\mathbb{D}A) + \tfrac{\rho}{2}|\nabla^\alpha z|^2\,\mathbb{D}x - \langle \ell(t),q\rangle \text{ for } q\in\mathcal{Q}_0 \tag{26}$$

and $\mathcal{E}(t,q) = +\infty$ for $q = (\varphi,\mu,z) \in \mathcal{Q}\backslash\mathcal{Q}_0$. Here $\rho > 0$ and $\nabla^\alpha z$, $\alpha > 0$, denotes a (fractional) derivative, for instance for $\alpha \in (0,1)$ we have

$$\int_\Omega |\nabla^\alpha z|^2\,\mathbb{D}x = \int_\Omega\int_\Omega \tfrac{|z(x)-z(\widetilde{x})|^2}{|x-\widetilde{x}|^{d+2\alpha}}\,\mathbb{D}x\,\mathbb{D}\widetilde{x}\ .$$

This regularizing term allows us to choose the strong topology in $\mathrm{L}^1(\Omega,Z_N)$ as this space is compactly embedded into $\mathrm{W}^{\alpha,2}(\Omega,\mathbb{R}^N)$ for $\alpha > 0$. Nevertheless, for $\alpha < 1/2$ the functions in $\mathrm{W}^{\alpha,2}(\Omega,\mathbb{R}^N)$ may have jumps along smooth interfaces such as the habit plane between austenite and martensite.

The dissipation distance is defined as

$$\mathcal{D}(z,\widetilde{z}) = \int_\Omega D(x,z(x),\widetilde{z}(x))\,\mathbb{D}x = \int_\Omega R(x,\widetilde{z}(x)-z(x))\,\mathbb{D}x\ . \tag{27}$$

Since $\mathcal{D}$ is (strongly) continuous on $\mathcal{Z}$ the crucial closedness condition (9) of the stable sets is easily obtained via Lemma 3.1 by taking $\widetilde{q}_k = \widetilde{q}$.

The following existence theorem is established in [37]. The earlier version in [54] was based on the much stronger assumption that $\mathcal{E}(t,(\cdot,\cdot,z))$ has a unique minimizer, but this condition is no longer needed because of the abstract developments in [22, 49].

Theorem 5.1. *Let $p \in (1,\infty)$, $\alpha \in (0,1)$, and $\rho > 0$. Assume that there exists $C > 0$ such that for $j = 1,\dots,N$, we have*

$$\forall\, A\in\mathbb{R}^{d\times d}:\ \tfrac{1}{C}|A|^P - C \le W_j(x,A) \le C|A|^P + C\ . \tag{28}$$

Further assume $\ell \in \mathrm{C}^1([0,T],\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)^*)$ and that $q_0 \in \mathcal{S}(0)$. Then, the energetic formulation (S) and (E) associated with $\mathcal{E}$ and $\mathcal{D}$ from (26) *and* (27)*, respectively, has a solution $q : [0,T] \to \mathcal{Q}_0$.*

The theory of Γ-convergence discussed in Sect. 4.1 can also be used to show that space-time discretizations of the energetic formulation contain subsequences which converge to energetic solutions. For this we use triangulations $\mathcal{T}_h$ of Ω. Moreover, we approximate gradient Young measures by sequential laminates of order $\kappa \in \mathbb{N}$, see [3, 8, 38, 72] and Sect. 5.4 in [48] for an introduction. Fixing $\kappa \in \mathbb{N}$ we define $\mathcal{Q}^h_\kappa$ as the space of functions $q = (\varphi,\mu,z) \in \mathcal{Q}$ for which $\nabla\varphi, \mu$ and z are constant on each simplex and μ is a laminate of order at most κ. Using the penalization parameter $\varepsilon > 0$ we let

$$\mathcal{E}_{h,\varepsilon}(t,q) = \mathcal{E}(t,q) + \tfrac{1}{\varepsilon}\int_\Omega |z-\zeta\bullet\mu|^2\,\mathbb{D}x \text{ for } q\in\mathcal{Q}^n_\kappa \text{ with } \nabla\varphi = \mathrm{id}\bullet\mu$$

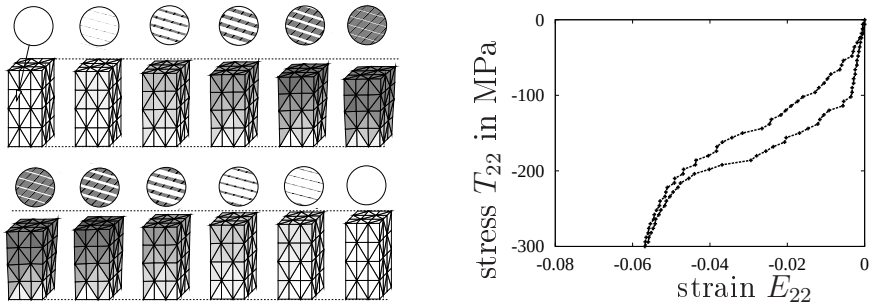

Figure 2: Left: 12 steps of the compression cycle. The greyscale indicates the volume fraction of martensite in each element. Additionally, in the discs the microstructure on the indicated element is indicated (white=austenite, grey=martensite2, black=martensite3). Right: hysteresis in the overall stress-strain relation

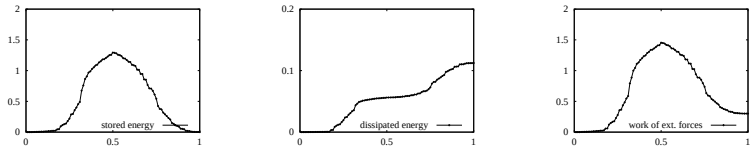

Figure 3: Energies (in Joule) during the cycle. Left: stored energy $\mathcal{E}(t, q(t))$, middle: dissipated energy $\mathrm{Diss}_{\mathcal{D}}(q, [0, t])$ and right: work of external forces $\int_0^t \partial_s \mathcal{E}(s, q(s))\, \mathbb{D}s$

and $\mathcal{E}_{h,\varepsilon}(t, q) = +\infty$ otherwise on $\mathcal{Q}$.

In [37] a function $H : (0, \infty) \to (0, \infty)$ with $H(\varepsilon) \to 0$ for $\varepsilon \searrow 0$ is constructed such that the following holds: *If* $(\Pi_k)_{k\in\mathbb{N}}$ *is a sequence of partitions of* $[0, T]$ *with* $\phi(\Pi_k) \to 0$, *if* $(\mathcal{T}_{h_k})$ *is a sequence of triangulations of* Ω *and if* $\varepsilon_k \to 0$ *with* $h_k \leq H(\varepsilon_k)$, *then the incremental solutions* $\overline{q}_k$ *associated with* $(\mathcal{E}_{h_k,\varepsilon_k}, \mathcal{D})$ *on* $\mathcal{Q}^h_\kappa$ *have a subsequence which converges to an energetic solution* $q : [0, T] \to \mathcal{Q}$ *for* $(\mathcal{E}, \mathcal{D})$.

Figures 2 and 3 show results from a numerical simulation from [37] for a sample of 4×4×9 mm single-crystal alloy of CuAlNi. It has a cubic-to-orthorhombic phase transition with one austenite and 6 variants of a martensite (i.e., $N = 7$).

All phases are modelled by a Saint-Venant-Kirchhoff material $W_j(x, A) = \frac{1}{2}(A^\top A - C_j) : \mathbb{C}_j : (A^\top A - C_j) + d_j$, where $C_j \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$, $\mathbb{C}_j \in \mathrm{Lin}(\mathbb{R}^{3\times3}_{\mathrm{sym}})$, and d_j are the experimentally measured values for each $j \in \{1, \ldots, 7\}$ at a fixed temperature of 312 K. The dissipation constants $D(e_j, e_\ell)$ are chosen to be 0.5 MPa for transformations between martensite and austenite or vice versa. Transformations between different variants of austenite are assumed to have much lower dissipation thresholds.

The discretization involves 180 tetrahedrons and second-order laminates. This leads to 20 degrees of freedom in each element, which lie in a nonlinear manifold with boundary (box constraints). The minimization technique for solving the highly nonconvex incremental problem is described in Sect. 6.3 in [37].

A microscopic model that does not allow for microstructure and uses only pure phases is developed and analyzed in [41, 42]. It is based on the usage of an interface

energy that is proportional to the surface of the interfaces. We denote the set of pure phases by $P_N = \{e_1, \dots, e_N\} \subset Z_N \in \mathbb{R}^N$ and let $\mathcal{Z}_{\text{pure}} = \mathrm{L}^1(\Omega, P_N)$ equipped with the strong L^1-topology. The space $\mathcal{F}$ of admissible deformations remains as above, whereas the energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ takes the form

$$\mathcal{E}(t,\varphi,z) = \int_\Omega W_{j(x)}(x, \nabla\varphi(x))\, \mathbb{D}x + \rho \int_\Omega |\mathrm{D}z| - \langle \ell(t), \varphi \rangle \ ,$$

where $j(x) = k \ \Leftrightarrow \ z(x) = e_k$ and where $\int_\Omega |\mathrm{d}z|$ denotes the total variation

$$\int_\Omega |\mathrm{D}z| \stackrel{\text{def}}{=} \sup\{ \int_\Omega z \cdot \mathrm{div}\psi\, \mathbb{D}x \mid \psi \in \mathrm{C}^1(\Omega, \mathbb{R}^{n\times d}),\ |||\psi(x)||| \le 1 \text{ on } \Omega \} \ .$$

The norm $|||\cdot|||$ on $\mathbb{R}^{N\times d}$ can be adjusted to anisotropies in $\Omega \subset \mathbb{R}^d$ and to different weights for the interfaces between phases j and k, see [41, 42] for the details and for generalizations.

Using the same dissipation distance $\mathcal{D}$ as above an existence theory as in Theorem 5.1 can be derived, since $\mathrm{BV}(\Omega)$ embeds compactly into $\mathrm{L}^1(\Omega)$. The solution $q = (\varphi, z) : [0,T] \to \mathcal{Q}$ now satisfies

$$\begin{aligned} &\varphi \in \mathrm{L}^\infty([0,T], \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)) \text{ and} \\ &z \in \mathrm{BV}([0,T], \mathrm{L}^1(\Omega, P_N)) \cap \mathrm{L}^\infty_{\text{weak}}([0,T], \mathrm{BV}(\Omega, \mathbb{R}^N)) \ . \end{aligned}$$

5.3 Mesoscopic Models

Often it is not desirable or prohibitly costly to calculate the evolution of the microstructure during the hysteretic evolution process. If these details are not needed and if volume fractions or other effective quantities are sufficient, then simpler models may be used.

If we only care about volume fractions, then the *mixture function* can be used to describe the effective behavior of phase mixtures. Let $W_j(x,\cdot)$ be given as above for $j = 1, \dots, N$. For $z \in Z_N$ and $A \in \mathbb{R}^{d\times d}$ we let

$$\mathbb{W}(x,A,z) = \inf \Big\{ \int\limits_{(0,1)^d} W_{J(y)}(x, A + \nabla\psi(y))\, \mathbb{D}y \ \Big|\ J \in \mathrm{L}^1((0,1)^d, \{1,...,N\}), \\ \int\limits_{(0,1)^d} e_{J(y)}\, \mathbb{D}y = z,\ \psi \in \mathrm{W}^{1,\infty}_0((0,1)^d, \mathbb{R}^d) \Big\}$$

where $(0,1)^d$ is a microscopic representative volume element, J a microscopic phase indicator, and $\nabla\psi$ microscopic fluctuation of the gradient. In [40] $\mathbb{W}$ is also called *cross-quasiconvexification* and in [25] the *free energy of mixing*.

Unfortunately, in general situations it is almost impossible to calculate $\mathbb{W}$ explicitly. Nevertheless $\mathbb{W}$ is cross-quasiconvex and hence, for each $x \in \Omega$ and $A \in \mathbb{R}^{d\times d}$, the function $\mathbb{W}(x,A,\cdot) : Z_N \to [0,\infty)$ is convex and, for each $x \in \Omega$ and $z \in Z_N$, the function $\mathbb{W}(x,\cdot,z) : \mathbb{R}^{d\times d} \to [0,\infty)$ is quasiconvex. Explicit formulas are only

available in dimension $d = 1$ or if each W_j is quadratic with an elastic tensor independent of j, viz.,

$$W_j(x, A) = \tfrac{1}{2}(\varepsilon(A) - \varepsilon_j(x)) : \mathbb{C}(x) : (\varepsilon(A) - \varepsilon_j(x)) + d_j(x)$$

where $\varepsilon(A) = \frac{1}{2}(A + A^\top - 2I)$. Then,

$$\mathbb{W}(x, A, z) = \textstyle\sum_{j=1}^N z_j W_j(x, A) + w_{\mathrm{mix}}(x, z) \ ,$$

where $w_{\mathrm{mix}}(x, e_k) = 0$ and $w_{\mathrm{mix}}(x, \cdot) : Z_N \to \mathbb{R}$ is convex. See [25, 47] for cases where w_{mix} can be calculated or estimated efficiently.

The advantage of the mixture theory is that we are not forced to work with quasiconvexity. We are able to use polyconvexity as well. Hence it is possible to use energy densities that take the value $+\infty$, as for instance in finite-strain elasticity where $W(x, A) = +\infty$ for $\det A \leq 0$. Instead of cross-quasiconvexity we may use cross-polyconvexity, namely $\mathbb{W}(x, \cdot, \cdot) : \mathbb{R}^{d\times d} \times Z_N \to [0, \infty]$ is called *cross-polyconvex*, if there exists a function $g(x, \cdot) : \mathbb{R}^{m_d+N} \to [0, \infty]$ that is convex, lower semi-continuous and satisfies

$$\mathbb{W}(x, A, z) = g(x, \mathcal{M}(A), z) \ ,$$

where $\mathcal{M}(A) \in \mathbb{R}^{m_d}$ is the set of all minors.

We now define the state space $\mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ for classical functions φ only, namely

$$\mathcal{F} = \{\, \varphi \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d) \mid \varphi|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id} \,\} \subset \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)$$

equipped with the weak topology. The stored-energy functional takes the form

$$\mathcal{E}(t, \varphi, z) = \textstyle\int_\Omega \mathbb{W}(x, \nabla\varphi(x), z(x)) + \frac{\rho}{2}|\nabla^\alpha z|^2 \,\mathbb{D}x - \langle \ell(t), \varphi\rangle \ . \tag{29}$$

For $\rho > 0$ and $\alpha > 0$ we take $\mathcal{Z} = \mathrm{L}^1(\Omega, Z_N)$ equipped with the strong topology. Under suitable coercivity and (poly)quasiconvexity assumptions on $\mathbb{W}(x, \cdot, z)$ it can then be shown that the sublevels of $\mathcal{E}(t, \cdot)$ are compact in $\mathcal{Q}$, which is our basic condition (5). In the case $\rho = 0$, this is more difficult, since $\mathcal{Z}$ then has to be equipped with the weak topology. Then, cross-(poly)quasiconvexity is necessary for weak lower semi-continuity of $\mathcal{E}$. However, for the case without regularization ($\rho = 0$) the best we can hope for is that solutions for the incremental problem $(\mathrm{IP})_\Pi$ exist. The passage to the limit of vanishing time incrementals strongly relies on the closedness condition (9) for the stable sets which, so far, cannot be established in cases without regularization.

The following result is a slight variant of the existence results in [22, 41, 49].

Theorem 5.2. *Let $p \in (1, \infty)$, $\alpha, \rho > 0$ and $\ell \in \mathrm{C}^1([0, T], \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)^*)$. Moreover, let $\mathcal{D}$ be given as in Sect. 5.2 and assume that $\mathcal{E}$ in (29) has compact sublevels in $\mathcal{Q} \subset \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)_{\mathrm{weak}} \times \mathrm{L}^1(\Omega, Z_N)_{\mathrm{strong}}$. Then, for each stable initial state $q_0 = (\varphi_0, z_0) \in \mathcal{Q}$ there exists an energetic solution $q = (\varphi, z) : [0, T] \to \mathcal{Q}$ for $(\mathcal{E}, \mathcal{D})$ with $\varphi \in \mathrm{L}^\infty([0, T], \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d))$ and $z \in \mathrm{BV}([0, T], \mathrm{L}^1(\Omega, Z_N)) \cap \mathrm{L}^\infty([0, T], \mathrm{H}^\alpha(\Omega, \mathbb{R}^N))$.*

The associated numerical convergence results are discussed in [55]. But all the above models have the disadvantage that the solutions are not unique. Hence, it is not possible to show that numerical solutions converge.

The next model goes back to [77] and was further developed in [4, 5]. This model is based on the linearized strain tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ and the mesoscopic transformation strain $z \in Z = \{ A \in \mathbb{R}^{d\times d} \mid A = A^\top, \operatorname{tr} A = 0 \}$. The dissipation is simply a multiple of the L^1-norm:

$$\mathcal{D}(z,\widetilde{z}) = \mathcal{R}(\widetilde{z}-z) = \int_\Omega c_d|\widetilde{z}(x)-z(x)|\,\mathbb{D}x \ .$$

The energy functional takes the form

$$\mathcal{E}(t,u,z) = \int_\Omega W(x,\varepsilon(u),z) + h(|z|) + \tfrac{\rho}{2}|\nabla^\alpha z|^2\,\mathbb{D}x - \langle \ell(t),u\rangle \ ,$$

with $W(x,\varepsilon,z) = \frac{1}{2}(\varepsilon-z){:}\mathbb{C}{:}(\varepsilon-z)$. Again, the classical model has no regularization, i.e., $\rho = 0$. The hardening function $h : [0,\infty) \to [0,\infty]$ equals

$$h(r) = \begin{cases} c_1\sqrt{\delta^2+r^2} + c_2 r^2 & \text{for } r \in [0,r_*]\,, \\ \infty & \text{otherwise}\ , \end{cases}$$

in [5] and has $\delta = 0$ in [77]. In these cases it is easy to solve the incremental problems $(\mathrm{IP})_\Pi$ in the space $\mathcal{Q} = \mathrm{H}^1(\Omega,\mathbb{R}^d) \times \mathrm{H}^\alpha(\Omega,Z)$, for all $\alpha \geq 0$. However, for obtaining energetic solutions we again need ρ and α strictly positive, to make $\mathcal{D}$ weakly continuous on $\mathcal{Z} = \mathrm{H}^\alpha(\Omega,Z)$.

A further variation is considered in [4], where h is replaced by a smooth, convex function taking finite values and growing at most quadratically, e.g.,

$$h(r) = c_1\sqrt{\delta^2+r^2} + c_2 r^2 + \tfrac{c_3}{\delta}\tfrac{\max\{0,r-r_*\}^4}{r_*^2+r^2} \ .$$

Then, for $\alpha \geq d/6$ it can be shown that $\mathcal{E}(t,\cdot) : \mathrm{H}^1(\Omega,\mathbb{R}^d) \times \mathrm{H}^\alpha(\Omega,Z) \to \mathbb{R}$ is three-times differentiable and uniformly convex. Hence, the theory of Sect. 7 in [59] is applicable. This allows us to conclude uniqueness of the solutions as well as strong convergence of the solutions of the incremental problem. In fact, the convergence rate is $(\phi(\Pi_k))^{1/2}$. In [4] also the convergence of spatial discretization will be discussed.

5.4 Temperature-Induced Phase Transformation

The original shape-memory effect is based on cooling and heating to switch between martensite occurring in several variants and the single austenite phase. So far the energetic formulation is only available for the isothermal case and thus is suited for stress-induced phase transformations only.

There is at least one nonisothermal case that can be treated via the energetic formulation as well, namely if the temperature field is given a priori independent of the

solution to be calculated. This means that the deformation and phase transformation process is so slow that all latent heat which is either consumed or generated via phase transformation can be transported via heat conduction into the environment.

Thus, our model is based on a temperature dependent stored-energy density $W(x, A, z, \theta)$ which is assumed to satisfy

$$\begin{aligned} &\exists\, c_3^W \in \mathbb{R}\ \exists\, c_4^W > 0\ \forall\, x \in \Omega\ \forall\, A \in \mathbb{R}^{d\times d}\ \forall\, z \in Z_N\ \forall\, \theta > 0 : \\ &|\partial_\theta W(x, A, z, \theta)| \le c_4^W (W(x, A, z, \theta) + c_3^W)\ . \end{aligned} \tag{30}$$

The given temperature profile θ should satisfy $(\log \theta) \in \mathrm{C}^1([0, T] \times \overline{\Omega})$, then the energy potential, which for simplicity is now without external forcing, takes the form

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W(x, \nabla\varphi(x), z(x), \theta(t, x))\, \mathbb{D}x$$

and the power associated to the temperature changes is

$$\partial_t \mathcal{E}(t, \varphi, z) = \int_\Omega \partial_\theta W(x, \nabla\varphi(x), z(x), \theta(t, x)) \partial_t\theta(t, x)\, \mathbb{D}x\ .$$

Using (30) it is easy to establish the condition (6) and, under suitable additional assumptions, the conditions (7) and (8) hold as well. In [50] we will provide the detailed assumptions for a full existence theory.

5.5 Poling Induced Piezoelectricity

Multifunctional materials derive their functionality from the combination of several properties such as elasticity, polarizability, and magnetizability. For such materials the polarization p or the magnetization m may be considered as the variable z used above. However, in addition we have to take the relevant version of the Maxwell equation into account.

In the quasi-static setting either the electric or the magnetic field vanishes such that we obtain two clearly distinguished cases, which are dual in a certain sense. Throughout we will restrict to the case of small strains, since otherwise the Maxwell equations have to be solved in the deformed configuraton, see the references at the beginning of Sect. 5.6.

The electric field E and the dielectric displacement D are defined on all of $\mathbb{R}^d$ whereas the polarization $P : \Omega \to \mathbb{R}^d$ on the body only. These fields are related by the constitutive relation

$$D = \varepsilon_0 E + P \text{ in } \Omega \quad \text{and} \quad D = \varepsilon_0 E \text{ in } \mathbb{R}^d \backslash \Omega\ .$$

The reduced Maxwell equations are

$$\mathrm{div} D = 0 \text{ and } \mathrm{curl}(E - E_{\mathrm{ext}}(t, \cdot)) = 0 \text{ in } \mathbb{R}^d\ , \tag{31}$$

where $\mathrm{curl}\, \widetilde{E} = \nabla\widetilde{E} - (\nabla\widetilde{E})^\top$. We will implement these equations as part of the energetic formulation.

We consider the displacement $u : \Omega \to \mathbb{R}^d$ and the dielectric displacement D as variables in the space

$$\begin{aligned} \mathcal{F} &= \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega, \mathbb{R}^d) \times \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d, \mathbb{R}^d) \\ \text{with } \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d, \mathbb{R}^d) &= \{\, D \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{R}^d) \mid \mathrm{div} D = 0 \,\} \,. \end{aligned}$$

The internal variable $p \in \mathcal{Z} = H^1(\Omega, \mathbb{R}^d)$ is the remanent polarization. For $q = (u, D, p) \in \mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ and $t \in [0, T]$ the energy potential $\mathcal{E}$ is defined via

$$\begin{aligned} \mathcal{E}(t, q) = \textstyle\int_\Omega W(x, \varepsilon(u), p) - \frac{1}{\varepsilon_0} D \cdot P(x, \varepsilon(u), p) + \frac{\rho}{2} |\nabla p|^2 \, \mathbb{D}x \\ + \textstyle\int_{\mathbb{R}^d} \frac{1}{2\varepsilon_0} |D|^2 \, \mathbb{D}x - \langle \ell(t), (u, D) \rangle \end{aligned}$$

where the external forcing occurs via mechanical volume and surface loadings and via an external electric field

$$\langle \ell(t), (u, D) \rangle = \textstyle\int_\Omega f_{\mathrm{ext}}(t) \cdot u \, \mathbb{D}x + \int_{\Gamma_{\mathrm{Neu}}} g_{\mathrm{ext}}(t) \cdot u \, \mathbb{D}a + \int_{\mathbb{R}^d} E_{\mathrm{ext}}(t) \cdot D \, \mathbb{D}x \,.$$

The electric field is the dual variable to the dielectric displacement D, i.e.,

$$E = \tfrac{1}{\varepsilon_0}(D - P(x, \varepsilon(u), p)) \text{ in } \Omega \quad \text{and} \quad E = \tfrac{1}{\varepsilon_0} D \text{ in } \mathbb{R}^d \backslash \Omega \,. \tag{32}$$

The polarization is given as a constitutive function and poling induced piezoelectricity means that the piezoelectric tensor $\partial_\varepsilon P$ does not vanish.

Following [32, 70] the dissipation distance is the Legendre transform of the so-called switching function, namely

$$\mathcal{D}(p, \widetilde{p}) = \mathcal{R}(\widetilde{p} - p) = \textstyle\int_\Omega R(x, \widetilde{p}(x) - p(x)) \, \mathbb{D}x$$

for some Caratheodory function $R : \Omega \times \mathbb{R}^d \to [0, \infty)$ with $R(x, \cdot)$ being convex and 1-homogeneous. Under the assumption that $W(x, \cdot, \cdot, p) : \mathbb{R}^{d \times d}_{\mathrm{sym}} \times \mathbb{R}^d \to \mathbb{R}$ is convex and that W satisfies suitable upper and lower bounds, it is now straight forward to prove the existence of energetic solutions $(u, D, p) : [0, T] \to \mathcal{Q}$ with $(u, D) \in \mathrm{L}^\infty([0, T], \mathcal{F})$ and $p \in \mathrm{BV}([0, T], \mathrm{L}^1(\Omega, \mathbb{R}^d)) \cap \mathrm{L}^\infty([0, T], H^1(\Omega, \mathbb{R}^d))$.

To see the compatibility with the Maxwell equations (31) we note that the stability condition (S) implies that for all $t \in [0, T]$ we have

$$\mathrm{D}_D \mathcal{E}(t, u(t), D(t), p(t))[\widehat{D}] = 0 \text{ for all } \widehat{D} \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d, \mathbb{R}^d) \,.$$

In Proposition 2.1 of [61] it is shown that the latter relation is equivalent to the Maxwell equations (31), if the definition (32) is used.

Moreover, in that work additional conditions are discussed which imply also uniqueness of solutions. For this the uniqueness theory of Sect. 7 in [59] is employed. However, the resulting conditions seem very restrictive.

5.6 Magnetostrictive Materials

We summarize the theory of [17] which is based on small-strain elasticity, see also [55]. For the much more complicated constitutive theory in the case of finite-strain elasticity we refer to [16, 30] and for some analysis for the static problem with second-order regularization of the deformation we refer to [74]. For small strain-models including microstructure via Young measure (like in Sect. 5.2) we refer to [72, 73].

In analogy to the case of polarizable materials we use the magnetization $m : \Omega \to \mathbb{R}^d$ as an internal variable. Usually the saturation assumption $|m(x)| = m_{\rm sat} > 0$ is added which we impose by letting $Z = \{\, m \in \mathbb{R}^d \mid |m| = m_{\rm sat} \,\}$. The magnetic induction $B : \mathbb{R}^d \to \mathbb{R}^d$ and the magnetic field $H : \mathbb{R}^d \to \mathbb{R}^d$ are related via the constitutive law

$$B = \mu_0(H + m) \text{ in } \Omega \quad \text{and} \quad B = \mu_0 H \text{ in } \mathbb{R}^d \backslash \Omega \,.$$

In this quasistatic setting Maxwell's equation reduces to

$$\operatorname{div} B = 0 \quad \text{and} \quad \operatorname{curl} H = 0 \text{ in } \mathbb{R}^d \,. \tag{33}$$

We choose $\mathcal{F} = \mathrm{H}^1_{\Gamma_{\rm Dir}}(\Omega, \mathbb{R}^d) \times \mathrm{L}^2_{\rm div}(\mathbb{R}^d, \mathbb{R}^d)$ equipped with the weak topology and $\mathcal{Z} = \mathrm{L}^1(\Omega, Z)$ with the strong topology. The energy potential reads

$$\begin{aligned}\mathcal{E}(t, u, B, m) = {} & \textstyle\int_\Omega W(x, \varepsilon(u), m) - B\cdot m + \frac{\rho}{2}|\nabla m|^2 \,\mathbb{D}x \\ & + \textstyle\int_{\mathbb{R}^d} \frac{1}{2\mu_0}|B|^2 \,\mathbb{D}x - \langle \ell(t), (u, B)\rangle\end{aligned}$$

with an external forcing of the form

$$\langle \ell(t), (u, B)\rangle = \textstyle\int_\Omega f_{\rm ext}(t)\cdot u \,\mathbb{D}x + \int_{\Gamma_{\rm Neu}} g_{\rm ext}(t)\cdot u \,\mathbb{D}a + \int_{\mathbb{R}^d} H_{\rm ext}(t)\cdot B \,\mathbb{D}x \,.$$

The parameter $\sqrt{\rho}$ relates to the exchange length, which determines the scalings for the width of domain walls. The dissipation distance may be chosen via an arbitrary distance $D(x, \cdot, \cdot)$ on $Z = m_{\rm sat}\mathbb{S}^{d-1}$, e.g.,

$$D(x, m, \widetilde{m}) = c_1 \arccos\left(\tfrac{m\cdot\widetilde{m}}{m_{\rm sat}^2}\right) + c_2|\widehat{e}\cdot(m - \widetilde{m})|$$

where $\widehat{e}$ is an "easy" axis and $c_2 = 0$ in the isotropic case. We let $\mathcal{D}(m, \widetilde{m}) = \int_\Omega D(x, m(x), \widetilde{m}(x)) \,\mathbb{D}x$.

Using the standard coercivity assumptions on $W : \Omega \times \mathbb{R}^{d\times d}_{\rm sym} \times Z \to [0, \infty)$, convexity in $\varepsilon(u)$ and continuity in $m \in Z$ it is standard to show that $\mathcal{E}(t, \cdot) : \mathcal{Q} = \mathcal{F} \times \mathcal{Z} \to \mathbb{R}$ is lower semi-continuous with compact sublevels. Moreover $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is continuous in the strong L^1-topology (or in the weak H^1-topology). Thus, existence of energetic solutions for $(\mathcal{E}, \mathcal{D})$ can be easily obtained from Theorem 3.2.

Since the magnetic field H is the dual variable to B,

$$\mathrm{D}_B\mathcal{E}(t, u(t), B(t), m(t))[\widehat{B}] = 0 \text{ for all } \widehat{B} \in \mathrm{L}^2_{\rm div}(\mathbb{R}^d, \mathbb{R}^d)$$

is equivalent to (33) in the form

$$\mathrm{div} B = 0 \ \text{ and } \ \mathrm{curl}(\tfrac{1}{\mu_0} B - H_{\mathrm{ext}} - \chi_\Omega m) = 0 \ \text{ in } \mathbb{R}^d \,.$$

It is more common to formulate the problem of magnetostriction in terms of the potential U of the magnetic field H, i.e., $H = \nabla U$. In the above formulation we may then replace B via

$$B = \mu_0(\nabla U + H_{\mathrm{ext}}(t) + \chi_\Omega m) \tag{34}$$

in the energy $\mathcal{E}$ to arrive at

$$\begin{aligned}\widetilde{\mathcal{E}}(t,u,U,m) = {} & \textstyle\int_\Omega W(x,\varepsilon(u),m) - \frac{\mu_0}{2}|m|^2 - m\cdot H_{\mathrm{ext}}(t)\,\mathbb{D}x \\ & + \textstyle\int_{\mathbb{R}^d} \frac{\mu_0}{2}|\nabla U|^2 - \frac{\mu_0}{2}|H_{\mathrm{ext}}(t)|^2\,\mathbb{D}x - \langle \ell_{\mathrm{mech}}(t), u\rangle \,.\end{aligned}$$

Note that the Euler-Lagrange equation for U does not supply the desired Maxwell equation

$$\mathrm{div}(\nabla U + H_{\mathrm{ext}}(t,\cdot) + \chi_\Omega m) = 0 \ \text{ in } \mathbb{R}^d \,. \tag{35}$$

Thus, to derive an energetic formulation in this situation the variable U has to be taken as a function of $m \in L^1(\Omega, Z)$ and $t \in [0,T]$ via $H_{\mathrm{ext}}(t,\cdot)$, namely $U = \mathcal{U}(t,m)$ being the solution of (35).

Instead of simply replacing B by the corresponding variable, we might as well perform a partial Legendre transform such that $\widetilde{W}(f,x,\varepsilon(u),B,m)$ is replaced by

$$\widehat{W}(t,x,\varepsilon,H,m) = \widetilde{W}(t,x,\varepsilon,B,m) - B\cdot\partial_B\widetilde{W}(t,x,\varepsilon,B,m)$$

where B is again eliminated using (34). The corresponding energy $\widehat{\mathcal{E}}$ then contains the negative definite term $-\int_{\mathbb{R}^d} \frac{\mu_0}{2}|\nabla U|^2\,\mathbb{D}x$. Thus, we may use $\mathrm{D}_U\widehat{\mathcal{E}}(t,u,U,m)[\widehat{U}] = 0$ to obtain (35), but the saddle point structure of $\widehat{\mathcal{E}}$ does not allow us to introduce a stability condition in terms of (u,U,m). Thus, it is not possible to derive an energetic formulation either.

6 Conclusions

The energetic formulation of rate-independent processes was developed much further via the abstract approaches described in [22, 43, 49]. The major improvement occurred through finding abstract versions of the ideas in [15] for treating a rate-independent model for crack growth. Now it is possible to deal with problems where the energy $\mathcal{E}(t,\cdot,z) : \mathcal{F} \to \mathbb{R}_\infty$ is non-convex. In general, the abstract theory is available in topological spaces without any linear structure. Thus, it is possible to treat finite-strain elasticity (cf. [22, 33]) as well as internal variables which lie in general nonconvex sets such as in magnetism (cf. Sect. 5.6) or in finite-strain plasticity, see [46]. Moreover, it is possible to include Young measure into the state space as well [37, 54].

Further developments include the abstract theory of Γ-convergence and relaxations of the energetic formulation. This allows us, for instance, to treat numerical approximations, see [4, 37, 55]. However, the numerical analysis and efficient simulations still need a lot of further developments.

The major drawback of the energetic formulation is that there are only very few results on the uniqueness of solutions, see [10, 52, 59]. Another deficiency concerns the fact that the stability condition (S) involves a global stability condition. For a better physical modeling and for numerical implementation it would be desirable to replace this condition by a suitable local stability condition. First attempts are given in [18, 53], but a reasonable general theory is not yet developed. This is closely to the general problem how these rate-independent models can be embedded into more general dynamical problems, for instance including rate-dependent heat conduction, viscous effects or even kinetic terms.

On the side of material modeling there is now quite a variety of models for shape-memory materials. It is possible to describe models on many different length scales. However, the question of upscaling and deriving effective models on larger scales needs further investigations. The relaxations and Γ-convergence results in Sect. 4 will be a good basis for doing this, see also [57]. A first step in two-scale homogenization will be developed in [63]. Moreover, evolutionary models for microstructures and textures will certainly be important future areas where the energetic formulation can be helpful.

The strength of the energetic formulation is that it can model the statics extremely well by adjusting the energy-storage functional $\mathcal{E}$ according to experiments, see, e.g., [37]. However, the modeling of the dissipation distances, which contains the only information on the dynamics, is not supported very well by experiments. In this sense, the energetic formulation provides a first mathematical step to well-posed evolutionary models for complex material behavior.

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References

[1] J. Alberty and C. Carstensen. Numerical analysis of time-depending primal elastoplasticity with hardening. *SIAM J. Numer. Anal.*, 37:1271–1294 (electronic), 2000.

[2] L.-E. Andersson and A. Klarbring. A review of the theory of static and quasi-static frictional contact problems in elasticity. *R. Soc. Lond. Philos. Trans. Ser. A*, 359:2519–2539, 2001.

[3] S. Aubry, M. Fago, and M. Ortiz. A constrained sequential-lamination algorithm for the simulation of sub-grid microstructure in martensitic materials. *Comput. Methods Appl. Mech. Engrg.*, 192:2823–2843, 2003.

[4] F. Auricchio, A. Mielke, and U. Stefanelli. A rate-independent model for the isothermal quasi-static evolution of shape-memory materials. *In preparation*, 2006.

[5] F. Auricchio and L. Petrini. Improvements and algorithmical considerations on a recent three-dimensional model describing stress-induced solid phase transformations. *Int. J. Numer. Meth. Engng*, 55:1255–1284, 2002.

[6] J. Ball. Constitutive inequalities and existence theorems in nonlinear elastostatics. In *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium (Edinburgh, 1976), Vol. I*, pages 187–241. Res. Notes in Math., No. 17. Pitman, London, 1977.

[7] J. M. Ball and R. D. James. Fine phase mixtures as minimizers of energy. *Arch. Rational Mech. Anal.*, 100(1):13–52, 1987.

[8] S. Bartels, C. Carstensen, K. Hackl, and U. Hoppe. Effective relaxation for microstructure simulations: algorithms and applications. *Comput. Methods Appl. Mech. Engrg.*, 193:5143–5175, 2004.

[9] K. Bhattacharya. *Microstructure of Martensite. Why it Forms and How it Gives Rise to the Shape-Memory Effect.* Oxford University Press, New York, 2003.

[10] M. Brokate, P. Krejčí, and H. Schnabel. On uniqueness in evolution quasivariational inequalities. *J. Convex Analysis*, 11:111–130, 2004.

[11] M. Brokate and J. Sprekels. *Hysteresis and phase transitions.* Springer-Verlag, New York, 1996.

[12] A. Chambolle. A density result in two-dimensional linearized elasticity and applications. *Arch. Rat. Mech. Analysis*, 167:211–233, 2003.

[13] P. G. Ciarlet. *Mathematical Elasticity. Vol. I: Three-dimensional elasticity.* North-Holland Publishing Co., Amsterdam, 1988.

[14] S. Conti and F. Theil. Single-slip elastoplastic microstructures. *Arch. Rational Mech. Analysis*, 178:125–148, 2005.

[15] G. Dal Maso, G. Francfort, and R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, 176:165–225, 2005.

[16] A. Dorfmann and R. W. Ogden. Some problems in nonlinear magnetoelasticity. *Z. Angew. Math. Phys.*, 56:718–745, 2005.

[17] M. Efendiev and A. Mielke. An energetical model for evolution in magnetostriction. *In preparation*, 2006.

[18] M. Efendiev and A. Mielke. On the rate-independent limit of systems with dry friction and small viscosity. *J. Convex Analysis*, 13(1):151–167, 2006.

[19] M. Efendiev, A. Mielke, R. Rossi, and G. Savaré. On a metric approach to a class of doubly nonlinear evolution equations and applications to rate-independent problems. *In preparation*, 2006.

[20] G. Francfort and A. Garroni. A variational view of brittle damage evolution. *Arch. Rational Mech. Anal.*, 2006. To appear.

[21] G. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46:1319–1342, 1998.

[22] G. Francfort and A. Mielke. Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. reine angew. Math.*, 2006. In print.

[23] M. Frémond. *Non-smooth thermomechanics*. Springer-Verlag, Berlin, 2002.

[24] M. Frémond and S. Miyazaki. *Shape Memory Alloys*. Springer-Verlag, Wien, 1996.

[25] S. Govindjee, A. Mielke, and G. Hall. The free–energy of mixing for n–variant martensitic phase transformations using quasi–convex analysis. *J. Mech. Physics Solids*, 50:1897–1922, 2002. Erratum and Correct Reprinting: 51(4) 2003, pp. 763 & I-XXVI.

[26] K. Hackl. Generalized standard media and variational principles in classical and finite strain elastoplasticity. *J. Mech. Phys. Solids*, 45(5):667–688, 1997.

[27] K. Hackl and U. Hoppe. On the calculation of microstructures for inelastic materials using relaxed energies. In C. Miehe, editor, *IUTAM Symposium on Computational Mechanics of Solids at Large Strains*, pages 77–86. Kluwer, 2003.

[28] B. Halphen and Q. S. Nguyen. Sur les matériaux standards généralisés. *J. Mécanique*, 14:39–63, 1975.

[29] W. Han and B. D. Reddy. *Plasticity (Mathematical theory and numerical analysis)*, volume 9 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1999.

[30] R. D. James. Configurational forces in magnetism with application to the dynamics of a small-scale ferromagnetic shape memory cantilever. *Contin. Mech. Thermodyn.*, 14:55–86, 2002.

[31] C. Johnson. Existence theorems for plasticity problems. *J. Math. Pures Appl. (9)*, 55(4):431–444, 1976.

[32] M. Kamlah. Ferroelectric and ferroelastis piezoceramics-modelling of electromechanical hysteresis phenomena. *Continuum Mech.Thermodyn.*, 13(4):219–268, 2001.

[33] D. Knees and A. Mielke. Energy release rate for cracks in finite-strain elasticity. *Math. Methods Applied Sciences*, 2006. Submitted (WIAS Preprint 1100).

[34] M. Kočvara, A. Mielke, and T. Roubíček. A rate–independent approach to the delamination problem. *Math. Mech. Solids*, 2005. In print, online June 2005.

[35] M. A. Krasnosel'skiĭ and A. V. Pokrovskiĭ. *Systems with hysteresis*. Springer-Verlag, Berlin, 1989.

[36] P. Krejčí. Evolution variational inequalities and multidimensional hysteresis operators. In *Nonlinear differential equations (Chvalatice, 1998)*, volume 404, pages 47–110. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[37] M. Kružík, A. Mielke, and T. Roubíček. Modelling of microstructure and its evolution in shape-memory-alloy single-crystals, in particular in cualni. *Meccanica*, 40:389–418, 2005.

[38] M. Kružík and M. Luskin. The computation of martensitic microstructure with piecewise laminates. *J. Sci. Comp.*, 19:293–308, 2003.

[39] M. Kružík and T. Roubíček. Mesoscopic model of microstructure evolution in shape memory alloys with applications to NiMaGa. Technical Report no. 2003, IMA Minneapolis, 2004.

[40] H. Le Dret and A. Raoult. Variational convergence for nonlinear shell models with directors and related semicontinuity and relaxation results. *Arch. Ration. Mech. Anal.*, 154(2):101–134, 2000.

[41] A. Mainik. A rate-independent model for phase transformations in shape-memory alloys. PhD Thesis, IADM, Universität Stuttgart, 2005.

[42] A. Mainik. A rate-independent model for phase transformations in shape-memory alloys. *Submitted*, 2006. Preprint Universität Stuttgart, SFB404 2006/04.

[43] A. Mainik and A. Mielke. Existence results for energetic models for rate-independent systems. *Calc. Var. PDEs*, 22:73–99, 2005.

[44] J. A. C. Martins, A. Pinto da Costa, and F. M. F. Simões. Some notes on friction and instabilities. In *Friction and instabilities (Udine, 2000)*, volume 457 of *CISM Courses and Lectures*, pages 65–136. Springer, 2002.

[45] G. A. Maugin. *The Thermomechanics of Plasticity and Fracture*. Cambridge University Press, Cambridge, 1992.

[46] C. Miehe and A. Mielke. Finite-strain elastoplasticity. In *SFB404-Abschlussband*. Springer-Verlag, 2006.

[47] A. Mielke. Estimates on the mixture function for multiphase problems in elasticity. In A.-M. Sändig, W. Schiehlen, and W. Wendland, editors, *Multifield problems*, pages 96–103, Berlin, 2000. Springer–Verlag.

[48] A. Mielke. Deriving new evolution equations for microstructures via relaxation of variational incremental problems. *Comput. Methods Appl. Mech. Engrg.*, 193:5095–5127, 2004.

[49] A. Mielke. Evolution in rate-independent systems (ch. 6). In C. Dafermos and E. Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, pages 461–559. Elsevier B.V., 2005.

[50] A. Mielke. Temperature-induced phase transformations in shape-memory alloys. In preparation, 2006.

[51] A. Mielke and M. Ortiz. A class of minimum principles for characterizing the trajectories of dissipative systems. *In preparation*, 2006.

[52] A. Mielke and R. Rossi. Existence and uniqueness results for a class of rate-independent hysteresis problems. *M^3AS Math. Models Methods Appl. Sci.*, 2006. To appear (WIAS Preprint 1078).

[53] A. Mielke, R. Rossi, and G. Savaré. On a metric approach to a class of rate-independent problems. *In preparation*, 2006.

[54] A. Mielke and T. Roubíček. A rate–independent model for inelastic behavior of shape–memory alloys. *Multiscale Model. Simul.*, 1:571–597, 2003.

[55] A. Mielke and T. Roubíček. Numerical approaches to rate-independent processes and applications in inelasticity. *In preparation*, 2005.

[56] A. Mielke and T. Roubíček. Rate-independent damage processes in nonlinear elasticity. *M^3AS Math. Models Methods Appl. Sci.*, 16:177–209, 2006.

[57] A. Mielke, T. Roubíček, and U. Stefanelli. Γ-limits and relaxations for rate-independent evolution equations. *In preparation*, 2006.

[58] A. Mielke and F. Theil. A mathematical model for rate-independent phase transformations with hysteresis. In H.-D. Alber, R. Balean, and R. Farwig, editors, *Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering"*, pages 117–129. Shaker-Verlag, 1999.

[59] A. Mielke and F. Theil. On rate-independent hysteresis models. *Nonl. Diff. Eqns. Appl. (NoDEA)*, 11:151–189, 2004. (Accepted July 2001).

[60] A. Mielke, F. Theil, and V. I. Levitas. A variational formulation of rate-independent phase transformations using an extremum principle. *Arch. Rational Mech. Anal.*, 162:137–177, 2002.

[61] A. Mielke and A. Timofte. An energetic material model for time-dependent ferroelectric behavior: existence and uniqueness. *Math. Meth. Appl. Sciences*, 2005. In print (WIAS preprint no. 1014).

[62] A. Mielke and A. M. Timofte. Modeling and analytical study for ferroelectric materials. *Mech. Advanced Materials Structures*, 2006. In print.

[63] A. Mielke and A. M. Timofte. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *In preparation*, 2006.

[64] J.-J. Moreau. On unilateral constraints, friction and plasticity. In *New variational techniques in mathematical physics (Centro Internaz. Mat. Estivo (C.I.M.E.), II Ciclo, Bressanone, 1973)*, pages 171–322. Edizioni Cremonese, Rome, 1974.

[65] J.-J. Moreau. Application of convex analysis to the treatment of elastoplastic systems. In P. Germain and B. Nayroles, editors, *Applications of methods of functional analysis to problems in mechanics*, pages 56–89. Springer-Verlag, 1976. Lecture Notes in Mathematics, 503.

[66] S. Müller. Variational models for microstructure and phase transitions. In *Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996)*, pages 85–210. Springer, Berlin, 1999.

[67] M. Ortiz and E. Repetto. Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids*, 47(2):397–462, 1999.

[68] L. Prigozhin. On the bean critical-state model in superconductivity. *European J. of Appl. Math.*, 7:237–248, 1996.

[69] J. Rice. Inelastic constitutive relations for solids: an internal-variable theory and its applications to metal plasticity. *J. Mech. Phys. Solids*, 19:203–240, 1971.

[70] H. Romanowski and J. Schröder. Coordinate invariant modelling of the ferroelectric hysteresis within a thermodynamically consistent framework. A mesoscopic approach. In Y. Wang and K. Hutter, editors, *Trends in Applications of Mathematics to Mechanics*, pages 419–428. Shaker Verlag, 2005.

[71] T. Roubíček. Models of microstructure evolution in shape memory alloys. In P. Ponte Castaneda, J. Telega, and B. Gambin, editors, *Nonlinear Homogenization and its Applications to Composites, Polycrystals and Smart Materials*, pages 269–304. Kluwer, 2004. NATO Sci. Series II/170.

[72] T. Roubíček and M. Kružík. Mircrostructure evolution model in micromagnetics. *Zeits. angew. Math. Physik*, 55:159–182, 2004.

[73] T. Roubíček and M. Kružík. Mesoscopic model for ferromagnets with isotropic hardening. *Zeits. angew. Math. Physik*, 56:107–135, 2005.

[74] P. Rybka and M. Luskin. Existence of energy minimizers for magnetostrictive materials. *SIAM J. Math. Anal.*, 36:2004–2019, 2005.

[75] F. Schmid. A local existence result for a finite-dimensional model in dry friction. *In preparation*, 2006.

[76] F. Schmid and A. Mielke. Vortex pinning in super-conductivity as a rate-independent process. *Europ. J. Appl. Math.*, 2005. In print.

[77] A. Souza, E. Mamiya, and N. Zouain. Three-dimensional model for solids undergoing stress-induced phase transformations. *Europ. J. Mech., A/Solids*, 17:789–806, 1998.

[78] P.-M. Suquet. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mécanique*, 20(1):3–39, 1981.

[79] F. Theil. Relaxation of rate-independent evolution problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 132:463–481, 2002.

[80] A. Visintin. *Differential models of hysteresis*. Springer-Verlag, Berlin, 1994.

[81] H. Ziegler and C. Wehrli. The derivation of constitutive relations from the free energy and the dissipation function. In *Advances in applied mechanics, Vol. 25*, pages 183–237. Academic Press, Orlando, FL, 1987.