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On the energetic stability of solitary water waves

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Dedicated to Klaus Kirchgässner on the occasion of his seventieth birthday

1 Introduction

The water–wave problem has attracted the attention of many researchers within the last three decades. Significant advances have been made in the local existence theory for the time–dependent problem, cf. [Wu97, Wu99], and in the analysis of steadily traveling waves in general cf. [Tol96, BDT00a, BDT00b]. The local bifurcation theory for traveling waves was greatly forwarded the Kirchgässner reduction, also called the spatial center manifold approach, cf. [Kir82, AK89, IK92, GM01, ILS01]. The connection between the water–wave problem and model equations like the sine–Gordon, Boussinesq or Korteweg–de Vries equations was made rigorous, cf. [Cra85, SW00]. Finally, the KAM (Kolmogorov, Arnold, Moser) method for small–divisor problems was applied to obtain bifurcation results for traveling waves (cf. [CN00]) as well as for standing waves (cf. [PT01]).

Here we want to popularize another tool in the study of the water waves, namely the stability theory for Hamiltonian systems. It was realized in [Mie91] that the spatial dynamics approach to the steady water–wave problem leads to an ill–posed Hamiltonian system whose Kirchgässner reduction to the center manifold leads to classical finite–dimensional Hamiltonian systems. From this, the bifurcation theory for steady waves got a new foundation (cf. [BM92, GT97, GM01] and the references therein). On the other hand the time–dependent problem has a Hamiltonian structure as well which was discovered by Zakharov [Zak68]. In [Bri92, BM95, Bri97] the joint symplectic structure in space and time was exploited.

However, the stability theory for Hamiltonian systems with symmetries, which was largely developed in the 1980's and 1990's, has not yet been systematically employed in the classical field of water waves. We refer to [Arn66, LMR87, Lew89, KS98] and the references therein for energetic stability proofs for shear flows and for rotating liquid drops. It is the purpose of this work to employ some of these methods to the stability problem for (i) the flat surface of a infinitely extended fluid layer over a flat bottom and for (ii) solitary waves on such a layer. All our results concern conditional stability as defined in

[Lew89] since global existence results for the water–wave problem are not available even under the assumption of small and smooth initial data. We will make the definition of conditional stability precise in our results.

We now discuss our problem in more detail. Consider the two–dimensional water–wave problem under the presence of gravity g and surface tension b. The undisturbed fluid fills an unbounded strip of depth h and disturbances are assumed to travel with speed close to c. The problem consists in determining the motion of the fluid surface $\mathbb{R} \ni x \mapsto \eta(t,x)$ as a function of time t. The full problem involves the potential $\phi : (t,x,y) \mapsto \phi(t,x,y) \in \mathbb{R}$ for the velocity $v = (\partial_x\phi, \partial_y\phi)^{\mathsf{T}}$. The Hamiltonian Zakharov formulation ([Zak68]) is given in terms of η and ξ with $\xi(t,x) := \phi(t,x,\eta(t,x))$ only. It is a canonical Hamiltonian system, i.e., the time evolution is governed by the PDE

$$\eta_t = \mathrm{D}_\xi \mathcal{H}(\eta,\xi), \quad \xi_t = -\mathrm{D}_\eta \mathcal{H}(\eta,\xi), \tag{1.1}$$

where the Hamiltonian energy $\mathcal{H}$ is given by the sum of kinetic and potential energy: $\mathcal{H}(\eta,\xi) = \mathcal{K}(\eta,\xi) + \mathcal{V}(\eta)$ with

$$\mathcal{K}(\eta,\xi) = \int_{\mathbb{R}} \tfrac{1}{2}\xi(x)[G(\eta)\xi](x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{V}(\eta) = \int_{\mathbb{R}} \Big[\tfrac{g}{2}\eta(x)^2 + b\big(\sqrt{1+\eta_x(x)^2}-1\big)\Big]\,\mathrm{d}x.$$

and the phase space is an open subset of the Hilbert space $Z = \mathrm{H}^1(\mathbb{R}) \times \mathrm{H}^{1/2}_*(\mathbb{R})$.

The potential energy $\mathcal{V}$ is a sum of the gravitational potential and the surface energy due to stretching the surface. The kinetic energy is quadratic in the velocity and has a complicated dependence on the surface η via the Dirichlet–Neumann operator $G(\eta)$. It is this operator which plays a central role and gives rise to technical difficulties. From the work [CM85, Wu93, CN00, NR01] it is known that $G(\eta)$ as an operator in $\mathrm{Lin}(\mathrm{H}^1(\mathbb{R}), \mathrm{L}^2(\mathbb{R}))$ depends analytically on η considered as an element of $\mathrm{C}^{\mathrm{Lip}}(\mathbb{R})$. The boundedness of $G(\eta)$ as mapping from $\mathrm{H}^1(\mathbb{R})$ into $\mathrm{L}^2(\mathbb{R})$ is already a nontrivial fact for general Lipschitz domains. It relies on cancellation effects but it can be obtained by elementary methods, cf. [Cos88].

Since the energetic approach only controls the H^1–norm of η we need to weaken this result. A major rôle in our analysis plays the augmented potential $\mathcal{V}^{\mathrm{aug}}_c(\eta) := \mathcal{V}(\eta) - \frac{c^2}{2}\Pi(\eta)$. The locked inertia tensor reads $\Pi(\eta) = \int_{\mathbb{R}} \eta\,[K(\eta)\eta]\,\mathrm{d}x$ with $K(\eta)w = -(G(\eta)^{-1}w')'$. In the appendix we show that

$$K : E_R \to \mathrm{Lin}(\mathrm{H}^1(\mathbb{R}), \mathrm{H}^{-1}(\mathbb{R})); \eta \mapsto K(\eta),$$

is Hölder continuous if $E_R = \{\, \eta \in \mathrm{H}^1(\mathbb{R}) \mid \eta(x) \in [-h+1/R, R], \|\eta'\|_\infty \le R \,\}$ is equipped with the H^1–norm. This result is based on the $\mathrm{W}^{1,p}$–regularity result in [Grö89] for the Dirichlet problem in the Lipschitz domain $\Omega(\eta) := \{\, (x,y) \,|\, x \in \mathbb{R},\ y \in (-h, \eta(x)) \,\}$, where p depends on the Lipschitz bound $\|\eta'\|_\infty \le R$.

This technical result allows us to prove the following stability results. First we consider the trivial flow $(\eta,\xi) \equiv (0,0)$ with respect to perturbations traveling with speed c. Then the quadratic expansion of the augmented Hamiltonian $\mathcal{H}_c = \mathcal{H} + c\mathcal{P}$, where $\mathcal{P}(\eta,\xi) = \int_{\mathbb{R}} \eta'\xi\,\mathrm{d}x$ is the conserved horizontal linear momentum, shows that $(0,0)$ is a local minimum, if the dimensionless parameters $\lambda = gh/c^2$ and $\beta = b/(hc^2)$ lie in the

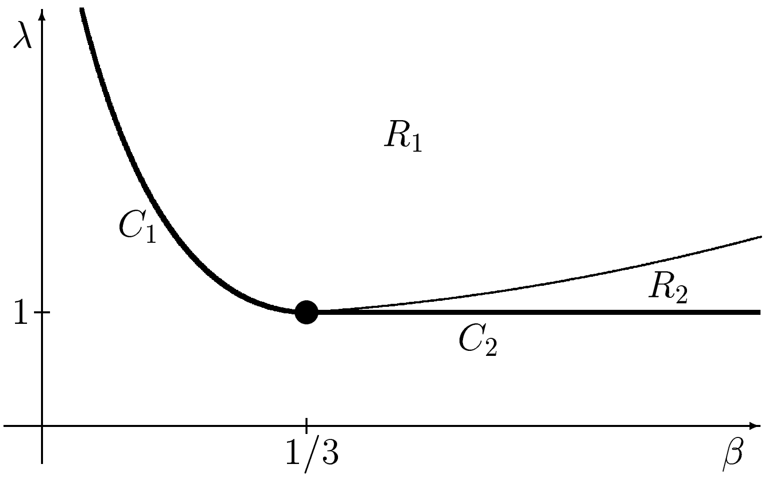

Figure 1.1: Regions in the (λ,β)–plane. (i) The parallel flow is energetically stable in R_1, which is the whole region above the curves C_1 and C_2. (ii) Small amplitude solitary waves, which are energetically stable, exist in R_2, which is a small wedge above C_2.

subregion R_1 of the positive quadrant of the (λ,β)–plane, see Figure 1.1. In the interior of this region the zero solution is conditionally energetically stable. This means roughly that solutions η starting close to $(0,0)$ in the topology of $\mathrm{H}^1 \times \mathrm{H}^{1/2}_*$ stay close to $(0,0)$ as long as they exist, are continuous and remain inside of E_R.

Second we study the nonlinear stability of solitary waves. We start by considering the small amplitude solitary waves for $\beta > 1/3$ and $\lambda = 1+\varepsilon^2$, which were first constructed in [AK89]. The second derivative of $\mathcal{H}_c$ at this solution defines a self–adjoint operator $\mathcal{A}_\varepsilon$ which after rescaling with ε converges to a limit which is the second derivative of the Korteweg–de Vries energy at the corresponding solitary wave. Hence, the spectrum consists of one negative, one zero and one positive discrete eigenvalue (all three are simple) and the continuous spectrum arising from the trivial solution $(0,0)$. Thus, the energetic stability method, as devised by Benjamin [Ben72] for the Korteweg–de Vries equation, can be generalized since the conservation of the horizontal momentum $\mathcal{P}$ can be used to compensate for the negative eigenvalue of the second derivative of the augmented Hamiltonian. In fact, our conditional stability result is not restricted to small amplitude solitary waves but holds also for any families of solitary waves satisfying the same spectral assumptions.

2 Formulation of the problem

The water–wave problem consists in solving the free boundary problem:

$$\begin{array}{rl} \Delta\phi = 0 & \text{for } t,x\in\mathbb{R},\ y\in(-h,\eta(t,x)), \\ \phi_y = 0 & \text{for } t,x\in\mathbb{R} \text{ and } y=-h, \\ \left.\begin{array}{r} \eta_t + \eta_x\phi_x - \phi_y = 0 \\ \phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) + g\,\eta - b\left(\eta_x/\sqrt{1+\eta_x^2}\right)_x = 0 \end{array}\right\} & \text{for } t,x\in\mathbb{R} \text{ and } y=\eta(t,x). \end{array} \qquad (2.1)$$

Here h is the depth of the undisturbed fluid, g is the acceleration due to gravity and b measures the surface tension.

We use the fact that this system has the form of a canonical Hamiltonian system which is also called the Zakharov equation [Zak68]. Introducing $\xi(t,x) = \phi(t,x,\eta(t,x))$ we obtain

$$\eta_t = \mathrm{D}_\xi \mathcal{H}(\eta,\xi), \quad \xi_t = -\mathrm{D}_\eta \mathcal{H}(\eta,\xi). \tag{2.2}$$

Here the Hamiltonian function $\mathcal{H}$ is defined as the sum of the kinetic energy $\mathcal{K}$ and the potential energy $\mathcal{V}$:

$$\begin{array}{rcl} \mathcal{K}(\eta,\xi) &=& \int_{\mathbb{R}} \frac{1}{2}\xi(x)[G(\eta)\xi](x)\,\mathrm{d}x \\ \mathcal{V}(\eta) &=& \int_{\mathbb{R}}[\frac{g}{2}\eta(x)^2 + b(\sqrt{1+\eta_x(x)^2}-1)]\,\mathrm{d}x. \end{array} \tag{2.3}$$

An important ingredient of this theory is the Dirichlet–Neumann operator $G(\eta)$ associated to the η–dependent domain

$$\Omega(\eta) = \{\, (x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ y \in (-h,\eta(x)) \,\}.$$

Here $\gamma = G(\eta)\xi$ is defined by solving the linear problem

$$\Delta\phi = 0 \text{ on } \Omega(\eta), \quad \phi_y(x,-h) = 0 \quad \text{and} \quad \phi(x,\eta(x)) = \xi(x) \text{ for } x \in \mathbb{R},$$

and then letting

$$\gamma(x) = \sqrt{1+\eta_x(x)^2}\,\nabla\phi(x,\eta(x))\cdot\nu = \phi_y(x,\eta(x)) - \eta_x(x)\phi_x(x,\eta(x)),$$

where ν is the outward unit normal to $\Omega(\eta)$ at $(x,\eta(x))$.

For the subsequent analysis we need the following expressions for the derivatives of the mapping $\eta \mapsto G(\eta)$. We use the notation $\langle v,w\rangle = \int_{\mathbb{R}} vw\,\mathrm{d}x$ to denote the scalar product in $\mathrm{L}^2(\mathbb{R})$ as well as the $\mathrm{H}^s \times \mathrm{H}^{-s}$ duality pairing.

Proposition 2.1 *Let $\mathcal{N} \subset \mathrm{W}^{1,\infty}(\mathbb{R})$ be the open set of η satisfying $\inf\eta > -h$. Then, $G : \mathcal{N} \to \mathrm{Lin}(\mathrm{H}^1(\mathbb{R}),\mathrm{L}^2(\mathbb{R}))$ is analytic. The first derivative satisfies*

$$\begin{array}{rcl} \langle \widehat{v}, \mathrm{D}G(\eta)[w]v\rangle &=& \langle w, a_0(\eta,v,\widehat{v})\rangle \\ &=& \langle w, a_1(\eta,v)\widehat{v}'+a_2(\eta,v)G(\eta)\widehat{v}\rangle, \end{array} \tag{2.4}$$

with $a_0(\eta,v,\widehat{v}) = \frac{1}{1+\eta'^2}[\widehat{v}'v' - (G(\eta)\widehat{v})G(\eta)v - \eta'(\widehat{v}'G(\eta)v+v'G(\eta)\widehat{v})]$, $a_1(\eta,v) = (v'-\eta'G(\eta)v)/(1+\eta'^2)$, *and* $a_2(\eta,v) = -(\eta'v'+G(\eta)v)/(1+\eta'^2)$.

For the second derivative we have

$$\langle v, \mathrm{D}^2G(\eta)[w,w]v\rangle = \langle w, -2a_3(\eta,v)w'+2a_2(\eta,v)\mathrm{D}G(\eta)[w]v\rangle \tag{2.5}$$

with $a_3(\eta,v) = \frac{1}{(1+\eta'^2)^2}[\eta'v'^2 - \eta'(G(\eta)v)^2+(1-\eta'^2)v'G(\eta)v]$. *This can be rewritten as* $\langle v, \mathrm{D}^2G(\eta)[w,w]v\rangle = \langle w, a_4w\rangle + 2\langle w, a_2G(\eta)(a_2w)\rangle$ *with* $a_4 = a_3' + a_1a_2' - a_1'a_2$.

The analyticity was proved in [CM85], see also [Wu93, Wu97, NR01]. Formula (2.4) for the first derivative is well–established in the literature, cf. [Zak68, BO82, CS93]. The formulae for the second derivative follow by differentiating (2.4) in the direction w, then applying (2.4) again to eliminate terms involving $\mathrm{D}G$, and using integration by parts.

Combining (2.2), (2.3) and (2.4) we find Zakharov's equation

$$\begin{aligned} \eta_t &= G(\eta)\xi \\ \xi_t &= -g\eta + b\Big(\eta_x/\sqrt{1{+}\eta_x^2}\Big)_x - \tfrac{1}{2}\Gamma(\eta,\xi) \end{aligned} \tag{2.6}$$

with $\Gamma(\eta,\xi) = (\xi_x^2 - [G(\eta)\xi]^2 - 2\eta_x\xi_x[G(\eta)\xi])/(1{+}\eta_x^2)$.

This system is invariant under the horizontal translations T_α, under time translations and under addition Φ_β of a constant to the surface trace ξ of the potential

$$T_\alpha\begin{pmatrix}\eta\\ \xi\end{pmatrix}(x) = \begin{pmatrix}\eta(x{-}\alpha)\\ \xi(x{-}\alpha)\end{pmatrix}, \quad \Phi_\beta\begin{pmatrix}\eta\\ \xi\end{pmatrix} = \begin{pmatrix}\eta\\ \xi{+}\beta\end{pmatrix}.$$

We use here that $G(\eta)\beta = 0$. Associated with these three symmetries are the three first integrals

$$\mathcal{P}(\eta,\xi) := \textstyle\int_{\mathbb{R}} \eta_x\xi \,\mathrm{d}x, \quad \mathcal{H}(\eta,\xi) := \mathcal{K}(\eta,\xi) + \mathcal{V}(\eta), \quad \mathcal{M}(\eta,\xi) := \textstyle\int_{\mathbb{R}} \eta \,\mathrm{d}x,$$

which are called horizontal momentum, energy, and mass, respectively.

Our aim is to investigate the stability of traveling waves which can be understood as relative equilibria with respect to the translation group T_α:

$$\eta(t,x) = \overline{\eta}(x{-}ct), \quad \xi(t,x) = \overline{\xi}(x{-}ct).$$

The pair $(\overline{\eta},\overline{\xi})$ defines a solution of the Zakharov equation if and only if it is a critical point of the augmented Hamiltonian $\mathcal{H}_c := \mathcal{H} + c\mathcal{P}$, i.e.,

$$\begin{aligned} 0 &= \mathrm{D}_\xi\mathcal{H}_c(\eta,\xi) = G(\eta)\xi + c\eta_x, \\ 0 &= \mathrm{D}_\eta\mathcal{H}_c(\eta,\xi) = g\eta - b\Big(\eta_x/\sqrt{1{+}\eta_x^2}\Big)_x + \tfrac{1}{2}\Gamma(\eta,\xi) - c\xi_x, \end{aligned} \tag{2.7}$$

where we have dropped the overbar for notational simplicity.

The stability of the relative equilibria $\overline{\zeta} = (\overline{\eta},\overline{\xi})^\mathsf{T}$ can now be obtained by showing that $\mathcal{H}_c$ has a local minimum in $\overline{\zeta}$ under the constraint $\mathcal{P}(\zeta) = \mathcal{P}(\overline{\zeta})$.

Using the fact that ξ appears only in quadratic form due to the kinetic energy we obtain the augmented potential

$$\mathcal{V}_c^{\mathrm{aug}}(\eta) := \min\{\, \mathcal{H}_c(\eta,\xi) \mid \xi \in \mathrm{H}_*^{1/2}(\mathbb{R}) \,\} = \mathcal{V}(\eta) - \tfrac{c^2}{2}\Pi(\eta),$$

where $\Pi(\eta) := \int_{\mathbb{R}} \eta_x[G(\eta)^{-1}\eta_x]\,\mathrm{d}x$ is called the locked inertia tensor, cf. [SLM91]. For the definition of $\mathrm{H}_*^{1/2}(\mathbb{R})$ see (5.1).

By construction, the critical points $\overline{\eta}$ of $\mathcal{V}_c^{\mathrm{aug}}$ are in one–to–one correspondence to the traveling waves $(\overline{\eta},\overline{\xi})$ with speed c, where $\overline{\xi} = -c\,G(\overline{\eta})^{-1}\overline{\eta}'$. The stability of these traveling waves is then decided using the second variation $\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\overline{\eta})$ which is an unbounded self–adjoint operator on $\mathrm{L}^2(\mathbb{R})$. For small amplitude solitary waves we will show that the critical part of the spectrum (i.e. that in $(-\infty,\delta)$ for $0<\delta\ll 1$) is governed by that of the KdV approximation. There are three discrete eigenvalues: by translation invariance one eigenvalue is 0 and there is one small negative and one small positive eigenvalue, above which the continuous spectrum starts.

3 Calculation of the second variation

In this section we make the form of $\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\eta)$ more explicit. To this end we assume that η is a critical point of $\mathcal{V}_c^{\mathrm{aug}}$, i.e., $\mathrm{D}\mathcal{V}_c^{\mathrm{aug}}(\eta)=0$. To simplify the notation we reintroduce the variable ξ given by $G(\eta)\xi+c\eta'=0$, moreover we use $'$ in place of ∂_x.

The first and second variation of the potential $\mathcal{V}$ are easily calculated

$$\begin{aligned} \mathrm{D}\mathcal{V}(\eta)[v] &= \int_{\mathbb{R}}[g\eta v+b\tfrac{\eta'}{\sqrt{1+\eta'^2}}v']\,\mathrm{d}x \\ \mathrm{D}^2\mathcal{V}(\eta)[v,v] &= \int_{\mathbb{R}}[gv^2+b\tfrac{(v')^2}{(1+\eta'^2)^{3/2}}]\,\mathrm{d}x. \end{aligned} \tag{3.1}$$

The variations of the locked inertia tensor are much more complicated.

Lemma 3.1 *For general $\eta\in\mathrm{H}^1(\mathbb{R})\cap\mathrm{C}^{\mathrm{Lip}}(\mathbb{R})$ and $v\in\mathrm{H}^1(\mathbb{R})$ we have*

$$\mathrm{D}\Pi(\eta)[v]=2\int_{\mathbb{R}}v'[G(\eta)^{-1}\eta']\,\mathrm{d}x-\int_{\mathbb{R}}\tfrac{v}{1+\eta'^2}[(G(\eta)^{-1}\eta')^2-\eta'^2(1+[G(\eta)^{-1}\eta']')]\,\mathrm{d}x. \tag{3.2}$$

Proof: The definition of Π gives

$$\begin{aligned}\mathrm{D}\Pi(\eta)[v]&=2\langle v',G(\eta)^{-1}\eta'\rangle+\langle\eta',\mathrm{D}(G(\eta)^{-1})[v]\eta'\rangle\\ &=2\langle v',G(\eta)^{-1}\eta'\rangle-\int_{\mathbb{R}}(G(\eta)^{-1}\eta')\mathrm{D}G(\eta)[v](G(\eta)^{-1}\eta')\,\mathrm{d}x,\end{aligned}$$

where we have used that $G(\eta)^{-1}$ is self–adjoint. Employing (2.4) now gives the result. ∎

Lemma 3.2 *If $\zeta=(\eta,\xi)$ is a critical point of $\mathcal{H}_c$, then for all v we have*

$$\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\eta)[v,v]=\mathrm{D}^2\mathcal{V}(\eta)[v,v]+\tfrac{1}{2}\langle\mathrm{D}^2G(\eta)[v,v]\xi,\xi\rangle-\langle\mathcal{L}_\zeta v,G(\eta)^{-1}L_\zeta v\rangle \tag{3.3}$$

where $\mathcal{L}_\zeta v=cv'+\mathrm{D}G(\eta)[v]\xi$.

Proof: This formula is best obtained by returning to $\mathcal{H}_c$ and not by differentiating (3.2). By the definition of $\mathcal{V}_c^{\mathrm{aug}}$ the formula (3.3) follows from

$$\mathrm{D}^2\mathcal{H}_c(\zeta)[\tbinom{v}{w},\tbinom{v}{w}]=\mathrm{D}^2\mathcal{V}(\eta)[v,v]+\tfrac{1}{2}\langle\mathrm{D}^2G(\eta)[v,v]\xi,\xi\rangle+2\langle\mathcal{L}_\zeta v,w\rangle+\langle G(\eta)w,w\rangle \tag{3.4}$$

via minimization with respect to w. ∎

The final step in simplifying $\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}$ is obtained by introducing pointwise functions as far as possible. This also provides more insight as the coefficients show their physical meaning.

Theorem 3.3 *If $\zeta=(\eta,\xi)$ is a critical point of $\mathcal{H}_c$, then*

$$\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\eta)[v,v]=\int_{\mathbb{R}}[\alpha\,v'^2+\kappa\,v^2-v\,(M_\eta v)]\,\mathrm{d}x \tag{3.5}$$

where $\alpha=b(1+\eta'^2)^{-3/2}$ and $\kappa=g+\gamma(\eta'\gamma)'$ with $\gamma=(c-\xi')/(1+\eta'^2)$. The operator M_η is self–adjoint on $\mathrm{L}^2(\mathbb{R})$, positive and defined via

$$M_\eta v=-\gamma\,(G(\eta)^{-1}(\gamma v)')'. \tag{3.6}$$

Remark: The function γ gives the negative of the relative horizontal velocity. To see this, we use $\phi_y - \eta'\phi_x = G(\eta)\xi$ and $\phi(x,\eta(x)) = \xi(x)$ to obtain

$$\nabla\phi|_{y=\eta} = \frac{1}{1+\eta'^2}\begin{pmatrix} 1 & -\eta' \\ \eta' & 1 \end{pmatrix}\begin{pmatrix} \xi' \\ G(\eta)\xi \end{pmatrix}.$$

For traveling waves with $G(\eta)\xi = -c\eta'$ this implies

$$\nabla\phi|_{y=\eta} - \begin{pmatrix} c \\ 0 \end{pmatrix} = \frac{\xi'-c}{1+\eta'^2}\begin{pmatrix} 1 \\ \eta' \end{pmatrix} = -\gamma\begin{pmatrix} 1 \\ \eta' \end{pmatrix}. \tag{3.7}$$

Proof: We specialize the formulae of Proposition 2.1 by using $v = \xi$ and $G(\eta)\xi = -c\eta'$. The coefficient functions a_j are then denoted by ρ_j:

$$\langle \widehat{v}, \mathrm{D}G(\eta)[w]\xi\rangle = \langle w, \rho_1\widehat{v}' + \rho_2 G(\eta)\widehat{v}\rangle \tag{3.8}$$

with $\rho_1 = (\xi'+c\eta'^2)/(1+\eta'^2)$ and $\rho_2 = \eta'(c-\xi')/(1+\eta'^2)$. Because of $\rho_3 = -\rho_1\rho_2$ we have

$$\langle \xi, \mathrm{D}^2G(\eta)[w,w]\xi\rangle = \langle w, \rho_4 w + 2\rho_2 G(\eta)(\rho_2 w)\rangle \tag{3.9}$$

with $\rho_4 = -2\rho_1'\rho_2$. Employing (3.8) and (3.9) to eliminate the terms $\mathrm{D}G(\eta)[w]\xi$ we find

$$\tfrac{1}{2}\langle \xi, \mathrm{D}^2G(\eta)[w,w]\xi\rangle - \langle \mathcal{L}_\zeta w, G(\eta)^{-1}\mathcal{L}_\zeta w\rangle = \langle w, \rho_5 w - M_\eta w\rangle$$

with $\rho_5 = (c-\rho_1)\rho_2' = \gamma(\eta'\gamma)'$ and M_η as defined in (3.6). ∎

Associated with the quadratic form $\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\eta)$ is the self–adjoint operator

$$\begin{aligned} A(\eta)v &= -(\alpha v')' + \kappa v - M_\eta v, \\ \mathcal{D}(A(\eta)) &= \mathrm{H}^2(\mathbb{R}) \subset \mathrm{L}^2(\mathbb{R}). \end{aligned} \tag{3.10}$$

The equation (2.7) for traveling waves (η,ξ) now takes the form

$$\begin{aligned} 0 &= G(\eta)\xi + c\eta', \\ 0 &= g\eta - b\left(\eta'/\sqrt{1+\eta'^2}\right)' + \tfrac{1}{2}(\xi'-c)^2/(1+\eta'^2) - \tfrac{c^2}{2}, \end{aligned} \tag{3.11}$$

where we have used (3.7). Differentiating both equations with respect to x we obtain for $(\widehat{v}_1, \widehat{w}_1) = (\eta', \xi')$ the equation

$$0 = \mathrm{D}G(\eta)[\widehat{v}_1]\xi + G(\eta)\widehat{w}_1 + c\widehat{v}_1', \quad 0 = g\widehat{v}_1 - (\alpha\widehat{v}_1')' - \gamma\widehat{w}_1' - \gamma^2\eta'\widehat{v}_1'.$$

Using (3.8) the first equation results in $\widehat{w}_1 = -G(\eta)^{-1}([\gamma\widehat{v}_1]') - \rho_3\widehat{v}_1$ and inserting this into the second equation gives

$$0 = g\widehat{v}_1 - (\alpha\widehat{v}_1')' - M_\eta\widehat{v}_1 + \gamma\rho_2'\widehat{v}_1 = A(\eta)\widehat{v}_1,$$

where we have used $\gamma\rho_2 = \gamma^2\eta'$. Thus, we have reconfirmed the well–known fact that by translation invariance $\widehat{v}_1 = \eta'$ has to lie in the kernel of $A(\eta)$.

We now collect a few general facts about the operator $A(\eta)$.

Theorem 3.4 *Assume that $\eta \in \mathrm{H}^r(\mathbb{R})$ with $r > 5/2$ and that $\inf\{\,\eta(x) \mid x \in \mathbb{R}\,\} > -h$. Then, the operator $A(\eta)$ defined in (3.10) is self–adjoint on $\mathrm{L}^2(\mathbb{R})$ with domain $\mathcal{D}(A(\eta)) := \mathrm{H}^2(\mathbb{R})$ and has the same continuous spectrum as $A(0)$, namely $[\sigma_*, \infty)$ with*

$$\sigma_* = g - \tfrac{g}{\lambda}\begin{cases} 1 & \text{for } \beta \geq 1/3, \\ \max_{s\in\mathbb{R}}\left(\frac{s}{\tanh s} - \beta s^2\right) & \text{for } \beta < 1/3, \end{cases} \tag{3.12}$$

where the dimensionless parameters are $\lambda = gh/c^2$ and $\beta = b/(hc^2)$.

Proof: We start with the operator $\widehat{A}w = -(\alpha w')'$ which is a self–adjoint operator on $\mathrm{L}^2(\mathbb{R})$ with domain $\mathcal{D}(\widehat{A}) = \mathrm{H}^2(\mathbb{R})$, since $\alpha \in \mathrm{C}^1_{\mathrm{bd}}(\mathbb{R})$ and $\alpha(x) \geq \alpha_{\min} > 0$ for all $x \in \mathbb{R}$. Now, $\kappa \in \mathrm{C}^0_{\mathrm{bd}}(\mathbb{R})$ and $M_\eta \in \mathrm{Lin}(\mathrm{H}^1(\mathbb{R}), \mathrm{L}^2(\mathbb{R}))$ imply the self–adjointness of $A(\eta)$ on the same domain.

The equality of the continuous spectra of $A(\eta)$ and $A(0)$ follows since $\eta(x) \to 0$ for $|x| \to \infty$. As $A(0)$ is translation invariant the whole spectrum is continuous and can be calculated by Fourier transform:

$$A(0)\mathrm{e}^{\mathrm{i}kx} = \sigma(k)\mathrm{e}^{\mathrm{i}kx} \quad \text{with } \sigma(k) = g + k^2 b - c^2 k/\tanh(hk).$$

The continuous spectrum of $A(0)$ is the range of $k \mapsto \sigma(k)$ and $\sigma_* = \min\{\,\sigma(k) \mid k \in \mathbb{R}\,\}$. For $\beta \geq 1/3$ this minimum is attained at $k = 0$ whereas for $\beta < 1/3$ the minimum selects a wave length $k \neq 0$. ∎

The region R_1 in the (λ, β)–plane where the continuous spectrum is bounded away from 0 via $\sigma_* > 0$ is depicted in Figure 1.1. For this region we show that the trivial parallel flow is energetically stable, see Theorem 5.1. For nontrivial traveling waves $\eta \neq 0$, instability may arise through discrete spectrum as well, thus we need additional spectral assumptions which are proved to hold for the small amplitude solitary waves.

4 Small amplitude solitary waves

In this section we investigate the second variation $A(\eta)$ at the small amplitude solitary waves for $\beta = b/(hc^2) > 1/3$ and $0 < \lambda - 1 \ll 1$ (region R_2 in Figure 1.1). A first rigorous existence proof for these solutions was given in [AK89], see also [Mie91] and [IK92].

We now fix $\beta > 1/3$, $c = h = 1$, and consider $\lambda = gh/c^2$ in the form $\lambda = 1 + \varepsilon^2$ as varying with the small parameter $\varepsilon \in (0, \varepsilon_0)$. The solitary waves under consideration have the expansion

$$\eta_\varepsilon(x) = \frac{-\varepsilon^2}{(\cosh(\varepsilon\mu x))^2} + O(\varepsilon^4 \mathrm{e}^{-2\varepsilon\delta|x|}), \quad \xi_\varepsilon(x) = -\frac{\varepsilon}{\mu}\tanh(\varepsilon\mu x) + O(\varepsilon^3), \tag{4.1}$$

where $\mu = (2\sqrt{\beta - 1/3})^{-1}$.

For the investigation of the associated operator $A(\eta_\varepsilon)$ we note that the critical part of the spectrum goes to 0 for $\varepsilon \to 0$ since $A(\eta_\varepsilon) \to A(0)$ which has the continuous spectrum $[0, \infty)$. It is more convenient to define the rescaled operator

$$\mathcal{A}_\varepsilon = \tfrac{1}{\varepsilon^2} S_\varepsilon^{-1} A(\eta_\varepsilon) S_\varepsilon : \mathcal{D}(\mathcal{A}_\varepsilon) \subset \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$$

with $\mathcal{D}(\mathcal{A}_\varepsilon) = \mathrm{H}^2(\mathbb{R})$ and $(S_\varepsilon \widetilde{v})(x) = \widetilde{v}(\varepsilon x)$. The point is that $\mathcal{A}_\varepsilon$ has a limit $\mathcal{A}_0$ for $\varepsilon \to 0$ and thus the spectrum of $A(\eta_\varepsilon)$ can be controlled.

Proposition 4.1 *For $\widetilde{w} \in \mathcal{D}(\mathcal{A}_0) = \mathrm{H}^2(\mathbb{R})$ we have*

$$\lim_{\varepsilon \to 0} \mathcal{A}_\varepsilon \widetilde{w} = \mathcal{A}_0 \widetilde{w} := -4\mu^2 \widetilde{w}'' + \widetilde{w} + 3\widetilde{\eta}\, \widetilde{w} \quad \textit{where } \widetilde{\eta}(\widetilde{x}) = -[\cosh(\mu\widetilde{x})]^{-2}.$$

Proof: The operator $\mathcal{A}_\varepsilon$ can be written in the form $\mathcal{A}_\varepsilon \widetilde{w} = -(\widetilde{\alpha}_\varepsilon \widetilde{w}')' + \widetilde{\kappa}_\varepsilon \widetilde{w} - \mathcal{M}_\varepsilon \widetilde{w}$ where

$$\widetilde{\alpha}_\varepsilon(\widetilde{x}) = \alpha(\widetilde{x}/\varepsilon) = \beta + O(\varepsilon^4), \quad \widetilde{\kappa}_\varepsilon(\widetilde{x}) = \tfrac{\kappa(\widetilde{x}/\varepsilon)}{\varepsilon^2} = \tfrac{1}{\varepsilon^2} + 1 + O(\varepsilon^2).$$

To obtain an expansion for $\mathcal{M}_\varepsilon$ we consider $\mathcal{K}_\varepsilon$ defined via $\mathcal{K}_\varepsilon \widetilde{w} = -S_\varepsilon^{-1}(G(\eta_\varepsilon)^{-1}(S_\varepsilon \widetilde{w})')'$. Using $G(\eta_\varepsilon) = G(0) + \mathrm{D}G(0)[\eta_\varepsilon] + O(\varepsilon^4)$ we find $\mathcal{K}_\varepsilon \widetilde{w} = \widetilde{w} + \varepsilon^2(-\tfrac{1}{3}\widetilde{w}'' - \widetilde{\eta}\widetilde{w}) + O(\varepsilon^4)$. Since $\widetilde{\gamma} = S_\varepsilon^{-1}\gamma = 1 - \varepsilon^2\widetilde{\eta} + O(\varepsilon^4)$ and $\mathcal{M}_\varepsilon \widetilde{w} = \frac{\widetilde{\gamma}}{\varepsilon^2}\mathcal{K}_\varepsilon(\widetilde{\gamma}\widetilde{w})$ we arrive at

$$\mathcal{M}_\varepsilon \widetilde{w} = \tfrac{1}{\varepsilon^2}\widetilde{w} - (3\widetilde{\eta}\widetilde{w} + \tfrac{1}{3}\widetilde{w}'') + O(\varepsilon^2).$$

After the cancellation of the terms of order ε^{-2} we obtain the desired result. ∎

The operator $\mathcal{A}_0$ is exactly the one which is associated with the linearization of the solitary wave in the Korteweg–de Vries equation. It has the three discrete eigenvalues $\widetilde{\sigma}_1 = -5/4$, $\widetilde{\sigma}_2 = 0$, and $\widetilde{\sigma}_3 = 3/4$ which are all simple and have the associated eigenfunctions

$$\widetilde{\psi}_1(\widetilde{x}) = [\cosh(\mu\widetilde{x})]^{-3}, \quad \widetilde{\psi}_2(\widetilde{x}) = \widetilde{\eta}'(\widetilde{x}), \quad \widetilde{\psi}_3(\widetilde{x}) = 4[\cosh(\mu\widetilde{x})]^{-1} - 5[\cosh(\mu\widetilde{x})]^{-3}.$$

The continuous spectrum is $[1, \infty)$. By the classical perturbation theory for discrete eigenvalues and by the explicit knowledge of the continuous spectrum we obtain the following result.

Theorem 4.2 *For $h = c = 1$ and fixed $\beta > 1/3$ let $\lambda = 1 + \varepsilon^2$, then for sufficiently small ε the operator $\mathcal{A}_\varepsilon$ has three discrete eigenvalues*

$$\sigma_1(\varepsilon) = -\tfrac{5}{4}\varepsilon^2 + O(\varepsilon^3), \quad \sigma_2(\varepsilon) = 0, \quad \sigma_3(\varepsilon) = \tfrac{3}{4}\varepsilon^2 + O(\varepsilon^3)$$

and the continuous spectrum is given by $[\varepsilon^2, \infty)$. The eigenfunctions ψ_j associated with σ_j have the expansion $\psi_j = S_\varepsilon \widetilde{\psi}_j + O(\varepsilon^{1/2})$.

5 Stability results

We first derive a stability result for the parallel flow with constant horizontal velocity c. Second we provide stability results for solitary waves, in particular for the small amplitude waves discussed in the last section. All our nonlinear stability results suffer from the deficiency that we don't have any global existence results for the water–wave problem. Under general conditions we cannot expect global existence as wave breaking may occur (with $\eta_x(t, x_0) \to \infty$ for $t \nearrow t_0$). However, we are interested in solutions which stay "energetically" close to the parallel flow or to a solitary wave. For such solutions, under additional technical conditions on the initial data, it is known that the solutions exist on very large time intervals, see [Cra85] and [SW00].

We begin by specifying the functional analytic setup for the nonlinear problem. We define the Hilbert space

$$\mathrm{H}^s_*(\mathbb{R}) = \{\, \xi \in \mathrm{H}^s_{\mathrm{loc}}(\mathbb{R}) \mid \xi' \in \mathrm{H}^{s-1}(\mathbb{R}) \,\}/_{\mathbb{R}}, \tag{5.1}$$

which means that we don't distinguish functions which just differ by a constant. The norm in $\mathrm{H}^s_*(\mathbb{R})$ is defined via $\|\xi\|_{s,*} := \|\xi'\|_{s-1}$. The construction was done such that

$$\|\xi\|^2_{\mathrm{H}^{1/2}_*} = \min\{\, \textstyle\int_{\Omega(0)} |\nabla\phi|^2 \,\mathrm{d}y\,\mathrm{d}x \mid \phi \in \mathrm{H}^1_{\mathrm{loc}}(\Omega(0)),\ \phi|_{y=0} = \xi \,\}.$$

The kinetic energy $\mathcal{K}(\eta,\xi) = \frac{1}{2}\langle G(\eta)\xi,\xi\rangle$ induces a norm which is equivalent to that of $\mathrm{H}^{1/2}_*(\mathbb{R})$. This is reconfirmed as $\langle G(\eta)\xi,\xi\rangle = \int_{\Omega(\eta)} |\nabla\phi|^2 \,\mathrm{d}y\,\mathrm{d}x$ and the trace mapping maps $\mathrm{H}^1_*(\Omega(\eta))$ into $\mathrm{H}^{1/2}_*(\Gamma(\eta))$. The mapping from $\Gamma(\eta)$ onto $\Gamma(0) = \mathbb{R}$ via $(x,\eta(x)) \mapsto x$ leads us to $\mathrm{H}^{1/2}_*(\mathbb{R})$. Here and below we always assume that η is Lipschitz continuous by the following restriction, which will be essential for the sequel. For $R > 1/h$ we define the set

$$E_R = \{\, \eta \in \mathrm{H}^1(\mathbb{R}) \mid -h + 1/R \le \eta(x) \le R,\ \|\eta'\|_\infty \le R \,\} \tag{5.2}$$

which contains wave profiles which are in $\mathrm{H}^1(\mathbb{R})$ but have additional restrictions, in particular that the slopes η' are bounded. We always consider E_R as being equipped with the H^1–norm, which turns E_R into a complete metric space. Note that on E_R all the topologies induced by $\mathrm{W}^{1,p}$–norms with $p \in [1,\infty)$ coincide.

The basic function space for our investigation is $Z := \mathrm{H}^1(\mathbb{R}) \times \mathrm{H}^{1/2}_*(\mathbb{R})$ which will turn out to be the natural energy space associated to the Hamiltonian $\mathcal{H}_c$. The solutions we consider lie in the closed subset $Z_R := E_R \times \mathrm{H}^{1/2}_*(\mathbb{R})$. The important fact is that on Z_R the nonlinearities in $\mathcal{H}_c$ can be controlled. For all $R > 1/h$ there exist $C_{1,\mathcal{K}}, C_{2,\mathcal{K}} > 0$ such that

$$C_{1,\mathcal{K}}\|\xi\|^2_{1/2,*} \le \mathcal{K}(\eta,\xi) \le C_{2,\mathcal{K}}\|\xi\|^2_{1/2,*} \tag{5.3}$$

for all $\xi \in \mathrm{H}^{1/2}_*(\mathbb{R})$ and all $\eta \in E_R$, cf. Appendix A. Moreover, the potential $\mathcal{V}$ is analytic on E_R and the horizontal momentum $\mathcal{P}(\eta,\xi) = \int_{\mathbb{R}} \eta'\xi \,\mathrm{d}x = -\int_{\mathbb{R}} \eta\,\xi' \,\mathrm{d}x$ is well–defined on Z_R. The difficult term will be the locked inertia tensor.

For the following we consider the water–wave problem in coordinates moving with velocity c. It is described by the Hamiltonian system

$$\begin{array}{rclcl} \eta_t &=& \mathrm{D}_\xi \mathcal{H}_c(\eta,\xi) &=& c\eta_x + \mathrm{D}_\xi\mathcal{K}(\eta,\xi), \\ \xi_t &=& -\mathrm{D}_\eta \mathcal{H}_c(\eta,\xi) &=& c\xi_x - \mathrm{D}_\eta\mathcal{V}(\eta) - \mathrm{D}_\eta\mathcal{K}(\eta,\xi). \end{array} \tag{5.4}$$

By Galilean invariance the solution of this system can also be understood as perturbations of a parallel flow $(\eta,\xi) = (0,-cx)$ when considered in a fixed coordinate system. Note that our choice of the function space $\mathrm{H}^{1/2}_*(\mathbb{R})$ for ξ forbids underlying parallel flows.

Our first stability result concerns the parallel flow $\eta \equiv 0$.

Theorem 5.1 *Let $g,h,b > 0$ and $c \ge 0$ be given such that σ_* in (3.12) is positive. Choose a constant $R > 0$. Then, for all $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds:*

If $\zeta : [0,T) \to Z_R$ is a continuous, energy preserving solution of (5.4) with $\|\zeta(0)\|_Z < \delta$, then $\|\zeta(t)\|_Z < \varepsilon$ for $t \in [0,T)$.

Proof: There is a $C_{2,\mathcal{V}} > 0$ with $\mathcal{V}(\eta) \le C_{2,\mathcal{V}}\|\eta\|_1^2$ for all $\eta \in E_R$. With (5.3) we find

$$\mathcal{H}_c(\eta,\xi) \le C_{2,\mathcal{K}}\|\xi\|_{1/2,*}^2 + |c|\|\eta\|_{1/2}\|\xi\|_{1/2,*} + C_{2,\mathcal{V}}\|\eta\|_1^2 \quad \text{for } \eta \in E_R.$$

This implies $\mathcal{H}_c(\zeta(t)) = \mathcal{H}_c(\zeta(0)) \le C_{\mathcal{H}}\delta^2$ for a fixed δ–independent $C_{\mathcal{H}}$.

Using the augmented potential the energy can be written as

$$\mathcal{H}_c(\eta,\xi) = \tfrac{1}{2}\langle G(\eta)[\xi + cG(\eta)^{-1}\eta'], \xi + cG(\eta)^{-1}\eta'\rangle + \mathcal{V}_c^{\mathrm{aug}}(\eta). \tag{5.5}$$

For $\mathcal{V}_c^{\mathrm{aug}}$ we need a lower estimate in terms of $\|\eta\|_1$. By our assumption we have

$$\mathrm{D}\mathcal{V}_c^{\mathrm{aug}}(0) = 0 \quad \text{and} \quad \mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(0)[v,v] \ge \sigma_*\|v\|_1^2 \text{ with } \sigma_* > 0.$$

Using the results of Appendix A there exists $\alpha = \alpha(R)$ such that for the locked inertia tensor $\Pi(\eta) = \langle K(\eta)\eta, \eta\rangle$ the estimate

$$|\Pi(\eta) - \tfrac{1}{2}\mathrm{D}^2\Pi(0)[\eta,\eta]| = |\langle (K(\eta) - K(0))\eta, \eta\rangle| \le \|K(\eta) - K(0)\|_{\mathrm{H}^1 \to \mathrm{H}^{-1}}\|\eta\|_1^2 \le C\|\eta\|_1^{2+\alpha}$$

holds for $\eta \in E_R$, and with the analogous statement for $\mathcal{V}$ the augmented potential $\mathcal{V}_c^{\mathrm{aug}} = \mathcal{V} - c^2\Pi$ satisfies the estimate

$$\mathcal{V}_c^{\mathrm{aug}}(\eta) \ge \tfrac{1}{2}\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(0)[\eta,\eta] - C_3\|\eta\|_1^{2+\alpha} \ge \tfrac{\sigma_*}{3}\|\eta_1\|^2$$

for $\eta \in E_R$ with $\|\eta\|_1 \le \varepsilon_1$ where ε_1 is defined via $C_3\varepsilon_1^\alpha = \sigma_*/6$.

Since by (5.5) the energy $\mathcal{H}_c$ is the sum of two nonnegative terms, we know $\mathcal{V}_c^{\mathrm{aug}}(\eta(t)) \le C_{\mathcal{H}}\delta^2$ and obtain the bound $\|\eta(t)\|_1 \le (3C_{\mathcal{H}}/\sigma_*)^{1/2}\delta$ if $\|\eta\|_1 \le \varepsilon_1$ is known. Using the fact that the first term in $\mathcal{H}_c$ is also less than $C_{\mathcal{H}}\delta^2$ and employing the triangle inequality for the norm induced by $\langle G(\eta)\,\cdot\,,\cdot\rangle$ we find

$$\begin{aligned}\langle G(\eta)\xi,\xi\rangle^{1/2} &\le |c|\langle G(\eta)^{-1}\eta', \eta'\rangle^{1/2} + \sqrt{2C_{\mathcal{H}}}\delta \\ &= |c|\langle K(\eta)\eta,\eta\rangle^{1/2} + \sqrt{2C_{\mathcal{H}}}\delta \le C_4\delta\end{aligned}$$

and hence $\|\xi\|_{1/2,*} \le C_\xi\delta$. Thus, we arrive at $\|\eta(t)\|_1 \le \|\zeta(t)\|_Z \le C_\zeta\delta$ and choosing $\delta < \min\{\varepsilon_1, \varepsilon\}/C_\zeta$ proves the result. ■

A similar result holds for solitary waves, however, there the situation is more complicated as $\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\eta)$ has a negative eigenvalue which has to be treated using the first integral $\mathcal{P}$.

We start with the following assumptions on the existence of a family of solitary waves. All these assumptions are satisfied for small amplitude solitary waves, as is shown in Section 4.

Assumptions (A): *Fix g, h and b and an interval (c_1, c_2) of velocities such that the curve $c \mapsto (\beta, \lambda) = (b/(hc^2), gh/c^2)$ lies inside the region R_1, where $\sigma_*(\lambda, \beta) > 0$, see Figure 1.1. There exists a family of solitary waves $c \mapsto \overline{\zeta}(c) := (\overline{\eta}(c), \overline{\xi}(c))$ with $\overline{\eta} \in \mathrm{C}^1((c_1, c_2), \mathrm{H}^2(\mathbb{R}))$ which are even in $x \in \mathbb{R}$. Moreover, $\mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\overline{\eta}(c))$ has a simple negative discrete eigenvalue $\sigma_1(c) < 0$ associated with an even eigenfunction ψ_1, the simple eigenvalue $\sigma_2(c) = 0$ associated with $\overline{\eta}(c)'$, and is strictly positive definite on the L^2–orthogonal complement of these two eigenspaces. Moreover, the function $h : c \mapsto \mathcal{H}_c(\overline{\zeta}(c))$ satisfies $h''(c) > 0$ for $c \in (c_1, c_2)$.*

We choose a particular $c \in (c_1, c_2)$ and study the stability of the solitary wave $\overline{\zeta}(c)$. Linearizing (5.4) in $z = (v, w)^\mathsf{T} = \zeta - \overline{\zeta}(c)$ leads to

$$z_t = \overline{L}(c)z \quad \text{where } \overline{L}(c) := \mathbb{J}\mathbb{A} \quad \text{with } \mathbb{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \text{ and } \mathbb{A} := \mathrm{D}^2\mathcal{H}_c(\overline{\zeta}(c)). \tag{5.6}$$

Using (3.4) or (5.5) the selfadjoint operator $\mathbb{A}$ takes the form

$$\langle\!\langle \mathbb{A}z, z \rangle\!\rangle = \langle\!\langle \mathbb{A}\binom{v}{w}, \binom{v}{w} \rangle\!\rangle = \langle \overline{C}v, v \rangle + \langle \overline{G}(w+\overline{B}v), w+\overline{B}v \rangle \tag{5.7}$$

with $\overline{C} = \mathrm{D}^2\mathcal{V}_c^{\mathrm{aug}}(\overline{\eta}(c))$, $\overline{G} = G(\overline{\eta}(c))$ and $\overline{B} \in \mathrm{Lin}(\mathrm{H}^1(\mathbb{R}), \mathrm{H}_*^{1/2}(\mathbb{R}))$ where

$$\overline{B}v := c\overline{G}^{-1}[v' - \mathrm{D}G(\overline{\eta})[v]\overline{G}^{-1}\overline{\eta}'] = \overline{G}^{-1}\mathcal{L}_{\overline{\zeta}(c)}v.$$

This shows that $\mathbb{A}$ maps $Z = \mathrm{H}^1(\mathbb{R}) \times \mathrm{H}_*^{1/2}(\mathbb{R})$ continuously into Z^* (the dual of Z) and that Z is the natural energy space. We continue to use

$$\langle\!\langle \binom{\nu}{\mu}, \binom{v}{w} \rangle\!\rangle := \langle \nu, v \rangle + \langle \mu, w \rangle$$

as the dual pairing for $z = \binom{v}{w} \in Z$ and $\binom{\nu}{\mu} \in Z^*$.

The existence of the two–parameter family of traveling waves has important consequences for $\mathbb{A}$. Taking the derivative along the x–axis we obtain $\mathbb{A}(c)\widehat{z}_1 = 0$ with $\widehat{z}_1 := (\widehat{v}_1, \widehat{w}_1)^\mathsf{T} = \overline{\zeta}(c)'$. Taking the derivative of $\mathrm{D}\mathcal{H}_c(\overline{\zeta}(c)) = 0$ with respect to the parameter c along the solution family we obtain the relation $\mathrm{D}^2\mathcal{H}_c(\overline{\zeta}(c))\widehat{z}_2 + \frac{\partial}{\partial c}\mathrm{D}\mathcal{H}_c(\overline{\zeta}(c)) = 0$ with $\widehat{z}_2 := (\widehat{v}_2, \widehat{w}_2)^\mathsf{T} = \frac{\partial}{\partial c}\overline{\zeta}(c)$. Using $\frac{\partial}{\partial c}\mathrm{D}\mathcal{H}_c(\overline{\zeta}(c)) = \mathrm{D}\mathcal{P}(\overline{\zeta}(c)) =: -\widehat{p}_2 \in Z^*$ we arrive at

$$\mathbb{A}\widehat{z}_1 = 0, \quad \mathbb{A}\widehat{z}_2 = \widehat{p}_2 = \binom{\widehat{w}_1}{-\widehat{v}_1} = \mathbb{J}\widehat{z}_1. \tag{5.8}$$

This tells us that $\widehat{z}_1$ and $\widehat{z}_2$ lie in the generalized kernel of $\overline{L}$, as $\overline{L}\widehat{z}_1 = 0$ and $\overline{L}\widehat{z}_2 = -\widehat{z}_1$.

With $\widehat{p}_1 := (\widehat{v}_1, 0)^\mathsf{T} \in Z^*$ we define the subspaces $Z^{\mathrm{c}} := \mathrm{span}\{\widehat{z}_1, \widehat{z}_2\}$ and

$$Z^+ := \{\, z = \binom{v}{w} \in Z \mid \langle\!\langle \widehat{p}_1, z \rangle\!\rangle = \langle\!\langle \widehat{p}_2, z \rangle\!\rangle = 0 \,\}.$$

Proposition 5.2 *Let Assumptions (A) be satisfied. Then, the space Z decomposes into the direct sum $Z^{\mathrm{c}} \oplus Z^+$ and $\mathbb{A}$ has block structure:*

$$\langle\!\langle \mathbb{A}(\alpha_1\widehat{z}_1+\alpha_2\widehat{z}_2+z_+), \alpha_1\widehat{z}_1+\alpha_2\widehat{z}_2+z_+ \rangle\!\rangle = -h''(c)\alpha_2^2 + \langle\!\langle \mathbb{A}z_+, z_+ \rangle\!\rangle \tag{5.9}$$

for all $\alpha_j \in \mathbb{R}$ and $z_+ \in Z^*$. Moreover, there is a constant $\gamma_+ > 0$ such that

$$\langle\!\langle \mathbb{A}z_+, z_+ \rangle\!\rangle \geq \gamma_+\|z^+\|_Z^2 \quad \textit{for all } z_+ \in Z^+.$$

Proof: (i) By symmetry arguments (oddness and evenness) we see $\langle\!\langle \widehat{p}_1, \widehat{z}_2\rangle\!\rangle = \langle\!\langle \widehat{p}_2, \widehat{z}_1\rangle\!\rangle = 0$. Obviously, $\langle\!\langle \widehat{p}_1, \widehat{z}_1\rangle\!\rangle = \int_{\mathbb{R}} \widehat{v}_1^2 \,\mathrm{d}x > 0$. Finally, we have

$$\langle\!\langle \widehat{p}_2, \widehat{z}_2\rangle\!\rangle = -\mathrm{D}\mathcal{P}(\overline{\zeta}(c))[\tfrac{\partial}{\partial c}\overline{\zeta}(c)] = -h''(c) < 0.$$

Hence the the operator $Q \in \mathrm{Lin}(Z,Z)$ defined via $Qz = \frac{\langle\!\langle \widehat{p}_1, z\rangle\!\rangle}{\langle\!\langle \widehat{p}_1, \widehat{z}_1\rangle\!\rangle}\widehat{z}_1 + \frac{\langle\!\langle \widehat{p}_2, z\rangle\!\rangle}{\langle\!\langle \widehat{p}_2, \widehat{z}_2\rangle\!\rangle}\widehat{z}_2$ is a projection with range Z^{c} and kernel Z^+. This proves the decomposition $Z = Z^{\mathrm{c}} \oplus Z^+$. With (5.8) and the symmetry $\langle\!\langle \mathbb{A}z_+, \widehat{z}_j\rangle\!\rangle = \langle\!\langle \mathbb{A}\widehat{z}_j, z_+\rangle\!\rangle$ we obtain the block structure (5.9).

(ii) The spectral assumptions on $\overline{C}$ imply that $\overline{C} + (\varepsilon - \sigma_1)\psi_1 \otimes \psi_1 + \varepsilon \widehat{v}_1 \otimes \widehat{v}_1$ is positive definite for all $\varepsilon > 0$. Here ψ_1 and $\widehat{v}_1 = \overline{\eta}(c)'$ are the eigenfunctions of the two nonpositive eigenvalues. Using the positive definiteness of $\overline{G}$ on $\mathrm{H}^{1/2}_*(\mathbb{R})$ we find $\gamma_\varepsilon > 0$ such that

$$\langle\!\langle \mathbb{A}z, z\rangle\!\rangle + (\varepsilon - \sigma_1)\langle\!\langle \tbinom{\psi_1}{0}, z\rangle\!\rangle^2 + \varepsilon \langle\!\langle \widehat{p}_1, z\rangle\!\rangle^2 \geq \gamma_\varepsilon \|z\|_Z^2,$$

for all $z \in Z$. This shows that $\mathbb{A}$ is positive definite up to a suitable rank–two correction.

(iii) Now we assume that $\mathbb{A}$ restricted to Z^+ is not positive definite. By (ii) this implies that there exists $\widetilde{z}_+ \in Z^+$ with $\langle\!\langle \mathbb{A}\widetilde{z}_+, \widetilde{z}_+\rangle\!\rangle \leq 0$. By (i) we know that $\mathbb{A}$ restricted to $Z^{\mathrm{c}} = \mathrm{span}\{\widehat{z}_1, \widehat{z}_2\}$ is negative semidefinite. This implies that $\mathbb{A}$ is negative semidefinite in the three–dimensional space $\mathrm{span}\{\widetilde{z}_+\} \oplus Z^{\mathrm{c}}$, which clearly contradicts (ii). ■

With this proposition we are able to state our conditional stability result for solitary waves. Our method follows closely the one introduced in [Ben72] for the KdV equation: the momentum $\mathcal{P}$ is employed to control the negative eigenvalue in $\mathbb{A} = \mathrm{D}^2\mathcal{H}_c(\overline{\zeta}(c))$.

Theorem 5.3 *Let Assumptions (A) be satisfied. Choose any $c \in (c_1, c_2)$ and $R > 1$. Then, for all $\varepsilon > 0$ there exists $\delta > 0$ such that we have:*

> *If $\zeta : [0, T) \to Z_R$ is a continuous solution of (5.4) which preserves $\mathcal{H}$ and $\mathcal{P}$ and satisfies $\|\zeta(0) - \overline{\zeta}(c)\|_Z < \delta$, then it satisfies*
>
> $$\inf_{x_0 \in \mathbb{R}} \|\zeta(t, \cdot - x_0) - \overline{\zeta}(c)\|_Z < \varepsilon \quad \textit{for all } t \in [0, T). \tag{5.10}$$

Remark: The stability result (5.10) states that the shape of the solitary wave is stable, but it allows for a drift $x_0 = x_0(t)$. This drift is of course necessary, since $\zeta(0)$ could be a solitary wave $\overline{\zeta}(\widetilde{c})$ with $\widetilde{c} \approx c$. Then, the drift is $x_0 = (c - \widetilde{c})t$.

Proof: The proof relies heavily on the linear splitting of Proposition 5.2 and the first integrals $\mathcal{H}_c$ and $\mathcal{P}$. Without loss of generality we may assume $\mathcal{P}(\zeta(t)) = \mathcal{P}(\zeta(0)) = \mathcal{P}(\overline{\zeta}(c))$. If this is not the case we redefine c and use $\|\overline{\zeta}(c) - \overline{\zeta}(\widetilde{c})\|_Z \leq K|c - \widetilde{c}|$.

The assumption on $\zeta(0)$ implies $\mathcal{H}_c(\zeta(t)) = \mathcal{H}_c(\zeta(0)) \leq \mathcal{H}_c(\overline{\zeta}) + K\delta^2$, for a fixed K. To estimate

$$\rho(t) = \inf_{x_0 \in \mathbb{R}} \|\zeta(\cdot - x_0) - \overline{\zeta}\|_Z^2$$

we choose a ζ–dependent translation $\widehat{x}_\zeta$ such that $\langle\!\langle \widehat{p}_1, \zeta(\cdot - \widehat{x}_\zeta) - \overline{\zeta}\rangle\!\rangle = 0$. Locally around $\overline{\zeta}$ the function $\zeta \mapsto \widehat{x}_\zeta$ is obtained by the implicit function theorem, since $\langle\!\langle \widehat{p}_1, \widehat{z}_1\rangle\!\rangle > 0$.

Clearly, $\rho(t) \leq \|\zeta(t,\cdot)-\overline{\zeta}(\cdot+\widehat{x}_{\zeta(t)})\|^2$ and it suffices to estimate $z(t) = \zeta(t)-\overline{\zeta}(\cdot+\widehat{x}_{\zeta(t)})$. By translation invariance we may further assume $\widehat{x}_\zeta = 0$. Hence, we are left with estimating $\|z\|_Z$ when knowing $\mathcal{P}(\overline{\zeta}+z) = \mathcal{P}(\overline{\zeta})$, $\langle\!\langle \widehat{p}_1, z\rangle\!\rangle = 0$ and $\mathcal{H}_c(\overline{\zeta}+z) \leq \mathcal{H}_c(\overline{\zeta})+K\delta^2$.

We use the expansion $z = z_+ + \gamma_z \widehat{z}_2$ with $\gamma_z \in \mathbb{R}$ and $\gamma_z = O(\|z\|_Z^2) = O(\|z_+\|_Z^2)$ which follows from $\mathcal{P}(\overline{\zeta}) = \mathcal{P}(\overline{\zeta}+z) = \mathcal{P}(\overline{\zeta}) + \langle\!\langle \widehat{p}_2, z\rangle\!\rangle + O(\|z\|_Z^2)$. With $z = (v,w)^\mathsf{T}$ and $\Xi(v) = cG(\overline{\eta}+v)^{-1}(\overline{\eta}+v)' - cG(\overline{\eta})^{-1}\overline{\eta}'$ we obtain the estimate

$$K\delta^2 \geq \mathcal{H}_c(\overline{\zeta}+z) - \mathcal{H}_c(\overline{\zeta}) =: J_1 + J_2 + J_3 + J_4 \quad \text{with}$$

$$\begin{aligned}
J_1 &= \tfrac{1}{2}\langle\!\langle \mathbb{A}z, z\rangle\!\rangle \geq \tfrac{1}{2}\langle\!\langle \mathbb{A}z_+, z_+\rangle\!\rangle - O(\|z\|^4) \geq \tfrac{\gamma_+}{2}\|z_+\|_Z^2 - O(\|z\|_Z^4),\\
J_2 &= \mathcal{V}_c^{\mathrm{aug}}(\overline{\eta}+v) - \mathcal{V}_c^{\mathrm{aug}}(\overline{\eta}) - \tfrac{1}{2}\langle \overline{C}v, v\rangle = O(\|v\|_1^3),\\
J_3 &= \tfrac{1}{2}\langle G(\overline{\eta}+v)(\Xi(v)-\overline{B}v), w+\Xi(v)\rangle = O(\|v\|_1^{1+\alpha})\|z\|_Z,\\
J_4 &= \tfrac{1}{2}\langle (G(\overline{\eta}+v)-\overline{G})(w+\overline{B}v), w+\overline{B}v\rangle = \|z\|_Z^2 O(\|v\|_1^\alpha).
\end{aligned}$$

Here we have used $\|\Xi(v)-\overline{B}v\|_{1/2,*} = O(\|v\|^{1+a})$ which follows from Theorem A.3 since $(\Xi(v))' = K(\overline{\eta}+v)[\overline{\eta}+v] - K(\overline{\eta})\overline{\eta}$. As $\|z_+\|_Z \leq \|z\|_Z \leq 2\|z_+\|_Z$ in a suitable neighborhood of $z=0$ we conclude $K\delta^2 \geq \frac{\gamma_+}{9}\|z\|_Z^2$ as long as $\|z\|_Z \leq \varepsilon_1$. This gives us $\|z(t)\|_Z < \varepsilon$ if we choose $\delta < (\gamma_+/(9K))^{1/2}\min\{\varepsilon, \varepsilon_1\}$. ■

A Appendix

The purpose of this appendix is to provide the technical continuity result for the operator $K(\eta) : w \mapsto -(G(\eta)^{-1}w')'$ as a function of η. Recalling the definition

$$E_R = \{\, \eta \in \mathrm{H}^1(\mathbb{R}) \mid \eta(x) \in [-h+1/R, R] \text{ for } x \in \mathbb{R},\ \|\eta'\|_\infty \leq R \,\}$$

we start with the following nontrivial regularity result.

Proposition A.1 *For each $R > 1/h$ there exist $p > 2$ and $C > 0$ such that for all $\eta \in E_R$ the operator $N(\eta)$ defined via $\phi = N(\eta)w$ and*

$$\begin{aligned}
&\Delta\phi = 0 \text{ in } \Omega(\eta), \quad \phi_y = 0 \text{ on } y = -h,\\
&[\phi_y - \eta'\phi_x]\big|_{y=\eta(x)} = w'(x) \text{ for } x \in \mathbb{R},
\end{aligned} \tag{A.1}$$

maps $\mathrm{H}^1(\mathbb{R})$ into $\mathrm{W}^{1,p}_*(\Omega(\eta))$ (where $\|\phi\|_{\mathrm{W}^{1,p}_*} = \|\nabla\phi\|_{\mathrm{L}^p}$) and we have the uniform bound

$$\|N(\eta)w\|_{\mathrm{W}^{1,p}_*} \leq C\|w\|_{\mathrm{H}^1(\mathbb{R})} \text{ for all } w \in \mathrm{H}^1(\mathbb{R}) \text{ and all } \eta \in E_R. \tag{A.2}$$

Remark: The classical theory gives $\phi \in \mathrm{H}^1_*(\Omega(\eta)) = \mathrm{W}^{1,2}_*(\Omega(\eta))$. Higher regularity is obstructed by corner singularities. Having the bound $\|\eta'\|_\infty \leq R$ the largest possible interior corner angle is $\pi+2\arctan R$. Hence, the theory of corner singularities for Dirichlet problems (cf. [Gri85]) shows that $\mathrm{W}^{1,p}$–regularity is false for $p \geq 2+\pi/\arctan R$.

Proof: We use the results in [Grö89] which will provide the desired p. We repeat the main arguments since the notation is needed below and since we have to show uniformity in R. Throughout this proof constants C may depend on R but not on ϕ, η or w.

We use the Lipschitz mapping $\Psi : \Omega(0) \to \Omega(\eta)$ with inverse mapping $\Phi = \Psi^{-1}$ given by $\Psi(x, \widetilde{y}) = (x, Y(x, \widetilde{y}))$ where $Y(x, \widetilde{y}) = \eta + (h+\eta(x))\widetilde{y}/h$. Hence, $\mathrm{Lip}\,\Psi$ and $\mathrm{Lip}\,\Phi$ are bounded by C. The function ϕ solves (A.1) if and only if $\psi = \phi \circ \Psi$ solves

$$\int_{\Omega(0)} \langle U_\eta(x, \widetilde{y})\widetilde{\nabla}\psi, \widetilde{\nabla}\chi\rangle_{\mathbb{R}^2}\, \mathrm{d}(x, \widetilde{y}) = \int_{\mathbb{R}} w'(x)\chi(x, 0)\, \mathrm{d}x \quad \text{for all } \chi \in \mathrm{H}^1(\Omega(0)), \tag{A.3}$$

where the coefficient matrix U_η takes the form

$$U_\eta(x, \widetilde{y}) := \det(\mathrm{D}\Psi)\, [\mathrm{D}\Psi]^{-1}[\mathrm{D}\Psi]^{-\mathsf{T}} = \begin{pmatrix} \frac{h+\eta}{h} & -\frac{h+\widetilde{y}}{h}\eta' \\ -\frac{h+\widetilde{y}}{h}\eta' & \frac{h^2+(h+\widetilde{y})^2\eta'^2}{h(h+\eta)} \end{pmatrix} = \begin{pmatrix} Y_{\widetilde{y}} & -Y_x \\ -Y_x & \frac{1+Y_x^2}{Y_{\widetilde{y}}} \end{pmatrix}.$$

Clearly, $\|U_\eta\|_\infty \le C = O(R^3)$ and, with $Y_x = (h+\widetilde{y})\eta'/h$ and $Y_{\widetilde{y}} = \frac{h+\eta}{h}$ we find the ellipticity estimate

$$\langle U_\eta b, b\rangle_{\mathbb{R}^2} = Y_{\widetilde{y}}^{-1}\Big([Y_{\widetilde{y}} b_1 - Y_x b_2]^2 + b_2^2\Big) \ge \tfrac{Y_{\widetilde{y}}}{1+Y_{\widetilde{y}}^2+Y_x^2}|b|^2 \ge \tfrac{1}{C_{\mathrm{ell}}}|b|^2 \text{ for } b \in \mathbb{R}^2$$

with $C_{\mathrm{ell}} = O(R^2)$. Thus, Theorem 1 together with Remark 13 of [Grö89] is applicable to (A.3) and we obtain the existence of a $p > 2$ which solely depends on R. Transforming backwards via $\phi = \psi \circ \Phi$ we find $\nabla\phi \in \mathrm{L}^p(\Omega(\eta))$ as desired. ■

It is remarkable that this proof relies on a similar simple flattening procedure for the Lipschitz domain which was introduced in the water wave context in [Kir88, Mie88, NR01]. This is in contrast to more sophisticated methods using conformal mapping principles, cf. [CM85, AK89, Wu93, Wu97].

The second needed proposition concerns the boundedness which follows from the analyticity of the Dirichlet–Neumann operator $G(\eta) \in \mathrm{Lin}(\mathrm{H}^1_*(\mathbb{R}), \mathrm{L}^2(\mathbb{R}))$ with respect to $\eta \in \mathrm{C}^{\mathrm{Lip}}(\mathbb{R}) = \mathrm{W}^{1,\infty}(\mathbb{R})$, see [CM85, CN00, NR01].

Proposition A.2 *For each $R > 1/h$ there exists a constant C such that for all $\eta \in E_R$ we have $\|K(\eta)\|_{\mathrm{H}^1(\mathbb{R})\to\mathrm{L}^2(\mathbb{R})} \le C$.*

Our main result is now that $K(\eta)$ depends Hölder continuously on η, if we weaken the operator topology for $K(\eta)$ simultaneously to the topology for η.

Theorem A.3 *For each $R > 1/h$ there exist $\alpha \in (0, 1)$ and $C > 0$ such that for all $\eta, \widetilde{\eta} \in E_R$ we have*

$$\|K(\eta) - K(\widetilde{\eta})\|_{\mathrm{H}^1(\mathbb{R})\to\mathrm{H}^{-1}(\mathbb{R})} \le C\|\eta - \widetilde{\eta}\|^\alpha_{\mathrm{H}^1(\mathbb{R})}. \tag{A.4}$$

Proof: The definition of $K(\eta) : w \to -(G(\eta)^{-1}w')'$ can be written using the Neumann–Dirichlet operator $N(\eta)$ of (A.1) as $v = K(\eta)w$ with $v = [\phi_x + \eta'\phi_y]|_{y=\eta}$ where $\phi = N(\eta)w$. Moreover, we have $\langle K(\eta)w, w\rangle = \int_{\Omega(\eta)} |\nabla\phi|^2\, \mathrm{d}y\, \mathrm{d}x$ and, by symmetry,

$$\|K(\eta)\|_{\mathrm{H}^1\to\mathrm{H}^{-1}} = \max\{\, \langle K(\eta)w, w\rangle \mid \|w\|_{\mathrm{H}^1} \le 1 \,\}. \tag{A.5}$$

Since ϕ solves the Neumann problem (A.1) it is the minimizer of the quadratic functional

$$\mathcal{J}_{\eta,w} : \chi \mapsto \int_{\Omega(\eta)} |\nabla\chi|^2 \, \mathrm{d}y \, \mathrm{d}x - 2 \int_{\mathbb{R}} w'(x)\chi(x, \eta(x)) \, \mathrm{d}x.$$

With $\mathrm{D}\mathcal{J}_{\eta,w}(\phi)[\phi] = 0$ we find $\langle K(\eta)w, w\rangle = -\min\{ \mathcal{J}_{\eta,w}(\chi) \,|\, \chi \in \mathrm{H}^1_*(\Omega(\eta)) \}$. To compare the functionals $\mathcal{J}_{\eta,w}$ for different η we transform the integral from $\Omega(\eta)$ to $\Omega(0)$ as in the proof of Proposition A.1. This gives

$$\mathcal{J}_{\eta,w}(\chi) = \widetilde{\mathcal{J}}_{\eta,w}(\widetilde{\chi}) := \textstyle\int_{\Omega(0)} \langle U_\eta \widetilde{\nabla}\widetilde{\chi}, \widetilde{\nabla}\widetilde{\chi}\rangle_{\mathbb{R}^2} \, \mathrm{d}x \, \mathrm{d}\widetilde{y} - 2 \int_{\mathbb{R}} w'(x)\widetilde{\chi}(x, 0) \, \mathrm{d}x.$$

We now fix w with $\|w\|_{\mathrm{H}^1} = 1$ and choose $\eta_1, \eta_2 \in E_R$. Let $\widetilde{\phi}_j \in \mathrm{H}^1(\Omega(0))$ be the minimizer of $\widetilde{\mathcal{J}}_{\eta_j,w}$, then

$$\widetilde{\mathcal{J}}_{\eta_2,w}(\widetilde{\phi}_2) - \widetilde{\mathcal{J}}_{\eta_1,w}(\widetilde{\phi}_2) \leq \langle (K(\eta_1) - K(\eta_2))w, w\rangle \leq \widetilde{\mathcal{J}}_{\eta_2,w}(\widetilde{\phi}_1) - \widetilde{\mathcal{J}}_{\eta_1,w}(\widetilde{\phi}_1). \tag{A.6}$$

It is now sufficient to estimate the right–hand side from above, as a lower estimate for the left–hand side is obtained by changing signs and interchanging 1 and 2. Employing Proposition A.1 and choosing $q \geq 2$ with $1/q + 1/p + 1/p = 1$ we obtain

$$\begin{aligned} \widetilde{\mathcal{J}}_{\eta_2,w}(\widetilde{\phi}_1) - \widetilde{\mathcal{J}}_{\eta_1,w}(\widetilde{\phi}_1) &\leq C_1 \|U_{\eta_1} - U_{\eta_2}\|_{\mathrm{L}^q} \|\widetilde{\nabla}\widetilde{\phi}_1\|^2_{\mathrm{L}^p} \\ &\leq C_2 \|\eta_1 - \eta_2\|^\alpha_{\mathrm{H}^1} \|w\|^2_{\mathrm{H}^1} \text{ with } \alpha = 2/q. \end{aligned} \tag{A.7}$$

Here we have used that U_η depends Lipschitz continuously on η, η' with $\eta \in [-h + 1/R, R]$ and $|\eta'| \leq R$. Combining (A.5), (A.6) and (A.7) proves the result. ■

References

[AK89] C. J. Amick and K. Kirchgässner. A theory of solitary water-waves in the presence of surface tension. *Arch. Rational Mech. Anal.*, 105, 1–49, 1989.

[Arn66] V. Arnold. Sur un principe variationel pour les écoulements stationnaires des liquides parfaits es ses applications aux problèmes de stabilité non linéaires. *J. Mécanique*, 5, 29–43, 1966.

[BDT00a] B. Buffoni, E. N. Dancer, and J. F. Toland. The regularity and local bifurcation of steady periodic water waves. *Arch. Ration. Mech. Anal.*, 152, 207–240, 2000.

[BDT00b] B. Buffoni, E. N. Dancer, and J. F. Toland. The sub-harmonic bifurcation of Stokes waves. *Arch. Ration. Mech. Anal.*, 152, 241–271, 2000.

[Ben72] T. B. Benjamin. The stability of solitary waves. *Proc. Roy. Soc. (London) Ser. A*, 328, 153–183, 1972.

[BM92] C. Baesens and R. S. MacKay. Uniformly travelling water waves from a dynamical systems viewpoint: some insights into bifurcations from Stokes' family. *J. Fluid Mech.*, 241, 333–347, 1992.

[BM95] T. J. Bridges and A. Mielke. A proof of the Benjamin–Feir instability. *Arch. Rational Mech. Anal.*, 133(2), 145–198, 1995.

[BO82] T. B. Benjamin and P. J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.*, 125, 137–185, 1982.

[Bri92] T. J. Bridges. Spatial Hamiltonian structure, energy flux and the water-wave problem. *Proc. Roy. Soc. London Ser. A*, 439, 297–315, 1992.

[Bri97] T. J. Bridges. Multi-symplectic structures and wave propagation. *Math. Proc. Cambridge Philos. Soc.*, 121, 147–190, 1997.

[CM85] R. Coifman and Y. Meyer. Nonlinear harmonic analysis and analytic dependence. In *Pseudodifferential operators and applications (Notre Dame, Ind., 1984), Proc. Symposia in Pure Math. Vol. 43*, pages 71–78. Amer. Math. Soc., Providence, RI, 1985.

[CN00] W. Craig and D. P. Nicholls. Travelling two and three dimensional capillary gravity water waves. *SIAM J. Math. Anal.*, 32, 323–359 (electronic), 2000.

[Cos88] M. Costabel. Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.*, 19, 613–626, 1988.

[Cra85] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10, 787–1003, 1985.

[CS93] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.*, 108, 73–83, 1993.

[GM01] M. D. Groves and A. Mielke. A spatial dynamics approach to three-dimensional gravity-capillary steady water waves. *Proc. Roy. Soc. Edinburgh Sect. A*, 131, 83–136, 2001.

[Gri85] P. Grisvard. *Elliptic problems in nonsmooth domains.* Pitman (Advanced Publishing Program), Boston, MA, 1985.

[Grö89] K. Gröger. A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. *Math. Ann.*, 283, 679–687, 1989.

[GT97] M. D. Groves and J. F. Toland. On variational formulations for steady water waves. *Arch. Rational Mech. Anal.*, 137, 203–226, 1997.

[IK92] G. Iooss and K. Kirchgässner. Water waves for small surface tension: an approach via normal form. *Proc. Roy. Soc. Edinburgh Sect. A*, 122, 267–299, 1992.

[ILS01] G. Iooss, E. Lombardi, and S. Sun. Gravity travelling waves for two superposed fluid layers, one being of infinite depth: a new type of bifurcations. *Phil. Trans. Royal Soc. London*, 2001. Submitted.

[Kir82] K. Kirchgässner. Wave-solutions of reversible systems and applications. *J. Differential Equations*, 45, 113–127, 1982.

[Kir88] K. Kirchgässner. Nonlinearly resonant surface waves and homoclinic bifurcation. *Advances in Applied Mechanics*, 26, 135–181, 1988.

[KS98] H.-P. Kruse and J. Scheurle. On the bifurcation and stability of rigidly rotating inviscid liquid bridges. *J. Nonlinear Sci.*, 8, 215–232, 1998.

[Lew89] D. Lewis. Nonlinear stability of a rotating planar liquid drop. *Arch. Rational Mech. Anal.*, 106(4), 287–333, 1989.

[LMR87] D. Lewis, J. Marsden, and T. Ratiu. Stability and bifurcation of a rotating planar liquid drop. *J. Math. Phys.*, 28(10), 2508–2515, 1987.

[Mie88] A. Mielke. Reduction of quasilinear elliptic equations in cylindrical domains with applications. *Math. Methods Appl. Sci.*, 10(1), 51–66, 1988.

[Mie91] A. Mielke. *Hamiltonian and Lagrangian flows on center manifolds.* Springer-Verlag, Berlin, 1991. With applications to elliptic variational problems.

[NR01] D. P. Nicholls and F. Reitich. Stability of high-order perturbative methods for the computation of Dirichlet-Neumann operators. *J. Comput. Phys.*, 170, 276–298, 2001.

[PT01] P. I. Plotnikov and J. F. Toland. Nash-Moser theory for standing water waves. *Arch. Ration. Mech. Anal.*, 159, 1–83, 2001.

[SLM91] J. C. Simo, D. Lewis, and J. E. Marsden. Stability of relative equilibria. I. The reduced energy-momentum method. *Arch. Rational Mech. Anal.*, 115, 15–59, 1991.

[SW00] G. Schneider and C. E. Wayne. The long wave limit for the water wave problem. – I. the case of zero surface tension. *Comm. Pure Appl. Math.*, 53, 1475–1535, 2000.

[Tol96] J. F. Toland. Stokes waves. *Topol. Methods Nonlinear Anal.*, 7(1), 1–48, 1996.

[Wu93] S. J. Wu. Analytic dependence of Riemann mappings for bounded domains and minimal surfaces. *Comm. Pure Appl. Math.*, 46, 1303–1326, 1993.

[Wu97] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130, 39–72, 1997.

[Wu99] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12, 445–495, 1999.

[Zak68] V. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys*, 9, 190ff, 1968.