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Γ-convergence for evolutionary problems

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Many evolutionary problems, such as partial differential equations, display several temporal or spatial scales and it is desirable to find a suitable limit model that describes the macroscopic dynamics correctly. We want to address some general concepts that might be useful for deriving such effective models.

1. Geometric evolution via functionals

Consider a manifold $\mathcal{Q}$ and an energy-storage functional (potential energy) $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. For the dynamics we distinguish the dissipative situation and the Hamiltonian one.

In the first case we have Rayleigh's dissipation potential $\mathcal{R} : \mathrm{T}\mathcal{Q} \to \mathbb{R}_\infty$, where $\mathcal{R}(q,\cdot) : \mathrm{T}_q\mathcal{Q} \to \mathbb{R}_\infty$ is assumed to be convex. The evolution law is then given in terms of the balance of the dissipative forces $\partial_{\dot q}\mathcal{R}(q,\dot q)$ and the potential restoring forces $-\mathrm{D}\mathcal{E}(q)$, namely

$$0 \in \partial_{\dot q}\mathcal{R}(q,\dot q) + \mathrm{D}\mathcal{E}(t,q) \subset \mathrm{T}^*_q\mathcal{Q}. \tag{1}$$

(I) The viscous case corresponds to $\mathcal{R}$, which is given in terms of a Riemannian metric $\mathbf{g}$, i.e., $\mathcal{R}(q,v) = \frac{1}{2}\langle \mathbf{g}(q)v, v\rangle$, and leads to so-called gradient flows

$$\mathbf{g}(q)\dot q = -\mathrm{D}\mathcal{E}(t,q) \quad \Leftrightarrow \quad \dot q = -\nabla_{\mathbf{g}}\mathcal{E}(t,q).$$

(II) Another interesting dissipative situation is the case of rate-independent systems where $\mathcal{R}(q,\cdot)$ is homogeneous of degree 1. Then, $\partial_v\mathcal{R}(q,v) \subset \mathrm{T}_q\mathcal{Q}$ denotes the set-valued subdifferential of the convex function $\mathcal{R}(q,\cdot)$ and (1) is a differential inclusion, which may be reformulated as an evolutionary variational inequality, cf. [Mie05]. In fact, for the rate-independent case there is a weaker *energetic formulation* in terms of a global *stability condition* (S) and the *energy balance* (E). This formulation uses the dissipation distance $\mathcal{D} : \mathcal{Q}\times\mathcal{Q} \to \mathbb{R}_\infty$, that is associated with $\mathcal{R}$ via

$$\mathcal{D}(q_0,q_1) = \inf\{\, \textstyle\int_0^1 \mathcal{R}(\widetilde q(t), \dot{\widetilde q}(t))\,\mathrm{d}t \mid \widetilde q \in \mathrm{W}^{1,1}([0,1];\mathcal{Q}), \widetilde q(0) = q_0, \widetilde q(1) = q_1 \,\}.$$

We call a curve $q : [0,T] \to \mathcal{Q}$ an *energetic solution* associated with the functionals $\mathcal{E}$ and $\mathcal{D}$, if for all $t \in [0,T]$ we have

$$\begin{array}{ll} \text{(S)} & \mathcal{E}(t,q(t)) \le \mathcal{E}(t,\widetilde q) + \mathcal{D}(q(t),\widetilde q) \text{ for all } \widetilde q \in \mathcal{Q}, \\ \text{(E)} & \mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(q,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s. \end{array} \tag{2}$$

(III) Also classical Hamiltonian systems are driven by two functionals. In addition to the potential energy $\mathcal{E} : \mathcal{Q} \to \mathbb{R}_\infty$ we also have the kinetic energy $\mathcal{K} : \mathrm{T}\mathcal{Q} \to \mathbb{R}_\infty$, which is again given by a Riemannian metric $\mathbf{g}$ in the form $\mathcal{K}(q,\dot q) = \frac{1}{2}\langle \mathbf{g}(q)\dot q, \dot q\rangle$. The evolution equations in $\mathrm{T}\mathcal{Q}$ (the Lagrangian setting) then read

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{D}\mathcal{K}(q,\dot q)\big) + \mathrm{D}\mathcal{E}(q) = 0 \in \mathrm{T}^*_q\mathcal{Q}. \tag{3}$$

Often the canonical Hamiltonian form is preferred. It is based on the conjugate momentum $p = \mathbf{g}(q)\dot{q}$ and the Hamiltonian $\mathcal{H}(q,p) = \frac{1}{2}\langle p, \mathbf{g}(q)^{-1}p\rangle + \mathcal{E}(q)$:

$$\dot{q} = \mathrm{D}_p\mathcal{H}(q,p) = \mathbf{g}(q)^{-1}p \in \mathrm{T}_q\mathcal{Q}, \quad \dot{p} = -\mathrm{D}_q\mathcal{H}(q,p) = -\mathrm{D}\mathcal{E}(q) \in \mathrm{T}_q^*\mathcal{Q}. \tag{4}$$

2. Γ-convergence and the limit passage

We now consider sequences of pairs of functionals, namely $(\mathcal{E}_k, \mathcal{R}_k)$ for general dissipative systems, $(\mathcal{E}_k, \mathcal{D}_k)$ for the energetic formulation of the rate-independent case, and $(\mathcal{E}_k, \mathcal{K}_k)$ for Hamiltonian systems. Additionally we consider an associated sequence of solutions $q_k : [0,T] \to \mathcal{Q}$.

To define a convergence we equip $\mathcal{Q}$ with a Hausdorff topology and write "$\stackrel{\mathcal{Q}}{\to}$" for the corresponding convergence. The functional $\mathcal{E} : \mathcal{Q} \to \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$ is called Γ-*limit of the sequence* $(\mathcal{E}_k)_{k\in\mathbb{N}}$, if the following two conditions hold:

(i) lower estimate: $\quad q_k \stackrel{\mathcal{Q}}{\to} q \implies \mathcal{E}(q) \leq \liminf_{k\to\infty} \mathcal{E}_k(q_k)$,
(ii) upper estimate: $\quad \forall\, q \in \mathcal{Q}\ \exists\, \widehat{q}_k : \ \widehat{q}_k \stackrel{\mathcal{Q}}{\to} q$ and $\mathcal{E}(q) = \lim_{k\to\infty} \mathcal{E}_k(\widehat{q}_k)$.

Now assume that both functionals Γ-converge (independently) and that we have solutions q_k with a pointwise limit $q : [0,T] \to \mathcal{Q}$, i.e., $q_k(t) \stackrel{\mathcal{Q}}{\to} q(t)$. Then , it is a natural question whether q is a solution of the problem defined by the limit functionals. Of course, we cannot expect that the result holds true in sufficient generality. The real task is to identify conditions in the sense of a "joint Γ-convergence" for the two functionals that guarantee the desired result.

(0) In fact, Γ-convergence was introduced for *static problems*. It was developed over the last decade to provide very elegant and strong tools for deriving such macroscopic models, see [Dal95, Bra02]. In particular, it satisfies the desired convergence property in the following sense: If q_k is a minimizer of $\mathcal{E}_k$ and if $q_k \stackrel{\mathcal{Q}}{\to} q$, then q is a minimizer of the Γ-limit $\mathcal{E}$.

(I) For *gradient flows*, abstract positive results are contained in [SS04, Ort05]. They are based on specific assumptions on the gradients $\nabla\mathcal{E}_k(q_k)$. The following simple example in $\mathbb{R}^2$ shows that the desired result may even fail in finite dimensions. We let $\mathcal{Q} = \mathbb{R}^2$ and

$$\mathcal{E}_k(q) = \tfrac{1}{2}q_1^2 + \tfrac{k^\alpha}{2}(q_2 - q_1/k)^2 \quad\text{and}\quad \mathcal{R}_k(\dot{q}) = \tfrac{1}{2}\dot{q}_1^2 + \tfrac{k^\beta}{2}\dot{q}_2^2,$$

where α, β are positive constants. The Γ-limits $\mathcal{E}$ and $\mathcal{R}$ are easily obtained, namely $\mathcal{E}(q) = q_1^2/2$ for $q_2 = 0$ and ∞ otherwise and $\mathcal{R}(\dot{q}) = \dot{q}_1^2/2$ for $\dot{q}_2 = 0$ and ∞ otherwise. The solution with $q(0) = (1,0)^\top$ of the limit problem is obviously $q(t) = (\mathrm{e}^{-t}, 0)^\top$. The solution $q_k : [0,\infty) \to \mathbb{R}^2$ for the functionals $\mathcal{E}_k$ and $\mathcal{R}_k$ with $q_k(0) = (1,0)^\top$ can be written down explicitly in terms of the eigenvalues. The limit $k \to \infty$ shows that the correct limit solution is obtained only if $\min\{\alpha,\beta\} < 2$.

(II) For *rate-independent systems*, Γ-convergence is studied via the energetic formulation (2) in [MO06, MRS06]. Since rate-independent systems are very close to static problems (cf. (S), which is a purely static condition), the conditions can be formulated totally in terms of the functionals without using differentials. Again a simple example in $\mathbb{R}^2$ can be constructed to show that the limit passage is not true in general.

The main condition, which provides the positive result, is the existence of *joint recovery sequences*:

$$\begin{aligned}&\forall\, q_k \in \mathcal{S}_k(t) \text{ with } q_k \overset{\mathcal{Q}}{\to} q \;\forall\, \widehat{q}\;\exists\, \widehat{q}_k \text{ with } \widehat{q}_k \overset{\mathcal{Q}}{\to} \widehat{Q}:\\ &\limsup_{k\to\infty} \big(\mathcal{E}_k(t,\widehat{q}_k){+}\mathcal{D}_k(q_k,\widehat{q}_k){-}\mathcal{E}_k(t,q_k)\big) \leq \mathcal{E}(t,\widehat{q}){+}\mathcal{D}(q,\widehat{q}){-}\mathcal{E}(t,q),\end{aligned}$$

where $\mathcal{S}_k(t) = \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t,q) < \infty,\ \forall\, \widetilde{q} \in \mathcal{Q} : \mathcal{E}_k(t,q) \leq \mathcal{E}_k(t,\widetilde{q}) + \mathcal{D}_k(q,\widetilde{q})\,\}$ denotes the sets of stable states. Based on this condition several applications are given in [MRS06]. In [MT06] an application of two-scale homogenization for linearized elastoplasticity is derived.

(III) For *Hamiltonian systems* an abstract theory has not been developed.The oscillatory behavior of the solutions leads to an ongoing exchange between kinetic and potential energy, which is enforced by the underlying symplectic structure. So far, it is unclear how these structures can be used along with Γ-convergence. First preliminary results are given in [Mie06, GHM06]. There the passage from a discrete lattice system to a macroscopic elastodynamic wave equation is shown by different tools. As a result we obtain that the separate Γ-convergence of $\mathcal{K}_k$ and $\mathcal{E}_k$ in the Lagrangian setting (3) gives the correct limit equation. However, doing the corresponding Γ-limit in the canonical Hamiltonian system (4) leads to a limit equation, which, in general, does not characterize the limits of solutions.

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