Exponentially weighted L^∞–estimates and attractors for parabolic systems on unbounded domains[1]

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Abstract. We consider parabolic systems of reaction–diffusion type on large and unbounded domains $\Omega \subset \mathbb{R}^d$. Under the assumption that there is an absorbing ball in $\mathrm{L}^\infty(\Omega)$ we discuss the existence of global attractors $\mathcal{A}^\Omega$ and their dependence on the domain Ω. For this we derive continuity properties of the solution operator in exponentially weighted L^∞–norms.

1. Introduction

The dynamics of dissipative problems on unbounded domains has attracted a lot of attention over the last decade. In particular, the existence of attractors and their properties are of interest since the usual compactness methods do not apply, cf. [BaV90, MiS95, Mie97, GiV96, CoE99a, CoE99b, EfZ99].

Here we treat one aspect of this area, namely the theory in L^∞–spaces. Most previous work is based on L^p–spaces where the a–priori estimates are obtained by weighted energy estimates. Here we want to show how the maximum principle can be used even for a parabolic system of equations. This method is not useful to obtain a–priori bounds, it is meant instead to be used for finer qualitative analyses of the flow in an invariant set (e.g. the global attractor).

To be more precise, we consider parabolic systems of reaction–diffusion type

$$u_t = D\Delta u + f(u) \quad \text{with } x \in \Omega \subset \mathbb{R}^d, \quad u(t,\cdot)|_{\partial\Omega} = 0, \tag{1}$$

where $u(t,x) \in \mathbb{R}^m$. The eigenvalues of the diffusion matrix D have positive real part. The nonlinearity satisfies $f \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$ and is such that there exists a radius $Q_* > 0$ such that for all smooth domains Ω the ball $\{\, u \in \mathrm{L}^\infty(\Omega) \,:\, \|u\|_\infty \le Q_* \,\}$ is absorbing for the semigroup $(\mathcal{S}_t^\Omega)_{t\ge 0}$ associated to (1). See [Col94, Mie97, GiV96, Mie98] for a construction of such bounds Q_*. Alternatively we might assume the existence of an invariant compact region $K \subset \mathbb{R}^m$ such that the set $\{\, u \in \mathrm{L}^\infty(\Omega) \,:\, u(x) \in K \text{ for a.e. } x \in \Omega \,\}$ is invariant under the semiflow of (1), see [Smo83]. We then let $Q_* = \max\{\, |u| \,:\, u \in K \,\}$.

We are interested in the dependence of the global attractor $\mathcal{A}^\Omega \subset \mathrm{L}^\infty(\Omega)$ of (1). For bounded domains this means the classical compact attractor and for unbounded domains it is the attractor with respect to a weighted norm $\|\cdot\|_\rho$ defined via $\|u\|_\rho = \|\rho\, u\|_\infty$ (cf. Section 3), where the weight $\rho : \mathbb{R}^d \to (0,\infty)$ is continuous and satisfies $\rho(x) \to 0$ for $|x| \to \infty$. In case of the special weight $\rho_\alpha : x \mapsto \mathrm{e}^{-\alpha|x|}$ we shortly write $\|\cdot\|_{\{\alpha\}}$ for $\|\cdot\|_{\rho_\alpha}$. The associated functions spaces are denoted by $\mathrm{L}^\infty_\rho(\Omega)$ and $\mathrm{L}^\infty_{\{\alpha\}}(\Omega)$, respectively.

As a major tool for comparing semigroups on different domains and for analyzing the attractors in itself (like in [CoE99a, CoE99b]) it is useful to derive exponentially weighted

[1]Research partially supported by DFG–SPP 'Dynamische Systeme' under Mi 459/2-3.

estimates which allows us to control the influence of the initial data in one region on the solution in a distant place. In a certain sense this shows continuous dependence of the solution with respect to the domain. More precisely, assume that Ω_1 and Ω_2 are two domains in $\mathbb{R}^d$ satisfying $B(0,R{+}1) = \{\, x \in \mathbb{R}^d \;:\; |x| < R{+}1 \,\} \subset \Omega_1 \cap \Omega_2$. Then, we prove that there exists constants $C, \delta > 0$ (which are independent of $R \geq 1$) such that for each $\alpha > 0$ and any solutions $u_j(t) = \mathcal{S}_t^{\Omega_j}(v_j)$ with $\|u_j(t)\|_\infty \leq Q_*$ the following estimate holds:

$$\|u_1(t){-}u_2(t)\|\|_{\mathrm{L}^\infty_{\{\alpha\}}(B(0,R))} \leq C\mathrm{e}^{C(1+\alpha^2)t}\Big[\|u_1(0){-}u_2(0)\|_{\mathrm{L}^\infty_{\{\alpha\}}(B(0,R))} + t\mathrm{e}^{-\alpha R}\Big] \qquad (2)$$

See Theorem 4 for a more precise statement, particularly concerning the constants. The right–hand side in (2) shows two different influences on $u_1{-}u_2$: the first term gives Lipschitz continuity in the initial data in the exponentially weighted norm and the second term contains all the influence which comes from the different initial data outside the ball $B(0,R)$ and from the different shapes of the domains Ω_j.

We now consider a sequence of domains Ω_n which tend to $\mathbb{R}^d$ in the sense that $B(0,R_n) \subset \Omega_n$ and $R_n \to \infty$. The associated attractors $\mathcal{A}^{\Omega_n}$ of $\mathcal{S}_t^{\Omega_n}$ are embedded into $\mathrm{L}^\infty(\mathbb{R}^d)$ by continuing each function outside of Ω_n by 0. Theorem 6 states

$$\mathrm{dist}_\rho(\mathcal{A}^{\Omega_n}, \mathcal{A}^{\mathbb{R}^d}) \to 0 \quad \text{for } n \to \infty,$$

where $\mathrm{dist}_\rho(A,B) = \sup\{\inf\{\|a{-}b\|_\rho \;:\; b \in B\} \;:\; a \in A\}$ is the (nonsymmetric) semidistance for $A, B \subset \mathrm{L}^\infty(\mathbb{R}^d)$. This property is called upper semi–continuity of the sequence $(\mathcal{A}^{\Omega_n})_{n\to\infty}$ towards the limit $\mathcal{A}^{\mathbb{R}^d}$. This property is a standard consequence of the attractivity of $\mathcal{A}^{\mathbb{R}^d}$ and the continuity of the flows in the sense of (2). In general, we do not have lower semi–continuity, i.e., $\mathrm{dist}_\rho(\mathcal{A}^{\mathbb{R}^d}, \mathcal{A}^{\Omega_n}) \to 0$ which is shown by a simple example in [Mie99, Sect. 5.3].

Instead of considering the limit of the attractors $\mathcal{A}^{\Omega_n}$ we follow [HaR93] and define a double limit $(t, \Omega_n) \to (\infty, \mathbb{R}^d)$:

$$\begin{aligned}\mathcal{A}^* = \{\; u \in \mathrm{L}^\infty(\mathbb{R}^d) : \exists\, t_n \to \infty\; \exists\, u_n \in \mathrm{L}^\infty(\Omega_n) \text{ with } \sup_{n\in\mathbb{N}} \|u_n\|_\infty < \infty \\ \text{such that } \|u{-}u_n\|_\rho \to 0 \text{ for } n \to \infty \;\}.\end{aligned} \qquad (3)$$

Again relying on the weighted estimate (2) Theorem 7 shows that $\mathcal{A}^*$ coincides with $\mathcal{A}^{\mathbb{R}^d}$.

2. Exponentially weighted estimates

We want to compare two solutions on two different domains Ω_1 and Ω_2. Without loss of generality we take $\Omega_1 = \Omega$ with $B(0,R{+}1) \subset \Omega$ and $\Omega_2 = \mathbb{R}^d$. (The estimate for two general domains can be obtained by adding the two estimates between the domains and $\mathbb{R}^d$.) Assume we have two solutions $u_j(t) = \mathcal{S}^{\Omega_j}(u_j(0))$ with $\|u_j(t)\|_\infty \leq Q_*$. For a smooth function $\chi : \mathbb{R} \to [0,1]$ with $\chi(s) = 1$ for $s \geq 1$ and $\chi(s) = 0$ for $s \leq -1$ we let

$$w(t,x) = u_2(t,x) - \chi(R{-}|x|)u_1(t,x)$$

where we have extended $u_1(t,x)$ by 0 outside of Ω. The difference w solves

$$\partial_t w = D\Delta w + M(t,x)w + b(t,x), \quad t > 0,\; x \in \mathbb{R}^d, \qquad (4)$$

with $M = \int_{s=0}^{1} \mathrm{D}f(u_2{+}sw)\,\mathrm{d}s$ and $b = \chi\, f(u_1){-}f(\chi u_1){-}D[2\nabla\chi \cdot \nabla u_1{+}\Delta\chi\, u_1]$. The aim is to estimate $w(t,x)$ by its initial data and the inhomogeneity b as precisely as possible.

We denote by $G_D(t,z) \in \mathbb{R}^{m\times m}$ the Green's function associated to $\partial_t w = D\Delta w$, which is the leading part of (4). The solution w of (4) can be written in the form

$$\begin{aligned} w(t,x) \;&=\; \int_{y\in\mathbb{R}^d} G_D(t,x{-}y)w(0,y)\,\mathrm{d}y \\ &\quad+ \int_{s=0}^{t}\int_{y\in\mathbb{R}^d} G_D(t{-}s,x{-}y)[M(s,y)w(s,y){+}b(s,y)]\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{5}$$

Using Fourier transform the Green's function reads $G_D(t,x) = \int_{\xi\in\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\mathrm{e}^{-t|\xi|^2 D}\,\mathrm{d}\xi$. We introduce an important spectral quantity of D via

$$\delta_D = \Big[\min\{\,\mathrm{Re}(1/\lambda)\ :\ \lambda\in\mathrm{spec}(D)\,\}\Big]^{-1} = \max\{\,|\lambda|^2/(\mathrm{Re}\,\lambda)\ :\ \lambda\in\mathrm{spec}(D)\,\}.$$

Together with the scalar heat kernel H on $\mathbb{R}^d$, $H(t,x) = (4\pi t)^{-d/2}\mathrm{e}^{-|x|^2/(4t)}$, we can formulate the following result.

Proposition 1
For each $\delta > \delta_D$ there exists a constant C_δ such that

$$|G_D(t,x)| \le C_\delta H(\delta t,x) \quad \textit{for all } t>0,\ x\in\mathbb{R}^d. \tag{6}$$

If the eigenvalues λ of D with smallest $\mathrm{Re}(1/\lambda)$ are semi–simple, then it is also possible to choose $\delta = \delta_D$.

Proof: The result is obtained by bringing D into Jordan normal form and applying Fourier transform. For one Jordan block $J = \lambda I + N$ with $N^{k+1} = 0$ we have $\mathrm{e}^{-t|\xi|^2 J} = \mathrm{e}^{-t|\xi|^2\lambda}\sum_{m=0}^{k}(-t|\xi|^2 N)^m$. Fourier transform leads to the representation in x–space

$$\textstyle\sum_{m=0}^{k}(t\,\Delta_x)^m H(\lambda t,x)\,N^m = (4\pi t\lambda)^{-d/2}\sum_{m=0}^{k} W_m(|x|^2/(4\lambda t))N^m,$$

where $W_m(r) = p_m(r)\mathrm{e}^{-r}$ with p_m being a polynomial of degree m. Hence, each Jordan block can be estimated by $Ct^{-d/2}(1{+}|x|^2/t)^k\mathrm{e}^{-|x|^2/(4\delta_D t)}$ where $k{+}1$ is the length of the Jordan block. Since the semi–simple case means $k=0$ the result is proved. QED

Inserting the estimate (6) into the variation–of–constants formula (5) we can estimate the norm of any solution w of (4) by a solution of a scalar heat equation.

Proposition 2
Choose δ and C_δ as in Prop. 1. Then any solution $w : [0,\infty)\times\mathbb{R}^d$ of (4) satisfies $|w(t,x)| \le \psi(t,x)$ for all $t>0$ and $x\in\mathbb{R}^d$, where ψ is the unique bounded solution of

$$\partial_t\psi = \delta\Delta\psi + C_\delta|M(t,x)|\psi + C_\delta|b(t,x)|, \quad \psi(0,x) = C_\delta|w(0,x)|. \tag{7}$$

Hence, we obtain estimates on the solution $w(t,x)$ by finding a suitable estimate for the scalar function $\psi(t,x)$. The important point is that for the scalar problem (7) we can use the maximum principle.

Theorem 3
Let δ and C_δ as in Prop. 1 and take any weight function $\rho \in \mathcal{C}^2(\mathbb{R}^d, (0,\infty))$ for which there exists some $\beta > 0$ such that $\rho(x)\Delta(1/\rho)(x) \leq \beta$ for all x. Then, the solutions w of (4) satisfy

$$\|w(t)\|_\rho \leq C_\delta \mathrm{e}^{(C_\delta M_* + \delta\beta)t}\Big[\|w(0)\|_\rho + t\, b_*\Big] \quad \text{for all } t > 0, \tag{8}$$

where $M_* = \sup\{\, |M(t,x)| : t \geq 0,\ x \in \mathbb{R}^d \}$ and $b_* = \sup\{\, \rho(x)|b(t,x)| : t \geq 0,\ x \in \mathbb{R}^d \}$.

Proof: We estimate the function $\phi(t,x) = \rho(x)\psi(t,x)$, where ψ is the solution of (7). Clearly, it suffices to show $\|\phi(t)\|_\infty \leq \mathrm{e}^{(C_\delta M_* + \delta\beta)t}[\|\phi(0)\|_\infty + t\, C_\delta b_*]$, where ϕ satisfies

$$\partial_t \phi = \delta(\Delta\phi + 2\rho\nabla(1/\rho)\cdot\nabla\phi) + (C_\delta|M(t,x)| + \delta\rho\Delta(1/\rho))\phi + \rho|b(t,x)|.$$

Estimating $|M(t,x)| \leq M_*$, $\rho\Delta(1/\rho) \leq \beta$, $\rho|b| \leq b_*$ and employing the maximum principle gives the desired result. QED

We apply Theorem 3 with $\rho(x) = 1/\cosh(\alpha|x|)$ giving $\beta = d\alpha^2$. Since $\rho_\alpha \leq \rho \leq 2\rho_\alpha$ we arrive at the following result, which immediately implies (2).

Theorem 4
With δ and C_δ as in Prop. 1 we have, for each $\alpha \geq 0$, the exponentially weighted estimate

$$\|w(t)\|_{\{\alpha\}} \leq 2C_\delta \mathrm{e}^{(C_\delta M_* + \delta d\alpha^2)t}\Big[\|w(0)\|_{\{\alpha\}} + t\, B_\alpha\Big] \tag{9}$$

for all $t > 0$, where $B_\alpha = \sup\{\, \mathrm{e}^{-\alpha|x|}|b(t,x)| : t > 0,\ x \in \mathbb{R}^d \}$.

Earlier decay estimates in $\mathrm{L}^\infty(\mathbb{R}^d)$ were derived in [CoE99a, CoE99b], however, they do only provide algebraic decay in the spatial direction. In [Mie97, Mie98] exponential decay estimates in L^p–norms were obtained using weighted energy estimates as in [Col94]. This restricts p to $p \approx 2$ for general diffusion matrices D which are not close to a diagonal matrix. Also energy estimates involve the quantities

$$\delta_* = \min\{\, \nu : 2\nu \in \mathrm{spec}(D + D^{\mathrm{t}}) \}, \quad \text{and} \quad D_* = |D| = \max\{\, |Dv| : |v| = 1 \},$$

which depend on the chosen scalar product in $\mathbb{R}^m$, which is in contrast to our spectral quantity $\delta_D > 0$. Note that $\delta_* < 0$ is possible even if $\delta_D > 0$. If $\delta_* > 0$ we always have $\delta_D \leq D_*^2/\delta_*$ and we may approximate δ_D by D_*^2/δ_* as close as we like by choosing the scalar product appropriately. The present approach involving the Green's function includes the different choices of the scalar product in $\mathbb{R}^m$ in the constant C_δ.

3. Attractors

The reaction–diffusion system (1) defines a semiflow $\mathcal{S}^\Omega(t,\cdot)$ in $\mathrm{L}^\infty(\Omega)$. The weighted norm $\|\cdot\|_\rho$ is used to generate compactness. Indeed bounded sets in $\mathrm{W}^{1,\infty}(\Omega)$ are precompact in $\mathrm{L}^\infty_\rho(\Omega)$. For definiteness we use the special weight $\rho_\alpha(x) = \mathrm{e}^{-\alpha|x|}$ in this section, however, all the other norms are equally well possible.

One application of Theorem 4 is the (Lipschitz) continuity of the semiflow in the weighted norm. Thus, under the above assumptions we have the following existence result for a global attractor $\mathcal{A}^\Omega$. The case of bounded Ω is classical and that of unbounded Ω is treated in [MiS95, Mie97, EfZ99]. The modifications from L^p to L^∞ are immediate with the theory of Section 2.

Theorem 5
Under the above assumptions the semiflow $\mathcal{S}^\Omega(t,\cdot)$ has a global $(\mathrm{L}^\infty,\mathrm{L}^\infty_{\{\alpha\}})$–attractor $\mathcal{A}^\Omega$ with the following properties:
(a) it is flow invariant, i.e., $\mathcal{S}^\Omega(t,\mathcal{A}^\Omega)=\mathcal{A}^\Omega$ for all $t\geq 0$,
(b) it is nonempty, bounded and closed in $\mathrm{L}^\infty(\Omega)$, and compact in $\mathrm{L}^\infty_{\{\alpha\}}(\Omega)$,
(c) it attracts bounded sets B in $\mathrm{L}^\infty(\Omega)$ in the $\|\cdot\|_{\{\alpha\}}$–distance: $\mathrm{dist}_{\{\alpha\}}(\mathcal{S}^\Omega(t,B),\mathcal{A}^\Omega)\to 0$.

The attractor $\mathcal{A}^\Omega$ is independent of the chosen weight function ρ, see [MiS95].

Using the weighted estimates we compare the attractor $\mathcal{A}^\Omega$ for a large domain Ω with $\mathcal{A}^{\mathbb{R}^d}$. Therefore $\mathrm{L}^\infty(\Omega)$ is embedded into $\mathrm{L}^\infty(\mathbb{R}^d)$ by continuation with 0 outside of Ω.

Theorem 6
Under the above assumptions there exists for each $\varepsilon>0$ an $R>0$ such that for all domains $\Omega\subset\mathbb{R}^d$ with $B(0,R)\subset\Omega$ we have the estimate $\mathrm{dist}_\rho(\mathcal{A}^\Omega,\mathcal{A}^{\mathbb{R}^d})\leq\varepsilon$.

Proof: By our assumption we know that $\mathcal{A}^\Omega\subset\mathcal{B}=\{\,u\in\mathrm{L}^\infty(\mathbb{R}^d):\|u\|_\infty\leq Q_*\,\}$. Define $a(t)=\mathrm{dist}_{\{\alpha\}}(\mathcal{S}^{\mathbb{R}^d}(t,\mathcal{B}),\mathcal{A}^{\mathbb{R}^d})$. Then there exists $t_1>0$ such that $a(t_1)\leq\varepsilon/2$.

Now take any $u\in\mathcal{A}^\Omega$, then $u=\mathcal{S}^\Omega(t_1,v)$ for some $v\in\mathcal{A}^\Omega\subset\mathcal{B}$. Thus, applying Theorem 4 to $w=\mathcal{S}^\Omega(t,v)-\mathcal{S}^{\mathbb{R}^d}(t,v)$ we obtain $\|w(t_1)\|_{\{\alpha\}}\leq C\mathrm{e}^{Ct_1}[0+t_1\mathrm{e}^{-\alpha R}]$. Choosing R sufficiently large this term is less than $\varepsilon/2$ and we have proved $\mathrm{dist}_{\{\alpha\}}(u,\mathcal{A}^{\mathbb{R}^d})\leq\|u-\mathcal{S}^{\mathbb{R}^d}(t_1,v)\|_{\{\alpha\}}+\mathrm{dist}_{\{\alpha\}}(\mathcal{S}^{\mathbb{R}^d}(t_1,v),\mathcal{A}^{\mathbb{R}^d})\leq\|w(t_1)\|_{\{\alpha\}}+a(t_1)\leq\varepsilon$. QED

In general we cannot expect lower semi–continuity of the attractors, i.e., $\mathrm{dist}_\rho(\mathcal{A}^{\mathbb{R}^d},\mathcal{A}^\Omega)\leq\varepsilon$ for suitably large domains Ω, cf. [Mie99, Sect. 5.3]. This problem arises from the problem of interchanging the limit $t\to\infty$ in the definition of the attractors with the limit $\Omega_n\to\mathbb{R}^d$. The interchange of such limits would only be possible, if the attraction rate for the attractors $\mathcal{A}^{\Omega_n}$ would be uniform in n, however this is in general not the case.

For such situations a generalized limit attractor was introduced in [HaR93]. In our situation this leads to the definition of $\mathcal{A}^*$ as in (3). The motivation to consider $\mathcal{A}^*$ rather than the limit $\cap_{k\geq 1}\cup_{n\geq k}\mathcal{A}^{\Omega_n}$ is that for practical purposes (e.g., in experiments or numerics) we always have to consider finite time as well as bounded domains. Thus, one should neither prescribe the limit $t\to\infty$ before the limit $n\to\infty$ nor the opposite, see [HaR93] for further discussions.

Theorem 7
For all sequences $(\Omega_n)_{n\in\mathbb{N}}$ with $\Omega_n\to\mathbb{R}^d$ as above we have $\mathcal{A}^*=\mathcal{A}^{\mathbb{R}^d}$.

Proof: Let $R_n \to \infty$ be such that $B(0,R_n) \subset \Omega_n$.

We first show $\mathcal{A}^{\mathbb{R}^d} \subset \mathcal{A}^*$. To this end consider $u \in \mathcal{A}^{\mathbb{R}^d}$. Choose a sequence $t_n \to \infty$ such that the right–hand side in (2) with $\|u_1 - u_2\|_{\mathrm{L}^\infty_{\{\alpha\}}(B(0,R_n))} = 0$ tends to 0 for $n \to \infty$. By the invariance of $\mathcal{A}^{\mathbb{R}^d}$ there exist $v_n \in \mathcal{A}^{\mathbb{R}^d}$ such that $\mathcal{S}^{\mathbb{R}^d}(t_n, v_n) = u$. Defining $u_n \in \mathrm{L}^\infty(\Omega_n)$ via $u_n = v_n|_{\Omega_n}$ and using (2) we obtain the desired result $\|u - \mathcal{S}^{\Omega_n}(t_n, u_n)\|_{\{\alpha\}} \to 0$.

For the opposite inclusion we note that $\mathcal{A}^*$ is contained in $\mathcal{B}$ and hence it is a bounded set in $\mathrm{L}^\infty(\mathbb{R}^d)$. If we show that $\mathcal{A}^*$ is also negatively invariant (i.e., $\mathcal{A}^* \subset \mathcal{S}^{\mathbb{R}^d}(t, \mathcal{A}^*)$ for all $t > 0$), then we have the desired inclusion $\mathcal{A}^* \subset \mathcal{A}^{\mathbb{R}^d}$. Let $u \in \mathcal{A}^*$ and $t > 0$, then we have to find $\widetilde{u} \in \mathcal{A}^*$ such that $u = \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})$. Take the sequence (t_n, u_n) with $\|u - v_n\|_{\{\alpha\}} \to 0$ where $v_n = \mathcal{S}^{\Omega_n}(t_n, u_n)$ according to the definition of $\mathcal{A}^*$. The sequence $\widetilde{v}_n = \mathcal{S}^{\Omega_n}(t_n - t, u_n)$ is precompact in $\mathrm{L}^\infty_{\{\alpha\}}(\mathbb{R}^d)$ and thus, after taking a subsequence if necessary, we have $\|\widetilde{u} - \widetilde{v}_n\|_{\{\alpha\}} \to 0$. Using $v_n = \mathcal{S}^{\Omega_n}(t, \widetilde{v}_n)$ and (2) we conclude

$$\begin{aligned} \|u - \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})\|_{\{\alpha\}} &\le \|u - \mathcal{S}^{\Omega_n}(t, \widetilde{v}_n)\|_{\{\alpha\}} + \|\mathcal{S}^{\Omega_n}(t, \widetilde{v}_n) - \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})\|_{\{\alpha\}} \\ &\le \|u - v_n\|_{\{\alpha\}} + C\mathrm{e}^{Ct}\Big[\|\widetilde{v}_n - \widetilde{u}\|_{\{\alpha\}} + t\mathrm{e}^{-\alpha R_n}\Big] \ \to \ 0 \end{aligned}$$

for $n \to \infty$. This gives the desired result $u = \mathcal{S}^{\mathbb{R}^d}(t, \widetilde{u})$. QED

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