Multi–pulse solutions in problems with reflection symmetry in cylindrical domains

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Abstract. As an example a weakly-coupled reaction-diffusion system on the real line is treated which is invariant under reflection. If we suppose that the critical wave number is 0, then under non-degeneracy conditions the existence of time-periodic single and multiple pulse solutions is demonstrated. Preliminary statements on the behavior of the Floquet multipliers of the linearisation on single-pulses are given.

Multi-pulse solutions appear in a variety of equations governing physical problems with reflection symmetry. For instance one can consider hydromagnetic problems in 3D layers [Ch61]. For the study of 3D Poiseuille problem on viscous incompressible fluid flow between parallel plates we refer to [AfM99b]. In these problems multi-pulse solutions in transversal direction are downstream traveling waves. Unfortunately, so far no hydrodynamical example with the required properties was found where it is possible to avoid computer assistant study of the spectral problem of stability.

We prove the existence of periodic in time multi-pulses on a simple example of a weakly coupled reaction-diffusion system

$$\left.\begin{aligned} c\,\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial z^2} - (\beta{-}\mu)u - \sqrt{c_0^2{+}\beta^2}\, v - u^3 \\ c\,\frac{\partial v}{\partial t} &= d\frac{\partial^2 v}{\partial z^2} + \sqrt{c_0^2{+}\beta^2}\, u + (\beta{+}\mu)v - bu^2 v \end{aligned}\right\}, \qquad U = (u,v)' \in \mathbb{R}^2, \tag{1}$$

studied earlier in [Kir84, IoM92] where a suitable functional frame was given.

Parameters c, β, μ, c_0, d, b are real and $\mu, c - c_0$ are supposed to be small. Additional non-degeneracy conditions will be introduced later. Here, the reflection symmetry $z \to -z$ is obvious. The spectral problem of stability of the trivial solution $U = (0,0)'$ under the assumption of the spatial periodicity with the period $2\pi/k$ yields

$$c\,\sigma_\pm = \mu - \frac{d{+}1}{2}k^2 \pm \mathrm{i}\sqrt{c_0^2{+}\beta(d{-}1)k^2 - (d{-}1)k^4/4}. \tag{2}$$

For $\beta(d{-}1) > -2\sqrt{dc_0^2}$ the neutral curve of stability of the trivial steady state solution, given by $\operatorname{Re}\sigma_+(\mu,k) = 0$, takes the global minima at $k = 0$. Let us consider 2π-time-periodic solutions to problem (1).

Theorem 1
Fix parameters $\beta, b, d > 0, c_0 \neq 0$. Under non-degeneracy conditions on parameters given

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below there exists $\mu_0 > 0$ *and smooth function* $c^{(1)}(\cdot) : (-\mu_0, 0] \to \mathbb{R}$ *with* $c^{(1)}(0) = c_0$ *such that for* $\mu \in (-\mu_0, 0)$ *problem* (1) *with* $c = c^{(1)}(\mu)$ *has a family of* 2π*-time-periodic solutions (single-pulses)* $U^{(1)}(t, z)$ *with the following expansion*

$$U^{(1)}(t,z) = \tfrac{\eta}{s}\,\mathrm{Re}\Big(\cosh(\eta z)^{-1-\mathrm{i}\omega}\Psi_0(t)\Big) + \mathcal{O}(\eta^2\mathrm{e}^{-|\eta z|}),$$

where $\Psi_0(t) = \mathrm{e}^{\mathrm{i}t}(-\mathrm{i}\sqrt{c_0^2+(\beta-\mu)^2},\, \mathrm{i}(\beta-\mu) - c_0)'$ *and* $\eta, s, \omega \in \mathbb{R}$ *are defined from the relations*

$$\left.\begin{array}{rcl} 2c_0(\mu - \mathrm{i}(c^{(1)}(\mu)-c_0))/\rho &=& -\eta^2(1+\mathrm{i}\omega)^2 \\ (c_0^2+\beta^2)\left((b+3)c_0 + 3\mathrm{i}\beta(1-b)\right)/\rho &=& s^2(1+\mathrm{i}\omega)(2+\mathrm{i}\omega) \end{array}\right\} \qquad (3)$$

with $\rho = (d+1)c_0 + \mathrm{i}\beta(1-d)$.

Notice that the parameters are taken such that the bifurcation is subcritical since supercritical bifurcation clearly leads to unstable single-pulse solution.

Non-degeneracy condition. We suppose that transformation (3) has maximal rank and $\omega \notin \{0, \pm\omega_1, \pm\omega_2, \cdots\}$ with $\omega_1 \approx 8.03, \omega_2 \approx 9.51$ (cf. [AfM99a]).

Theorem 2

Take μ_0 *as in the theorem 1 and suppose that the non-degeneracy condition is fulfilled. Then for each integer* $n \geq 2$ *and each* $\mu \in (-\mu_0, 0)$ *there exists a sequence* $c_j^{(n)}(\mu)$ *with* $c_j^{(n)}(\mu) \to c^{(1)}(\mu)$ *for* $j \to \infty$ *such that problem* (1) *has a sequence of* 2π*-time-periodic* n*–pulse solutions* $U_j^{(n)}(t, z)$ *which satisfies*

$$\sup_{z\in\mathbb{R}} \Big\{ \|U_j^{(n)}(\cdot, z) - \sum_{k=1}^{n} U^{(1)}(\cdot - t_{j,k}^{(n)}, z - z_{j,k}^{(n)}) \Big\} \|_{[H^2_{per}(2\pi)]^2} \to 0 \quad \text{for } j \to \infty$$

with a suitable $t_{j,k}^{(n)} \in \mathbb{S}^1$ *and* $z_{j,k}^{(n)} \in \mathbb{R}$.

The proof of theorems 1,2 relays on the Kirchgässner reduction (cf. [Kir84, IoM92, AfM99b]) which leads to a 4–dimensional SO(2) invariant and reversible ODE. The analysis of this ODE yields the existence of homoclinic (single-pulse) solutions as well as multi-pulse solutions (cf. [AfM98, AfM99a, KaMP96]).

Stability of the single-pulse $U^{(1)}(t, z)$ solution to (1) is determined by the orbital stability of the Hocking-Stuartson pulse solution $H_\omega(x) = \mathrm{e}^{\mathrm{i}\gamma}(\cosh x)^{-1-\mathrm{i}\omega}, \gamma \in \mathbb{S}$ to the complex Ginzburg–Landau equation

$$\frac{\partial A}{\partial \tau} = (1 + \mathrm{i}\alpha)\left(\frac{\partial^2 A}{\partial x^2} - (1+\mathrm{i}\omega)^2 A + (1+\mathrm{i}\omega)(2+\mathrm{i}\omega)A|A|^2\right) \qquad (4)$$

with $\alpha = \frac{\beta}{c_0}\frac{1-d}{1+d}$, cf. [Sch99]. In fact, (4) can be understood as a modulation equation for problem (1), cf. [Mi99], where $x = \eta z$ is the large spatial scale and $\tau = \eta^2 t$ is the slow time. Clearly $H_\omega(x)$ can be stable only if $A \equiv 0$ is stable. Therefore analyzing continuous spectrum of the linearized problem we arrive at the necessary condition of stability $\omega^2 + 2\alpha\omega \leq 1$. Certainly there are discrete eigenvalues and in particular two eigenvalues 0 due to rotation and translation invariance. On the critical curve $\omega^2 + 2\alpha\omega = 1$ the eigenvalue 0 is in fact threefold since one can easily check the existence of the generalized eigenfunction $W(x) = [1 + (1 + \mathrm{i}\omega)x \tanh x]H_\omega(x)$ (cf. [Mi99]). There is no even numerical evidence of the existence of the stable $H_\omega(x)$ pulses [BL98], but the question remains open since for large $|\omega\alpha|$ the problem is not studied so far.

Notice that in the degenerate case for parameters $\beta = \mathcal{O}(|\mu|^{1/2})$ and $b+3 = \mathcal{O}(|\mu|^{1/2})$ problem (1) also has single and multi-pulse solutions, but of a different type, since the problem is reduced to the cubic-quintic complex Ginzburg–Landau equation. This problem can be also included into the framework of the papers [AfM98, AfM99a] and statements similar to theorems 1,2 are valid. In degenerate complex Ginzburg–Landau equation stable pulses are known to exist at least for the perturbed cubic-quintic nonlinear Schrödinger equation [KaS98]. Notice that some of the n-pulse solutions can only be stable if the single-pulse is stable. Stability of single-pulses in cubic-quintic complex Ginzburg–Landau equation and stability problem of multi-pulse solutions remains unstudied.

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