Pattern formation on large domains and diffusive mixing

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1 Introduction

Spontaneous pattern formation occurs in many physical and mechanical systems which are described by dissipative partial differential equations (PDE) on large domains $\Omega \subset \mathbb{R}^d$. Our aim is to introduce a mathematical concept which enables us to study whether pattern formation is independent of the boundary conditions at $\partial\Omega$ if we consider the system only away from the boundary.

The natural mathematical idealization is to consider the PDE on the full space $\mathbb{R}^d$ which means we have pushed the boundaries to infinity. An analysis of these systems leads to many new effects like nonuniform convergence and diffusive repair (see [CEE92, Sc96]) which arise from the fact that compactness is lost and linear operators have continuous spectra. We present a theory of global attractors for situations of large bounded domains as well as unbounded ones which was developed in [BV90, MS95, Fe96, Mi97]. In the unbounded case these attractors are usually infinite dimensional ([BV90, CE98]).

To illustrate these concept we consider parabolic systems of reaction–diffusion type

$$u_t = D\Delta u + f(u) \quad \text{with } x \in \Omega \subset \mathbb{R}^d,\ u(t,x) \in \mathbb{R}^m. \tag{1}$$

Under suitable assumptions on D and f (e.g. $D = D^T > 0$ and $f(u)\cdot u \leq C(1-|u|^{d+1})$) and with suitable boundary conditions 'bc' the system has a global attractor $\mathcal{A}^{\Omega}_{\mathrm{bc}}$. We say that the functions $u \in \mathcal{A}^{\Omega}_{\mathrm{bc}}$ form the set of all possible patterns of (1).

Consider a sequence Ω_n of domains in $\mathbb{R}^d$ with $\{\, x \,:\, |x| < n \,\} \subset \Omega_n$. In [Mi97] it is shown that $\mathrm{dist}_\rho(\mathcal{A}^{\Omega_n}_{\mathrm{bc}}, \mathcal{A}^{\mathbb{R}^d}) \to 0$ for $n \to \infty$, which means that the attractors are upper semicontinuous in the limit $\Omega \to \mathbb{R}^d$. It is important here that the distance measure involves a norm with a positive weight ρ decaying to 0 for $|x| \to \infty$. Convergence with respect to such a norm means convergence on every bounded subset of Ω. In loose terms this notion of upper semicontinuity says that all patterns which can be seen on large bounded domains are also present in the full space problem.

For practical purposes the opposite conclusion, called lower semicontinuity, is also desirable: All patterns in the full space problem can already be seen on sufficiently large domains. However, in Section 3 we present a counterexample which shows that in general lower semicontinuity (i.e., $\mathrm{dist}_\rho(\mathcal{A}^{\mathbb{R}^d}, \mathcal{A}^{\Omega_n}_{\mathrm{bc}}) \to 0$) does not hold. Moreover, the counterexample is such that the limit of the attractors $\mathcal{A}^{\Omega_n}_{\mathrm{bc}}$ for $n \to \infty$ depends on the chosen boundary conditions.

At the end of Section 2 we define a generalized limit attractor $\mathcal{A}^*$ which follows an idea in [HR93] where the limit of $\Omega_n \to \infty$ is taken simultaneous with the limit $t \to \infty$ in the definition of the attractor. Using exponential Lipschitz estimates for the solutions on different domains we are able to show that $\mathcal{A}^*$ in fact coincides with the full space attractor $\mathcal{A}^{\mathbb{R}^d}$. As $\mathcal{A}^*$ is constructed using the dynamics on a sequence of larger and larger bounded domains, we thus can find all patterns of the full space problem by considering bounded domains only, however, the attractors $\mathcal{A}^{\Omega_n}_{\rm bc}$ are too small as they do not contain enough information on the flow in the neighborhood of the attractor.

In Section 4 we discuss examples showing that the dynamics on unbounded domains may be very different to that on bounded domains. Associated new phenomena are diffusive repair and diffusive mixing. The latter occurs for PDEs which have a nontrivial family of steady states. Prescribing one of these states at $x \to -\infty$ and another at $x \to -\infty$ we are interested how the solutions mix these two states for time $t \to \infty$. We first display this phenomenon at the linear heat equation.

Finally we discuss a recent result in [GM98], generalizing results in [BK92], for the real Ginzburg–Landau equation $u_t = u_{xx} + u - |u|^2 u$, $x \in \mathbb{R}$, $u(t,x) \in \mathbb{C}$, where arbitrary Eckhaus stable steady states ($\eta_\pm^2 < 1/3$) of the form $U_{\eta_\pm}(x) = \sqrt{1-\eta_\pm^2}\,{\rm e}^{{\rm i}\eta_\pm x}$ are prescribed at $\pm\infty$. Then, there are functions $\eta_*(\eta_-,\eta_+)$ and $\phi_*(\eta_-,\eta_+)$ such that for suitable initial data the solution $u(t,x)$ satisfies for all $t>0$ the boundary conditions $u(t,x) - U_{\eta_\pm}(x) \to 0$ for $x \to \pm\infty$ and for every $\ell > 0$ we have

$$\sup_{|x|\leq \ell} |u(t,x) - {\rm e}^{{\rm i}\sqrt{t}\,\phi_*} U_{\eta_*}(x)| \;=\; \mathcal{O}(t^{-1/5}). \tag{2}$$

Thus, the patterns U_{η_-} and U_{η_+} mix diffusively to give the pattern U_{η_*} up to a phase translation.

2 Attractors

Since we are interested in large and even unbounded domains, we want to allow for solutions which are bounded but not necessarily decaying at infinity. To control such functions we introduce a localizing weighted norm as well as a uniform norm. Using the weight function $\rho(x) = {\rm e}^{-|x|}$ we consider the weighted ${\rm L}^p$–norm

$$\|u\|_{p,\rho} = \left(\int_\Omega \rho(x) |u(x)|^p \, dx \right)^{1/p}.$$

With ${\rm L}^p_\rho(\Omega)$ we denote the associated Banach space. With the translates $(T_y\rho)(x) = \rho(x-y)$, the uniformly local ${\rm L}^p$–norm on Ω is given as

$$\|u\|_{p,{\rm ul}} = \sup\{ \, \|u\|_{p,T_y\rho} \; : \; y \in \mathbb{R}^n \, \},$$

Let $\widetilde{\rm L}^p_{\rm ul}(\Omega) = \{ \, u \in {\rm L}^p_{\rm loc}(\Omega) \; : \; \|u\|_{p,{\rm ul}} < \infty \, \}$, then the uniformly local Lebesgue spaces are defined as

$${\rm L}^p_{\rm ul}(\Omega) = \text{closure of } \mathcal{C}^\infty_{\rm bdd}(\Omega) \text{ in } \widetilde{\rm L}^p_{\rm ul}(\Omega),$$

where $\mathcal{C}^\infty_{\rm bdd}(\Omega)$ is the set of all $\mathcal{C}^\infty$–functions which have all its derivatives bounded in Ω. This construction is necessary for unbounded domains in order to ensure density of ${\rm W}^{1,p}_{\rm ul}(\Omega)$ in ${\rm L}^p_{\rm ul}(\Omega)$, see [MS95]. Note the embeddings $\mathcal{C}_{\rm bdd,unif}(\overline{\Omega}) \subset {\rm L}^q_{\rm ul}(\Omega) \subset {\rm L}^p_{\rm ul}(\Omega)$

for $1 \le p < q < \infty$ with norm estimates $\|u\|_{p,\mathrm{ul}} \le C_d^{(q-p)/(qp)} \|u\|_{q,\mathrm{ul}} \le C_d^{1/p} \|u\|_\infty$ where $C_d = \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x$. The Sobolev spaces $\mathrm{W}^{k,2}_{\mathrm{ul}}(\Omega)$ are defined as the set of functions u which lie in $\mathrm{L}^2_{\mathrm{ul}}(\Omega)$ together with the first k distributional derivatives.

We now consider the parabolic PDE (1) on a large domain $\Omega \subset \mathbb{R}^d$ where large means $\{ x \in \mathbb{R}^d \ : \ |x| < \ell \} \subset \Omega$ for some $\ell \gg 1$. Note that this also includes general unbounded domains. Throughout we assume that the diffusion matrix D, the nonlinearity f, the boundary $\partial\Omega$ and the boundary conditions are such that we have global existence of smooth solutions such that a global semiflow $\mathcal{S}^{\Omega}_{\mathrm{bc}}(t,\cdot)$ exists on $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ for a suitable $p \in (1,\infty)$. Moreover, we assume that there exists a radius R which is independent of Ω such that $B_{\mathrm{abs}} = \{ u \in \mathrm{L}^p_{\mathrm{ul}}(\Omega) \ : \ \|u\|_{p,\mathrm{ul}} \le R \}$ is an absorbing set for the semigroup (see [Co94, Mi97, GV96, Mi98] for a construction of such bounds). Note that in the case of a bounded domain the set $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ and its topology coincide with the usual Lebesgue space $\mathrm{L}^p(\Omega)$.

Under these assumptions we have the following existence result for a global attractor $\mathcal{A}^{\Omega}_{\mathrm{bc}}$. The case of bounded Ω is classical ([BV92, Ha88, Te88]) and that of unbounded Ω is treated in [MS95, Mi97].

Theorem 1
Under the above assumptions the semiflow $\mathcal{S}^{\Omega}_{\mathrm{bc}}(t,\cdot)$ has a global $(\mathrm{L}^p_{\mathrm{ul}}, \mathrm{L}^p_\rho)$–attractor $\mathcal{A}^{\Omega}_{\mathrm{bc}}$ with the following properties:
(a) it is flow invariant, i.e., $\mathcal{S}^{\Omega}_{\mathrm{bc}}(t, \mathcal{A}^{\Omega}_{\mathrm{bc}}) = \mathcal{A}^{\Omega}_{\mathrm{bc}}$ for all $t \ge 0$,
(b) it is nonempty, bounded and closed in $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$, and compact in $\mathrm{L}^p_\rho(\Omega)$,
(c) it attracts bounded sets B in $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ in the L^p_ρ–distance, i.e.,

$$\mathrm{dist}_\rho(\mathcal{S}^{\Omega}_{\mathrm{bc}}(t,B), \mathcal{A}^{\Omega}_{\mathrm{bc}}) \to 0 \text{ for } t \to \infty$$

where $\mathrm{dist}_\rho(F,G) = \sup\{ \inf\{ \|u-v\|_{p,\rho} \ : \ v \in G \} \ : \ u \in F \}$.

The important feature here is that we use two different norms. The weighted norm localizes the distance measure and thus restores compactness whenever we are on an unbounded domain, viz., bounded sets in $\mathrm{W}^{1,p}(\Omega)$ are precompact in $\mathrm{L}^p_\rho(\Omega)$. For $\Omega = \mathbb{R}^d$ it is shown in [BV90, CE98] that the attractor $\mathcal{A}^{\mathbb{R}^d}$ is infinite dimensional.

Now we can compare attractors for different domains and boundary conditions as long as the PDE is the same inside both domains. To this end we simply embed $\mathrm{L}^p_{\mathrm{ul}}(\Omega)$ into $\mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ by extending all functions with 0 outside of Ω. Clearly this embedding preserves the norm.

The basic comparison result between the flows on different domains or with different boundary conditions can be roughly formulated as follows, see [Mi97, CE98] for more precise statements. Take two domains $\Omega_j \subset \mathbb{R}^d$ and associated boundary consitions (bc_j) such that the semiflows $\mathcal{S}^j = \mathcal{S}^{\Omega_j}_{\mathrm{bc}_j}(t,\cdot)$ are well–defined. Furthermore assume $\{ x \in \mathbb{R}^d \ : \ |x| < \ell \} \subset \Omega_j$ for $j = 1,2$. Then, for each $R > 0$ there exists a constant $C > 1$ which is independent of Ω_1 and Ω_2 such that

$$\|\mathcal{S}^1(t,u_1) - \mathcal{S}^2(t,u_2)\|_{p,\rho} \le C\mathrm{e}^{Ct}(\|u_1 - u_2\|_{p,\rho} + [\ell^{d-1}\mathrm{e}^{-\ell}]^{1/p}) \qquad (3)$$

for all $u_j \in \mathrm{L}^p_{\mathrm{ul}}(\Omega_j)$ with $\|u_j\|_\infty \leq R$. Here the second term on the right–hand side contains all the influence of the different domains and boundary conditions whereas the first term is a standard Lipschitz condition. In particular, if $u_1(x) = u_2(x)$ for $|x| \leq r < \ell$ then the first term is less than $C[r^{d-1}\mathrm{e}^{-r}]^{1/p}$ which provides a good control over the difference of the solutions over even smaller balls.

Using (3) and certain variants of it, it is not difficult to show the following upper semicontinuity results, see [Mi97].

Theorem 2
Under the above assumptions there exists for each $\varepsilon > 0$ an $\ell > 0$ such that for all domains $\Omega \subset \mathbb{R}^d$ with $\{ x : |x| < \ell \} \subset \Omega$ and all $\mathrm{bc} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{Per}\}$ we have the estimate

$$\mathrm{dist}_\rho(\mathcal{A}^\Omega_{\mathrm{bc}}, \mathcal{A}^{\mathbb{R}^d}) \leq \varepsilon.$$

The remaining open problem is under what conditions we also have lower semicontinuity of the attractor, i.e., $\mathrm{dist}_\rho(\mathcal{A}^{\mathbb{R}^d}, \mathcal{A}^\Omega_{\mathrm{bc}}) \leq \varepsilon$ for suitably large domains Ω. In the next section we give a simple example that shows that lower semicontinuity is in general false. In our example the limit of $\mathcal{A}^{(-\ell,\ell)}_{\mathrm{bc}}$ for $\ell \to \infty$ exists for each $\mathrm{bc} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{Per}\}$, but they depend on bc and are strictly contained in $\mathcal{A}^{\mathbb{R}}$.

The problem of convergence of $\mathcal{A}^{\Omega_n}_{\mathrm{bc}_n}$ to $\mathcal{A}^{\mathbb{R}^d}$ is a problem of interchanging the limit $t \to \infty$ in the definition of the attractors with the limit $\Omega_n \to \mathbb{R}^d$. The interchange of such limits would only be possible, if the attraction rate for the attractors $\mathcal{A}^{\Omega_n}_{\mathrm{bc}_n}$ would be uniform in n, however this is in general not the case. For such situations a generalized limit attractor was introduced in [HR93]. In our situation this leads to the following definition. Consider a sequence $(\mathrm{bc}_n, \Omega_n)_{n\in\mathbb{N}}$ of boundary conditions and domains which approach $\mathbb{R}^d$ in the sense that $\{ x : |x| < \ell_n \} \subset \Omega_n$ with $\ell_n \to \infty$. Now set

$$\mathcal{A}^* = \Big\{ u \in \mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d) \ : \ \exists\, t_n \text{ with } t_n \to \infty\ \exists\, w_n \in \mathrm{L}^p_{\mathrm{ul}}(\Omega_n) \text{ with } \sup\nolimits_{n\in\mathbb{N}} \|w_n\|_{p,\mathrm{ul}} < \infty \\ \text{such that } u = \rho\text{-}\lim\nolimits_{n\to\infty} \mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t_n, w_n) \Big\}$$

The motivation to consider $\mathcal{A}^*$ rather than $\rho\text{-}\lim \mathcal{A}^{\Omega_n}_{\mathrm{bc}_n}$ is that for practical purposes (e.g., in experiments or numerics) we always have to consider finite time and bounded domains. Thus, one should neither prescribe limit $t \to \infty$ before $n \to \infty$ nor the opposite, see [HR93] for a further discussion.

Theorem 3
For all sequences $(\mathrm{bc}_n, \Omega_n)_{n\in\mathbb{N}}$ as above we have $\mathcal{A}^* = \mathcal{A}^{\mathbb{R}^d}$.

Proof: We first show $\mathcal{A}^{\mathbb{R}^d} \subset \mathcal{A}^*$. To this end consider $u \in \mathcal{A}^{\mathbb{R}^d}$. As the sequence $\ell_n \to \infty$ for $n \to \infty$ is given as above, we can choose a sequence t_n with $t_n \to \infty$ such that the right–hand side in (3) tends to 0 if we assume $t = t_n$, $\ell = \ell_n$ and $u_1 = u_2$. By the invariance of $\mathcal{A}^{\mathbb{R}^d}$ there exist $v_n \in \mathcal{A}^{\mathbb{R}^d} \subset \mathrm{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ such that $\mathcal{S}^{\mathbb{R}^d}(t_n, v_n) = u$. Defining $w_n \in \mathrm{L}^p_{\mathrm{ul}}(\Omega_n)$ via $w_n = v_n|_{\Omega_n}$ and using (3) we obtain the desired result $u = \rho\text{-}\lim_{n\to\infty} \mathcal{S}^{\Omega_n}_{\mathrm{bc}_n}(t_n, w_n)$, since $\|v_n - w_n\|_{p,\rho}$ can also be estimated by $C[\ell_n^{d-1}\mathrm{e}^{-\ell_n}]^{1/d}$.

For the opposite inclusion we note that $\mathcal{A}^*$ is contained in B_{abs} and hence it is a bounded set in $\mathrm{L}^p_{\text{ul}}(\mathbb{R}^d)$. If we show that $\mathcal{A}^*$ is also negatively invariant (i.e., $\mathcal{A}^* \subset \mathcal{S}^{\mathbb{R}^d}(t,\mathcal{A}^*)$ for all $t>0$), then we have the desired inclusion $\mathcal{A}^* \subset \mathcal{A}^{\mathbb{R}^d}$. Let $u \in \mathcal{A}^*$ and $t>0$, then we have to find $\widetilde{u} \in \mathcal{A}^*$ such that $u = \mathcal{S}^{\mathbb{R}^d}(t,\widetilde{u})$. Take the sequence (t_n, w_n) with $u = \rho\text{-}\lim_{n\to\infty} v_n$ where $v_n = \mathcal{S}^{\Omega_n}_{\text{bc}_n}(t_n, w_n)$ according to the definition of $\mathcal{A}^*$. The sequence $\widetilde{v}_n = \mathcal{S}^{\Omega_n}_{\text{bc}_n}(t_n{-}t, w_n)$ is precompact in $\mathrm{L}^p_\rho(\mathbb{R}^d)$ and thus, after taking a subsequence if necessary, we have $\widetilde{u} = \rho\text{-}\lim_{n\to\infty}\widetilde{v}_n$. Using $v_n = \mathcal{S}^{\Omega_n}_{\text{bc}_n}(t,\widetilde{v}_n)$ and (3) we conclude

$$\begin{aligned} \|u - \mathcal{S}^{\mathbb{R}^d}(t,\widetilde{u})\|_{p,\rho} &\le \|u - \mathcal{S}^{\Omega_n}_{\text{bc}_n}(t,\widetilde{v}_n)\|_{p,\rho} + \|\mathcal{S}^{\Omega_n}_{\text{bc}_n}(t,\widetilde{v}_n) - \mathcal{S}^{\mathbb{R}^d}(t,\widetilde{u})\|_{p,\rho} \\ &\le \|u - v_n\|_{p,\rho} + C\mathrm{e}^{Ct}\Big[\|\widetilde{v}_n - \widetilde{u}\|_{p,\rho} + [\ell_n^{d-1}\mathrm{e}^{-\ell_n}]^{1/p}\Big] \;\to\; 0 \end{aligned}$$

for $n\to\infty$. This gives the desired result $u = \mathcal{S}^{\mathbb{R}^d}(t,\widetilde{u})$. QED

3 An example

We consider the scalar parabolic equation

$$u_t = u_{xx} + f(u) \quad \text{ with } f(u) = -u(1{-}u)^2 \tag{4}$$

for $x \in (-\ell,\ell)$ where $\ell > 0$. The essential feature of the nonlinearity f is that the ODE $\dot{u} = f(u)$ has exactly two equilibria, namely $u=0$ and $u=1$. While $u=0$ is a stable equilibrium, the second equilibrium $u=1$ is degenerate and semistable.

If $\ell < \infty$ we take one of the following boundary conditions:
(Dir): $u(-\ell) = u(\ell) = 0$,
(Neu): $u_x(-\ell) = u_x(\ell)$,
(Per): $u(-\ell) = u(\ell)$ and $u_x(-\ell) = u_x(\ell)$.
The initial value problem is well–posed and we have a global semiflow on $\mathrm{L}^2_{\text{ul}}((-\ell,\ell))$ which we denote by $(\mathcal{S}^\ell_{\text{bc}}(t,\cdot))_{t\ge 0}$. The associated attractors are denoted by $\mathcal{A}^\ell_{\text{bc}}$. More general boundary conditions could be allowed, but this would complicate the notations and the analysis considerably.

For $\ell < \infty$ we can study the attractor explicitly as the flow is a gradient flow with respect to the Liapunov function

$$I(u) = \int_{-\ell}^{\ell} \Big[u_x^2/2 - F(u)\Big]\,\mathrm{d}x \quad \text{with } F(u) = \int_0^u f(v)\,\mathrm{d}v = -(6u^2{-}8u^3{+}3u^4)/12.$$

The standard theory for attractors for flows with a Liapunov function (see [Ha88, Te88, BV92]) tells us that the attractor consists exactly of the equilibria and their heteroclinic connections. Denote by $\mathcal{E}^\ell_{\text{bc}}$ the set of equilibria of the semiflow $\mathcal{S}^\ell_{\text{bc}}$, then

$$\mathcal{E}^\ell_{\text{Dir}} = \{0\} \quad \text{and} \quad \mathcal{E}^\ell_{\text{Neu}} = \mathcal{E}^\ell_{\text{Per}} = \{0,1\}.$$

We obtain the following result for the global attractors.

Theorem 4
For all $0 < \ell < \infty$ we have $\mathcal{A}^\ell_{\rm Dir} = \{0\}$ and $\mathcal{A}^\ell_{\rm Neu} = \mathcal{A}^\ell_{\rm Per} = \mathcal{A}_*$ where $A_* = \{\, u \in \mathrm{L}^p((-\ell,\ell)) \,:\, \exists\, \theta \in [0,1] : u \equiv \theta \,\}$.

For (Dir) the result is clear since the only equilibrium $u \equiv 0$ is exponentially stable. For (Neu) and (Per) we use the fact that the linearization at $u \equiv 1$ has the eigenvalues $-(k\pi/\ell)^2$, $k \in \mathbb{Z}$. There is a one–dimensional center manifold associated to the eigenvalue 0, which, in fact, is given by the x–independent functions. Thus, the only solution connecting $u \equiv 1$ with $u \equiv 0$ is the spatially constant solution $u(t,x) = \theta(t)$, where $\dot\theta = f(\theta)$.

From Theorem 4 we immediately obtain the limits

$$\rho\text{-}\lim_{\ell\to\infty} \mathcal{A}^\ell_{\rm Dir} = \{0\}, \quad \rho\text{-}\lim_{\ell\to\infty} \mathcal{A}^\ell_{\rm Neu} = \rho\text{-}\lim_{\ell\to\infty} \mathcal{A}^\ell_{\rm Per} = \mathcal{A}_*\,. \tag{5}$$

Here $\rho\text{-}\lim_{\ell\to\infty} A(\ell) = B$ means $\mathrm{dist}_\rho(A(\ell), B) + \mathrm{dist}_\rho(B, A(\ell)) \to 0$ for $\ell \to \infty$.

Finally we show that the attractor $\mathcal{A}^\infty$ as defined above is larger than $\mathcal{A}_*$. This establishes the example where the limits of the attractors exist, depend on the boundary conditions and are strictly smaller than the limit attractor. Clearly we have $\mathcal{A}_* \subset \mathcal{A}^\infty$. However, there are more solution, for instance travelling waves of the form $u(t,x) = w(x{-}ct)$ with w satisfying the ODE

$$w'' + cw' + f(w) = 0. \tag{6}$$

For $c = 1/\sqrt{2}$ we have the explicit solution $w(x) = \mathrm{e}^{x/\sqrt{2}}/(1{+}\mathrm{e}^{x/\sqrt{2}})$.

Theorem 5
For $c \geq 1/\sqrt{2}$ equation (6) has a unique solution w_c which is monotone and satisfies $w_c(0) = 1/2$, $w_c(-\infty) = 0$ and $w_c(\infty) = 1$. For $|c| < 1/\sqrt{2}$ there are no nonconstant bounded solutions and for $c \leq -1/\sqrt{2}$ we have the solution $w_c(x) = w_{-c}(-x)$.

This results follows from a monotonicity argument. In fact, the existence result for $c \geq 1$ follows easily by considering the triangle $\Delta = \{\,(w,w') \,:\, w \geq 0,\ w' \geq 0,\ 2w'{+}w \leq 1\,\}$ which is positiv invariant for $c \geq 1$. Since the upper part of the unstable manifold of the origin enters Δ this orbit has to stay inside. As there is only one equilibrium at the corner $(w,w') = (1,0)$ it is clear that this must be the omega limit set.

This shows that the limit attractor $\mathcal{A}^\infty$ contains at least the two–dimensional set

$$\mathcal{B} = \mathcal{A}_* \cup \{\, w_c(\cdot - y) \,:\, |c| \geq c_*,\ y \in \mathbb{R}\,\}.$$

In the space $\mathrm{L}^2_\rho(\mathbb{R})$ this set forms a closed subset which is homeomorphic to the closed unit disk since $|w_c'(x)| \leq K/|c|$ implies that $w_c(\cdot - cz)$ converges for $c \to \infty$ to a constant function which depends smoothly and monotonically on z. Together we have shown

$$\rho\text{-}\lim_{\ell\to\infty} \mathcal{A}^\ell_{\rm Dir} \subsetneq \rho\text{-}\lim_{\ell\to\infty} \mathcal{A}^\ell_{\rm Neu} = \rho\text{-}\lim_{\ell\to\infty} \mathcal{A}^\ell_{\rm Per} \subsetneq \mathcal{B} \subset \mathcal{A}^\infty.$$

It remains open whether $\mathcal{A}^\infty$ coincides with $\mathcal{B}$ or whether it is strictly larger.

4 Diffusive mixing

In this section we report on results which show that PDEs on unbounded domains behave very different from those on bounded domains. The phenomenon which we discuss is called diffusive mixing, as it studies a problem on the infinite real line which has a family of spatially periodic steady states which are all linearly stable. We consider initial conditions which converge to one of these states for $x \to -\infty$ and to another for $x \to -\infty$. As these boundary conditions at infinity remain valid for all positive times, the question arises how the solution behaves in the middle region, that is on bounded sets independent of time.

The simplest case of diffusive mixing occurs in the linear heat equation $v_t = v_{xx}$. Consider any initial condition $v_0 \in \mathrm{L}^2_{\mathrm{ul}}(\mathbb{R})$ with $v_0(x) \to V_\pm \in \mathbb{R}$ for $x \to \pm\infty$. Using the explicit solution formula for $v(t) = \mathcal{S}^{\mathbb{R}}_{\mathrm{heat}}(t, v_0)$, it is easy to show that

$$\|v(t) - V_*\|_{2,\rho} \to 0 \quad \text{ for } t \to \infty \quad \text{ where } V_* = (V_+ + V_-)/2.$$

Of course, if $V_+ \neq V_-$ we cannot have convergence in $\|\cdot\|_{2,\mathrm{ul}}$. Considering the positively invariant set $X = \{\, v \in \mathrm{L}^2_{\mathrm{ul}}(\mathbb{R}) \,:\, \|v\|_\infty \leq 1 \,\}$ the global attractor $\mathcal{A}_X$ of $\mathcal{S}^{\mathbb{R}}_{\mathrm{heat}}(t, \cdot)$ on X consists exactly of the constant functions $\mathcal{A}_X = \{\, v \in X \,:\, \exists\, \theta \in [-1,1] : v \equiv \theta \,\}$. In contrast to the heat equation on bounded domains, it can be shown that there exists an initial condition $v_0 \in X$ whose ρ-omega limit set equals all of $\mathcal{A}_X$, that is, for all $\theta \in [-1,1]$ there exist $t_n \to \infty$ such that $\|\mathcal{S}^{\mathbb{R}}_{\mathrm{heat}}(t_n, v_0) - \theta\|_{2,\rho} \to 0$ for $n \to \infty$. See [CE92] for this construction.

We now consider a nonlinear generalization. The real Ginzburg–Landau equation

$$u_t = u_{xx} + u - |u|^2 u, \quad \text{ with } x \in \mathbb{R} \text{ and } u(t,x) \in \mathbb{C}, \tag{7}$$

has steady state solutions of the form $u(x) = \mathrm{e}^{\mathrm{i}\beta} U_\eta(x)$ with $\beta \in \mathbb{R}$ and

$$U_\eta(x) = \sqrt{1-\eta^2}\, \mathrm{e}^{\mathrm{i}\eta x}.$$

The classical Eckhaus instability theory [Ec65] tells us that the solutions U_η with $|\eta| > 1/\sqrt{3}$ are unstable while those with $|\eta| < 1/\sqrt{3}$ are linearly stable and also nonlinearly diffusive stable with respect to L^2–perturbations, see [CEE92, Sc96].

The structure of (7) becomes clearer when using polar coordinates $u = r\mathrm{e}^{\mathrm{i}\phi}$ giving

$$r_t = r_{xx} + r(1 - r^2 - \phi_x^2), \quad \phi_t = \phi_{xx} + 2 r_x \phi_x / r.$$

The underlying idea of diffusive repair and diffusive mixing is that for $t \to \infty$ the derivatives r_t and r_{xx} tend to 0 in suitable norms, see [CE92, GM98]. In particular, this implies that asymptotically the amplitude r is slaved to the phase derivative ϕ_x such that $r^2 = 1 - \phi_x^2 + \mathcal{O}(t^{-1/4})$. Hence, asymptocially the so–called phase diffusion equation

$$\phi_t = a(\phi_x)\phi_{xx} \quad \text{ with } a(s) = (1-3s^2)/(1-s^2) \tag{8}$$

is relevant.

The boundary conditions $u(t,x) - U_{\eta\pm}(x) \to 0$ for $x \to \pm\infty$ for all $t \geq 0$ imply $\phi_x(t,x) \to \eta_\pm$ for $x \to \pm\infty$. Hence, for $\eta(t,x) = \phi_x(t,x)$ we obtain from (8) the problem

$$\eta_t = (a(\eta)\eta_x)_x, \quad \eta(t,x) \to \eta_\pm \text{ for } x \to \pm\infty. \tag{9}$$

It is shown in [GM98], Thm. 3.1, that (9) has for each $\eta_+, \eta_- \in (-1/\sqrt{3}, 1/\sqrt{3})$ a unique similarity solution $\eta(t,x) = \widetilde{\eta}(x/\sqrt{t})$ where $\widetilde{\eta}$ is monotone and attains its limits η_+ and η_- faster than exponential.

With $\widetilde{N}(\xi) = \eta_-\xi + \int_{-\infty}^{\xi}[\widetilde{\eta}(s) - \eta_-]\,\mathrm{d}s$ we define the time–dependent profile

$$\widetilde{U}(t,x) = \sqrt{1 - \widetilde{\eta}(x/\sqrt{t})^2}\,\mathrm{e}^{\mathrm{i}\sqrt{t}\,\widetilde{N}(x/\sqrt{t})}. \tag{10}$$

The mathematical theory of diffusive mixing consists in showing that this profile is a good asymptotic approximation of the limiting behavior of suitable solutions of (7). First results for the case $\eta_\pm \approx 0$ were obtained in [BK92]. For the general case with $\eta_+, \eta_- \in (-1/\sqrt{3}, 1/\sqrt{3})$ the main result in [GM98] is the following.

Theorem 6
Let $\eta_+, \eta_- \in (-1/\sqrt{3}, 1/\sqrt{3})$ with $\eta_+ \neq \eta_-$. Then there exists $t_0 > 0$ and $\varepsilon > 0$ such that for all $u_0 \in \mathrm{W}^{2,2}_{\mathrm{ul}}(\mathbb{R})$ satisfying $\|u_0 - \widetilde{U}(t_0,\cdot)\|_{\mathrm{W}^{2,2}(\mathbb{R})} \leq \varepsilon$, the unique solution $u(t) \in \mathrm{W}^{2,2}_{\mathrm{ul}}(\mathbb{R})$ of (7) with $u(0) = u_0$ satisfies, for $t \to \infty$,

$$\|u(t) - \widetilde{U}(t_0{+}t,\cdot)\|_{\mathrm{W}^{1,2}_{\mathrm{ul}}(\mathbb{R})} = \mathcal{O}(t^{-1/5}) \quad \textit{and} \; \|\,|u(t)| - |\widetilde{U}(t_0{+}t,\cdot)|\,\|_{\mathrm{W}^{1,2}_{\mathrm{ul}}(\mathbb{R})} = \mathcal{O}(t^{-2/3}).$$

Note that the assumptions use the nonweighted classical $\mathrm{W}^{2,2}$ norm while the results are stated in the uniformly local norm. This is due to the diffusive character which transports initially localized perturbations out to infinity. The second result in the theorem manifests the fact that the slaving of the amplitude $r(t,x) = |u(t,x)|$ occurs faster than the diffusive mixing which is mainly driven by the phase diffusion which has the decay rate $1/\sqrt{t}$.

The above result says that the limit behavior of the solutions is completely determined by the time–dependent profile $\widetilde{U}$. From its explicit form we find the relation

$$\|\widetilde{U}(t,\cdot) - \mathrm{e}^{\mathrm{i}\phi_*\sqrt{t}}\,U_{\eta*}(\cdot)\|_{\infty,\rho} = \mathcal{O}(t^{-1/2}) \quad \text{for } t \to \infty.$$

Here we again have to use the weighted norm, since a similar estimate in $\mathrm{L}^2_{\mathrm{ul}}(\mathbb{R})$ cannot hold due to the different limit behavior of $\widetilde{U}(t)$ and $U_{\eta_*,\beta}$. The values η_* and ϕ_* are given via $\eta_* = \widetilde{\eta}(0)$ and $\phi_* = \widetilde{N}(0) \neq 0$ and hence are well–defined functions of the limit values η_+ and η_-. As a consequence we have uniform convergence on bounded intervals $(-\ell, \ell)$ as indicated in (2).

The results of Theorem 6 mean that the solutions $u(t)$ convergence to the circle

$$\mathbf{S} = \{\,\mathrm{e}^{\mathrm{i}\beta}\,U_{\eta_*} \in \mathrm{L}^2_{\mathrm{ul}}(\mathbb{R}) \,:\, \beta \in \mathbb{R}\,\}$$

of equilibria in the sense that $\mathrm{dist}_\rho(u(t,\cdot), \mathbf{S}) \to 0$ for $t \to \infty$. The solutions $u(t)$ do not converge to a particular point in $\mathbf{S}$ but they slide along $\mathbf{S}$ with speed $\frac{1}{2}\phi_* t^{-1/2}$ and thus never stettle down completely.

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