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Energy release rate for cracks in finite-strain elasticity

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Abstract

Griffith's fracture criterion describes in a quasistatic setting whether or not a pre-existing crack in an elastic body is stationary for given external forces. In terms of the energy release rate (ERR), which is the derivative of the deformation energy of the body with respect to a virtual crack extension, this criterion reads: If the ERR is less than a specific constant, then the crack is stationary, otherwise it will grow.

In this paper, we consider geometrically nonlinear elastic models with polyconvex energy densities and prove that the ERR is well defined. Moreover, without making any assumption on the smoothness of minimizers, we derive rigorously the well-known Griffith formula and the J-integral, from which the ERR can be calculated. The proofs are based on a weak convergence result for Eshelby tensors.

1 Introduction

In this paper we consider an elastic body Ω_0 with a pre-existing crack which is subjected to quasistatic external loadings. In the literature several fracture criteria are provided on the basis of which one can decide, whether or not the crack will propagate for the given forces. We consider here the Griffith criterion, which is an energetic criterion, and reads as follows [Gri20, Lie68, Che79]:

The crack is stationary for the given forces, if the total potential energy
of the actual configuration is minimal compared to the (1.1)
total potential energy of all admissible neighboring configurations.

The total energy is the sum of the deformation energy and a dissipative energy, which describes the energy which is needed to create the new crack surface. We treat here the simplest model for the dissipative energy and assume that this energy is proportional to the area or length of the crack surface. The deformation energy consists of the stored energy and the work of the external forces. In this paper we discuss models from finite-strain elasticity and assume that the stored energy density W is a polyconvex function with $W(A) = \infty$ for every matrix A with $\det A \leq 0$.

In general, it is not known in advance, which path the tip or front of a crack C_0 in a domain Ω_0 will follow during the propagation. In the most general case, a domain Ω_* with crack C_* is an *admissible neighboring configuration* with respect to Ω_0 and C_0 if $\overline{\Omega_*} = \overline{\Omega_0}$, $C_* \supset C_0$ and the area of $C_* \backslash C_0$ is small. This general point of view includes the kinking and branching of cracks. DalMaso et al. investigated an evolution problem for the development of cracks in nonlinear elastic materials with quasiconvex energy density based on this general point of view, [DFT05]. In this paper, we have a different point of view: We assume that the crack path is known a-priori and we are interested in a criterion on the basis of which one may decide whether or not the crack will propagate for given external loadings. We consider the simplest geometric situation, namely a straight crack in a two dimensional domain and we assume that the crack can grow straight on, only. However, the techniques developed in this paper can be extended to $\mathcal{C}^1$-smooth interface cracks, which will be investigated in a forthcoming paper.

Under these simplifying assumptions on the geometry of the crack and the possible crack path, Griffith's criterion can be reformulated in terms of the energy release rate: Let $\Omega_0 \subset \mathbb{R}^2$ describe the actual configuration with a crack which is part of a straight line. We assume that the crack can propagate straight on, only. Let furthermore $W : \mathbb{R}^{2\times 2} \to \mathbb{R}_\infty$ denote the polyconvex stored energy density and let $f : \Omega_0 \to \mathbb{R}^2$ be a given volume force density. The energy release rate $ERR(\Omega_0)$ is defined as the negative of the right derivative of the potential deformation energy with respect to the crack length, see e.g. [Che67, Gur79]:

$$ERR(\Omega_0) := -\frac{d}{d\delta}\left(\int_{\Omega_\delta} W(\nabla u_\delta)\,\mathrm{d}x - \int_{\Omega_\delta} f \cdot u_\delta \,\mathrm{d}x\right)\Big|_{\delta=0} \equiv -\frac{d}{d\delta} I(\Omega_\delta, u_\delta)\Big|_{\delta=0}. \tag{1.2}$$

Here, $\Omega_\delta \subset \Omega_0$ is a domain with an extended crack and the deformations u_δ are minimizers of the functional

$$I(\Omega_\delta, v) = \int_{\Omega_\delta} W(\nabla v)\,\mathrm{d}x - \int_{\Omega_\delta} f \cdot v \,\mathrm{d}x$$

on the domain Ω_δ. The fracture criterion now reads:

$$\textit{If } ERR(\Omega_0) < 2\gamma, \textit{ then the crack is stationary. Otherwise it will grow.} \tag{1.3}$$

The constant $\gamma > 0$ is the fracture toughness and depends on the material. Simple formulas are needed in order to calculate the energy release rate.

Let us give a short and not at all complete summary of known results from literature. There is a huge number of papers dealing with linear elastic models. There, the energy release rate is expressed by the Griffith formula, the J-integral or the Cherepanov-Rice integral, see e.g. [Esh51, Che67, Ric68, DD81, KS00], or by stress intensity factors, see e.g. [MN87]. Regularity results play an essential role in the derivation of the J-integral and the formulas involving the stress intensity factors. For nonlinear models similar formulas are given in the literature, as well. These formulas are derived assuming that the minimizers have a certain regularity or that the corresponding strain and stress fields have a special singular structure in a neighborhood of the crack tip. In general, however, such regularity results are not proved yet and, to our knowledge, a rigorous derivation of these formulas from definition (1.2) taking into account the known regularity and integrability properties of minimizers is, except for a class of power-law models [Kne05], not done yet for nonlinear elastic models. In particular, no results exist for nonconvex situations.

The goal of this paper is to describe sufficient conditions on the polyconvex energy density W which enable us to prove that the energy release rate (1.2) is well defined also in the nonlinear and nonconvex case and to derive the well-known formulas for the energy release rate rigorously. In particular, we prove the following formula for the energy release rate for energies with polyconvex density W and assuming that the crack is part of the x_1-axis (theorem 3.3):

$$ERR(\Omega_0) = \max\{\, G(u_0, \theta)\,;\; u_0 \text{ minimizes } I(\Omega_0, \cdot) \,\} \tag{1.4}$$

with

$$G(u_0, \theta) = \int_{\Omega_0} \left(\nabla u_0^\top DW(\nabla u_0) - W(\nabla u_0)\mathbf{1} \right) : \nabla \begin{pmatrix} \theta \\ 0 \end{pmatrix} \mathrm{d}x - \int_{\Omega_0} \theta f \cdot \partial_1 u_0 \, \mathrm{d}x. \tag{1.5}$$

Formula (1.5) is the well-known Griffith formula. Here, θ is a cut-off function which equals to 1 near the crack tip and $\mathbf{1}$ denotes the identity matrix in $\mathbb{R}^{2\times 2}$. The term $\nabla u_0^\top DW(\nabla u_0) - W(\nabla u_0)\mathbf{1}$ is the Eshelby or Hamilton tensor [Esh51, GH96]. We prove furthermore that the maximum in (1.4) is attained and that $G(u_0, \theta) = G(u_0)$ is independent of the cut-off function θ. Finally we show that the energy release rate can also be expressed through a path independent integral, the J-integral, in the following form (theorem 3.5):

$$G(u_0) = \int_{\Gamma_R} \left(\left(\nabla u_0^\top DW(\nabla u_0) - W(\nabla u_0)\mathbf{1} \right) \vec{n} \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (u_0 \cdot f) n_1 \, \mathrm{d}s$$

for almost every $R > 0$, where $\Gamma_R = \{\, x \in \mathbb{R}^2\,;\; |x| = R \,\}$ and $\vec{n} = (n_1, n_2)^\top$ is the interior unit normal vector on the path Γ_R. Let us note that it remains an open problem whether $G(u_0) = G(u_1)$ for two different minimizers u_0 and u_1 of $I(\Omega_0, \cdot)$. We give an interpretation of (1.4) at the end of section 3.

Like in the proofs for the linear case we derive relation (1.4) by transforming the domains Ω_δ with extended cracks back to the reference configuration Ω_0 through a diffeomorphism $T_\delta : \Omega_\delta \to \Omega_0$ which is defined with the help of the cut-off function θ through $T_\delta(x) = x - \delta(\theta(x), 0)^\top$. Let $\{u_0,\, u_\delta,\, \delta > 0\}$ be minimizers of $I(\Omega_0, \cdot)$ and $I(\Omega_\delta, \cdot)$, respectively. For every $\delta > 0$ we have

$$\delta^{-1} \left(I(\Omega_0, u_0) - I(\Omega_\delta, u_\delta) \right) \geq \delta^{-1} \left(I(\Omega_0, u_0) - I(\Omega_\delta, u_0 \circ T_\delta) \right), \tag{1.6}$$

$$\delta^{-1} \left(I(\Omega_0, u_0) - I(\Omega_\delta, u_\delta) \right) \leq \delta^{-1} \left(I(\Omega_0, u_\delta \circ T_\delta^{-1}) - I(\Omega_\delta, u_\delta) \right). \tag{1.7}$$

In order to prove (1.4), we calculate the limes inferior for $\delta \searrow 0$ of the right hand side in (1.6) and the limes superior of the right hand side in (1.7) and show that the limits exist and are equal. The main assumptions, which we need here in addition to polyconvexity, are the following estimates for the derivatives of W: there exist constants $\kappa_1, \kappa_2 > 0$ such that for every $A, B \in \mathbb{M}^{2\times 2}$ with $\det A > 0$

$$\left| A^\top DW(A) \right| \leq \kappa_1 (W(A) + 1) \tag{1.8}$$

$$\left| A^\top (D^2 W(A)[AB]) \right| \leq \kappa_2 (W(A) + 1) \left| B \right| . \tag{1.9}$$

Here, $D^2W(A)[B] \in \mathbb{M}^{2\times 2}$ with $D^2W(A)[B]_{ij} = \sum_{k,l=1}^{2} \frac{\partial^2 W(A)}{\partial A_{kl} \partial A_{ij}} B_{kl}$. Condition (1.8) was first introduced in [BOP91, Bal02], and guarantees that $G(u_0)$ from (1.5) is finite for minimizers u_0 of $I(\Omega_0, \cdot)$. The main tool for calculating the limes inferior on the right hand side in (1.6) is Lebesgue's convergence theorem, where assumption (1.8) leads to integrable majorants. In this part, we follow the considerations in [BOP91], where inner variations of the energy $I(\Omega_0, \cdot)$ with respect to vector fields $\vec{\theta} \in \mathcal{C}_0^\infty(\Omega_0)$ are investigated. In the notion of [GH96], $G(u_0, \theta)$ is the strong inner variation of the energy functional $I(\Omega_0, \cdot)$ in u_0 with respect to the vector field $(\theta, 0)^\top$.

In deriving the limes superior in (1.7) the main difficulty is that subsequences of minimizers $\{ u_\delta \circ T_\delta^{-1} \,;\, \delta > 0 \}$ converge weakly in $W^{1,p}(\Omega_0)$ to some minimizer u_0 of $I(\Omega_0, \cdot)$, only. Thus the usual theorems on interchanging the limit with integrals cannot be applied here. Based on assumptions (1.8) and (1.9) we deduce in section 2 a theorem which states that weak convergence of the deformation fields together with convergence of the corresponding energies implies weak convergence of the Eshelby tensor in L^1. The proof of this theorem relies on a proposition recently derived in [FM06], see also [DFT05], where the convergence of derivatives of parameter depending integrals is investigated. The limes superior in (1.7) is calculated on the basis of these weak convergence results.

The paper is organized as follows: In section 2 we prove the convergence results for the Eshelby tensor, proposition 2.2 and theorem 2.6, in a slightly more general setting where we make no restriction on the dimension of the domain. In section 3 we formulate the main results for the energy release rate on a two dimensional domain with a straight crack. Furthermore, we introduce a perturbed problem from which we deduce that $G(u_0)$ in (1.5) does not depend on the cut-off function θ. The proofs of the main results are given in section 4. In section 5 we indicate how these results can be carried over to functionals with quasi-convex energy densities. Moreover, we discuss shortly the case when non-interpenetration conditions are prescribed on the crack faces.

2 Weak convergence of the Eshelby tensor

The goal of this section is to provide some basic estimates which follow from assumptions (1.8) and (1.9) and to prove a convergence result for Eshelby tensors (theorem 2.6). The following notation is used: $\mathbb{M}^{m\times d}$ denotes the set of the real $m \times d$-matrices and $\mathbb{M}_+^{d\times d}$ are those with positive determinant. For elements $A, B \in \mathbb{M}^{m\times d}$ the inner product is denoted by $A : B = \sum_{k=1}^{m} \sum_{s=1}^{d} A_{ks} B_{ks}$. For a function $W : \mathbb{M}^{m\times d} \to \mathbb{R}$, $DW(A) \in \mathbb{M}^{m\times d}$ is the derivative of W with respect to $A \in \mathbb{M}^{m\times d}$, i.e. $DW(A)_{ks} = \frac{\partial W(A)}{\partial A_{ks}}$ for $1 \leq k \leq m$, $1 \leq$

$s \leq d$, and $D^2W(A) \in \mathbb{M}^{(m\times d)\times(m\times d)}$ is the Hessian of W with $(D^2W(A))_{ksjr} = \frac{\partial^2 W(A)}{\partial A_{ks}\partial A_{jr}}$, $1 \leq k,j \leq m$, $1 \leq s,r \leq d$. Furthermore, $D^2W(A)[B] \in \mathbb{M}^{m\times d}$ with $D^2W(A)[B]_{ks} = \sum_{j=m}^{d}\sum_{r=1}^{d} D^2W(A)_{ksjr}B_{jr}$.

We assume polyconvexity for the energy density W and adopt the notation from Dacorogna's book, [Dac89]: For $A \in \mathbb{M}^{d\times d}$, the vector $\mathcal{T}(A) = (A, \mathrm{adj}_2(A), \dots, \mathrm{adj}_d(A)) \in \mathbb{R}^{\tau(d)}$ denotes the vector of the minors of A, $\tau(d)$ is the number of minors and $\mathrm{adj}_s(A)$ is the adjugate matrix of A of order s.

A1 $W : \mathbb{M}^{d\times d} \to [0,\infty]$ is polyconvex, i.e. there exists a function $g : \mathbb{R}^{\tau(d)} \to [0,\infty]$ which is continuous and convex and $W(A) = g(\mathcal{T}(A))$ for every $A \in \mathbb{M}^{d\times d}$. Moreover, $W(A) = \infty$ if $\det A \leq 0$.

The following growth conditions are imposed on the derivatives of W:

A2 $W : \mathbb{M}^{d\times d} \to [0,\infty]$ is differentiable on $\mathbb{M}^{d\times d}_+$ and there exists a constant $\kappa_1 > 0$ such that for every $A \in \mathbb{M}^{d\times d}_+$

$$\left|A^\top DW(A)\right| \leq \kappa_1(W(A)+1). \tag{2.1}$$

A3 $W : \mathbb{M}^{d\times d} \to [0,\infty]$ is twice differentiable on $\mathbb{M}^{d\times d}_+$ and there exists a constant $\kappa_2 > 0$ such that for every $A \in \mathbb{M}^{d\times d}_+$ and every $B \in \mathbb{M}^{d\times d}$

$$\left|A^\top (D^2W(A)[AB])\right| \leq \kappa_2(W(A)+1)\left|B\right|. \tag{2.2}$$

At the end of this section we give an example for a polyconvex energy density satisfying **A1** to **A3**. Assumptions **A2** and **A3** imply the following estimates:

Lemma 2.1. *[BOP91, Bal02] Let $W : \mathbb{M}^{d\times d} \to [0,\infty]$ satisfy* **A2***. There exist constants $\gamma_1, c_1 > 0$ such that for every $A, C \in \mathbb{M}^{d\times d}_+$ with $|C - \mathbf{1}| \leq \gamma_1$ it holds*

$$W(AC) + 1 \leq c_1(W(A)+1), \tag{2.3}$$

$$\left|A^\top DW(AC)\right| \leq c_1(W(A)+1), \tag{2.4}$$

$$\left|W(AC) - W(A)\right| \leq c_1(W(A)+1)\left|C - \mathbf{1}\right|. \tag{2.5}$$

Let in addition assumption **A3** *be valid. Then there exist constants $\gamma_2, c_2 > 0$, $\gamma_2 \leq \gamma_1$, such that for every $A, C \in \mathbb{M}^{d\times d}_+$ with $|C - \mathbf{1}| \leq \gamma_2$ and for every $B \in \mathbb{M}^{d\times d}$ we have*

$$\left|A^\top (D^2W(AC)[AB])\right| \leq c_2(W(A)+1)\left|B\right|. \tag{2.6}$$

Finally, there exists a constant $c_3 > 0$ such that for every $A \in \mathbb{M}^{d\times d}_+$, $B \in \mathbb{M}^{d\times d}$ and every $s, t \in [-t_0, t_0]$ with $t_0 = \frac{\gamma_2}{2|B|}$ we have

$$\left|\big(DW(A(\mathbf{1}+tB)) - DW(A(\mathbf{1}+sB))\big) : (AB)\right| \leq c_3\left|t-s\right|\left|B\right|^2(W(A)+1). \tag{2.7}$$

Proof. Assertions (2.3)–(2.5) are derived in [BOP91, Bal02] and we prove here (2.6) and (2.7), only. Let $\gamma_2 = \min\{\frac{1}{2}, \gamma_1\}$. For every $C \in \mathbb{M}_+^{d\times d}$ with $|C - \mathbf{1}| \leq \gamma_2$, every $A \in \mathbb{M}_+^{d\times d}$ and $B \in \mathbb{M}^{d\times d}$ it holds

$$\begin{aligned} \left|A^\top D^2W(AC)[AB]\right| &= \left|C^{-\top}(AC)^\top D^2W(AC)[ACC^{-1}B]\right| \\ &\overset{(2.2)}{\leq} \kappa_2(W(AC)+1)\left|C^{-\top}\right|\left|C^{-1}B\right| \\ &\overset{(2.3)}{\leq} 4\kappa_2 c_1(W(A)+1)\,|B|\,. \end{aligned} \tag{2.8}$$

This proves (2.6). Let $B \in \mathbb{M}^{d\times d}$ and $A \in \mathbb{M}_+^{d\times d}$ be arbitrary, $t_0 = \frac{\gamma_2}{2|B|}$. For every $t \in [-t_0, t_0]$ it follows that $\mathbf{1} + tB$ is invertible (Neumann series) and has a positive determinant. Thus, for every $s, t \in [-t_0, t_0]$ we obtain

$$\begin{aligned} &\Big(DW(A(\mathbf{1}+tB)) - DW(A(\mathbf{1}+sB))\Big) : AB \\ &\qquad = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big(DW(A(\mathbf{1}+(s+\alpha(t-s))B))\Big) : AB\mathrm{d}\alpha \\ &\qquad = (t-s)\int_0^1 \left(A^\top D^2W(AC_\alpha)[AB]\right) : B\mathrm{d}\alpha, \end{aligned} \tag{2.9}$$

where $C_\alpha = \mathbf{1} + (s+\alpha(t-s))B$. Since $|\mathbf{1} - C_\alpha| \leq \gamma_2$ and $\det C_\alpha > 0$, we may apply (2.6) to estimate the right hand side in (2.9) and obtain finally (2.7). □

The main result of this section is the following proposition:

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and assume that $W : \Omega \to [0,\infty]$ fulfills* **A1**-**A3**. *Furthermore, let $\{ F_n\,;\, n \in \mathbb{N}_0 \} \subset L^1(\Omega, \mathbb{M}^{d\times d})$ be a sequence with*

$$\mathcal{T}(F_n) \rightharpoonup \mathcal{T}(F_0) \text{ weakly in } L^1(\Omega) \text{ for } n \to \infty, \tag{2.10}$$

$$J(F_0) := \int_\Omega W(F_0)\,\mathrm{d}x < \infty \text{ and } J(F_n) \to J(F_0) \text{ for } n \to \infty. \tag{2.11}$$

Then $F_n^\top DW(F_n) \rightharpoonup F_0^\top DW(F_0)$ weakly in $L^1(\Omega)$.

The proof of proposition 2.2 relies on a convergence lemma for parameter depending energies $E : [-t_0, t_0] \times \mathcal{Y} \to \mathbb{R}_\infty$. This lemma was recently derived in [FM06] and a variant fitting to our special situation reads as follows:

Lemma 2.3. *[FM06] Let $E : [-t_0, t_0] \times L^1(\Omega, \mathbb{M}^{d\times d}) \to [0,\infty]$ satisfy*

1. *For every $|t| \leq t_0$ and $\{ F_n\,;\, n \in \mathbb{N}_0 \} \subset L^1(\Omega, \mathbb{M}^{d\times d})$ with $\mathcal{T}(F_n) \rightharpoonup \mathcal{T}(F_0)$ weakly in $L^1(\Omega)$ it holds $\liminf_{n\to\infty} E(t, F_n) \geq E(t, F_0)$.*
2. *If $E(0, F) < \infty$ for a fixed $F \in L^1(\Omega, \mathbb{M}^{d\times d})$, then $E(\cdot, F) < \infty$ on the whole interval $[-t_0, t_0]$ and $t \mapsto E(t, F)$ is differentiable on $[-t_0, t_0]$.*

3. *For every $R > 0$ exists a modulus of continuity $\omega_R : [0, 2t_0] \to [0, \infty)$ such that for every $F \in L^1(\Omega)$ we have*

$$E(0, F) \leq R \Rightarrow \forall s, t \in [-t_0, t_0] : \; |\partial_t E(s, F) - \partial_t E(t, F)| \leq \omega_R(|s - t|). \tag{2.12}$$

Then for every $t \in (-t_0, t_0)$ the following implication holds:

$$\left.\begin{array}{rl} \mathcal{T}(F_n) & \rightharpoonup \mathcal{T}(F_0) \text{ weakly in } L^1(\Omega) \\ E(t, F_n) & \to E(t, F_0) < \infty \end{array}\right\} \Rightarrow \; \partial_t E(t, F_n) \to \partial_t E(t, F_0). \tag{2.13}$$

By modulus of continuity we mean a nondecreasing function $\omega_R : [0, 2t_0] \to [0, \infty)$ with $\omega_R(s) \to 0$ for $s \searrow 0$. For the proof of proposition 2.2 we consider the following parameter depending energy $E(t, F)$: Let $B \in L^\infty(\Omega, \mathbb{M}^{d\times d})$ and $t_0 = \gamma_2/(2\,\|B\|_{L^\infty(\Omega)})$ with γ_2 from lemma 2.1. For $F \in L^1(\Omega)$ and $|t| \leq t_0$ we define

$$E(t, F) = \int_\Omega W(F(x)(\mathbf{1} + tB(x)) \,\mathrm{d}x. \tag{2.14}$$

The properties of the energy $E(\cdot, \cdot)$ are summarized in the next lemma.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let the assumptions* **A1**-**A3** *be satisfied. For every $R > 0$ the above defined energy $E(\cdot, \cdot)$ is uniformly bounded on $[-t_0, t_0] \times \mathcal{M}_R$, where $\mathcal{M}_R = \{\, F \in L^1(\Omega)\,;\, E(0, F) \leq R \,\}$. Moreover the function $t \to E(t, F)$ is differentiable on $[-t_0, t_0]$ for every fixed $F \in \mathcal{M}_R$ and*

$$\partial_t E(t, F) = \int_\Omega (F^\top DW(F(\mathbf{1} + tB))) : B \,\mathrm{d}x. \tag{2.15}$$

Furthermore, there exists a constant $c(R, B) > 0$ such that for every $F \in \mathcal{M}_R$ and $|t| \leq t_0$

$$|\partial_t E(t, F)| \leq c(R, B). \tag{2.16}$$

There exists a constant $L = L(R, B) > 0$ such that for every $s, t \in [-t_0, t_0]$ and every $F \in \mathcal{M}_R$

$$|\partial_t E(t, F) - \partial_t E(s, F)| \leq L(R, B)\, |t - s| \,. \tag{2.17}$$

For fixed $t \in [-t_0, t_0]$ and every sequence $\{\, F_n\,;\, n \in \mathbb{N}_0 \,\} \subset L^1(\Omega)$ with $\mathcal{T}(F_n) \rightharpoonup \mathcal{T}(F_0)$ in $L^1(\Omega)$, the functional $E(t, \cdot)$ satisfies $\liminf_{n\to\infty} E(t, F_n) \geq E(t, F_0)$.

Proof. Let $B \in L^\infty(\Omega, \mathbb{M}^{d\times d})$. The uniform boundedness of $E(\cdot, \cdot)$ on $[-t_0, t_0] \times \mathcal{M}_R$ is an immediate consequence of estimate (2.3). Let $t \in [-t_0, t_0]$, $F \in \mathcal{M}_R$ and $h \in \mathbb{R}\backslash\{0\}$ such that $|t + h| \leq t_0$. Then

$$\frac{1}{h}(E(t + h, F) - E(t, F)) = \int_\Omega \int_0^1 DW(F(\mathbf{1} + (t + \alpha h)B)) : (FB)\mathrm{d}\alpha \,\mathrm{d}x. \tag{2.18}$$

Since W is twice differentiable on $\mathbb{M}^{d\times d}_+$, DW is in particular continuous on $\mathbb{M}^{d\times d}_+$ and thus $DW(F(x)(\mathbf{1} + (t + \alpha h)B(x))) \to DW(F(x)(\mathbf{1} + tB(x)))$ almost everywhere in Ω for

$h \to 0$. Moreover, for every $F \in \mathcal{M}_R$, $|t| \le t_0$, $\alpha \in [0,1]$, $h \in \mathbb{R}\backslash\{0\}$ with $|t+h| \le t_0$ and almost every $x \in \Omega$ assumption (2.4) implies that

$$\Big|F(x)^\top DW(F(x)(\mathbf{1} + (t+\alpha h)B(x)))\Big| \le c_1(W(F(x)) + 1). \tag{2.19}$$

Together with (2.18), the dominated convergence theorem now leads to (2.15). Estimate (2.16) follows from (2.19), and (2.17) is an immediate consequence of (2.7).

For the proof of the last assertion in lemma 2.4 let $\{ F_n \,;\, n \in \mathbb{N}_0 \} \subset L^1(\Omega)$ be a sequence with $\mathcal{T}(F_n) \rightharpoonup \mathcal{T}(F_0)$ weakly in $L^1(\Omega)$. Taking into account that the multiplicativity of adjugate matrices, $\operatorname{adj}_s(AB) = \operatorname{adj}_s(B)\operatorname{adj}_s(A)$ for $A, B \in \mathbb{M}^{d\times d}$ and $1 \le s \le d$, [Rei57, Chapter 7], it follows that $\mathcal{T}(F_n(\mathbf{1}+tB)) \rightharpoonup \mathcal{T}(F_0(\mathbf{1}+tB))$ weakly in $L^1(\Omega)$ for $n \to \infty$ as well. Since the energy density W is polyconvex, we obtain finally that $\liminf_{n\to\infty} E(t, F_n) \ge E(t, F_0)$. $\square$

Proposition 2.2 is now a combination of lemmata 2.3 and 2.4:

Proof of proposition 2.2. Let $B \in L^\infty(\Omega, \mathbb{R}^{d\times d})$, $t_0 = \gamma_2/(2\,\|B\|_{L^\infty(\Omega)})$ and define $E(t, F) = \int_\Omega W(F(\mathbf{1}+tB))\,\mathrm{d}x$ for $F \in L^1(\Omega)$. Let furthermore $\{ F_n \,;\, n \in \mathbb{N}_0 \} \subset L^1(\Omega)$ be a sequence satisfying (2.10)-(2.11). Lemma 2.4 shows that $E(\cdot,\cdot)$ satisfies the assumptions of convergence lemma 2.3 and thus (2.13) implies for $t = 0$ that

$$\int_\Omega \big(F_n^\top DW(F_n)\big) : B\,\mathrm{d}x \to \int_\Omega \big(F_0^\top DW(F_0)\big) : B\,\mathrm{d}x.$$

Since $B \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ is arbitrary, the proof of proposition 2.2 is finished. $\square$

Condition (2.2) on the second derivatives of the energy density W is a sufficient condition for obtaining (2.12) in lemma 2.3 via estimate (2.7). One could relax (2.2) by replacing it with a weaker assumption of the following type: For every $r > 0$ exists a modulus of continuity $\omega_r : [0, 2t_r] \to [0, \infty)$ with $t_r = \gamma_1/(2r)$ such that

$$\Big|\big(A^\top DW(A(\mathbf{1}+tB)) - A^\top DW(A(\mathbf{1}+sB))\big) : B\Big| \le \omega_r(|t-s|)(W(A)+1)$$

for every $s, t \in [-t_r, t_r]$, $A \in \mathbb{M}_+^{d\times d}$ and $B \in \mathbb{M}^{d\times d}$ with $|B| < r$.

In addition to proposition 2.2 we have the following general lemma on the weak convergence of energy densities:

Lemma 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^d$ and let $h : \Omega \times \mathbb{R}^m \to [0, \infty]$ satisfy*

1. *$h(x, \cdot) : \mathbb{R}^m \to [0, \infty]$ is convex and continuous for almost every $x \in \Omega$,*
2. *$h(\cdot, \xi) : \Omega \to [0, \infty]$ is measurable for every $\xi \in \mathbb{R}^m$.*

Let furthermore $\{ v_n \,;\, n \in \mathbb{N}_0 \} \subset L^1(\Omega, \mathbb{R}^m)$ be a sequence with

$$v_n \rightharpoonup v_0 \text{ weakly in } L^1(\Omega), \tag{2.20}$$

$$J(v_0) = \int_\Omega h(x, v_0(x))\,\mathrm{d}x < \infty \text{ and } J(v_n) \to J(v_0). \tag{2.21}$$

Then $h(\cdot, v_n(\cdot)) \rightharpoonup h(\cdot, v_0(\cdot))$ weakly in $L^1(\Omega)$.

Proof. Let $\varphi \in L^\infty(\Omega)$ be arbitrary and $\beta = \|\varphi\|_{L^\infty(\Omega)}$. The energy density $\tilde{h}(x, v) := (\varphi(x) + \beta)h(x, v)$ is nonnegative and satisfies *1.* and *2.* of lemma 2.5. Thus the functional $v \mapsto J_1(v) = \int_\Omega \tilde{h}(x, v(x))\,\mathrm{d}x$ is weakly lower semi-continuous on $L^1(\Omega)$, see e.g. [Cia93, Thm. 7.3-1], and for the sequence $\{ v_n \,;\, n \in \mathbb{N}_0 \}$ from Lemma 2.5 we obtain

$$\liminf_{n\to\infty} \int_\Omega \varphi(x)h(x, v_n(x))\,\mathrm{d}x = \liminf_{n\to\infty} \big(J_1(v_n) - \beta J(v_n)\big) \geq \int_\Omega \varphi(x)h(x, v_0(x))\,\mathrm{d}x.$$

Moreover, the functional $J_2(v) = \int_\Omega (\beta - \varphi(x))h(x, v(x))\,\mathrm{d}x$ is weakly lower semi-continuous as well and thus

$$\limsup_{n\to\infty} \int_\Omega \varphi(x)h(x, v_n(x))\,\mathrm{d}x = -\liminf_{n\to\infty} \big(J_2(v_n) - \beta J(v_n)\big) \leq \int_\Omega \varphi(x)h(x, v_0(x))\,\mathrm{d}x.$$

The assertion of lemma 2.5 follows since $\varphi \in L^\infty(\Omega)$ is arbitrary. □

Combining proposition 2.2 and lemma 2.5 we obtain a convergence result for Eshelby tensors.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which has the interior cone property. Let further $W : \mathbb{M}^{d\times d} \to [0, \infty]$ satisfy* **A1-A3***, $p > d$ and let $\{ u_n \,;\, n \in \mathbb{N}_0 \} \subset W^{1,p}(\Omega)$ be a sequence with*

$$u_n \rightharpoonup u_0 \text{ weakly in } W^{1,p}(\Omega), \tag{2.22}$$

$$J(u_n) \to J(u_0) = \int_\Omega W(\nabla u_0)\,\mathrm{d}x < \infty \text{ for } n \to \infty. \tag{2.23}$$

Then $\nabla u_n^\top DW(\nabla u_n) - W(\nabla u_n)\mathbf{1} \rightharpoonup \nabla u_0^\top DW(\nabla u_0) - W(\nabla u_0)\mathbf{1}$ weakly in $L^1(\Omega)$.

Proof. For $p > d$ and Lipschitz domains condition (2.22) implies that $\mathrm{adj}_s(\nabla u_n) \rightharpoonup \mathrm{adj}_s(\nabla u_0)$ weakly in $L^{p/s}(\Omega)$ for $1 \leq s \leq d$ and thus the minors $\mathcal{T}(\nabla u_n)$ converge to $\mathcal{T}(\nabla u_0)$ weakly in $L^1(\Omega)$, [Mor52, Res67]. Since bounded domains with interior cone property are a union of a finite number of Lipschitz domains, see e.g. [Wlo82], the convergence result for minors from [Mor52, Res67] on Lipschitz domains can be immediately carried over to domains with cone property. Lemma 2.5 and proposition 2.2 now imply theorem 2.6. □

Bounded domains with Lipschitz boundaries as well as the domains with cracks, which we introduce in section 3, satisfy the interior cone property.

Example 2.7. Let $d \geq 2$. The following energy density is polyconvex and satisfies **A2** and **A3**:

$$W(A) = \begin{cases} W_1(A) + \Gamma(\det A) & \text{for } A \in \mathbb{M}_+^{d\times d}, \\ \infty & \text{else,} \end{cases}$$

where $W_1 : \mathbb{M}^{d\times d} \to [0, \infty)$ is a convex and twice differentiable function of p-growth for some $p > 1$. This means that there exist constants $c_i > 0$ such that $c_1 |A|^p - c_2 \leq W(A) \leq$

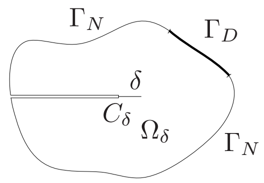

Figure 1: Domain Ω_δ with crack C_δ

$c_3(1+|A|^p)$, $|DW(A)| \leq c(1+|A|^{p-1})$ and $\left|D^2W(A)\right| \leq c(1+|A|^{p-2})$. Furthermore, $\Gamma : (0,\infty) \to [0,\infty)$ is convex, twice differentiable, $\Gamma(s) \to \infty$ for $s \to 0$ and satisfies $|s\Gamma'(s)| + \left|s^2\Gamma''(s)\right| \leq c(\Gamma(s)+1)$ for every $s > 0$ and some constant $c > 0$. For example, $\Gamma(s) = s^r + s^{-r}$ for some $r > 1$. The proofs of **A2** and **A3** are based on the following relations for $A \in \mathbb{M}^{d\times d}_+$, $B \in \mathbb{M}^{d\times d}$:

$$
\begin{aligned}
D_A(\det A) &= \operatorname{cof} A, \quad A^\top \operatorname{cof} A = (\det A)\mathbf{1},\\
\big(D_A(\Gamma'(\det A)\operatorname{cof} A)\big)[AB]&\\
&= \Gamma''(\det A)(\operatorname{cof} A : (AB))\operatorname{cof} A + \Gamma'(\det A)(D_A \operatorname{cof} A)[AB],\\
(D_A \operatorname{cof} A)[AB] = D_A(\operatorname{cof} A &: (AB)) - (\operatorname{cof} A)B^\top = (\operatorname{cof} A)((\operatorname{tr} B)\mathbf{1} - B^\top).
\end{aligned}
$$

Here, $\operatorname{cof} A \in \mathbb{M}^{d\times d}$ denotes the cofactor matrix of A.

3 Energy release rate, Griffith formula and J-integral

As already discussed in the introduction we consider the simplest geometrical situation, namely a straight crack in a two dimensional body. Let $S_\delta = \{\, x \in \mathbb{R}^2\,;\, x_2 = 0, x_1 \leq \delta\,\}$ for $\delta \in \mathbb{R}$.

A4 $\tilde\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary, $0 \in \tilde\Omega$ and there exists a constant $\delta_0 > 0$ such that $\partial\tilde\Omega \cap S_\delta$ is a single point for every $|\delta| \leq \delta_0$. Let $\Omega_\delta = \tilde\Omega \backslash S_\delta$ and $C_\delta = \overline{\tilde\Omega} \cap S_\delta$ for $|\delta| \leq \delta_0$. The boundary of Ω_δ is split as follows: $\partial\Omega_\delta = C_\delta \cup \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where $C_\delta, \Gamma_D, \Gamma_N$ are pairwise disjoint, Γ_D and Γ_N are open and independent of δ and $\Gamma_D \neq \emptyset$, see figure 1.

The domains Ω_δ satisfy the interior cone condition. We call Ω_0 reference configuration with initial crack C_0. Γ_D and Γ_N denote the Dirichlet and Neumann boundary, respectively. For $\delta > 0$, the pairs $(\Omega_\delta, C_\delta)$ describe admissible neighboring configurations. The following conditions are imposed on the volume force density and the Dirichlet- and Neumann data, where we adopt the usual notation for Sobolev-Slobodeckij spaces [Ada92, Gri85]:

A5 $p \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $g_D \in W^{1-\frac{1}{p},p}(\Gamma_D, \mathbb{R}^2)$, $h \in \left(W^{1-\frac{1}{p},p}(\Gamma_N, \mathbb{R}^2)\right)'$ and $f \in L^q(\tilde\Omega, \mathbb{R}^2)$. Furthermore, $\theta \in \mathcal{C}_0^\infty(\tilde\Omega)$ is a cut-off function with $\theta = 1$ in a neighborhood of the origin.

For $|\delta| \le \delta_0$, $p > 1$ and g_D from **A5** we define

$$V^p(\Omega_\delta) = \left\{ u \in W^{1,p}(\Omega_\delta, \mathbb{R}^2) \, ; \, u\big|_{\Gamma_D} = g_D \right\}.$$

The minimization problem for determining the deformation fields u_δ corresponding to Ω_δ now reads for $|\delta| \le \delta_0$:

$\mathbf{P}_\delta$ Find $u_\delta \in V^p(\Omega_\delta)$ such that for every $v \in V^p(\Omega_\delta)$

$$I(\Omega_\delta, u_\delta) \le I(\Omega_\delta, v) := \int_{\Omega_\delta} W(\nabla v)\,\mathrm{d}x - \int_{\Omega_\delta} f \cdot v\,\mathrm{d}x - \langle h, v\big|_{\Gamma_N} \rangle_{\Gamma_N}. \tag{3.1}$$

Here, $\langle \cdot, \cdot \rangle_{\Gamma_N}$ denotes the dual pairing in $(W^{1-\frac{1}{p},p}(\Gamma_N))'$ and $W^{1-\frac{1}{p},p}(\Gamma_N)$. In order to obtain existence of minimizers we need a coercivity assumption on the energy density W.

A6 $p \ge 2$ and there exist constants $\beta \in \mathbb{R}$, $r > 1$, $\alpha_1 > 0$ and α_2 with $\alpha_2 = 0$ if $p > 2$ and $\alpha_2 > 0$, else, such that $W(A) \ge \alpha_1 |A|^p + \alpha_2 |\det A|^r + \beta$ for every $A \in \mathbb{M}^{2\times 2}$.

The energy densities from example 2.7 satisfy **A1-A3** and **A6**. The following existence theorem is due to Ball [Bal77], see also [Cia93].

Theorem 3.1. *Let $d = 2$, $p \ge 2$ and assume that* **A1** *and* **A4-A6** *are satisfied. Assume in addition that* $\inf_{v \in V^p(\Omega_0)} I(\Omega_0, v) < \infty$*. Then problem* $\mathbf{P}_\delta$ *has a solution* $u_\delta \in V^p(\Omega_\delta)$ *for every* $|\delta| \le \delta_0$*.*

Note that minimizers of $I(\Omega_\delta, \cdot)$ need not be unique.

Definition 3.2. For $|\delta| \le \delta_0$ let $I(\Omega_\delta) = \min\{ I(\Omega_\delta, v) \, ; \, v \in V^p(\Omega_\delta) \}$. The energy release rate $ERR(\Omega_0)$ for the domain Ω_0 with crack C_0 and data f, g_D, h is defined by

$$ERR(\Omega_0) = \lim_{\substack{\delta \to 0,\\ \delta > 0}} \frac{I(\Omega_0) - I(\Omega_\delta)}{\delta}. \tag{3.2}$$

The next assumption is a condition on sequences of minimizer u_δ of $I(\Omega_\delta, \cdot)$:

A7 $u_* \in V^p(\Omega_0)$ is a minimizer of $I(\Omega_0, \cdot)$ and for every $\delta > 0$ exists a minimizer u_δ of $I(\Omega_\delta, \cdot)$ such that the whole sequence $\{ u_\delta \circ T_\delta^{-1} \, ; \, \delta > 0 \}$ converges weakly to u_* in $W^{1,p}(\Omega_0)$ for $\delta \to 0$ and, if $p = 2$, then $\det \nabla(u_\delta \circ T_\delta^{-1}) \rightharpoonup \det \nabla u_*$ weakly in $L^r(\Omega_0)$ with $r > 1$ from **A6**.

We are now ready to formulate our main result.

Theorem 3.3. *Let $d = 2$, $p \ge 2$ and* **A1-A6** *be satisfied,* $\inf_{v \in V^p(\Omega_0)} I(\Omega_0, v) < \infty$*. Then the energy release rate* $ERR(\Omega_0)$ *is well defined which means that the limit in* (3.2) *exists and is finite. Moreover, a generalized Griffith formula is valid:*

$$ERR(\Omega_0) = \max\left\{ G(u_0, \theta) \, ; \, u_0 \textit{ minimizes } I(\Omega_0, \cdot) \textit{ over } V^p(\Omega_0) \right\}, \tag{3.3}$$

where

$$G(u_0,\theta) = \int_{\Omega_0} \big(\nabla u_0^\top DW(\nabla u_0) - W(\nabla u_0)\mathbf{1}\big) : \nabla \begin{pmatrix}\theta\\0\end{pmatrix} \,\mathrm{d}x - \int_{\Omega_0} \theta f \cdot \frac{\partial}{\partial x_1} u_0 \,\mathrm{d}x. \tag{3.4}$$

The function θ is an arbitrary cut-off function from **A5**. *Moreover, $G(u_0,\theta) \equiv G(u_0)$ is independent of the choice of θ for every minimizer u_0. Let finally $T_\delta : \Omega_\delta \to \Omega_0$, $T_\delta(x) = x - \delta \left(\begin{smallmatrix}\theta(x)\\0\end{smallmatrix}\right)$. Then, for minimizers u_0 of $I(\Omega_0,\cdot)$ the following identity holds:*

$$G(u_0) = \lim_{\delta\to 0} \delta^{-1}\big(I(\Omega_0,u_0) - I(\Omega_\delta, u_0\circ T_\delta)\big). \tag{3.5}$$

For every minimizer u_* of $I(\Omega_0,\cdot)$ with property **A7** *we have $ERR(\Omega_0) = G(u_*)$.*

We postpone the proof of this theorem to section 4.3. Let us emphasize that the maximum in (3.3) is attained. It is an open question whether every minimizer of $I(\Omega_0,\cdot)$ has property **A7**. If this would be the case, then $ERR(\Omega_0) = G(u_0)$ for every minimizer u_0 of $I(\Omega_0,\cdot)$. Note that $ERR(\Omega_0) \geq 0$, since $V^p(\Omega_0) \subset V^p(\Omega_\delta)$ for $\delta > 0$ and therefore $I(\Omega_\delta) \leq I(\Omega_0)$. In the case of unique minimizers, (3.3) corresponds to formulas for the energy release rate in literature on fracture mechanics, see e.g. [Gur79, Che67].

Furthermore, we have the following behavior of the energy $I(\Omega_\delta)$ with respect to the parameter δ:

Theorem 3.4. *Let the assumptions* **A1**-**A6** *be satisfied and $I(\Omega_0) < \infty$. There exists a constant $\delta_0 > 0$ such that the function $E : [-\delta_0,\delta_0] \to \mathbb{R}$, $\delta \mapsto I(\Omega_\delta)$ is Lipschitz continuous, not increasing and for every $|\delta| < \delta_0$ the left and right derivatives exist and equal to*

$$\lim_{h\searrow 0} (-h)^{-1}(E(\delta - h) - E(\delta)) = \max\{\, -G(u_\delta)\,;\, u_\delta \textit{ minimizes } I(\Omega_\delta,\cdot)\,\},$$
$$\lim_{h\searrow 0} h^{-1}(E(\delta + h) - E(\delta)) = \min\{\, -G(u_\delta)\,;\, u_\delta \textit{ minimizes } I(\Omega_\delta,\cdot)\,\} = -ERR(\Omega_\delta).$$

The next theorem relates Griffith formula (3.4) with the J-integral.

Theorem 3.5. *Let assumptions* **A1**-**A6** *be satisfied and let $R_0 > 0$ such that $B_{R_0}(0) \Subset \tilde{\Omega}$. Assume furthermore that $\partial_1 f = 0$ on $B_{R_0}(0)$. For $R \in (0,R_0)$ let $\Gamma_R = \{\, x \in \mathbb{R}^2\,;\, |x| = R\,\}$ be a circular path around the crack tip 0 with interior unit normal vector $\vec{n} = (n_1,\, n_2)^\top$. For every minimizer u_0 of $I(\Omega_0,\cdot)$ and almost every $R \in (0,R_0)$ we have*

$$G(u_0) = \int_{\Gamma_R} \Big(\big(\nabla u_0^\top DW(\nabla u_0) - W(\nabla u_0)\mathbf{1}\big)\,\vec{n}\Big) \cdot \begin{pmatrix}1\\0\end{pmatrix} + (u_0 \cdot f) n_1 \,\mathrm{d}s. \tag{3.6}$$

The integrand in (3.6) is an element of $L^1(\Gamma_R)$ for almost every R.

The proof of this theorem is given in section 4.4 and uses the fact that $G(u_0,\theta)$ is independent of the cut-off function θ for minimizers u_0.

We will now give a comment on formula (3.3) for the energy release rate. In the case of several minimizers it is an open question whether $G(u_0) = G(u_1)$ for different minimizers u_0, u_1 of $I(\Omega_0, \cdot)$. One could interpret $G(u_0)$ as a local energy release rate, where local means that not only the geometry Ω_δ has to be close to Ω_0 but also that deformations on Ω_δ are considered which are in some sense "close" to u_0. In contrast to that, definition 3.2 describes the global energy release rate. Let u_0 be an arbitrary minimizer of $I(\Omega_0, \cdot)$ and assume that $ERR(\Omega_0) \geq 2\gamma$. The Griffith-criterion in the form (1.3) predicts that the crack will grow even though $G(u_0)$ might be less than 2γ. This means that the criterion in this formulation allows for a jump from minimizer u_0 to a minimizer u_1 with $G(u_1) = ERR(\Omega_0)$ and the crack will develop starting from the configuration defined through u_1. It is a modeling assumption whether one formulates Griffith's criterion in the global version (1.3) with $ERR(\Omega_0)$ from (3.3) and allows for jumps between minimizers, or whether one trusts in a local version of the type

$$\textit{If } G(u_0) < 2\gamma, \textit{ then the crack is stationary.} \tag{3.7}$$

In fact, (1.3) and (3.7) are based on different interpretations of the notion *admissible neighboring configuration* in (1.1).

This discussion might become clearer by considering a perturbed problem. Let $|\delta| \leq \delta_0$, $\varepsilon \geq 0$ and $\alpha \in (1, p]$. Let finally $u_0 \in V^p(\Omega_0)$ be a minimizer of $I(\Omega_0, \cdot)$. We define

$\mathbf{P}_\delta^\varepsilon$ Find $u_\delta^\varepsilon \in V^p(\Omega_\delta)$ such that for every $v \in V^p(\Omega_\delta)$

$$I_\delta^\varepsilon(u_0; u_\delta^\varepsilon) \leq I_\delta^\varepsilon(u_0; v) := I(\Omega_\delta, v) + \varepsilon \int_{\Omega_\delta} |v - u_0|^\alpha \, \mathrm{d}x. \tag{3.8}$$

Obviously, $\mathbf{P}_\delta^0 = \mathbf{P}_\delta$.

Theorem 3.6. *Let* **A1**-**A6** *be satisfied,* $p \geq 2$, $\alpha \in (1, p]$, $\inf_{v \in V^p(\Omega_0)} I(\Omega_0, v) < \infty$ *and* u_0 *a minimizer of* $I(\Omega_0, \cdot)$.

1. *Problem* $\mathbf{P}_\delta^\varepsilon$ *has a solution for every* $\varepsilon \geq 0$, $|\delta| \leq \delta_0$. *Moreover,* u_0 *is the unique minimizer of* $\mathbf{P}_0^\varepsilon$ *for every* $\varepsilon > 0$ *and* $I_0^\varepsilon(u_0; u_0) = I(\Omega_0, u_0) = I(\Omega_0)$.

2. *Let* $I_\delta^\varepsilon(u_0) = \min\{\, I_\delta^\varepsilon(u_0; v)\, ;\, v \in V^p(\Omega_\delta)\, \}$. *For every* $\varepsilon > 0$ *the function* $\delta \to I_\delta^\varepsilon(u_0)$ *is differentiable in* $\delta = 0$ *and*

$$\lim_{\delta \to 0} \delta^{-1} \big(I_0^\varepsilon(u_0) - I_\delta^\varepsilon(u_0) \big) = G(u_0, \theta) = G(u_0) \tag{3.9}$$

with $G(\cdot, \cdot)$ *from* (3.4). *The function* θ *is arbitrary as long as* **A5** *holds.*

This theorem will be proved in section 4.2. Theorem 3.6 reveals that $G(u_0)$ describes a local energy release rate whereas $ERR(\Omega_0)$ from (3.3) is a global one. It follows from (3.9) that $G(u_0)$ is independent of the choice of the cut-off function θ since the left hand side in (3.9) does not depend on θ.

4 Proofs

Like in the case of linear elastic models the proofs of our main results are based on the diffeomorphism $T_\delta : \Omega_\delta \to \Omega_0$, which we already introduced in theorem 3.3. We apply the mapping T_δ in order to transform the integral expressions in the difference quotient $\delta^{-1}\big(I(\Omega_0, u_0) - I(\Omega_\delta, u_\delta)\big)$ to the fixed domain Ω_0. The limit for $\delta \to 0$ is then calculated in the domain Ω_0. This section is organized as follows: In section 4.1 we summarize the properties of the mapping $T_\delta : \Omega_\delta \to \Omega_0$, derive convergence results for sequences $u_\delta \circ T_\delta^{-1}$ and prove two lemmata on the derivative of $I_\delta^\varepsilon(u_0; v_\delta)$ with respect to special paths v_δ which are parameterized by T_δ. Here, the convergence results from section 2 are essential. In sections 4.2 (for $\varepsilon > 0$) and 4.3 (for $\varepsilon = 0$) we combine these lemmata and prove theorems 3.3, 3.4 and 3.6 on the energy release rate and its connection with the Griffith formula. In section 4.4 we prove theorem 3.5 on the J-integral.

4.1 Convergence results based on the inner variation T_δ

The domains Ω_δ are mapped to Ω_0 in the following way: Let $\theta \in \mathcal{C}_0^\infty(\tilde{\Omega})$ be a function according to **A5**. Choose $0 < \delta_0 = \delta_0(\theta) < \frac{1}{2} \|\nabla\theta\|_{L^\infty(\tilde{\Omega})}^{-1}$ in such a way that $\theta = 1$ on the line $\{\, x \in \mathbb{R}^2 \,;\, x_2 = 0, |x_1| \le \delta_0 \,\}$. For $|\delta| \le \delta_0$ we define

$$T_\delta : \Omega_\delta \to \Omega_0, \quad x \mapsto y = T_\delta(x) = x - \delta \begin{pmatrix} \theta(x) \\ 0 \end{pmatrix} \tag{4.1}$$

and use the notation

$$q_\delta(x) := \det \nabla_x T_\delta(x) = 1 - \delta\theta_{,1}(x), \tag{4.2}$$

where $\theta_{,i}(x) = \frac{\partial}{\partial x_i}\theta(x)$ for $i \in \{1, 2\}$. The mapping T_δ is an element of $\mathcal{C}^\infty(\overline{\Omega_\delta})$ and $\det \nabla_x(T_\delta(x)) \ge c > 0$ for every $|\delta| \le \delta_0$ and $x \in \overline{\tilde{\Omega}}$. Moreover, T_δ is a diffeomorphism and maps the crack C_δ to C_0, see e.g. [DD81, GH96]. For functions $v_\delta : \Omega_\delta \to \mathbb{R}^2$ we introduce the notation $v^\delta(y) = v_\delta(T_\delta^{-1}(y))$ for $y \in \Omega_0$. Furthermore,

$$\nabla_x\theta^\delta(y) := \nabla_x\theta(T_\delta^{-1}(y)),\ \nabla_x T^\delta(y) := \nabla_x T_\delta(\cdot)\big|_{T_\delta^{-1}(y)},\ q^\delta(y) := \det \nabla_x T^\delta(y).$$

Derivatives are transformed as follows for $x \in \Omega_\delta$ and $y \in \Omega_0$:

$$\nabla_x v_\delta(T_\delta^{-1}(y)) = \nabla_y v^\delta(y)\nabla_x T^\delta(y), \quad \nabla_y v^\delta(T_\delta(x)) = \nabla_x v_\delta(x)(\nabla_x T_\delta(x))^{-1}.$$

Elementary calculations show that T_δ induces an isomorphism $\mathbf{T}_\delta$ between the spaces $W^{1,p}(\Omega_\delta)$ and $W^{1,p}(\Omega_0)$ for every $p \in (1, \infty)$ via

$$\mathbf{T}_\delta : W^{1,p}(\Omega_0) \to W^{1,p}(\Omega_\delta) : \ u \mapsto u \circ T_\delta.$$

For fixed $p \in (1, \infty)$, the operator norms of $\mathbf{T}_\delta$ and $\mathbf{T}_\delta^{-1}$ are bounded with respect to $|\delta| \le \delta_0$. The same holds for $L^p(\Omega_\delta)$. Moreover, $\mathbf{T}_\delta$ is a bijection between $V^p(\Omega_\delta)$ and $V^p(\Omega_0)$ since $T_\delta\big|_{\partial\tilde{\Omega}} = \mathrm{id}$ and therefore, $\mathbf{T}_\delta$ keeps the boundary conditions unchanged. Let us finally remark that the constants in the Poincaré/Friedrichs inequality as well as the constants in embedding theorems for Sobolev spaces are uniformly bounded with respect to $|\delta| \leq \delta_0$.

The next technical lemma can be seen as an analog to convergence theorems for difference quotients of $W^{1,p}$-functions.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^2$ be open and bounded and $\theta \in \mathcal{C}_0^\infty(\Omega, \mathbb{R})$. Let furthermore $\delta_0 > 0$ be small enough such that the family $T_\delta : \Omega \to \Omega$, $x \mapsto x - (\delta\theta(x), 0)^\top$ is a diffeomorphism for every $|\delta| \leq \delta_0$. We define $\varphi^\delta(y) = \varphi(T_\delta^{-1}(y))$ for $y \in \Omega$. Then it holds:*

1. *Let $p \in [1, \infty)$ and $\varphi \in L^p(\Omega)$. Then $\varphi^\delta \to \varphi$ strongly in $L^p(\Omega)$ for $\delta \to 0$.*
2. *Assume in addition that $\partial\Omega$ is Lipschitz. For $p \in [1, \infty)$ we define $L^p(\Omega, \partial_1) = \{\, \varphi \in L^p(\Omega, \mathbb{R})\,;\, \partial_1\varphi \in L^p(\Omega)\,\}$. Then there exists a constant $c = c(\theta) > 0$ such that for every $\varphi \in L^p(\Omega, \partial_1)$ and $|\delta| \leq \delta_0$*
$$\left\| \delta^{-1}(\varphi^\delta - \varphi) \right\|_{L^p(\Omega)} \leq c(\theta) \left\| \partial_1 \varphi \right\|_{L^p(\Omega)} . \tag{4.3}$$
Moreover, $\delta^{-1}(\varphi^\delta - \varphi) \to \theta\partial_1\varphi$ strongly in $L^p(\Omega)$.
3. *Assume that $\partial\Omega$ is Lipschitz, $p \in (1, \infty)$. Let $\{\, \varphi, \varphi^{\delta_n}\,;\, |\delta_n| \leq \delta_0,\, n \in \mathbb{N}\,\} \subset L^p(\Omega, \partial_1)$ be a sequence with $\delta_n \to 0$ for $n \to \infty$ and $\varphi^{\delta_n} \rightharpoonup \varphi$ weakly in $L^p(\Omega, \partial_1)$. Then $\delta_n^{-1}(\varphi^{\delta_n} - \varphi^{\delta_n} \circ T_{\delta_n}) \rightharpoonup \theta\partial_1\varphi$ weakly in $L^p(\Omega)$.*

Proof. The first assertion of lemma 4.1 can be proved completely analogously to the corresponding assertion on the strong convergence of finite differences of L^p functions, see e.g. [Neč67]. We prove the second and third assertion in detail.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. The set $L^p(\Omega, \partial_1)$ with the norm $\|\varphi\|_{L^p(\Omega,\partial_1)} = \|\varphi\|_{L^p(\Omega)} + \|\partial_1\varphi\|_{L^p(\Omega)}$ is a reflexive Banach space since it is a closed subspace of the reflexive Banach space $L^p(\Omega, \operatorname{div}) = \{\, f \in L^p(\Omega, \mathbb{R}^2)\,;\, \operatorname{div} f \in L^p(\Omega)\,\}$ via $\varphi \mapsto (\varphi, 0)^\top$. The set $\mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^2)$ is dense in $L^p(\Omega, \operatorname{div})$, see e.g. [GS86], and therefore it follows that $\mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ is dense in $L^p(\Omega, \partial_1)$ as well. We prove first that the second part of lemma 4.1 is valid for $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$. By density, this result is then carried over to the whole space $L^p(\Omega, \partial_1)$.

For $|\delta| \leq \delta_0$ and $\varphi \in L^p(\Omega, \partial_1)$ we define $L_\delta(\varphi) = \frac{1}{\delta}(\varphi \circ T_\delta^{-1} - \varphi)$, $L_0(\varphi) = \theta\partial_1\varphi$. Obviously, $L_\delta : L^p(\Omega, \partial_1) \to L^p(\Omega)$ is a linear and bounded operator for every $|\delta| \leq \delta_0$. We prove now that the operator norms are uniformly bounded.

Let $|\delta| \leq \delta_0$ and $h \in \mathbb{R}$ small. Let further $x = T_\delta^{-1}(y)$, $x_h = T_{\delta+h}^{-1}(y)$ for every $y \in \Omega$. Then it holds

$$\lim_{h\to 0} \frac{1}{h}(x_h - x) = \lim_{h\to 0} \frac{1}{h}(T_{\delta+h}^{-1}(y) - T_\delta^{-1}(y)) = \frac{\theta(x)}{q_\delta(x)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.4}$$

with $q_\delta = \det \nabla_x T_\delta$ from (4.2). Relation (4.4) can be derived as follows: By the definition of x and x_h we have $0 = T_\delta(x) - T_{\delta+h}(x_h)$ which implies

$$\begin{pmatrix} \theta(x_h) \\ 0 \end{pmatrix} = \frac{1}{h}(x_h - x) - \frac{\delta}{h}\begin{pmatrix} \theta(x_h) - \theta(x) \\ 0 \end{pmatrix} = \frac{1}{h}\nabla T_\delta(x_s)(x_h - x),$$

where $x_s = x + s(x_h - x)$ for some $s = s(x,h) \in [0,1]$. By assumption, the matrix $\nabla T_\delta(x_s)$ is invertible for every x_s and $|\delta| \le \delta_0$ and thus

$$\frac{1}{h}(x_h - x) = (\nabla T_\delta(x_s))^{-1}\left(\begin{smallmatrix} \theta(x_h) \\ 0 \end{smallmatrix}\right) = \frac{\theta(x_h)}{q_\delta(x_s)}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $x_h \to x$ and $x_s \to x$ for $|h| \to 0$, we obtain (4.4). Let now $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$. For every $y \in \Omega$ it follows with (4.4) that

$$L_\delta(\varphi)(y) = \frac{1}{\delta}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\varphi(T_{s\delta}^{-1}(y))\,\mathrm{d}s = \int_0^1 \frac{\theta^{s\delta}(y)\partial_1\varphi^{s\delta}(y)}{q^{s\delta}(y)}\,\mathrm{d}s. \tag{4.5}$$

Hölder's inequality and a transformation of coordinates ($y = T_{s\delta}(x)$) lead to

$$\|L_\delta(\varphi)\|_{L^p(\Omega)}^p \le \int_0^1\int_\Omega \left|\frac{\theta(x)\partial_1\varphi(x)}{q_{s\delta}(x)}\right|^p q_{s\delta}(x)\,\mathrm{d}x\,\mathrm{d}s \le c(\theta)\,\|\varphi\|_{L^p(\Omega,\partial_1)}^p. \tag{4.6}$$

Since $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $L^p(\Omega,\partial_1)$, we obtain (4.6) immediately for arbitrary functions from $L^p(\Omega,\partial_1)$ and thus $\sup_{|\delta|\le\delta_0}\|L_\delta\| \le c$. Let again $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$. Relation (4.5) implies that

$$\|L_\delta(\varphi) - L_0(\varphi)\|_{L^p(\Omega)}^p \le \int_0^1\int_\Omega \left|\frac{\theta^{s\delta}(y)\partial_1\varphi^{s\delta}(y)}{q^{s\delta}(y)} - \theta(y)\partial_1\varphi(y)\right|^p \mathrm{d}y\,\mathrm{d}s. \tag{4.7}$$

Taking into account that $\theta^{s\delta}(y)\partial_1\varphi^{s\delta}(y)/q^{s\delta}(y)$ converges for fixed $(s,y) \in [0,1]\times\Omega$ to $\theta(y)\partial_1\varphi(y)$, the dominated convergence theorem implies that the right hand side in (4.7) converges to 0. Since the operators L_δ are uniformly bounded and since $L_\delta(\varphi) \to L_0(\varphi)$ in $L^p(\Omega)$ on the dense subset $\mathcal{C}^\infty(\overline{\Omega})$, we obtain immediately that $L_\delta(\varphi) \to L_0(\varphi)$ for every $\varphi \in L^p(\Omega,\partial_1)$. This finishes the proof of part two.

Let $L_\delta^* : L^p(\Omega,\partial_1) \to L^p(\Omega)$ be defined by $L_\delta^*(\varphi) = \frac{1}{\delta}(\varphi - \varphi\circ T_\delta)$ for $\delta \ne 0$. A transformation of coordinates leads to

$$\langle L_\delta^*(\varphi), v\rangle = -\langle L_\delta(v),\varphi\rangle - \frac{1}{\delta}\int_\Omega \Big(\frac{1}{\det\nabla T^\delta} - 1\Big)\varphi v\circ T_\delta\,\mathrm{d}x \tag{4.8}$$

for every $\varphi \in L^p(\Omega,\partial_1)$, $v \in \mathcal{C}_0^\infty(\Omega)$ and $\delta \ne 0$. Here, $\langle u,v\rangle = \int_\Omega uv\,\mathrm{d}x$. Note that $\delta^{-1}((\det\nabla T^\delta)^{-1} - 1) = \theta^\delta_{,1}/q^\delta$ and therefore, together with the first and second part of this lemma, we have

$$L_\delta(v) + \frac{1}{\delta}\left(\frac{1}{\det\nabla T^\delta} - 1\right) v\circ T_\delta \;\to\; \theta\partial_1 v + v\partial_1\theta \tag{4.9}$$

strongly in $L^q(\Omega)$ for every $v \in \mathcal{C}_0^\infty(\Omega)$ and $q \in [1,\infty)$. Let now $\{\, \varphi^{\delta_n}\,;\, n \in \mathbb{N}\,\} \subset L^p(\Omega, \partial_1)$ be a sequence which converges weakly to $\varphi \in L^p(\Omega,\partial_1)$ for some $p \in (1,\infty)$. Relations (4.8) and (4.9) show that

$$\lim_{\delta_n \to 0} \langle L^*_{\delta_n}(\varphi^{\delta_n}), v\rangle = -\int_\Omega \varphi \partial_1(\theta v)\,\mathrm{d}x = \int_\Omega v\theta\partial_1\varphi\,\mathrm{d}x \tag{4.10}$$

for every $v \in \mathcal{C}_0^\infty(\Omega)$. In the last equality we have used the Gauss theorem, which is applicable to elements from $L^p(\Omega,\partial_1)$. Since $L^*_\delta(\varphi) = L_\delta(\varphi \circ T_\delta)$, estimate (4.3) and the weak convergence of the sequence $\{\,\varphi^{\delta_n}\,;\, n\in\mathbb{N}\,\}$ imply that there is a constant $c > 0$ with $\left\| L^*_{\delta_n}(\varphi_{\delta_n})\right\|_{L^p(\Omega)} \le c$ for every $n \in \mathbb{N}$. Combining this estimate with (4.10) and taking into account that $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^{p'}(\Omega)$ finally proves the last assertion of lemma 4.1. $\square$

The next lemma states that the mapping $\delta \to I^\varepsilon_\delta(u_0)$ is Lipschitz continuous. This result is based on the mapping properties of T_δ, the coercivity of W and assumption **A2**. We recall that $I^\varepsilon_\delta(u_0) = \min\{\, I^\varepsilon_\delta(u_0; v)\,;\, v \in V^p(\Omega_\delta)\,\}$ for $\varepsilon \ge 0$ with $I^\varepsilon_\delta(u_0;\cdot)$ from (3.8). For $\varepsilon = 0$, it is $I^0_\delta(u_0) \equiv I(\Omega_\delta) = \inf\{\, I(\Omega_\delta, v)\,;\, v \in V^p(\Omega_\delta)\,\}$.

Lemma 4.2. *Let $p \ge 2$, $\varepsilon \ge 0$, $\alpha \in (1,p]$ and let* **A1**, **A2** *and* **A4**-**A6** *be satisfied. Assume further that $\inf_{v\in V^p(\Omega_0)} I(\Omega_0, v) < \infty$ and that u_0 minimizes $I(\Omega_0,\cdot)$. Then there exists a constant $\delta_0 > 0$ such that $\inf_{v\in V^p(\Omega_\delta)} I^\varepsilon_\delta(u_0; v) < \infty$ for every $|\delta| \le \delta_0$. Furthermore, the set*

$$L := \{\, u \in V^p(\Omega_0)\,;\, \exists\, |\delta| \le \delta_0 \textit{ such that } u \circ T_\delta \textit{ minimizes } I^\varepsilon_\delta(u_0;\cdot)\,\} \tag{4.11}$$

is bounded in $W^{1,p}(\Omega_0)$. If $p = 2$, then the set $\{\, \det \nabla u\,;\, u \in L\,\}$ is bounded in $L^r(\Omega_0)$ with r from **A6**. *Moreover, the set*

$$\{\, \int_{\Omega_\delta} W(\nabla u_\delta)\,\mathrm{d}x\,;\, u_\delta \textit{ minimizes } I^\varepsilon_\delta(u_0;\cdot),\ |\delta| \le \delta_0\,\} \tag{4.12}$$

is bounded in $\mathbb{R}$. Finally there exists a constant $c > 0$ such that

$$\left| I^\varepsilon_{\delta_1}(u_0) - I^\varepsilon_{\delta_2}(u_0)\right| \le c\,|\delta_1 - \delta_2|\,, \tag{4.13}$$

$$0 \le I^\varepsilon_0(u_0; u_\delta \circ T_\delta^{-1}) - I^\varepsilon_0(u_0) \le c\,|\delta|\,, \tag{4.14}$$

for every $\delta, \delta_1, \delta_2 \in [-\delta_0, \delta_0]$ and every minimizer u_δ of $I^\varepsilon_\delta(u_0;\cdot)$.

In particular, lemma 4.2 implies that every sequence $\{\, u_\delta \circ T_\delta^{-1}\,;\, |\delta| \le \delta_0\,\}$, where u_δ minimizes $I^\varepsilon_\delta(u_0;\cdot)$, is a minimizing sequence for $I^\varepsilon_0(u_0,\cdot)$ for $\delta \to 0$.

Proof. Let $\varepsilon \ge 0$ be arbitrary and let $T_\delta : \Omega_\delta \to \Omega_0$ be the mapping defined in (4.1). Chose $\delta_0 \le 1/(2\,\|\nabla\theta\|_{L^\infty(\tilde{\Omega})})$ small such that $|\nabla T_{2\delta_0}(x) - \mathbf{1}| \le \gamma_1$ and $\left|(\nabla T_{2\delta_0}(x))^{-1} - \mathbf{1}\right| \le \gamma_1$ with γ_1 from lemma 2.1. Let $v \in V^p(\Omega_0)$ with $I(\Omega_0, v) < \infty$. It follows from estimate (2.3) for the energy density W that $I^\varepsilon_\delta(u_0; v \circ T_\delta) < \infty$ for every $|\delta| \le \delta_0$. This proves the first assertion of lemma 4.2.

For $\delta_1 \leq \delta_2$ we have $V^p(\Omega_{\delta_1}) \subset V^p(\Omega_{\delta_2})$ and thus

$$I^\varepsilon_{\delta_0}(u_0) \leq I^\varepsilon_\delta(u_0) \leq I^\varepsilon_{-\delta_0}(u_0) < \infty \tag{4.15}$$

for every $|\delta| \leq \delta_0$. Coercivity assumption **A6** implies that the set

$$\{\, v \in V^p(\Omega_{\delta_0})\,;\, I^\varepsilon_{\delta_0}(u_0; v) \leq I^\varepsilon_{-\delta_0}(u_0) \,\} \tag{4.16}$$

is bounded in $V^p(\Omega_{\delta_0})$. Since the minimizers u_δ of $I^\varepsilon_\delta(u_0; \cdot)$ are contained in this set, we get together with the mapping properties of T_δ that the set L from (4.11) is bounded in $V^p(\Omega_0)$. Estimate (4.15) and the boundedness of the set L finally imply that the set $\{\, \int_{\Omega_\delta} W(\nabla u_\delta)\,\mathrm{d}x\,;\, u_\delta$ minimizes $I^\varepsilon_\delta(u_0; \cdot)$, $|\delta| \leq \delta_0 \,\}$ is bounded as well.

We will now prove the Lipschitz continuity of the mapping $\delta \to I^\varepsilon_\delta(u_0)$. Let $|\delta_1|, |\delta_2| \leq \delta_0$ with $\delta_1 \leq \delta_2$. Then

$$0 \leq I^\varepsilon_{\delta_1}(u_0) - I^\varepsilon_{\delta_2}(u_0) \leq I^\varepsilon_{\delta_1}(u_0; u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1}) - I^\varepsilon_{\delta_2}(u_0; u_{\delta_2}), \tag{4.17}$$

where u_{δ_2} is an arbitrary minimizer of $I^\varepsilon_{\delta_2}(u_0; \cdot)$. Note that $T_{\delta_2-\delta_1} : \Omega_{\delta_2} \to \Omega_{\delta_1}$ is a diffeomorphism. After a transformation of coordinates ($y = T_{\delta_2-\delta_1}(x)$) in the terms with the energy density W we obtain

$$\begin{aligned}
&I^\varepsilon_{\delta_1}(u_0; u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1}) - I^\varepsilon_{\delta_2}(u_0; u_{\delta_2}) \\
&\quad = \int_{\Omega_{\delta_2}} W(\nabla_y u_{\delta_2}(x)) q_{\delta_2-\delta_1}(x)\,\mathrm{d}x - \int_{\Omega_{\delta_2}} W(\nabla_x u_{\delta_2}(x))\,\mathrm{d}x \\
&\qquad - \int_{\Omega_{\delta_1}} f \cdot u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1}\,\mathrm{d}y + \int_{\Omega_{\delta_2}} f \cdot u_{\delta_2}\,\mathrm{d}x \\
&\qquad + \varepsilon \int_{\Omega_{\delta_1}} |u_0 - u_{\delta_2} \circ T_{\delta_2-\delta_1}|^\alpha\,\mathrm{d}y - \varepsilon \int_{\Omega_{\delta_2}} |u_0 - u_{\delta_2}|^\alpha\,\mathrm{d}x \\
&\quad = I_1 + \ldots + I_6.
\end{aligned} \tag{4.18}$$

Here, $q_{\delta_2-\delta_1}(x) = \det \nabla T_{\delta_2-\delta_1}(x)$. Our next task is to show that

$$|I_1 + \ldots I_6| \leq c\,|\delta_1 - \delta_2| \tag{4.19}$$

with a constant c which is independent of u_{δ_2}. For the estimate of $I_5 + I_6$ we apply the following inequality, see e.g. [Kne05]: for every $\beta > 0$ exists a constant $c > 0$ such that for every $A, B \in \mathbb{R}^s$:

$$\left| |A|^\beta - |B|^\beta \right| \leq c(|A| + |B|)^{\beta-1}\,|A - B|\,. \tag{4.20}$$

The previous inequality, Hölder's inequality and the triangle inequality show that

$$\begin{aligned}
I_5 + I_6 &\leq \varepsilon \int_{\tilde\Omega} \left| |u_0 - u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1}|^\alpha - |u_0 - u_{\delta_2}|^\alpha \right| \mathrm{d}y \\
&\leq c\varepsilon \left(\left\| u_0 - u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1} \right\|_{L^\alpha(\tilde\Omega)} + \|u_0 - u_{\delta_2}\|_{L^\alpha(\tilde\Omega)} \right)^{\alpha-1} \left\| u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1} - u_{\delta_2} \right\|_{L^\alpha(\tilde\Omega)}.
\end{aligned} \tag{4.21}$$

The mapping properties of T_δ and the boundedness of the set L imply that the first factor in (4.21) is bounded independently of δ_1, δ_2 and u_{δ_2}. Since $\alpha \le p$, lemma 4.1 and the boundedness of the set L lead to

$$\left\| u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1} - u_{\delta_2} \right\|_{L^\alpha(\tilde\Omega)} \le c(\theta)\,|\delta_2 - \delta_1|\,\|\partial_1 u_{\delta_2}\|_{L^\alpha(\tilde\Omega)} \le c\,|\delta_1 - \delta_2|\,,$$

where c is independent of δ_1, δ_2 and u_{δ_2}. Altogether we have $|I_5 + I_6| \le c\varepsilon\,|\delta_1 - \delta_2|$. The terms $I_3 + I_4$ can be treated similarly: Hölder's inequality and lemma 4.1 yield

$$|I_3 + I_4| \le \int_{\tilde\Omega} |f|\,\left| u_{\delta_2} \circ T^{-1}_{\delta_2-\delta_1} - u_{\delta_2} \right|\,\mathrm{d}y \le |\delta_1 - \delta_2|\,\|f\|_{L^q(\tilde\Omega)}\,\|\partial_1 u_{\delta_2}\|_{L^q(\tilde\Omega)}$$

and thus $|I_3 + I_4| \le c\,|\delta_1 - \delta_2|$ for a constant c which is independent of δ_1, δ_2 and u_{δ_2}. Note that

$$\begin{aligned} I_1 + I_2 = &\int_{\Omega_{\delta_2}} W(\nabla_x u_{\delta_2} (\nabla_x T_{\delta_2-\delta_1})^{-1}) - W(\nabla_x u_{\delta_2})\,\mathrm{d}x \\ &- (\delta_2 - \delta_1)\int_{\Omega_{\delta_2}} \theta_{,1} W(\nabla_x u_{\delta_2} (\nabla_x T_{\delta_2-\delta_1})^{-1})\,\mathrm{d}x. \end{aligned}$$

The assumption on δ_0 entails that $\left|(\nabla T_{\delta_2-\delta_2}(x))^{-1} - \mathbf{1}\right| \le \gamma_1$ and therefore, the inequalities from lemma 2.1 are applicable to $I_1 + I_2$ and lead to

$$|I_1 + I_2| \le c\,|\delta_1 - \delta_2| \int_{\Omega_{\delta_2}} \big(W(\nabla u_{\delta_2}) + 1\big)\,\mathrm{d}x.$$

Since the set in (4.12) is bounded, we arrive finally at $|I_1 + I_2| \le c\,|\delta_1 - \delta_2|$. Altogether we have shown that estimate (4.19) is valid, which yields (4.13). For $|\delta| \le \delta_0$, estimate (4.14) follows from

$$\begin{aligned} 0 &\le I_0^\varepsilon(u_0; u_\delta \circ T_\delta^{-1}) - I_0^\varepsilon(u_0) \\ &\le \left| I_0^\varepsilon(u_0; u_\delta \circ T_\delta^{-1}) - I_\delta^\varepsilon(u_0; u_\delta) \right| + \left| I_\delta^\varepsilon(u_0; u_\delta) - I_0^\varepsilon(u_0) \right| \end{aligned}$$

and (4.17)-(4.19) with $\delta_1 = 0$ and $\delta_2 = \delta$. □

The next lemma provides a formula for the derivative of the energy $I_\delta^\varepsilon(u;\cdot) = I(\Omega_\delta,\cdot) + \varepsilon\int_{\Omega_\delta} |\cdot - u|^\alpha\,\mathrm{d}x$ with respect to the path $\{\, u \circ T_\delta\,;\, |\delta| \le \delta_0 \,\}$ for general $u \in V^p(\Omega_0)$.

Lemma 4.3. *Let* **A1**,**A2** *and* **A4**-**A6** *be satisfied,* $\varepsilon \ge 0$ *and* $\alpha \in (1,p]$. *Furthermore, let* $u \in V^p(\Omega_0)$ *with* $I(\Omega_0, u) < \infty$ *and let* θ *be chosen according to* **A5**, $T_\delta(x) = x - \delta(\theta(x), 0)^\top$. *Then the function* $\delta \mapsto I_\delta^\varepsilon(u; u \circ T_\delta)$ *is differentiable in* $\delta = 0$ *and*

$$-\left(\frac{\mathrm{d}}{\mathrm{d}\delta} I_\delta^\varepsilon(u; u \circ T_\delta)\right)\Big|_{\delta=0} = \lim_{\delta\to 0} \delta^{-1}\left(I_0^\varepsilon(u;u) - I_\delta^\varepsilon(u; u \circ T_\delta)\right) = G(u,\theta) \qquad (4.22)$$

with $G(\cdot,\cdot)$ *from* (3.4).

Note that the formula on the right hand side of (4.22) is independent of $\varepsilon \geq 0$. In the notion of [GH96], $G(u,\theta)$ is the (strong) inner variation of $I_0^\varepsilon(u;\cdot)$ at u in the direction of the vector field $(\theta,0)^\top$. The proof of lemma 4.3 relies on assumption **A2** and the dominated convergence theorem and we follow the arguments in [BOP91].

Proof. Assumption **A2** implies that $|I_\delta^\varepsilon(u;u\circ T_\delta)| < \infty$ for $|\delta| \leq \delta_0$ if δ_0 is small enough. A transformation of coordinates in $I_\delta^\varepsilon(u;u\circ T_\delta)$ leads to

$$\begin{aligned} I_0^\varepsilon(u;u) - I_\delta^\varepsilon(u;u\circ T_\delta) &= \int_{\Omega_0} \frac{1}{q^\delta}\big(W(\nabla_y u) - W(\nabla_x u)\big)\,\mathrm{d}y - \delta \int_{\Omega_0} \frac{\theta^\delta_{,1}}{q^\delta} W(\nabla_y u)\,\mathrm{d}y \\ &\quad - \int_{\tilde{\Omega}} \frac{1}{q^\delta} f^\delta \cdot (u^\delta - u)\,\mathrm{d}y - \varepsilon \int_{\Omega_0} \frac{1}{q^\delta} \left| u - u^\delta \right|^\alpha \,\mathrm{d}y \\ &= I_1 + \ldots + I_4, \end{aligned}$$

where $f^\delta(y) = f \circ T_\delta^{-1}(y)$ and where we have used that $\int_{\Omega_0} f\cdot u\,\mathrm{d}x - \int_{\Omega_\delta} f\cdot u_\delta\,\mathrm{d}x = \int_{\tilde{\Omega}} f\cdot(u-u_\delta)\,\mathrm{d}x$. Lemma 4.1 applied to I_4 yields

$$\left|\delta^{-1} I_4\right| \leq \frac{\varepsilon}{|\delta|} c(\theta) \left\| u - u^\delta \right\|^\alpha_{L^\alpha(\tilde{\Omega})} \leq \varepsilon\,|\delta|^{\alpha-1}\,c(\theta)\,\|\partial_1 u\|^\alpha_{L^\alpha(\tilde{\Omega})}$$

and thus $\delta^{-1} I_4 \to 0$ for $\delta \to 0$ since $\alpha \in (1,p]$. Again by lemma 4.1 we obtain for I_3:

$$\delta^{-1} I_3 \to -\int_{\Omega_0} \theta f \cdot \partial_1 u\,\mathrm{d}y.$$

Since $\theta^\delta_{,1}/q^\delta$ converges uniformly to $\theta_{,1}$ on Ω_0 for $\delta \to 0$, we have

$$\lim_{\delta\to 0} \delta^{-1} I_2 = -\int_{\Omega_0} \theta_{,1} W(\nabla_y u)\,\mathrm{d}y.$$

For the convergence of $\delta^{-1} I_1$ note that $\nabla_x u(y) = \nabla_y u(y) \nabla_x T_\delta(\cdot)\big|_{T_\delta^{-1}(y)}$ and

$$|\mathbf{1} - \nabla_x T_\delta(x)| \leq |\delta|\,\|\nabla\theta\|_{L^\infty(\tilde{\Omega})} \leq \gamma_1 \tag{4.23}$$

with γ_1 from lemma 2.1, if $|\delta|$ is small enough. The term $\delta^{-1} I_1$ can be written as follows:

$$\begin{aligned} &\frac{1}{\delta}\int_\Omega \frac{1}{q^\delta}\big(W(\nabla_y u(y)) - W(\nabla_x u(y))\big)\,\mathrm{d}y \\ &= \frac{1}{\delta}\int_\Omega\int_0^1 \frac{1}{q^\delta}\frac{\mathrm{d}}{\mathrm{d}s} W(\nabla_x u(y) + s(\nabla_y u(y) - \nabla_x u(y)))\,\mathrm{d}s\mathrm{d}y \\ &= \int_\Omega\int_0^1 \frac{1}{q^\delta} DW\big(\nabla_y u(y)(\nabla_x T^\delta(y) + s(\mathbf{1} - \nabla_x T^\delta(y)))\big) : \Big(\nabla_y u(y)\,\nabla_x \Big(\begin{smallmatrix}\theta^\delta(y)\\ 0\end{smallmatrix}\Big)\Big)\,\mathrm{d}s\mathrm{d}y. \end{aligned} \tag{4.24}$$

The integrand of (4.24) converges pointwise to $DW(\nabla_y u(y)) : (\nabla_y u(y) \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes \nabla\theta)$ for almost every s and y as $\delta \to 0$. Furthermore, estimate (2.4) is applicable to the integrand of (4.24) due to inequality (4.23) and implies that the integrand in (4.24) is bounded by the integrable function $c(W(\nabla_y u(y))+1)\,\|\nabla\theta\|_{L^\infty(\tilde\Omega)}$. The constant c is independent of δ, s and y. Thus the dominated convergence theorem leads to

$$\delta^{-1}\int_\Omega \frac{1}{q^\delta}\big(W(\nabla_y u(y))-W(\nabla_x u(y))\big)\,\mathrm{d}y \to \int_\Omega \nabla_y u^\top(y)\,DW(\nabla_y u(y)) : \nabla\begin{pmatrix}\theta(y)\\0\end{pmatrix}\mathrm{d}y$$

for $\delta\to 0$. This finishes the proof of lemma 4.3. $\square$

In the proof of lemma 4.3 it is important that the parameter δ occurs in the mapping T_δ, only, which enables us to derive pointwise convergence of the energy density and its derivatives and to apply the dominated convergence theorem. In the next lemma we discuss the case, where a sequence $\{\, u^{\delta_n}\,;\, n\in\mathbb{N}\,\}$ converges weakly to u_0, only. Here, the convergence results for Eshelby tensors, which we deduced in section 2, play an important role.

Lemma 4.4. *Let* **A1**-**A6** *be satisfied,* $p\geq 2$*,* $\alpha\in(1,p]$ *and* $\varepsilon\geq 0$*. Let further* $u_0\in V^p(\Omega_0)$ *with* $I(\Omega_0,u_0)<\infty$ *and let* $\{\, u^{\delta_n}\,;\, n\in\mathbb{N}\,\}\subset V^p(\Omega_0)$ *be a sequence with* $\delta_n\to 0$ *for* $n\to\infty$ *and*

$$u^{\delta_n}\rightharpoonup u_0 \ \textit{weakly in}\ W^{1,p}(\Omega_0)\ \textit{for}\ n\to\infty, \tag{4.25}$$

$$\det\nabla u^{\delta_n}\rightharpoonup \det\nabla u_0\ \textit{weakly in}\ L^r(\Omega_0)\ \textit{with}\ r\ \textit{from}\ \mathbf{A6}\ \textit{if}\ p=2, \tag{4.26}$$

$$I_0^\varepsilon(u_0;u^{\delta_n})\to I_0^\varepsilon(u_0;u_0)=I(\Omega_0,u_0)\ \textit{for}\ n\to\infty. \tag{4.27}$$

Then, for every θ *from* **A5***,*

$$\lim_{n\to\infty}\frac{I_0^\varepsilon(u_0;u^{\delta_n})-I_{\delta_n}^\varepsilon(u_0;u^{\delta_n}\circ T_{\delta_n})}{\delta_n}=G(u_0,\theta) \tag{4.28}$$

with $G(\cdot,\cdot)$ *from* (3.4)*.*

Later we choose $u^{\delta_n}=u_{\delta_n}\circ T_{\delta_n}^{-1}$, where the u_{δ_n} are minimizers of $I_{\delta_n}^\varepsilon(u_0;\cdot)$.

Proof. We skip the index n in the proof. Let $\{\, u^\delta\,;\,\delta\neq 0\,\}$ and u_0 be given according to lemma 4.4. After a transformation of coordinates in $I_\delta^\varepsilon(u_0;u^\delta\circ T_\delta)$ we obtain

$$\begin{aligned}
&I_0^\varepsilon(u_0;u^\delta)-I_\delta^\varepsilon(u_0;u^\delta\circ T_\delta)\\
&\quad=\int_{\Omega_0}\frac{1}{q^\delta}\big(W(\nabla_y u^\delta)-W(\nabla_x u^\delta)\big)\,\mathrm{d}y+\int_{\Omega_0}(1-\frac{1}{q^\delta})W(\nabla_y u^\delta)\,\mathrm{d}y\\
&\quad\quad-\int_{\tilde\Omega} f\cdot(u^\delta-u_\delta)\,\mathrm{d}y+\frac{\varepsilon}{\delta}\int_{\tilde\Omega}(\left|u_0-u^\delta\right|^\alpha-|u_0-u_\delta|^\alpha)\,\mathrm{d}y\\
&\quad=I_1+\ldots+I_4.
\end{aligned}\tag{4.29}$$

Here, $u_\delta=u^\delta\circ T_\delta$. Inequality (4.20), Hölder's inequality, the triangle inequality and lemma 4.1 lead to

$$\begin{aligned}
\left|\delta^{-1}I_4\right|&\leq\frac{c\varepsilon}{|\delta|}\int_{\tilde\Omega}(\left|u_0-u^\delta\right|+|u_0-u_\delta|)^{\alpha-1}\left|u^\delta-u_\delta\right|\,\mathrm{d}y\\
&\leq\varepsilon c(\left\|u_0-u^\delta\right\|_{L^\alpha(\tilde\Omega)}+\|u_0-u_\delta\|_{L^\alpha(\tilde\Omega)})^{\alpha-1}\,\|\partial_1 u_\delta\|_{L^\alpha(\tilde\Omega)}\,.
\end{aligned}\tag{4.30}$$

It follows from the mapping properties of T_δ, assumption (4.25) and $\alpha \in (1,p]$ that the last factor in (4.30) is uniformly bounded with respect to $|\delta| \le \delta_0$. The triangle inequality and lemma 4.1 imply

$$\begin{aligned}\|u_0 - u_\delta\|_{L^\alpha(\tilde\Omega)} &\le \left\|u_0 - u^\delta\right\|_{L^\alpha(\tilde\Omega)} + \left\|u^\delta - u_\delta\right\|_{L^\alpha(\tilde\Omega)} \\ &\le \left\|u_0 - u^\delta\right\|_{L^\alpha(\tilde\Omega)} + |\delta| \,\|\partial_1 u_\delta\|_{L^\alpha(\tilde\Omega)}.\end{aligned}$$

Together with the weak convergence $u^\delta \rightharpoonup u_0$ in $W^{1,p}(\Omega_0) \Subset L^\alpha(\Omega_0)$ we obtain that the first factor in (4.30) tends to 0 for $\delta \to 0$. Altogether we have shown that

$$\delta^{-1} I_4 \to 0$$

for $\delta \to 0$. Again by lemma 4.1 we obtain

$$\delta^{-1} I_3 = -\frac{1}{\delta}\int_{\tilde\Omega} f \cdot (u^\delta - u_\delta)\,\mathrm{d}y \to -\int_{\tilde\Omega} \theta f \cdot \partial_1 u_0 \,\mathrm{d}y$$

and it remains to calculate the limits of $\delta^{-1} I_1$ and $\delta^{-1} I_2$. We have

$$\delta^{-1} I_2 = -\int_{\Omega_0} \frac{\theta^\delta_{,1}}{q^\delta} W(\nabla_y u^\delta)\,\mathrm{d}y.$$

Assumptions (4.25)–(4.27) together with lemma 2.5 imply that $W(\nabla_y u^\delta)$ converges weakly to $W(\nabla_y u_0)$ in $L^1(\Omega_0)$. Since $\theta^\delta_{,1}/q^\delta$ converges uniformly to $\theta_{,1}$, we arrive at

$$\delta^{-1} I_2 \to -\int_{\Omega_0} \theta_{,1} W(\nabla_y u_0)\,\mathrm{d}y.$$

The term $\delta^{-1} I_1$ can be written as follows:

$$\begin{aligned} I_1 &= \int_{\Omega_0} W(\nabla_y u^\delta) - W(\nabla_x u^\delta)\,\mathrm{d}y + \int_{\Omega_0} \big(\frac{1}{q^\delta} - 1\big)\big(W(\nabla_y u^\delta) - W(\nabla_x u^\delta)\big)\,\mathrm{d}y \\ &= I_{11} + I_{12}.\end{aligned}$$

Note that $\nabla_x u^\delta(y) = \nabla_y u^\delta(y)\,\nabla_x T^\delta$ and therefore, by estimate (2.5),

$$\left|\delta^{-1} I_{12}\right| \le |\delta|\, c(\theta)\big(1 + \int_{\Omega_0} W(\nabla_y u^\delta)\,\mathrm{d}y\big).$$

Thus, $\left|\delta^{-1} I_{12}\right| \to 0$ for $|\delta| \to 0$. We treat the remaining term I_{11} as follows:

$$\begin{aligned}\delta^{-1} I_{11} &= \delta^{-1}\int_{\Omega_0} W(\nabla_y u^\delta) - W(\nabla_y u^\delta \nabla_x T^\delta)\,\mathrm{d}y \\ &= \delta^{-1}\int_{\Omega_0}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} W(\nabla_y u^\delta(\nabla_x T^\delta + s(\mathbf{1} - \nabla_x T^\delta)))\,\mathrm{d}s\,\mathrm{d}y \\ &= \int_{\Omega_0} \nabla_y u^{\delta\top} DW(\nabla_y u^\delta) : \big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) \otimes \nabla_x \theta^\delta \,\mathrm{d}y \\ &\quad + \int_{\Omega_0}\int_0^1 \nabla_y u^{\delta\top}\big(DW(\nabla_y u^\delta C(y,s,\delta)) - DW(\nabla_y u^\delta)\big) : \big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) \otimes \nabla_x \theta^\delta \,\mathrm{d}s\,\mathrm{d}y \\ &= I_{111} + I_{112}. \end{aligned} \tag{4.31}$$

Here, $C(y,s,\delta) = \nabla_x T^\delta(y) + s(\mathbf{1} - \nabla_x T^\delta(y)) = \mathbf{1} - \delta(1-s)\nabla\left(\begin{smallmatrix}\theta^\delta\\0\end{smallmatrix}\right)$. For small enough $|\delta|$, estimate (2.7) leads to

$$|I_{112}| \leq c(\theta)\,|\delta| \int_{\Omega_0} (1 + W(\nabla_y u^\delta)\,\mathrm{d}y$$

and thus $I_{112} \to 0$ for $\delta \to 0$. From proposition 2.2 it follows that $\nabla u^{\delta\top} DW(\nabla u^\delta) \rightharpoonup \nabla u_0^\top DW(\nabla u_0)$ weakly in $L^1(\Omega_0)$. Since $\nabla_x \theta^\delta$ converges uniformly to $\nabla\theta$, we finally arrive at

$$I_{111} \to \int_{\Omega_0} \nabla u_0^\top DW(\nabla u_0) : \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes \nabla\theta\,\mathrm{d}y.$$

This finishes the proof of lemma 4.4. □

4.2 Prof of theorem 3.6 for $\varepsilon > 0$

We prove now theorem 3.6 on the energy release rate for the functional $I_\delta^\varepsilon(u_0;\cdot)$ for fixed $\varepsilon > 0$. Let the assumptions of theorem 3.6 be satisfied, $\varepsilon > 0$ and let $u_0 \in V^p(\Omega_0)$ be a minimizer of $I(\Omega_0,\cdot)$. We recall that u_0 is the unique minimizer of $I_0^\varepsilon(u_0;\cdot)$. Our aim is to show that

$$\lim_{\delta\to 0} \delta^{-1}(I_0^\varepsilon(u_0) - I_\delta^\varepsilon(u_0)) = G(u_0,\theta) \tag{4.32}$$

with $G(\cdot,\cdot)$ from (3.4) and arbitrary θ from **A5**. Let $\{\,\delta_n\,;\,n\in\mathbb{N},\,\delta_n > 0\,\}$ be a sequence with $\delta_n \searrow 0$ for $n\to\infty$ and let $\{\,u_{\delta_n}\,;\,n\in\mathbb{N}\,\}$ be minimizers of $I_{\delta_n}^\varepsilon(u_0;\cdot)$. Then for every $n\in\mathbb{N}$

$$\begin{aligned}\delta_n^{-1}(I_0^\varepsilon(u_0;u_0) - I_{\delta_n}^\varepsilon(u_0;u_0\circ T_{\delta_n})) &\leq \delta_n^{-1}(I_0^\varepsilon(u_0) - I_{\delta_n}^\varepsilon(u_0))\\ &\leq \delta_n^{-1}(I_0^\varepsilon(u_0;u_{\delta_n}\circ T_{\delta_n}^{-1}) - I_{\delta_n}^\varepsilon(u_0;u_{\delta_n})).\end{aligned} \tag{4.33}$$

Here, $T_\delta(x) = x - \delta(\theta(x),0)^\top$ is the diffeomorphism introduced in section 4.1. Lemma 4.3 guarantees for the left hand side in (4.33) that

$$\lim_{\delta_n\searrow 0} \delta_n^{-1}(I_0^\varepsilon(u_0;u_0) - I_{\delta_n}^\varepsilon(u_0;u_0\circ T_{\delta_n})) = G(u_0,\theta). \tag{4.34}$$

From lemma 4.2 we know that the sequence $\{\,u_{\delta_n}\circ T_{\delta_n}^{-1}\,;\,n\in\mathbb{N}\,\}$ is a minimizing sequence for $I_0^\varepsilon(u_0;\cdot)$ and contains therefore a subsequence which converges weakly in $W^{1,p}(\Omega_0)$ to the unique minimizer u_0 of $I_0^\varepsilon(u_0;\cdot)$. By contradiction we conclude that also the whole sequence converges weakly to u_0. Lemma 4.4 implies now for the right hand side in (4.33) that

$$\lim_{\delta_n\searrow 0} \delta_n^{-1}(I_0^\varepsilon(u_0;u_{\delta_n}\circ T_{\delta_n}^{-1}) - I_{\delta_n}^\varepsilon(u_0;u_{\delta_n})) \to G(u_0,\theta). \tag{4.35}$$

Combining (4.34) and (4.35) implies (4.32) for the whole sequence $\delta \searrow 0$.

The case $\delta_n \nearrow 0$ can be treated completely analogously to the case $\delta_n \searrow 0$, we only have to replace $\leq$ in (4.33) by $\geq$. Since the left hand side in (4.32) is independent of the cut-off function θ, it follows that $G(u_0,\theta)$ is also independent of the choice of θ. This finishes the proof of theorem 3.6.

4.3 Proof of theorems 3.3 and 3.4

Let the assumptions of theorem 3.3 be satisfied and $\varepsilon = 0$. The goal is to show that

$$\lim_{\delta \searrow 0} \delta^{-1}(I(\Omega_0) - I(\Omega_\delta)) = \max\{\, G(u_0, \theta)\,;\, u_0 \text{ minimizes } I(\Omega_0, \cdot)\,\} \tag{4.36}$$

with $G(u_0, \theta)$ from (3.4). The major difference to the case with $\varepsilon > 0$ (theorem 3.6) is that $I(\Omega_0, \cdot)$ may have several minimizers. Moreover, we cannot prove that every minimizer u_0 of $I(\Omega_0, \cdot)$ has property **A7**, which means that it is not clear in general, whether for every minimizer u_0 there exists a sequence of minimizers u_δ of $I(\Omega_\delta, \cdot)$ such that the whole sequence $\{\, u_\delta \circ T_\delta^{-1}\,;\, \delta > 0\,\}$ converges weakly to u_0 for $\delta \searrow 0$. If this would be the case, we could argue completely analogously to the case $\varepsilon > 0$ and would obtain $ERR(\Omega_0) = G(u_0, \theta)$ for every minimizer u_0 of $I(\Omega_0, \cdot)$.

We will now prove (4.36). Let θ be chosen according to **A5**. For every $\delta > 0$ and every minimizer u_0 of $I(\Omega_0, \cdot)$ we have

$$\delta^{-1}(I(\Omega_0) - I(\Omega_\delta)) \geq \delta^{-1}(I(\Omega_0, u_0) - I(\Omega_\delta, u_0 \circ T_\delta)) \tag{4.37}$$

and, by lemma 4.3, the right hand side in (4.37) converges to $G(u_0, \theta)$ for $\delta \searrow 0$. Since the minimizer u_0 in (4.37) is arbitrary, we may then take the supremum over all minimizers. This leads to

$$\liminf_{\delta \searrow 0} \delta^{-1}(I(\Omega_0) - I(\Omega_\delta)) \geq \sup\{\, G(u_0, \theta)\,;\, u_0 \text{ minimizes } I(\Omega_0, \cdot)\,\}. \tag{4.38}$$

It follows from the boundedness of the set L from (4.11) and the set defined in (4.12) (see lemma 4.2) together with assumption **A2** that the supremum in (4.38) is finite. Let now $\{\, \delta_n\,;\, n \in \mathbb{N}\,\}$ be a sequence with $\delta_n \searrow 0$ for $n \to \infty$ and let $\{\, u_{\delta_n}\,;\, n \in \mathbb{N}\,\}$ be minimizers of $I(\Omega_{\delta_n}, \cdot)$. By lemma 4.2 there exists a subsequence $\{\, \delta_{n_k}\,;\, k \in \mathbb{N}\,\}$ and a minimizer u_0 of $I(\Omega_0, \cdot)$ with $u_{\delta_{n_k}} \circ T_{\delta_{n_k}}^{-1} \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega_0)$. With lemma 4.4 it follows for this subsequence

$$\begin{aligned}\limsup_{\delta_{n_k} \searrow 0} \delta_{n_k}^{-1}(I(\Omega_0) - I(\Omega_{\delta_{n_k}})) &\leq \lim_{\delta_{n_k} \searrow 0} \delta_{n_k}^{-1}(I(\Omega_0, u_{\delta_{n_k}} \circ T_{\delta_{n_k}}^{-1}) - I(\Omega_{\delta_{n_k}}, u_{\delta_{n_k}})) \\ &= G(u_0, \theta) \leq \sup\{\, G(v, \theta)\,;\, v \text{ minimizes } I(\Omega_0, \cdot)\,\}.\end{aligned}$$

Thus we have shown that every sequence $\delta_n \searrow 0$ contains a subsequence $\delta_{n_k} \searrow 0$ with

$$\limsup_{\delta_{n_k} \searrow 0} \delta_{n_k}^{-1}(I(\Omega_0) - I(\Omega_{\delta_{n_k}})) \leq \sup\{\, G(u_0, \theta)\,;\, u_0 \text{ minimizes } I(\Omega_0, \cdot)\,\}. \tag{4.39}$$

The usual proof by contradiction shows that (4.39) holds also for the whole sequence δ_n. Combining (4.38) and (4.39) leads to

$$\lim_{\delta \searrow 0} \delta^{-1}(I(\Omega_0) - I(\Omega_\delta)) = \sup\{\, G(u_0, \theta)\,;\, u_0 \text{ minimizes } I(\Omega_0, \cdot)\,\}. \tag{4.40}$$

It remains to show that the supremum in (4.40) is attained. Coercivity assumption **A6** and the weak lower semi-continuity of $I(\Omega_0, \cdot)$ guarantee that the set of minimizers of $I(\Omega_0, \cdot)$ is weakly compact in $V^p(\Omega_0)$. Let $\{ u_n \,;\, n \in \mathbb{N} \}$ be a sequence of minimizers of $I(\Omega_0, \cdot)$ with $G(u_n, \theta) \to \sup\{ G(v, \theta) \,;\, v \text{ minimizes } I(\Omega, \cdot) \}$. Then there exists a subsequence, which we also denote by $\{ u_n \,;\, n \in \mathbb{N} \}$, and a minimizer u_0 of $I(\Omega_0, \cdot)$ such that $u_n \rightharpoonup u_0$ weakly in $V^p(\Omega_0)$ and, if $p = 2$, $\det \nabla u_n \rightharpoonup \det \nabla u_0$ weakly in $L^r(\Omega_0)$ with $r > 1$ from **A6**. Since $I(\Omega_0, u_n) = I(\Omega_0, u_0)$ for every n, it follows in particular that $\int_{\Omega_0} W(\nabla u_n) \,\mathrm{d}y \to \int_{\Omega_0} W(\nabla u_0) \,\mathrm{d}y$ for $n \to \infty$. Thus, proposition 2.2 and lemma 2.5 imply that $G(u_n, \theta) \to G(u_0, \theta)$ and therefore u_0 is a maximizer of $G(\cdot, \theta)$ with respect to the set $\{ v \in V^p(\Omega_0) \,;\, v \text{ minimizes } I(\Omega_0, \cdot) \}$. This proves (3.3).

Let finally $u_* \in V^p(\Omega_0)$ be a minimizer of $I(\Omega_0, \cdot)$ which satisfies property **A7**. This means that for every $\delta > 0$ there exists a minimizer u_δ of $I(\Omega_\delta, \cdot)$ such that the whole sequence $\{ u_\delta \circ T_\delta^{-1} \,;\, \delta > 0 \}$ converges weakly to u_*. As before, lemma 4.4 implies now that

$$\limsup_{\delta \searrow 0} \delta^{-1}(I(\Omega_0) - I(\Omega_\delta)) \leq \limsup_{\delta \searrow 0} \delta^{-1}(I(\Omega_0, u^\delta) - I(\Omega_\delta, u_\delta)) = G(u_*, \theta)$$

and together with (4.38) we obtain the last assertion of theorem 3.3.

For the proof of theorem 3.4 note that for $\delta < 0$ and minimizers u_0 of $I(\Omega_0, \cdot)$ we have $\delta^{-1}(I(\Omega_0) - I(\Omega_\delta)) \leq \delta^{-1}(I(\Omega_0, u_0) - I(\Omega_\delta, u_0 \circ T_\delta))$. We obtain therefore completely analogous to the previous considerations that

$$\lim_{\substack{\delta \to 0 \\ \delta < 0}} \delta^{-1}(I(\Omega_0) - I(\Omega_\delta)) = \min\{ G(u_0, \theta) \,;\, u_0 \text{ minimizes } I(\Omega_0, \cdot) \}$$

with $G(\cdot, \cdot)$ from (3.4).

4.4 Proof of theorem 3.5 on the J-integral

The proof of theorem 3.5 on the J-integral is based on the lemma of Du Bois-Reymond, [Hör83]. Let $f \in L^q(\tilde{\Omega})$ with $\partial_1 f = 0$ and let u_0 be an arbitrary minimizer of $I(\Omega_0, \cdot)$. The functions u_0 and f are elements of $L^p(\tilde{\Omega}, \partial_1; \mathbb{R}^2)$ and $L^q(\tilde{\Omega}, \partial_1; \mathbb{R}^2)$, respectively. Applying Green's formula to the last term in the Griffith formula (3.4) leads to

$$-\int_{\tilde{\Omega}} \theta f \cdot \partial_1 u_0 \,\mathrm{d}y = \int_{\tilde{\Omega}} \partial_1 \theta \, u_0 \cdot f \,\mathrm{d}y.$$

Thus, the Griffith formula (3.4) can be rewritten as follows for arbitrary $\theta \in \mathcal{C}_0^\infty(\tilde{\Omega})$ with $\theta = 1$ near the crack tip:

$$G(u_0) = \int_{\Omega_0} F(x) \cdot \nabla \theta(x) \,\mathrm{d}x, \tag{4.41}$$

where $F \in L^1(\Omega_0)$ and

$$F(x) = (\nabla u_0^\top DW(\nabla u_0))^\top \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) - W(\nabla u_0) \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + (u_0 \cdot f) \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right). \tag{4.42}$$

Inserting two different θ_1 and θ_2 into (4.41) and taking the difference implies

$$0 = \int_{\Omega_0} F(x) \cdot \nabla\tilde{\theta} \, \mathrm{d}x \tag{4.43}$$

for every $\tilde{\theta} \in \mathcal{C}_0^\infty(\tilde{\Omega})$ with $\tilde{\theta} = 0$ near the crack tip 0. Let $R_0 >$ such that $B_{R_0}(0) \Subset \tilde{\Omega}$. In particular, we may choose $\tilde{\theta}(x) = \hat{\theta}(|x|)$ with $\hat{\theta} \in \mathcal{C}_0^\infty((0, R_0))$. With this choice, equation (4.43) reads in polar coordinates

$$0 = \int_0^{R_0} \int_{-\pi}^{\pi} F(x(r,\varphi)) \cdot \left(\begin{smallmatrix} \cos\varphi \\ \sin\varphi \end{smallmatrix}\right) \mathrm{d}\varphi \, r\hat{\theta}'(r) \, \mathrm{d}r \equiv \int_0^{R_0} g(r) r\hat{\theta}'(r) \, \mathrm{d}r. \tag{4.44}$$

Since $F \in L^1(\Omega_0)$, Fubini's theorem guarantees that $g \in L^1(0, R_0)$ as well. The lemma of Du Bois-Reymond implies now that there exists a constant κ such that $g(r)r = \kappa$ for almost every $r \in (0, R_0)$. Together with (4.41) we obtain finally that $g(r)r = \kappa = -G(u_0)$ for almost every r, which finishes the proof of theorem 3.5.

5 Concluding remarks

5.1 Non-interpenetration conditions

Up to now the two sides of the crack are allowed to penetrate into each other, which may lead to unphysical solutions. To overcome this problem one has to include non-interpenetration conditions in the set $V^p(\Omega_\delta)$ of admissible deformation fields. One possibility is to replace $V^p(\Omega_\delta)$ by

$$\tilde{V}^p(\Omega_\delta) = \{ \, u \in W^{1,p}(\Omega_\delta) \, ; \, u\big|_{\Gamma_D} = g_D, \, (u_+ - u_-) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \geq 0 \text{ on } C_\delta \, \},$$

where $u_\pm$ is the trace of u on the crack face C_δ with respect to the upper and lower half plane, respectively. Since the mapping $T_\delta : \Omega_\delta \to \Omega_0$ induces an isomorphism between $\tilde{V}^p(\Omega_\delta)$ and $\tilde{V}^p(\Omega_0)$, all results from section 3 on the energy release rate remain valid.

A further possibility to exclude self-interpenetration is proposed by Ciarlet and Nečas, [CN87]. Here, the set of admissible deformation fields is defined by

$$V^p_{\mathrm{CN}}(\Omega_\delta) = \Big\{ u \in W^{1,p}(\Omega_\delta) \, ; \, u\big|_{\Gamma_D} = g_D, \, \det \nabla u \geq 0 \text{ a.e. in } \Omega_\delta, \\ \int_{\Omega_\delta} \det \nabla u \, \mathrm{d}x \leq \mathrm{vol}(u(\Omega_\delta)) \Big\} \tag{5.1}$$

for $p > d$. Again, the mapping $T_\delta : \Omega_\delta \to \Omega_0$ induces an isomorphism between the spaces $V^p_{\mathrm{CN}}(\Omega_\delta)$ and $V^p_{\mathrm{CN}}(\Omega_0)$. In [CN87], it is proved that for bounded Lipschitz domains Ω and $p > d$ the set $V^p_{\mathrm{CN}}(\Omega)$ is closed in the weak topology of $W^{1,p}(\Omega)$. Taking into account that bounded domains with cone property are a union of a finite number of Lipschitz domains, this result can be carried over to such domains and it is in particular valid for the domains with cracks, which we introduced in section 3. The results from section 3 on the energy release rate can now be proved without essential changes for problems with a non-interpenetration condition of the form (5.1) and $p > 2$.

5.2 Quasiconvex energy densities

The results from the previous sections remain true for continuous, quasiconvex energy densities $W : \mathbb{M}^{d\times d} \to [0,\infty)$, $d \geq 2$, satisfying the following additional assumptions:

AQ W is twice differentiable and there exist constants $c_i > 0$ and $p \in (1,\infty)$ such that for every $A, B \in \mathbb{M}^{d\times d}$

$$0 \leq W(A) \leq c_1(1 + |A|^p), \tag{5.2}$$

$$\left| A^\top D^2 W(A)[AB] \right| \leq c_2(W(A) + 1)\,|B|\,. \tag{5.3}$$

Note that inequality (5.2) implies in particular that $|DW(A)| \leq c(1+|A|^{p-1})$, see [Dac89], and therefore, estimate (2.1) is valid. Under assumption (5.2) with $p \in (1,\infty)$, proposition 2.2 remains valid if one replaces (2.10) by $\{\, u_n \,;\, n \in \mathbb{N}_0 \,\} \subset W^{1,p}(\Omega)$, $u_n \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega)$ and $F_n = \nabla u_n$. Lemma 2.5 can be modified in the following way:

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Assume in addition that $W : \mathbb{M}^{d\times d} \to [0,\infty)$ is quasiconvex and that (5.2) holds for some $p \in (1,\infty)$. Let $\{\, u_n \,;\, n \in \mathbb{N}_0 \,\} \subset W^{1,p}(\Omega)$ be a sequence with*

$$u_n \rightharpoonup u_0 \text{ weakly in } W^{1,p}(\Omega) \text{ and } \int_\Omega W(\nabla u_n)\,\mathrm{d}x \to \int_\Omega W(\nabla u_0)\,\mathrm{d}x.$$

Then $W(\nabla u_n) \rightharpoonup W(\nabla u_0)$ weakly in $L^1(\Omega)$.

Proof. For every $\varphi \in L^\infty(\Omega)$ with $\varphi \geq 0$, Theorem 1.1 in [Mar85] guarantees that the functional $J : W^{1,p}(\Omega) \to \mathbb{R}$, $u \mapsto \int_\Omega \varphi(x) W(\nabla u(x))\,\mathrm{d}x$ is weakly lower semicontinuous. The lemma can now be proved in the same way as lemma 2.5. □

Assuming **AQ** and **A4–A6**, the convergence results from section 4.1 as well as the results on the energy release rate in section 3 can now be derived in the same way as for the polyconvex case on the basis of the modified proposition 2.2 and lemma 5.1.

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