Necessary and sufficient conditions for polyconvexity of isotropic functions*

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3 December 2002 / Revised 16 March 2004; 29 July 2004

1 Introduction

In elastostatics and in incremental elasto–plasticity (cf. [CHM02, Mie03, Mie04]) the aim is to find global minimizers for functionals of the form

$$I : \mathrm{W}^{1,p}(\Omega) \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}; \quad u \mapsto \int_\Omega W(\mathrm{D}u(x)) - f(x) \cdot u(x) \, \mathrm{d}x.$$

Important properties in this context are the lower semicontinuity of I which is strongly linked to the quasiconvexity of the function $W : \mathbb{R}^{d\times d} \to \mathbb{R}$, that is, for all $F \in \mathbb{R}^{d\times d}$ we have

$$W(F) \leq \int_{(0,1)^d} W(F + \mathrm{D}v(y)) \, \mathrm{d}y \quad \text{ for all } v \in \mathrm{C}_0^\infty((0,1)^d, \mathbb{R}^d).$$

A major problem with quasiconvexity is that so far there are no suitable methods to treat quasiconvex functions W which attain the value $+\infty$. We refer to the survey [Bal02]. However, in nonlinear elasticity physical considerations force us to consider the case $W(F) = \infty$ for all F with $\det F \leq 0$ together with $W(F) \to +\infty$ for $\det F \searrow 0$, see [Bal76, Bal77]. To circumvent this difficulty, in the latter two papers a stronger property was introduced which is called polyconvexity. It implies quasiconvexity of W and hence, under a few further conditions, the lower semicontinuity of I.

We denote by $\mathbb{M}(F) \in \mathbb{R}^{m(d)}$ the vector of all minors (subdeterminants) of F including 1 as the minor of order 0. A function $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ is called *polyconvex* if there exists a lower semi–continuous (lsc), convex function $g : \mathbb{R}^{m(d)} \to \mathbb{R}_\infty$ such that

$$W(F) = g(\mathbb{M}(F)) \; \text{ for all } F \in \mathbb{R}^{d\times d}.$$

The functions $p_\beta : F \mapsto \langle \beta, \mathbb{M}(F) \rangle$ for $\beta \in \mathbb{R}^{m(d)}$ are called *polyaffine* and we denote the set of all these functions by $\mathbb{PA}(\mathbb{R}^{d\times d})$. (We use $\langle \cdot, \cdot \rangle$ to denote the scalar product in $\mathbb{R}^{m(d)}$

*Research partially supported by DFG under SFB 404 *Multifield Problems in Continuum Mechanics* within the subproject C11

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and $h{\cdot}\nu$ for the scalar product of $h, \nu \in \mathbb{R}^d$.) An equivalent definition of polyconvexity for lower semi–continuous functions W is that W is the pointwise supremum of polyaffine functions, i.e.,

$$W(F) = \sup\{\, p(F) \mid p \in \mathbb{PA}(\mathbb{R}^{d\times d}),\ p \le W \,\}.$$

For W taking only finite values polyconvexity is equivalent to

$$\forall\, G \in \mathbb{R}^{d\times d}\ \exists\, \beta \in \mathbb{R}^{m(d)}\ \forall\, F \in \mathbb{R}^{d\times d}\ : W(F) \ge W(G) + \langle \beta, \mathbb{M}(F) - \mathbb{M}(G)\rangle. \tag{1.1}$$

Here the last term may also be rewritten as $\langle \widehat{\beta}, \mathbb{M}(F-G)\rangle$.

The aim of this paper is to connect the notion of polyconvexity with that of isotropy. A function $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ is called isotropic if

$$W(R_1 F R_2) = W(F) \qquad \text{for all } F \in \mathbb{R}^{d\times d} \text{ and all } R_1, R_2 \in \mathrm{SO}(d).$$

Isotropic functions W with $W(F) = \infty$ for $\det F \le 0$ can be written in terms of the singular values

$$\nu = \sigma(F) \in \mathcal{V}_d = \{\, \gamma \in (0,\infty)^d \mid \gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_d > 0 \,\}.$$

For matrices F with nonnegative determinant we have $F = R_1(\operatorname{diag}\nu)R_2$ with $R_j \in \mathrm{SO}(d)$. Throughout this work will be restricted to the case $\det F > 0$ which is the relevant case for elastostatics with finite strains. We use the following notation. A function $\Phi : \mathcal{V}_d \to \mathbb{R}_\infty$ generates the function $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ through

$$W = \Phi \circ \sigma : \begin{cases} \quad \infty & \text{for } \det F < 0, \\ \Phi(\sigma(F)) & \text{for } \det F \ge 0. \end{cases}$$

We will shortly write $W = \Phi \circ \sigma$, which is supposed to include the definition $W(F) = \infty$ for $\det F < 0$. We shortly say that Φ is *singular–value polyconvex* if W is polyconvex. (For a theory allowing for finite values in the region $\det F < 0$ we refer to [Šil00], where signed singular values are used.)

For applications in nonlinear elasticity it is now of great interest to give necessary and sufficient conditions on the function Φ for obtaining a polyconvex function $W = \Phi \circ \sigma$. Very useful, sufficient conditions were already provided in [Bal76, Bal77, Bal84]. If $\Psi : (0,\infty)^{d+1} \to \mathbb{R}_\infty$ is convex, symmetric and (separately) monotone increasing in its first d arguments, then $W : F \mapsto \Psi(\sigma(F), \det F)$ is polyconvex. Here the monotonicity in the variables $\sigma_j(F)$ is not necessary and it is the purpose of this work to give a better characterization which shows how much "nonmonotonicity" is allowed. We refer to [Bul02, Bul01], where also nonmonotone functions are constructed which give quasiconvex densities W (which are not polyconvex in general).

While polyconvexity is a stronger notion than quasiconvexity, the notion of strong ellipticity or, equivalently, rank–one convexity is a weaker condition. Necessary and sufficient conditions for rank–one convexity of W in the isotropic case, i.e., for Φ, were derived about twenty–five years ago. The two–dimensional case was treated in [KS77, AT80]. The three–dimensional case was solved in [ZS83] under the incompressiblity contraint and in [SS83, AT85, Aub88] for the general case, see also [DH98] for a survey.

Corresponding characterizations for the stronger notions of polyconvexity of isotropic functions exist sofar only in the two–dimensional case, see in [Ros98, Šil99a]: If $d = 2$ and if Φ does not take the value $+\infty$, then $W = \Phi \circ \sigma$ is polyconvex if and only if there exists a convex function $\psi : (0,\infty)^3 \to \mathbb{R}$, such that

$$\begin{aligned}
&\Phi(\nu) = \psi(\nu_1,\nu_2,\nu_1\nu_2) = \psi(\nu_1,\nu_2,\nu_1\nu_2) \quad \text{for all } \nu \in \mathcal{V}_2,\\
&\forall\, \widehat{\omega} \in (0,\infty)^3\ \exists\, \beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{R}^3 \text{ with } \beta_1{+}\beta_2 \geq 0 :\\
&\qquad\qquad \psi(\omega) \geq \psi(\widehat{\omega}) + \beta\cdot(\omega - \widehat{\omega}) \quad \text{for all } \omega \in (0,\infty)^3.
\end{aligned} \tag{1.2}$$

The goal of this work is a corresponding generalization of the two–dimensional results to dimension three. We emphasize that our approach is rather simple and self–contained. In particular we do not need any of the delicate results from linear algebra on the dependence of the singular values $\sigma_j(F)$ in F.

Since we also want to allow for functions which may take the value ∞ we formulate our result in terms suprema of functions. To this end we translate the notion of polyaffine functions into the context of isotropic functions. We define the set $\mathbb{F}_d$ of *singular–value affine functions* as follows:

$$\begin{aligned}
&\mathbb{F}_d := \{\, B(\beta,\cdot) : \mathcal{V}_d \to \mathbb{R} \mid \beta \in \mathbb{R}^{m(d)} \,\}\\
&\text{with} \quad B(\beta,\nu) := \max\{\, \langle \beta, \mathbb{M}(F)\rangle \mid F \in \mathcal{E}(\nu) \,\},
\end{aligned}$$

where $\mathcal{E}(\nu) := \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0,\ \sigma(F) = \nu \,\} = \{\, R_1(\operatorname{diag}\nu)R_2 \mid R_1, R_2 \in \mathrm{SO}(d) \,\}$. These functions play the same role on $\mathcal{V}_d$ as the polyaffine functions of $\mathbb{R}^{d\times d}$. In particular, we have the following abstract characterization of polyconvex, isotropic functions. A lower semi–continuous function $\Phi : \mathcal{V}_d \to \mathbb{R}_\infty$ is singular–value polyconvex if and only if

$$\Phi(\nu) = \sup\{\, s(\nu) \mid s \in \mathbb{F}_d,\ s \leq \Phi \,\},$$

cf. Theorem 2.2.

The usefulness of this characterization depends on the ability to characterize the functions in $\mathbb{F}_d$ more precisely. This is in fact possible for the case $d \leq 3$. Let

$$\mathcal{D}(\nu) := \{\, F \in \mathcal{E}(\nu) \mid F \text{ diagonal} \,\} = \{\, \operatorname{diag}(S\nu) \mid S \in \mathcal{S}_d \,\},$$

where $\mathcal{S}_d := \{\, P \operatorname{diag}\varepsilon \mid P \in \mathrm{Perm}(d),\ \varepsilon \in \{-1,1\}^d,\ \overset{\bullet}{\varepsilon} = 1 \,\}$ with $\overset{\bullet}{\nu} = \prod_1^d \nu_j$. We define

$$\begin{aligned}
&\mathbb{F}_d^* := \{\, D(\beta,\cdot) : \mathcal{V}_d \to \mathbb{R} \mid \beta \in \mathbb{R}^{m(d)} \,\}\\
&\text{with} \quad D(\beta,\nu) := \max\{\, \langle \beta, \mathbb{M}(F)\rangle \mid F \in \mathcal{D}(\nu) \,\}\\
&\qquad\qquad\qquad\quad = \max\{\, \langle \beta, \mathbb{M}(\operatorname{diag}(S\nu))\rangle \mid S \in \mathcal{S}_d \,\}.
\end{aligned}$$

The set $\mathcal{S}_d$ is finite with $2^{d-1}d!$ elements, hence $\mathcal{D}(\nu)$ is finite and $D(\beta,\cdot)$ is a maximum over finitely many values. This is in contrast to $\mathcal{E}(\nu)$ which a smooth manifold, in general. Thus, it is reasonable to expect that the set of all functions $D(\beta,\cdot)$ is easier to characterize than that of all $B(\beta,\cdot)$.

Moreover, for all $d \in \mathbb{N}$ we provide the necessary condition (cf. Proposition 3.1)

$$\begin{aligned} &\Phi : \mathcal{V}_d \to \mathbb{R}_\infty \text{ is polyconvex} \\ &\Longrightarrow \quad \forall\, \nu \in \mathcal{V}_d :\ \Phi(\nu) = \sup\{\, s^*(\nu) \mid s^* \in \mathbb{F}_d^*,\ s^* \leq \Phi \,\}. \end{aligned} \tag{1.3}$$

In Section 3 we also show that $\mathbb{F}_d^* \subset \mathbb{F}_d$ for $d \leq 3$ which implies that the condition in (1.3) is also sufficient for $d \leq 3$. Moreover, $\mathbb{F}_2^*$ and $\mathbb{F}_3^*$ have a simple characterization, see (1.4) and (1.5). So far, the opposite implication in (1.3) is established only for $d \leq 3$ but it is conceivable, that the result also holds in higher dimensions.

The method relies on an observation which has some interest in itself. The functions $B(\beta,\cdot)$ are defined by maximizing the function $F \mapsto \langle \beta, \mathbb{M}(F)\rangle$ over F with given singular values, i.e., $B(\beta,\nu) = \max\{\, \langle \beta, \mathbb{M}(R_1 \operatorname{diag} \nu R_2)\rangle \mid R_1, R_2 \in \mathrm{SO}(d) \,\}$. For $d \leq 3$ this can be reduced to minimizing $(R_1 \odot R_2){:}K_\beta$ where $\odot$ denotes the Schur product (elementwise multiplication). Thus, it suffices to characterize the extremal points of the convex hull of the sets $\mathcal{T}_d = \{\, R_1 \odot R_2 \in \mathbb{R}^{d\times d} \mid R_1, R_2 \in \mathrm{SO}(d) \,\}$. For $d = 2$ and $d = 3$ we find that the extremal points are given by $\mathcal{S}_d$ as defined above. For $d = 2$ there are 4 extremal points and for $d = 3$ there are 24 extremal points. Moreover, these extremal points correspond to $F = R_1(\operatorname{diag} \nu)R_2$ being diagonal and we find $B(\beta^*,\cdot) = D(\beta^*,\cdot)$ for a suitable set of $\beta^* \in \mathbb{R}^{m(d)}$.

Our analysis also explains why the case $d = 3$ is much more difficult than $d = 2$. From

$$\begin{aligned} &\mathbb{F}_2^* \ = \ \{\, s^{(2)}_{a,h,c} \mid a, c \in \mathbb{R},\ h = (h_1, h_2)^\mathsf{T} \in \mathbb{R}^2 \text{ with } h_1 \geq |h_2| \,\} \\ &\text{where } s^{(2)}_{a,h,c}(\nu) := a + h\cdot\nu + c\,\nu_1\nu_2, \end{aligned} \tag{1.4}$$

we see that all $s^{(2)}$ are polynomial and only a restriction on the subgradient appears (cf. (1.2) where $\omega = (\nu_1, \nu_2, \nu_1\nu_2) \in (0,\infty)^3$ is considered). For $d = 3$ we obtain

$$\begin{aligned} &\mathbb{F}_3^* = \{\, s^{(3)}_{a,h,k,c} \mid a, c \in \mathbb{R},\ h, k \in \mathbb{R}^3 \,\} \\ &\text{where } s^{(3)}_{a,h,k,c}(\nu) := a + \max\{\, \langle Sh, \nu\rangle + \langle Sk, \widetilde{\nu}\rangle \mid S \in \mathcal{S}_d \,\} + c\,\overset{\bullet}{\nu}, \end{aligned} \tag{1.5}$$

where $\widetilde{\nu} = (\nu_2\nu_3, \nu_1\nu_3, \nu_1\nu_2)^\mathsf{T} \in (0,\infty)^3$. Hence, most functions in $\mathbb{F}_3^*$ are only piecewise polynomials (with up to seven different polynomial regions in $\mathcal{V}_3$).

Using the characterization via $\mathbb{F}_d^*$ we give several nontrivial examples of isotropic, polyconvex functions. In particular, we produce examples in the form $\Phi(\nu) = \Psi(\nu, \widetilde{\nu}, \overset{\bullet}{\nu})$ where Ψ has decreasing parts in the first six arguments. In the last section we analyze the incompressible case, where in the three–dimensional case Φ is given in the form

$$\Phi(\nu) = \begin{cases} \infty & \text{for } \overset{\bullet}{\nu} \neq 1, \\ \widehat{\varphi}(\nu_1, 1/\nu_3) & \text{for } \overset{\bullet}{\nu} = 1. \end{cases}$$

We give a necessary and sufficient condition of $\widehat{\varphi} : \{\, \mu \in [1,\infty)^2 \mid \sqrt{\mu_1} \leq \mu_2 \leq \mu_1^2 \,\} \to \mathbb{R}_\infty$ which guarantees that Φ is singular–value polyconvex. This conditions implies that $\widehat{\varphi}$ is nondecreasing in each μ_j, but it allows for nonconvex functions.

2 Notations and basic facts

For $d \in \mathbb{N}$ we denote by $\mathrm{GL}_+(d) = \{ F \in \mathbb{R}^{d\times d} \mid \det F > 0 \}$ the group of matrices on $\mathbb{R}^d$ with positive determinant. The vector $\nu = \sigma(F)$ denotes the ordered d–tuple of singular values of F; i.e. $0 < \nu_d \leq \ldots \leq \nu_2 \leq \nu_1$ and

$$F = R_1 \operatorname{diag}(\sigma(F)) R_2 \quad \text{for some } R_1, R_2 \in \mathrm{SO}(d). \tag{2.1}$$

We let

$$\mathcal{V}_d = \{ \nu \in (0,\infty)^d \mid \nu_1 \geq \ldots \geq \nu_d > 0 \},$$

then $\sigma : \mathrm{GL}_+(d) \to \mathcal{V}_d$ is a continuous and surjective mapping.

In nonlinear elasticity one considers stored–energy densities $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ in the form $W = \Phi \circ \sigma$, i.e.,

$$W(F) = \begin{cases} \infty & \text{for } \det F < 0, \\ \Phi(\sigma(F)) & \text{for } \det F \geq 0, \end{cases} \tag{2.2}$$

where $\Phi : \mathcal{V}_d \to \mathbb{R}$ is assumed to be lower semi–continuous (lsc), i.e.,

$$\nu^{(k)} \to \overline{\nu} \quad \Longrightarrow \quad \Phi(\overline{\nu}) \leq \liminf_{k\to\infty} \Phi(\nu^{(k)}).$$

The aim of this work is to provide conditions on Φ that guarantee that the function W is polyconvex. This notion is defined using the vector $\mathbb{M}(F) \in \mathbb{R}^{m(d)}$ of all minors (subdeterminants) of $F \in \mathrm{GL}_+(d)$ including 1 as the trivial minor of order 0. Here $m(d) = \sum_{j=0}^{d} \binom{d}{j}\binom{d}{j}$ such that $m(1) = 2, m(2) = 6$ and $m(3) = 20$. (Please be aware of the fact, that most other works do not include the minor of order $j = 0$.) We let

$$\mathbb{PA}(\mathbb{R}^{d\times d}) := \{ p_\beta \mid \beta \in \mathbb{R}^{m(d)} \} \quad \text{with } p_\beta(F) = \langle \beta, \mathbb{M}(F) \rangle.$$

Definition 2.1 *A lsc function $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ is called* ***polyconvex****, if*

$$\forall\, F \in \mathbb{R}^{d\times d} : \quad W(F) = \sup\{ p(F) \mid p \in \mathbb{PA}(\mathbb{R}^{d\times d}),\ p \leq W \}, \tag{2.3}$$

viz., W is the pointwise supremum of polyaffine functions.

A function $\Phi : \mathcal{V}_d \to \mathbb{R}_\infty$ is called **singular–value polyconvex**, *if W in* (2.2) *is polyconvex.*

A well–established sufficient condition for $\Phi : \mathcal{V}_3 \to \mathbb{R}_\infty$ generating a polyconvex function (cf. [Bal77, Dac89]) is that it has the form

$$\Phi(\nu) = \Psi(\nu, \widetilde{\nu}, \overset{\bullet}{\nu}) \quad \text{with } \widetilde{\nu} = (\nu_2\nu_3, \nu_1\nu_3, \nu_1\nu_2)^\mathsf{T} \text{ and } \overset{\bullet}{\nu} = \nu_1\nu_2\nu_3, \tag{2.4}$$

where $\Psi : (0,\infty)^7 \to \mathbb{R}_\infty$ is symmetric (w.r.t. permutations of ν_j), convex and nondecreasing in the first 6 arguments. We will show in this work that the monotonicity is not necessary.

For $\nu \in \mathcal{V}_d$ and $\beta \in \mathbb{R}^{m(d)}$ we introduce the notations

$$\begin{aligned} \mathcal{E}(\nu) &:= \{ F \in \mathrm{GL}_+(d) \mid \sigma(F) = \nu \} = \{ R_1 (\operatorname{diag} \nu) R_2 \mid R_1, R_2 \in \mathrm{SO}(d) \}, \\ B(\beta, \nu) &:= \max\{ \langle \beta, \mathbb{M}(F) \rangle \mid F \in \mathcal{E}(\nu) \} \end{aligned}$$

By their definition, many of the function coincide. In fact, take $\beta \in \mathbb{R}^{m(d)}$, $R_1, R_2 \in \mathrm{SO}(d)$, then there exists a $\widehat{\beta}_{R_1,R_2} \in \mathbb{R}^{m(d)}$ such that $\langle \beta, \mathbb{M}(R_1 F R_2)\rangle = \langle \widehat{\beta}_{R_1,R_2}, \mathbb{M}(F)\rangle$ for all $F \in \mathbb{R}^{d\times d}$. From this we conclude

$$B(\beta, \cdot) = B(\widehat{\beta}_{R_1,R_2}, \cdot) \quad \text{for all } R_1, R_2 \in \mathrm{SO}(d). \tag{2.5}$$

Combining the isotropic form (2.2) of W and the definition (2.3) of polyconvexity we immediately obtain the following characterization.

Theorem 2.2 (Necessary and sufficient condition for $d \in \mathbb{N}$)
The function Φ singular–value polyconvex if and only if

$$\Phi(\nu) = \sup\{\, B(\beta, \nu) \mid B(\beta, \cdot) \leq \Phi \,\} \quad \textit{for all } \nu \in \mathcal{V}_d, \tag{2.6}$$

i.e., Φ is the pointwise supremum of functions in $\mathbb{F}_d := \{\, B(\beta, \cdot) \mid \beta \in \mathbb{R}^{m(d)} \,\}$.

Proof: By $W = \Phi \circ \sigma$ we have $\langle \beta, \mathbb{M}(\cdot)\rangle \leq W$ if and only if $B(\beta, \cdot) \leq \Phi$. Moreover,

$$\widehat{W} : F \mapsto \sup\{\, \langle \beta, \mathbb{M}(F)\rangle \mid B(\beta, \cdot) \leq \Phi \,\}$$

is a polyconvex, isotropic function, defining $\widetilde{\Phi}$ via $\widehat{W} = \widehat{\Phi} \circ \sigma$. Because of (2.5) we have $\widetilde{\Phi} = \widehat{\Phi}$. Clearly, W is polyconvex if and only if $W = \widehat{W}$, which is the same as (2.6). ■

The functions $B(\beta, \cdot) : \mathcal{V}_d \to \mathbb{R}$ play a major role in the present theory. In fact, they are the exact counterpart of the polyaffine functions $\langle \beta, \mathbb{M}(\cdot)\rangle : \mathbb{R}^{d\times d} \to \mathbb{R}$ of the classical theory.

Corollary 2.3
(a) For all $\beta \in \mathbb{R}^{m(d)}$ the function $B(\beta, \cdot) : \mathcal{V}_d \to \mathbb{R}$ is singular–value polyconvex.
(b) If $\Phi : \mathcal{V}_d \to \mathbb{R}_\infty$ is singular–value polyconvex, then

$$\forall\, \widehat{\nu} \in \mathcal{V}_d \textit{ with } \Phi(\widehat{\nu}) < \infty\ \exists\, \beta \in \mathbb{R}^{m(d)} :\ \Phi(\nu) \geq \Phi(\widetilde{\nu}) + B(\beta, \nu) - B(\beta, \widehat{\nu}). \tag{2.7}$$

Proof: We show that $W_\beta(F) := B(\beta, \sigma(F))$ is the pointwise supremum of polyaffine functions and, hence, it is polyconvex. Indeed, we have

$$W_\beta(F) = \max_{G \in \mathcal{E}(\sigma(F))} \langle \beta, \mathbb{M}(G)\rangle = \max_{R_1, R_2 \in \mathrm{SO}(d)} \langle \beta, \mathbb{M}(R_1 F R_2)\rangle,$$

where $F \mapsto \langle \beta, \mathbb{M}(R_1 F R_2)\rangle = \langle \widehat{\beta}_{R_1,R_2}, \mathbb{M}(F)\rangle$ is polyaffine. This shows (a).

Part (b) follows from the standard fact, that for finite values the supremum in (2.3) is in fact a maximum. ■

The remainder of this paper is devoted to finding a characterization of the functions $B(\beta, \cdot)$ which is simple enough and useful in applications.

3 Necessary and sufficient conditions

The criteria of the previous section are useless as long as we are not able to calculate $B(\beta,\nu)$. Hence, we first construct comparison functions $D(\beta,\cdot)$ which generally are smaller but are easier to evaluate. They allow us to formulate a necessary condition for polyconvexity. We then show that for $d=2$ and 3 and certain β we have $D(\beta,\cdot)=B(\beta,\cdot)$ and thus obtain a sufficiency result.

Roughly spoken, the necessary condition is obtained by testing (1.1) only for diagonal matrices, i.e., $G=\operatorname{diag}\gamma$ and $F\in\mathcal{D}(\nu)=\{\,F\in\mathcal{E}(\nu)\mid F\text{ diagonal}\,\}=\{\,\operatorname{diag}(S\nu)\mid S\in\mathcal{S}_d\,\}$ with $\mathcal{S}_d:=\{\,P\operatorname{diag}\varepsilon\in\mathrm{O}(d)\mid P\in\mathrm{Perm}(d),\ \varepsilon\in\{-1,1\}^d,\ \prod_1^d\varepsilon_j=1\,\}$. In exact mathematical terms, we define the restricted maximum

$$D(\beta,\nu):=\max_{F\in\mathcal{D}(\nu)}\langle\beta,\mathbb{M}(F)\rangle=\max_{S\in\mathcal{S}_d}\langle\beta,\mathbb{M}(\operatorname{diag}(S\nu))\rangle\le B(\beta,\nu). \tag{3.1}$$

The important point is that most components of β are irrelevant in $D(\beta,\cdot)$. Denote by $\mathbb{Q}_d:\mathbb{R}^{m(d)}\to\mathbb{R}^{m(d)}$ the projection which sets all those components of β equal to 0 which do not belong to determinants of submatrices which are symmetric to the diagonal of F. We have $\dim\mathbb{Q}_d\mathbb{R}^{m(d)}=2^d$ and $\mathbb{Q}_d\in\{\,\operatorname{diag}b\mid b\in\{0,1\}^{m(d)}\,\}$, however the special form of $\mathbb{Q}_d$ depends on the chosen ordering of $\mathbb{M}(F)$. For instance, $d=2$ and $\mathbb{M}(F)=(1,F_{11},F_{12},F_{21},F_{22},\det F)$ gives $\mathbb{Q}_2\beta=(\beta_1,\beta_2,0,0,\beta_5,\beta_6)$. The function D in (3.1) satisfies $D(\beta,\nu)=D(\mathbb{Q}_d\beta,\nu)$ for all β and ν. We let

$$\mathbb{F}_d^*\ :=\ \{\,D(\beta^*,\cdot):\mathcal{V}_d\to\mathbb{R}\mid\beta^*\in\mathbb{Q}_d\mathbb{R}^{m(d)}\,\},$$

which is a set with 2^d parameters.

Proposition 3.1 (Necessary condition for $d\in\mathbb{N}$) *If Φ is singular–value polyconvex, then for all $\nu\in\mathcal{V}_d$ we have*

$$\Phi(\nu)=\sup\{\,s^*(\nu)\mid s^*\in\mathbb{F}_d^*,\ s^*\le\Phi\,\}=\sup\{\,D(\beta^*,\nu)\mid D(\beta^*,\cdot)\le\Phi\,\}, \tag{3.2}$$

viz., Φ is the pointwise supremum of functions in $\mathbb{F}_d^*$.

Proof: Denote the function on the right–hand side by $\widehat{\Phi}$. Since every lower envelope satisfies $\widehat{\Phi}\le\Phi$, we only have to show $\widehat{\Phi}\ge\Phi$.

If $\Phi(\nu)<\infty$, then polyconvexity gives $W(G)\ge W(\operatorname{diag}\nu)+\langle\beta,\mathbb{M}(G)-\mathbb{M}(\operatorname{diag}\nu)\rangle$ for some $\beta\in\mathbb{R}^{m(d)}$. Taking the maximum over $G\in\mathcal{D}(\gamma)$ gives

$$\Phi(\gamma)\ge\Phi(\nu)+D(\beta^*,\gamma)-\langle\beta^*,\mathbb{M}(\operatorname{diag}\nu)\rangle\ge\Phi(\nu)+D(\beta^*,\gamma)-D(\beta^*,\nu)$$

for all $\gamma\in\mathcal{V}_d$, where $\beta^*=\mathbb{Q}_d\beta$, since only diagonal matrices are involved. We conclude $\Phi(\nu)-D(\beta^*,\nu)+D(\beta^*,\cdot)\le\Phi$ which implies $\widehat{\Phi}(\nu)\ge\Phi(\nu)$ as desired.

If $\Phi(\nu)=\infty$, then polyconvexity implies that for each $\varepsilon>0$ there exists β_ε such that $W(G)\ge1/\varepsilon+\langle\beta_\varepsilon,\mathbb{M}(G)-\mathbb{M}(\operatorname{diag}\nu)\rangle$. As above we find

$$\Phi(\gamma)\ge1/\varepsilon+D(\beta_\varepsilon^*,\gamma)-\langle\beta_\varepsilon^*,\mathbb{M}(\operatorname{diag}\nu)\rangle\ge1/\varepsilon+D(\beta_\varepsilon^*,\gamma)-D(\beta_\varepsilon^*,\nu)$$

for all $\gamma\in\mathcal{V}_d$. Again, we conclude $\widehat{\Phi}(\nu)\ge1/\varepsilon$ which implies the desired result $\widehat{\Phi}(\nu)=\infty\ge\Phi(\nu)$. ∎

Remark 3.2 This necessary condition can be generalized by using any subgroup $\widehat{\mathcal{S}}$ of $\mathrm{SO}(d)\times\mathrm{SO}(d)$ instead of $\mathcal{S}_d$ from above. We then obtain functions $D_{\widehat{\mathcal{S}}}(\beta,\cdot)$ and a singular–value polyconvex function Φ must be a pointwise supremum of such functions. For the trivial group $\widehat{\mathcal{S}}=\{\mathbb{I}\}$ we obtain the trivial fact, that for each singular–value polyconvex function Φ there exists a convex function $\Psi:\mathbb{R}^{2^d-1}\to\mathbb{R}_\infty$, such that $\Phi(\nu)=\Psi(\nu,\nu_1\nu_2,\ldots,\overset{\bullet}{\nu})$.

Next we show that for $d\leq 3$ this necessary condition is also sufficient, by establishing that $D(\beta^*,\nu)=B(\beta^*,\nu)$ whenever $\beta^*=\mathbb{Q}_d\beta^*$. We believe the result to be true for all $d\geq 2$, however the involved algebra is not yet understood.

For $d=2$ we use the notation $\mathbb{M}(F)=(1,F,\det F)\in\mathbb{R}\times\mathbb{R}^{2\times2}\times\mathbb{R}\cong\mathbb{R}^6$ and $\beta=(a,H,c)$. For $d=3$ we use the adjunct matrix $\operatorname{adj}F=(\det F)F^{-\mathsf{T}}$ (which is polynomial of degree 2 and hence can be continued to all matrices) to denote the minors of 2×2 matrices and let $\mathbb{M}(F)=(1,F,\operatorname{adj}F,\det F)\in\mathbb{R}\times\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3}\times\mathbb{R}\cong\mathbb{R}^{20}$ and $\beta=(a,H,K,c)$. For rotations $R_1,R_2\in\mathrm{SO}(3)$ we have

$$\mathbb{M}(R_1FR_2)=(1,R_1FR_2,R_1(\operatorname{adj}F)R_2,\det F). \tag{3.3}$$

The scalar products and the projections $\mathbb{Q}_d$ are given by

$$\begin{aligned}
d=2:&\qquad \langle\beta,\mathbb{M}(F)\rangle=a{+}H{:}F{+}c\det F,\quad \mathbb{Q}_2(a,H,c)=\left(a,\begin{pmatrix}H_{11}&0\\0&H_{22}\end{pmatrix},c\right)\\
d=3:&\qquad \left\{\begin{aligned}\langle\beta,\mathbb{M}(F)\rangle&=a{+}H{:}F+K{:}(\operatorname{adj}F){+}c\det F,\\ \mathbb{Q}_3(a,H,K,c)&=(a,\operatorname{diag}(H_{ii}),\operatorname{diag}(K_{ii}),c).\end{aligned}\right.
\end{aligned}$$

Here $H{:}F=\sum_{i,j=1}^d H_{ij}F_{ij}$ denotes the scalar product in $\mathbb{R}^{d\times d}$. In this situation we write

$$\beta^*=(a,\operatorname{diag}h,c)\text{ for }d=2\qquad\text{and}\qquad\beta^*=(a,\operatorname{diag}h,\operatorname{diag}k,c)\text{ for }d=3, \tag{3.4}$$

respectively.

The restriction to $d\leq3$ emanates from the usage of the following result, which involves the Schur product $\odot$ for matrices which is given by simple componentwise multiplication

$$A{\odot}B=(A_{ij}B_{ij})_{i,j=1,\ldots d}\qquad\text{(no summation)}$$

Lemma 3.3 *For $d=2$ and $d=3$ and with β^* as in* (3.4) *we have*

$$\begin{aligned}
&\langle\beta^*,\mathbb{M}(R_1(\operatorname{diag}\nu)R_2)\rangle=a+(R_1{\odot}R_2^{\mathsf{T}}){:}N+c\overset{\bullet}{\nu}\\
&\text{with }N=\left\{\begin{array}{ll}h{\otimes}\nu & \text{for }d=2,\\ h{\otimes}\nu+k{\otimes}\widetilde{\nu} & \text{for }d=3,\end{array}\right.\qquad\text{where }\widetilde{\nu}=(\nu_2\nu_3,\nu_1\nu_3,\nu_1\nu_2)^{\mathsf{T}}.
\end{aligned}\tag{3.5}$$

Proof: This formula follows from simple rearrangements of the definition of the formula using (3.3) and $\operatorname{adj}(\operatorname{diag}\nu)=\operatorname{diag}\widetilde{\nu}$ for $d=3$. ∎

Hence, $B(\beta^*,\nu)$ is obtained by maximizing the linear function $A\mapsto a+A{:}N+c\overset{\bullet}{\nu}$ over the set

$$\mathcal{T}_d=\{\,A=R_1{\odot}R_2\mid R_1,R_2\in\mathrm{SO}(d)\,\}. \tag{3.6}$$

A classical fact from optimization tells us that the maximum of a linear function over a compact set T is always attained on $\mathrm{ex}(\mathrm{conv}\, T)$, the set of extremal points of the convex hull of T. For $C, T \subset \mathbb{R}^{d\times d}$ with C convex we have

$$\begin{aligned} \mathrm{conv}\, T &:= \{\, A = \textstyle\sum_{j=1}^{d^2+1} \lambda_j A_j \mid \lambda_j \geq 0, \sum_{j=1}^{d^2+1} \lambda_j = 1, A_j \in T \,\}, \\ \mathrm{ex}\, C &:= \{\, A \in C \mid C \backslash \{A\} \text{ is convex} \,\}. \end{aligned}$$

The following result is central for our sufficient condition.

Proposition 3.4 *For $d \in \{2,3\}$ we have* $\mathrm{ex}(\mathrm{conv}\, \mathcal{T}_d) = \mathcal{S}_d$.

Proof: We first show $\mathcal{S}_d \subset \mathcal{T}_d$. Choose any $S = P\,\mathrm{diag}\,\varepsilon \in \mathcal{S}_d$ and define $\varepsilon_P = (\varepsilon_1 \det P, \varepsilon_2, \ldots, \varepsilon_d)$, $\delta_P = (\det P, 1, \ldots, 1)$, $R_1 = P\,\mathrm{diag}\,\varepsilon_P$ and $R_2 = P\,\mathrm{diag}\,\delta_P$. Then, $R_1, R_2 \in \mathrm{SO}(d)$ and $R_1 \odot R_2 = S$ which shows $S \in \mathcal{T}_d$ as desired.

Next we show $\mathcal{T}_d \subset \mathrm{conv}\, \mathcal{S}_d$. Consider $A = R \odot \widetilde{R}$ with $R, \widetilde{R} \in \mathrm{SO}(d)$. For the j–th column we obtain

$$\textstyle\sum_{i=1}^d |A_{ij}| = \sum_{i=1}^d |R_{ij}||\widetilde{R}_{ij}| \leq (\sum_{i=1}^d R_{ij}^2)^{1/2} (\sum_{k=1}^d \widetilde{R}_{kj}^2)^{1/2} = 1$$

and similar for the rows $\sum_{j=1}^d |A_{ij}| \leq 1$. Moreover, we have the estimate

$$\sum_{i,j=1}^{d} \varepsilon_i A_{ij} \delta_j \leq d{-}2 \text{ for all } \varepsilon, \delta \in \{-1,1\}^d \text{ with } \overset{\bullet}{\varepsilon}\, \overset{\bullet}{\delta} = -1. \tag{3.7}$$

This follows from $\sum \varepsilon_i A_{ij} \delta_j = \mathrm{trace}\, Q$ where $Q = (\mathrm{diag}(\varepsilon) R)^{\mathsf{T}} \widetilde{R}\, \mathrm{diag}(\delta)$ and $A = R \odot \widetilde{R}$. Since $Q \in \mathrm{O}(d)$ with $\det Q = -1$ we have $\mathrm{trace}\, Q \leq d{-}2$. (See also eqn. (6.15) in [Šil02] for a closely related estimate for arbitrary d.)

We define the polyhedron

$$\mathcal{A}_d = \{\, A \in \mathbb{R}^{d\times d} \mid \textstyle\sum_{j=1}^d |A_{ij}| \leq 1, \sum_{k=1}^d |A_{ki}| \leq 1, \text{ (3.7) holds} \,\}.$$

Obviously, $\mathcal{S}_d \subset \mathcal{T}_d \subset \mathcal{A}_d$ and the proof is completed if we show $\mathcal{S}_d = \mathrm{ex}\, \mathcal{A}_d$, which implies $\mathcal{T}_d = \mathcal{A}_d$.

For the case $d = 2$ the set $\mathcal{S}_2$ has the four elements $\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\pm \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Condition (3.7) applied to $A \in \mathcal{A}_2$ gives $A = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ with $|a| + |b| \leq 1$. Hence, $\mathcal{A}_2$ is a square and its corners are exactly the 4 points of $\mathcal{S}_2$.

The case $d = 3$ is more difficult, since $\mathcal{S}_3$ consists of 24 points and $\mathcal{A}_3$ is defined via the 16 linear inequalities of (3.7). It is easy to see that the conditions $\sum_{k=1}^3 |A_{kj}| \leq 1$ and $\sum_{k=1}^3 |A_{ik}| \leq 1$ are consequences of (3.7). (In fact, fix j, ε_j and δ and add the two possible inequalities (by varying ε) to obtain $\varepsilon_j \sum_{i=1}^3 A_{ij} \delta_j \leq 1$. The arbitrariness of ε_j and δ implies $\sum_{i=1}^3 |A_{ij}| \leq 1$.)

We solve the problem by mapping $\mathcal{A}_3$ affinely into the well–known polyhedron

$$\mathcal{D}_4 = \{\, C \in \mathbb{R}^{4\times 4} \mid C_{ij} \geq 0, \sum_{k=1}^{3} C_{kj} = 1, \sum_{k=1}^{3} C_{ik} = 1 \text{ for } i,j = 1,2,3 \,\}$$

of doubly stochastic matrices. By Birkhoff's theorem we know $\operatorname{ex}\mathcal{D}_4 = \mathrm{Perm}(4)$, which has 24 elements. To construct the desired mapping we number the four $\varepsilon \in \{-1,1\}^3$ with $\overset{\bullet}{\varepsilon} = -1$ by $\varepsilon^{(m)}, m = 1,\ldots,4$, and let $\delta^{(n)} = -\varepsilon^{(n)}$. Then, condition (3.7) reads

$$\forall\, m,n \in \{1,\ldots,4\} :\ s(m,n;A) := \sum_{i,j=1}^{3} \varepsilon_i^{(m)} A_{ij} \delta_j^{(n)} \leq 1.$$

Obviously, $s(m,n;A) \geq -3$ and $\sum_{k=1}^3 s(m,k;A) = \sum_{k=1}^3 s(k,n;A) = 0$ for $m,n = 1,\ldots,4$. Thus, define

$$\mathcal{M} : \begin{cases} \mathcal{A}_3 \to \mathcal{D}_4, \\ \quad A \mapsto C = (C_{mn}(A)), \end{cases} \quad \text{with } C_{mn}(A) = (1 - s(m,n;A))/4,$$

which makes $\mathcal{M}$ affine. A simple calculation shows $\mathcal{M}^{-1}(\mathrm{Perm}(4)) = \mathcal{S}_3$ which implies that $\mathcal{M}$ is surjective. Using $\mathcal{A}_3 \subset \mathbb{R}^{3\times 3}$ and $\mathcal{D}_4 \subset C_* + \mathcal{U}$ with $\dim \mathcal{U} = 9$ a dimension count shows that $\mathcal{M}$ is also injective. Thus, $\mathcal{A}_3 = \mathcal{M}^{-1}(\mathcal{D}_4)$ and affinity of $\mathcal{M}^{-1}$ implies

$$\operatorname{ex}(\mathcal{A}_3) = \mathcal{M}^{-1}(\operatorname{ex}\mathcal{D}_4) = \mathcal{M}^{-1}(\mathrm{Perm}(4)) = \mathcal{S}_3.$$

This proves the result. ∎

Proposition 3.5 *For $d \leq 3$ and $\beta^* = \mathbb{Q}_d\beta^*$ we have $B(\beta^*,\cdot) = D(\beta^*,\cdot)$.*

Proof: As a consequence of Lemma 3.3, Proposition 3.4, and the fact that linear functionals attain their extrema on extremal points (Krein–Milman theorem), we find the relation

$$B(\beta^*,\nu) = \sup_{R_1,R_2 \in \mathrm{SO}(d)} R_1 \odot R_2^{\mathsf{T}} {:} N + c\overset{\bullet}{\nu} = \sup_{A \in \mathcal{A}_d} A{:}N + c\overset{\bullet}{\nu} = \max_{S \in \mathcal{S}_d} S{:}N + c\overset{\bullet}{\nu},$$

where $N \in \mathbb{R}^{d\times d}$ is given in (3.5). For $d = 2$ we have $S{:}N = S{:}(h\otimes\nu) = h{\cdot}S\nu = (\operatorname{diag} h){:}\operatorname{diag}(S\nu)$. This proves $B(\beta^*,\nu) = D(\beta^*,\nu)$.

For $d = 3$ we have $S{:}N = S{:}(h\otimes\nu + k\otimes\widetilde{\nu})$ where $\widetilde{\nu} = (\nu_2\nu_3, \nu_1\nu_3, \nu_1\nu_2)$. Using $S\widetilde{\nu} = \widetilde{S\nu}$ we find $S{:}N = (\operatorname{diag} h){:}\operatorname{diag}(S\nu) + (\operatorname{diag} k){:}\operatorname{diag}(\widetilde{S\nu})$ and hence $B(\beta^*,\nu) = D(\beta^*,\nu)$. ∎

Clearly, Lemma 3.3 and Proposition 3.4 don't have simple counterparts in dimensions $d \geq 4$. However, it is not unlikely that Proposition 3.5 still holds, and this is all we need below.

Thus, combining Theorem 2.2 and the Propositions 3.1, and 3.5 (showing $\mathbb{F}_d^* \subset \mathbb{F}_d$) we arrive at our main result of the paper, which says that the necessary condition (3.2) is in fact also sufficient for $d \leq 3$.

Theorem 3.6 *Let $d \in \{2,3\}$. Then,*
(a) for all $\beta^* \in \mathbb{Q}_d\mathbb{R}^{m(d)}$ the function $D(\beta^*,\cdot)$ is singular–value polyconvex, i.e., $\mathbb{F}_d^* \subset \mathbb{F}_d$;
(b) a function $\Phi : \mathcal{V}_d \to \mathbb{R}_\infty$ is singular–value polyconvex if and only if for all $\nu \in \mathcal{V}_d$:

$$\Phi(\nu) = \sup\{\, D(\beta^*,\nu) \mid D(\beta^*,\cdot) \leq \Phi \,\} = \sup\{\, s^*(\nu) \mid s^* \in \mathbb{F}_d^*,\ s^* \leq \Phi \,\}$$

viz., Φ is the pointwise supremum of functions in $\mathbb{F}_d^*$.
(c) a function $\Phi : \mathcal{V}_d \to \mathbb{R}$ (finite values only) is singular–value polyconvex if and only if

$$\forall\, \gamma \in \mathcal{V}_d\ \exists\, \beta^* \in \mathbb{Q}_d\mathbb{R}^{m(d)}\ \forall\, \nu \in \mathcal{V}_d :\ \Phi(\nu) \geq \Phi(\gamma) + D(\beta^*,\nu) - D(\beta^*,\gamma).$$

A further advantage of the reduction from the set of functions $\mathbb{F}_d$ to $\mathbb{F}_d^*$ is that these sets can be characterized easily. Using the specific form of $\beta^* = \mathbb{Q}_d\beta^*$, we have

$$\begin{aligned}\mathbb{F}_2^* &= \{\, \nu \mapsto a + \max_{S\in\mathcal{S}_2} Sh\cdot\nu + c\overset{\bullet}{\nu} \mid a, c \in \mathbb{R},\ h \in \mathbb{R}^2 \,\} \\ &= \{\, \nu \mapsto a + h\cdot\nu + c\overset{\bullet}{\nu} \mid a, h_1, h_2, c \in \mathbb{R},\ h_1 \geq |h_2| \,\}.\end{aligned}$$

This equality is simply obtained by using the four elements in $\mathcal{S}_2$ to find $\widehat{h} = Sh$ such that $\widehat{h}\cdot\nu = \max_{S\in\mathcal{S}_2} Sh\cdot\nu$. The nice fact is, that $\widehat{h}$ can be chosen independently of ν which shows that all the functions in $\mathbb{F}_2^*$ are affine in ν_1, ν_2 and $\nu_1\nu_2$.

For $d = 2$ we find the following simple result, which was already established in [Šil97, Ros98, Šil99a].

Theorem 3.7 *The following three conditions are equivalent:*
(i) $\Phi : \mathcal{V}_2 \to \mathbb{R}$ *(finite values only) is singular–value polyconvex;*
(ii) $\forall\, \gamma \in \mathcal{V}_2\ \exists\, c \in \mathbb{R},\ h = (h_1, h_2)^{\mathsf{T}} \in \mathbb{R}^2$ *with* $h_1 \geq |h_2|\ \forall\, \nu \in \mathcal{V}_2$ *:*

$$\Phi(\nu) \geq \Phi(\gamma) + h\cdot(\nu-\gamma) + c(\nu_1\nu_2-\gamma_1\gamma_2);$$

(iii) there exists a convex function $\Psi : \mathcal{V}_2 \times (0, \infty) \to \mathbb{R}$ *such that* $\Phi(\nu) = \Psi(\nu, \nu_1\nu_2)$ *and that for each* $\alpha, \delta > 0$ *the functions* $[0, \infty) \ni t \mapsto \Psi(\alpha+t, t, \delta)$ *and* $[0, \alpha] \ni t \mapsto \Psi(\alpha+t, \alpha-t, \delta)$ *are nondecreasing.*

Our theory shows the equivalence of (i) and (ii), and we refer to the above–mentioned literature for the equivalence with (iii).

The situation for $d = 3$ is more difficult, since the functions in the set $\mathbb{F}_3^*$ are more difficult. Using (3.4) we find

$$\mathbb{F}_3^* = \{\, s^* : \nu \mapsto a + \max_{S\in\mathcal{S}_d} \big(\langle Sh, \nu\rangle + \langle Sk, \widetilde{\nu}\rangle\big) + c\,\overset{\bullet}{\nu} \mid a, c \in \mathbb{R},\ h, k \in \mathbb{R}^3 \,\}$$

where $\widetilde{\nu} = (\nu_2\nu_3, \nu_1\nu_3, \nu_1\nu_2)^{\mathsf{T}} \in (0, \infty)^3$ and $\overset{\bullet}{\nu} = \nu_1\nu_2\nu_3$. (We used here that $\widetilde{S\nu} = S\widetilde{\nu}$, $\overset{\bullet}{S\nu} = \overset{\bullet}{\nu}$, and $S^{\mathsf{T}} \in \mathcal{S}_3$ for all $\nu \in \mathcal{V}_3$ and $S \in \mathcal{S}_3$.)

Hence, the functions involve 8 real parameters and are piecewise polynomials. For example, choosing $h = (2/3, 0, -1)^{\mathsf{T}}$ and $k = (1/5, 1/2, -1)^{\mathsf{T}}$ it can be shown that the maximum in the definition of $D(\beta^*, \cdot)$ is attained in at least seven different matrices $S \in \mathcal{S}_3$. To see this, just evaluate $D(\beta^*, \nu)$ at the following seven points:

$$\begin{pmatrix} 0.3 \\ 0.2 \\ 0.1 \end{pmatrix}, \begin{pmatrix} 0.62 \\ 0.61 \\ 0.6 \end{pmatrix}, \begin{pmatrix} 1.2 \\ 1.1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0.1 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

We will continue the study of the functions $D(\beta^*, \cdot)$ at the end of Section 4 when we have discussed the necessary and sufficient conditions for rank–one convexity. In particular, we work out that the different polynomials regions in $\mathcal{V}_3$ are mostly separated by hyperplanes which are parallel to the coordinates planes. However, hyperbolas may also appear as interfaces.

Remark 3.8 The set $\mathbb{F}_3^*$ is defined via the 8 real parameters in $a, c \in \mathbb{R}$ and $h, k \in \mathbb{R}^3$. Of course there are redundancies, since $\widehat{\beta}^* \cong (\widehat{a}, \widehat{h}, \widehat{k}, \widehat{c}) = (a, Sh, Sk, c) \cong \beta^*$ for some $S \in \mathcal{S}_3$ implies that the corresponding functions $D(\widehat{\beta}^*, \cdot)$ and $D(\beta^*, \cdot)$ are the same. There are no further redundancies, which means that we can't make $\mathbb{F}_3^*$ substantially smaller without losing the necessity property in Proposition 3.1. (Of course, we may omit a countable subset which is then recovered by taking the supremum.)

To see that redundancies are impossible, we have to show that no function s^* is a lower envelope of function in $\mathbb{F}_3^* \backslash \{s^*\}$. Let γ be a point in the interior of $\mathcal{V}_3$ where $s^* = D(\beta^*, \cdot)$ is smooth. By Theorem 3.6(c) there is a $\widehat{\beta}^*$ such that

$$s^*(\nu) = D(\beta^*, \nu) \geq D(\beta^*, \gamma) + D(\widehat{\beta}^*, \nu) - D(\widehat{\beta}^*, \gamma) \quad \text{for all } \nu \in \mathcal{V}_3.$$

We need to show that $\widehat{\beta}^* = \beta^*$ which means that s^* is the only function in $\mathbb{F}_3^*$ which supports itself. Both functions can be rewritten in terms of $\kappa = \nu - \gamma$ giving $D(\beta^*, \gamma + \nu) = D(\eta^*, \kappa)$ and $D(\widehat{\beta}^*, \gamma + \nu) = D(\widehat{\eta}^*, \kappa)$. We identify $\eta^* \in \mathbb{Q}_3\mathbb{R}^{20}$ by (a, h, k, c) and similarly for $\widehat{\eta}^*$. Using standard arguments for local minima of $D(\eta^*, \cdot) - D(\eta^*, \cdot)$ at $\kappa = 0$ we find

$$\begin{aligned}
&a = D(\eta^*, 0) = D(\widehat{\eta}^*, 0) = \widehat{a}, \quad h = \nabla D(\eta^*, 0) = \nabla D(\widehat{\eta}^*, 0) = \widehat{h} \in \mathbb{R}^3, \\
&B(k) = \mathrm{D}^2 D(\eta^*, 0) \geq \mathrm{D}^2 D(\widehat{\eta}^*, 0) = B(\widehat{k}) \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},
\end{aligned}$$

where $B(k)$ is given in (4.6), and $B(k) \geq B(\widehat{k})$ is equivalent to $B(k - \widehat{k}) \geq 0$ which implies $k = \widehat{k}$. Thus, $(a, h, k) = (\widehat{a}, \widehat{h}, \widehat{k})$ and we are left with the inequality $c\overset{\bullet}{\kappa} \geq \widehat{c}\overset{\bullet}{\kappa}$ for all small $\kappa \in \mathbb{R}^3$. This, gives $c = \widehat{c}$ and we have shown $\eta^* = \widehat{\eta}^*$ which is the same as $\beta^* = \widehat{\beta}^*$.

Very often it is common to represent isotropic functions W via symmetric functions $\Psi : (0, \infty)^d \to \mathbb{R}_\infty$, i.e., we set $\Psi(P\nu) = \Phi(\nu)$ for $\nu \in \mathcal{V}_d$ and $P \in \mathrm{Perm}(d)$. Since this representation is somewhat more suggestive we give the next example in this form. We simply consider the case $d = 3$ and the function $\varphi : \nu \mapsto \varphi(\nu) = D(\beta^*, \nu)$ with β^* given by $h = (1, 0, 0), k = (-1, 0, 0)$ and $c = 0$. Hence,

$$\Psi(\nu) = \max\{|\nu_1 - \nu_2\nu_3|, |\nu_2 - \nu_1\nu_3|, |\nu_3 - \nu_1\nu_2|\} \tag{3.8}$$

defines via $W = \Psi \circ \sigma$ an isotropic polyconvex function. Note that $(0, \infty)^3$ decomposes into six polyhedral regions such that each of the six functions $\pm(\nu_{\pi(1)} - \nu_{\pi(2)}\nu_{\pi(3)})$ is valid in exactly one region, see Figure 1. Clearly, Ψ is nonconvex in $\nu \in (0, \infty)^3$ and not monotone in any ν_j. The function $\varphi = \Psi|_{\mathcal{V}_3} : \mathcal{V}_3 \to \mathbb{R}$ is piecewise polynomial only, viz., $\varphi(\nu) = \max\{\nu_1 - \nu_2\nu_3, \nu_1\nu_2 - \nu_3\}$.

Our theory proposes that the symmetric representation on $(0, \infty)^d$ is just an intermediate compromise. Since the set $\mathcal{D}(\nu) = \{\, \mathrm{diag}(S\nu) \mid S \in \mathcal{S}_d \,\}$ of diagonal matrices involves also negative eigenvalues, we suggest the following representation. Let

$$\begin{aligned}
&\mathbb{R}^d_{\mathrm{sym}} := \{\, \nu \in \mathbb{R}^d \mid \overset{\bullet}{\nu} > 0 \,\}, \\
&\mathbb{P}^{\mathrm{sym}}_d := \{\, p : \mathbb{R}^d_{\mathrm{sym}} \to \mathbb{R} \mid \exists\, \beta^* \in \mathbb{Q}_d\mathbb{R}^{m(d)} : p(\nu) = \langle \beta^*, \mathbb{M}(\mathrm{diag}\,\nu) \rangle \,\}.
\end{aligned}$$

The polynomials $p \in \mathbb{P}^{\mathrm{sym}}_d$ now replace the role of the functions $s^* \in \mathbb{F}_d^*$, where we have factored out the reflection and permutation symmetries. For each function $\Phi : \mathcal{V}_d \to \mathbb{R}_\infty$

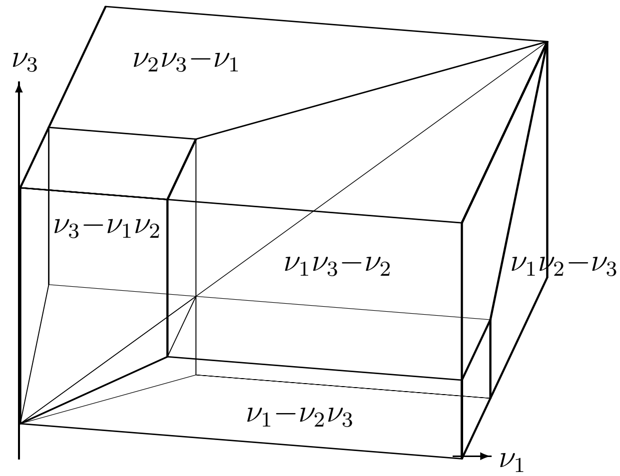

Figure 1: The function $\Psi : \nu \mapsto \max\{|\nu_1-\nu_2\nu_3|, |\nu_2-\nu_1\nu_3|, |\nu_3-\nu_1\nu_2|\}$.

we define the unique symmetric extension $\Phi^{\rm sym} : \mathbb{R}^d_{\rm sym} \to \mathbb{R}_\infty$ via $\Phi^{\rm sym}(\nu) := \Phi(S\nu)$ whenever $S\nu \in \mathcal{V}_d$.

This symmetric representation in $\mathbb{R}^d_{\rm sym}$ is opposite to our above representation on $\mathcal{V}_d$. There we have used the minimal representation by factoring $\mathbb{R}^d_{\rm sym}$ with respect to the symmetry group $\mathcal{S}_d$. Hence, Φ has no more symmetry, but the singular–value polyconvex functions arise as maximum over the action of the symmetry group over the functions $p \in \mathbb{P}^{\rm sym}_d$. For the next result, we consider $\Phi^{\rm sym}$ with the full symmetry and hence, do not need to deal with piecewise polynomial functions. In this sense the following result is a simple rewriting of Proposition 3.1 and Theorem 3.6(b).

Theorem 3.9 *Let $\Phi : \mathcal{V}_d \to \mathbb{R}_\infty$ and its symmetric extension $\Phi^{\rm sym} : \mathbb{R}^d_{\rm sym} \to \mathbb{R}_\infty$ be given. Then, the following holds:*
(a) (Necessary condition) If Φ is singular–value polyconvex, then

$$\Phi^{\rm sym}(\nu) = \sup\{\, p(\nu) \mid p \in \mathbb{P}^{\rm sym}_d,\ p \leq \Phi^{\rm sym} \,\} \quad \text{for all } \nu \in \mathbb{R}^d_{\rm sym}, \tag{3.9}$$

i.e., $\Phi^{\rm sym}$ is a pointwise supremum over functions in $\mathbb{P}^{\rm sym}_d$.
(b) (Sufficiency) For $d \leq 3$ the condition (3.9) is also sufficient.

4 The differentiable case

The criterion of Theorem 3.6 can be simplified if the function $\Phi : \mathcal{V}_d \to \mathbb{R}$ is differentiable. It is a common fact for differentiable convex functions the subdifferential contains exactly one point. Hence, the freedom to choose $(a, h, c) \in \mathbb{R}^4$ if $d = 2$ or $(a, h, k, c) \in \mathbb{R}^8$ if $d = 3$ is reduced by $d{+}1$ coordinates using $\Phi(\gamma) \in \mathbb{R}$ and $f = \mathrm{D}\Phi(\gamma) \in \mathbb{R}^d$. The following result for $d = 2$ is also contained in [Šil99a] as Prop. 4.1.

Theorem 4.1 *The function $\Phi \in \mathrm{C}^1(\mathcal{V}_2, \mathbb{R})$ is singular–value polyconvex if and only if*

$$\begin{aligned} &\forall\, \gamma \in \mathcal{V}_2\ \exists\, c \in [-\tfrac{f_1-f_2}{\gamma_1-\gamma_2}, \tfrac{f_1+f_2}{\gamma_1+\gamma_2}]\ \forall\, \nu \in \mathcal{V}_2 : \\ &\qquad \Phi(\nu) \geq \Phi(\gamma) + f\cdot(\nu-\gamma) + c(\nu_1-\gamma_1)(\nu_2-\gamma_2), \end{aligned} \tag{4.1}$$

where $f = \mathrm{D}\Phi(\gamma) \in \mathbb{R}^2$.

Since the estimate in (4.1) is linear in c, we would like to choose c either equal to $-\frac{f_1-f_2}{\gamma_1-\gamma_2}$ or equal to $\frac{f_1+f_2}{\gamma_1+\gamma_2}$ if $(\nu_1-\gamma_1)(\nu_2-\gamma_2)$ is positive or negative, respectively. However, c may depend only on γ but not on ν.

Proof: This is a direct consequence of Theorem 3.7(ii) where now $h = f + c\binom{\nu_2}{\nu_1}$. The restriction on c is just the one which guarantees $h_1 \geq |h_2|$. ■

The restriction on c in (4.1) implicitly states that the interval is non–empty, which is equivalent to the well–known Baker–Ericksen condition $\frac{\gamma_1 f_1-\gamma_2 f_2}{\gamma_1-\gamma_2} \geq 0$. In analogy to Aubert's criterion (cf. [Aub87]), for $\gamma \in \mathcal{V}_2$ we may define

$$C_\gamma^+ = \inf\{ \tfrac{\Phi(\nu)-\Phi(\gamma)-f\cdot(\nu-\gamma)}{(\nu_1-\gamma_1)(\nu_2-\gamma_2)} \mid \nu \in \Sigma_\gamma^+ \},\ C_\gamma^- = \sup\{ \tfrac{\Phi(\nu)-\Phi(\gamma)-f\cdot(\nu-\gamma)}{(\nu_1-\gamma_1)(\nu_2-\gamma_2)} \mid \nu \in \Sigma_\gamma^- \},$$

where $\Sigma_\gamma^\pm = \{ \nu \in \mathcal{V}_2 | \pm(\nu_1-\gamma_1)(\nu_2-\gamma_2) > 0 \}$. Then, the conditions in (4.1) are equivalent to

$$[-\tfrac{f_1-f_2}{\gamma_1-\gamma_2}, \tfrac{f_1+f_2}{\gamma_1+\gamma_2}] \cap [C_\gamma^-, C_\gamma^+] \neq \emptyset.$$

Expanding the inequality in (4.1) to second power in $\kappa = \nu-\gamma$ using $H = \mathrm{D}^2\Phi(\gamma)$ we obtain the necessary condition

$$\tfrac{1}{2}H\kappa\cdot\kappa - c\kappa_1\kappa_2 \geq 0 \text{ for all } \kappa \in \mathbb{R}^2. \tag{4.2}$$

This is equivalent to $|H_{12}-c| \leq \sqrt{H_{11}H_{22}}$ where $H_{ij} = \frac{\partial^2}{\partial\gamma_i\partial\gamma_j}\Phi(\gamma)$. There exists $c \in [-\frac{f_1-f_2}{\gamma_1-\gamma_2}, \frac{f_1+f_2}{\gamma_1+\gamma_2}]$ which satisfies (4.2) if and only if H satisfies

$$H_{11}, H_{22}, \tfrac{\gamma_1 f_1-\gamma_2 f_2}{\gamma_1^2-\gamma_2^2} \geq 0 \quad \text{and} \quad |\tfrac{\gamma_1 f_2-\gamma_2 f_1}{\gamma_1^2-\gamma_2^2} - H_{12}| \leq \sqrt{H_{11}H_{22}} + \tfrac{\gamma_1 f_1-\gamma_2 f_2}{\gamma_1^2-\gamma_2^2}. \tag{4.3}$$

These are the classical conditions for rank–one convexity for $d = 2$, see [KS77, AT80, MS98, Dav91, DH98]. This is not surprising since for quadratic functionals on $\mathbb{R}^{2\times2}$ it is well–known that rank–one convexity and polyconvexity coincide, see [Dac89].

However, there is even further similarity between rank–one convexity and polyconvexity for isotropic functions which stems from the theory in [AT87, CT00, CT02]. There it is shown that compact, connected, and isotropic subsets of $\{ A \in \mathbb{R}^{2\times2} | \det A > 0 \}$ are rank–one convex if and only if they are polyconvex. So we conjecture that all isotropic, rank–one convex functions are in fact polyconvex, if all the sublevel sets $\{ F \in \mathbb{R}^{2\times2} \mid \det F > 0,\ \Phi(\sigma(F)) \leq t \}$, $t \in \mathbb{R}$, are connected. The last condition rules out the famous counter example in [Aub87] given via

$$\Phi(\nu) = \tfrac{1}{3}(\nu_1^4+\nu_2^4) + \tfrac{1}{2}\nu_1^2\nu_2^2 - \tfrac{2}{3}\nu_1\nu_2(\nu_1+\nu_2).$$

Example 4.2 We consider the density $W = \Phi \circ \sigma$ with $\Phi : \mathcal{V}_2 \to \mathbb{R}_\infty$ defined via

$$\Phi(\nu) = \phi(\nu)+\psi(\nu_1\nu_2) \quad \text{where } \phi(\nu) = \begin{cases} \frac{2}{\alpha}\sqrt{\nu_1^\alpha(\nu_2^\alpha+b)} & \text{for } \nu_1 \geq (\nu_2^\alpha+b)^{1/\alpha}, \\ \frac{1}{\alpha}(\nu_1^\alpha+\nu_2^\alpha+b) & \text{for } \nu_1 \in [\nu_2, (\nu_2^\alpha+b)^{1/\alpha}], \end{cases} \tag{4.4}$$

$\alpha \geq 2$, and $\psi : (0,\infty) \to \mathbb{R}_\infty$ is a lower semicontinuous, convex function. This density was obtained in [Mie03] in connection with elasto–plasticity (see also [CHM02, Mie04]).

It is easy to show by a direct computation that the conditions (4.3) for rank–one convexity are satisfied. In Appendix B we use Theorem 4.1 to establish that $\phi : \mathcal{V}_2 \to \mathbb{R}$ is singular–value polyconvex for all $\alpha \geq 2$. Since the additive term $\nu \mapsto \psi(\nu_1\nu_2)$ is also singular–value polyconvex, this proves that Φ generates a polyconvex density $W = \Phi \circ \sigma$.

For the three–dimensional case a similar result can be derived by using $f = \mathrm{D}\Phi(\gamma) \in \mathbb{R}^3$. We find by comparing zeroth and first order terms

$$h{\cdot}\gamma + k{\cdot}\widetilde{\gamma} = \max_{S\in\mathcal{S}_3}(h{\cdot}S\gamma + k{\cdot}S\widetilde{\gamma}) \tag{4.5}$$

and

$$f = h + B(k)\gamma + c\widetilde{\gamma} \quad \text{where } B(k) = \begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix}. \tag{4.6}$$

The relation (4.6) enables us to eliminate h from the condition in Theorem 3.6 which gives the following result.

Theorem 4.3 *The function $\Phi \in \mathrm{C}^1(\mathcal{V}_3, \mathbb{R})$ is singular–value polyconvex if and only if*

$$\begin{aligned} &\forall\, \gamma \in \mathcal{V}_3\ \exists\, k \in \mathbb{R}^3\ \exists\, c \in \mathbb{R}\ \forall\, \nu \in \mathcal{V}_3 : \\ &\Phi(\nu) \geq \Phi(\gamma) + \max\nolimits_{S\in\mathcal{S}_3}[(f{-}B(k)\gamma{-}c\widetilde{\gamma}){\cdot}(S\nu) + k{\cdot}(S\widetilde{\nu})] \\ &\qquad - f{\cdot}\gamma + k{\cdot}\widetilde{\gamma} + c(\nu_1\nu_2\nu_3{+}2\gamma_1\gamma_2\gamma_3) \end{aligned} \tag{4.7}$$

This result is not as useful as the two–dimensional one, since we are not able to simplify the functions $D(\beta^*, \cdot)$ further. However, we are able to derive some more explicit necessary conditions for polyconvexity. For $\gamma \in \mathbb{R}^3$ define

$$\begin{aligned} \mathcal{G}(\gamma) = \{\, (f,k,c) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \mid \forall\, S \in \mathcal{S}_3 \backslash \{\mathbb{I}\} : (f{-}B(k)\gamma{-}c\widetilde{\gamma}){\cdot}\gamma + k{\cdot}\widetilde{\gamma} \geq \\ (f{-}B(k)\gamma{-}c\widetilde{\gamma}){\cdot}(S\gamma) + k{\cdot}(S\widetilde{\gamma}) \,\} \end{aligned}$$

Hence, the set $\mathcal{G}(\gamma)$ is a convex polyhedron in $\mathbb{R}^7$ which is characterized by 23 linear constraints. Moreover we define

$$\begin{aligned} \mathcal{F}(\gamma) &= \{\, f \in \mathbb{R}^3 \mid \exists\, (k,c) : (f,k,c) \in \mathcal{G}(\gamma) \,\}, \\ \mathcal{M}(f,\gamma) &= \{\, k{+}c\gamma \in \mathbb{R}^3 \mid (f,k,c) \in \mathcal{G}(\gamma) \,\}, \end{aligned}$$

which are polyhedra in $\mathbb{R}^3$, respectively. Using only the six matrices

$$S_1^\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \pm1 \\ 0 & \pm1 & 0 \end{pmatrix},\ S_2^\pm = \begin{pmatrix} 0 & 0 & \pm1 \\ 0 & 1 & 0 \\ \pm1 & 0 & 0 \end{pmatrix},\ S_3^\pm = \begin{pmatrix} 0 & \pm1 & 0 \\ \pm1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in the constraints defining $\mathcal{G}(\gamma)$ we obtain the six restrictions

$$r_i^- = r_i^-(f,\gamma) := -\frac{f_{i+2} - f_{i+1}}{\gamma_{i+2} - \gamma_{i+1}} \leq k_i + c\gamma_i \leq r_i^+ = r_i^+(f,\gamma) := \frac{f_{i+2} + f_{i+1}}{\gamma_{i+2} + \gamma_{i+1}}, \tag{4.8}$$

where the indices are taken modulo 3. These restrictions imply that all $f \in \mathcal{F}(\gamma)$ satisfy the Baker–Ericksen inequalities

$$r_i^+ - r_i^- = \frac{\gamma_{i+2} f_{j+2} - \gamma_{i+1} f_{i+1}}{\gamma_{i+2}^2 - \gamma_{i+1}^2} \geq 0 \quad \text{for } i = 1, 2, 3,$$

and that $\mathcal{M}(f, \gamma)$ is contained in the rectangular box

$$\mathcal{R}(f, \gamma) := [r^-, r^+] = [r_1^-, r_1^+] \times [r_2^-, r_2^+] \times [r_3^-, r_3^+] \subset \mathbb{R}^3,$$

where the $r_i^\pm$ are defined in (4.8). We conjecture that $f \in \mathcal{F}(\gamma)$ implies $\mathcal{R}(f, \gamma) = \mathcal{M}(f, \gamma)$.

To derive our final necessary condition we go to second order terms after assuming that Φ is twice differentiable. We compare our necessary condition for polyconvexity with the necessary and sufficient condition for rank–one convexity. For this we recall that a matrix $M \in \mathbb{R}^{d \times d}$ is called *co–positive* if

$$M\kappa \cdot \kappa \geq 0 \quad \text{for all } \kappa \in [0, \infty)^d.$$

Clearly, this condition is strictly weaker that positive semi–definiteness.

To compare rank–one convexity, which is a local concept by means of the Legendre–Hadamard condition, we introduce a local version of condition (4.7). The function Φ is called *locally singular–value polyconvex at* $\gamma \in \mathcal{V}_3$, if

$$\exists\, m \in \mathcal{M}(f, \gamma) : \Phi(\gamma{+}\kappa) \geq \Phi(\gamma) + f \cdot \kappa + m \cdot \widetilde{\kappa} + O(|\kappa|^3) \quad \text{for } \kappa \to 0. \tag{4.9}$$

This condition is a consequence of (4.7), since $m = k{+}c\gamma$ and $\overset{\bullet}{\kappa} = O(|\kappa|^3)$. This is the strongest local condition which can be derived from (4.7). It is reformulated in part (b) below whereas part (a) gives the classical conditionfor rank–one convexity, which is the same as strong ellipticity or the Legendre–Hadamard condition, see [SS83, AT85, Aub88].

Theorem 4.4 *Consider* $\Phi \in \mathrm{C}^2(\mathcal{V}_3, \mathbb{R})$*. For* $\gamma \in \mathcal{V}_3$ *set* $f = \mathrm{D}\Phi(\gamma)$*,* $H = \mathrm{D}^2\Phi(\gamma)$ *and* $r^\pm$ *according to* (4.8)*.*

(a) $W = \Phi \circ \sigma$ *satisfies the Legendre–Hadamard conditions at* F *with* $\gamma = \sigma(F)$ *if and only if the Baker–Ericksen conditions* $r_i^- \leq r_i^+$*,* $i = 1, 2, 3$*, hold and, for all* $\varepsilon \in \{-1, 1\}^3$*, the symmetric matrix* $\operatorname{diag}\varepsilon\, [H{-}B(\widehat{r}^\varepsilon)] \operatorname{diag}\varepsilon$ *is co–positive, where*

$$B(r) = \begin{pmatrix} 0 & r_3 & r_2 \\ r_3 & 0 & r_1 \\ r_2 & r_1 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{r}^\varepsilon = (r_1^{(-\varepsilon_2\varepsilon_3)}, r_2^{(-\varepsilon_1\varepsilon_3)}, r_3^{(-\varepsilon_1\varepsilon_2)})^\mathsf{T}.$$

(b) Φ *is locally singular–value polyconvex at* γ*, if and only if there exists a vector* $m \in \mathcal{M}(f, \gamma) \subset \mathcal{R}(f, \gamma) \subset \mathbb{R}^3$ *such that the matrix* $H{-}B(m)$ *is positive semi–definite.*

Proof: Part (a) is a rewriting of the conditions given in [Dac01] or in Sect. 6 of [Šil99b]. For part (b) we simply use $\Phi(\gamma{+}\kappa) = \Phi(\gamma) + f \cdot \kappa + \frac{1}{2} H\kappa \cdot \kappa + O(|\kappa|^3)$ in (4.9) to obtain

$$H\kappa \cdot \kappa = \mathrm{D}^2\Phi(\gamma)\kappa \cdot \kappa \geq 2m \cdot \widetilde{\kappa} = B(m)\kappa \cdot \kappa,$$

which gives the desired result. ∎

In Appendix A we show that the conditions in (a) and (b) are in fact equivalent if $\mathcal{M}(f,\gamma) = \mathcal{R}(f,\gamma)$. In particular, (b) always implies (a). We conjecture that $\mathcal{M}(f,\gamma) = \mathcal{R}(f,\gamma)$ is true in general which would show that local polyconvexity does not enforce any stronger condition on the function Φ than rank–one convexity. We believe that this is one of the major reasons, why many relaxation results in three–dimensional elasticity show that the rank–one and the polyconvex hulls are equal. It seems conceivable that the results in [AT87, CT00, CT02] have analogous counterparts in dimension $d = 3$.

We conclude the discussion with a discussion of the basic functions

$$P_{h,k} : \mathcal{V}_3 \to \mathbb{R};\ \nu \mapsto h\cdot\nu + k\cdot\widetilde{\nu},$$

from which the functions $D(\beta^*,\cdot)$ are composed.

Lemma 4.5 *Let $h, k \in \mathbb{R}^3$ be given. The function $W = P_{h,k} \circ \sigma$ satisfies the Legendre–Hadamard condition at F with $\gamma = \sigma(F)$ if and only if*

$$h_1+\gamma_3 k_2 \geq |h_2+\gamma_3 k_1|, \quad h_1+\gamma_2 k_3 \geq |h_3+\gamma_2 k_1|, \quad h_2+\gamma_1 k_3 \geq |h_3+\gamma_1 k_2|. \tag{4.10}$$

Proof: We have $f = \mathrm{D}P_{h,k}(\gamma) = h+B(k)\gamma$ and $H = \mathrm{D}^2 P_{h,k}(\gamma) = B(k)$. Since $B(k)-B(r) = B(k-r)$ is positive semi–definite if and only if $r = k$, we see by Theorem 4.4 that the Legendre–Hadamard conditions is equivalent to $k \in \mathcal{R}(f,\gamma)$. Inserting the formula for f into the definitions of r^+ and r^- gives the result. ■

For given $h, k \in \mathbb{R}^3$, conditions (4.10) define subsets of $\mathcal{V}_3$ which are bounded by hyperplanes parallel to the coordinate axis. Let β^* be defined via h, k and $a, c = 0$ such that $D(\beta^*,\nu) = P_{\widehat{S}(\nu)h,\widehat{S}(\nu)k}(\nu)$ where the function $\widehat{S} : \mathcal{V}_3 \to \mathcal{S}_3$ is piecewise constant. We see that regions where $\widehat{S}$ is constant must be contained in the regions where (4.10) holds for (h,k) replaced by $(\widehat{S}(\nu)h, \widehat{S}(\nu)k)$. Upon crossing an interface which is such a hyperplane the signed permutation $\widehat{S}$ will change by a transposition of the two components which are parallel to the hyperplane (in one of the three conditions changes the direction of the inequality sign).

The conditions (4.10) are the only conditions which are imposed by rank–one convexity. However, polyconvexity imposes more restrictions.

Example 4.6 As in Section 3 considering $P_{h,k}$ with

$$h = \begin{pmatrix} 2/3 \\ 0 \\ -1 \end{pmatrix}, \quad k = \begin{pmatrix} 1/5 \\ 1/2 \\ -1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \text{and } S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we find $\widehat{S}(\gamma) = S_j$ for $\gamma \in V_j$ where rank–one convexity gives

$$V_1 = \{\, \nu \in V_3 \mid \nu_1 \leq 20/21,\ \nu_2 \leq 2/3,\ \nu_3 \leq 5/12 \,\},$$
$$V_2 = \{\, \nu \in V_3 \mid \nu_1 \geq 2/3,\ \nu_2 \leq 5/12,\ \nu_3 \leq 20/21 \,\}.$$

The intersection $V_1 \cap V_2$ has nonempty interior. This proves that polyconvexity imposes a stronger condition which selects the correct $\widehat{S}(\nu)$ inside of $V_1 \cap V_2$. A calculation shows that the interface between the two regions where S_1 and S_2 are valid is the hyperbola

$$(S_1 h - S_2 h)\cdot\nu + (S_1 k - S_2 k)\cdot\widetilde{\nu} = (\tfrac{1}{3}, \tfrac{2}{3}, -1)^{\mathsf{T}}\cdot\nu + (\tfrac{4}{5}, \tfrac{7}{10}, \tfrac{-3}{2})^{\mathsf{T}}\cdot\widetilde{\nu} = 0.$$

In particular, the two functions $p_j : \mathcal{V}_3 \to \mathbb{R}_\infty$ with

$$p_j(\nu) := \begin{cases} P_{S_j h, S_j k}(\nu) & \text{for } \nu \in V_j, \\ \infty & \text{else,} \end{cases}$$

generate a rank–one convex density $W = p_j \circ \sigma$ which is not polyconvex.

5 The incompressible case

In this last section we treat the incompressible case which is an often used idealization in elasto–plasticity. It is implemented by assuming that $\det \mathrm{D}u(x) = 1$ for almost all $x \in \Omega$. In our isotropic setting the stored–energy density takes the form

$$W(F) = \Phi(\sigma(F)) \quad \text{with} \quad \Phi(\nu) = \begin{cases} \varphi_{\text{inc}}(\nu) & \text{for } \overset{\bullet}{\nu} = 1, \\ \infty & \text{else.} \end{cases} \tag{5.1}$$

The aim is to find necessary and sufficient conditions on the function $\varphi_{\text{inc}} : \mathcal{I}_d \to \mathbb{R}_\infty$ such that W is polyconvex. The domain $\mathcal{I}_d = \mathcal{V}_d \cap \{\, \nu \mid \overset{\bullet}{\nu} = 1 \,\}$ has dimension $d-1$ and can be parameterized as follows:

$$\begin{aligned} \mathcal{I}_2 &= \{\, (\mu, 1/\mu)^\mathsf{T} \in \mathcal{V}_2 \mid \mu \in \widehat{\mathcal{I}}_2 = [1, \infty) \,\}, \\ \mathcal{I}_3 &= \{\, (\mu_1, \mu_2/\mu_1, 1/\mu_2)^\mathsf{T} \in \mathcal{V}_3 \mid \mu \in \widehat{\mathcal{I}}_3 \,\}, \\ &\quad \text{where } \widehat{\mathcal{I}}_3 = \{\, \mu \in [1, \infty)^2 \mid \sqrt{\mu_1} \le \mu_2 \le \mu_1^2 \,\}. \end{aligned}$$

In the three–dimensional case we have chosen the parameterization $\mu_1 = \nu_1 = \nu_{\max}$ and $\mu_2 = 1/\nu_3 = 1/\nu_{\min}$, which leads to most symmetric statements, see [DD02, Šil01].

The two–dimensional case leads to a simple characterization.

Theorem 5.1 *For $\widehat{\varphi} : \widehat{\mathcal{I}}_2 = [1, \infty) \to \mathbb{R}_\infty$ define the functions $\varphi_{\text{inc}} : \mathcal{I}_2 \to \mathbb{R}_\infty; (\mu, 1/\mu)^\mathsf{T} \mapsto \widehat{\varphi}(\mu)$ and $\Phi : \mathcal{V}_2 \to \mathbb{R}_\infty$ as in (5.1). Then, the following three statements are equivalent.*

(i) Φ is singular–value polyconvex;

(ii) $\varphi_{\text{inc}}(\mu) = \sup\{\, s_{\alpha,\eta}(\mu) \mid s_{\alpha,\eta} \le \varphi_{\text{inc}},\ \alpha \in \mathbb{R},\ \eta \ge 0 \,\}$, where $s_{\alpha,\eta}(\mu) = \alpha + \eta(\mu - 1/\mu)$;

(iii) the function $\varphi_{\text{inc}} \circ A : [0, \infty) \to \mathbb{R}_\infty$ is nondecreasing and convex, where $A : [0, \infty) \to \widehat{\mathcal{I}}_2; \rho \mapsto \rho/2 + \sqrt{1 + \rho^2/4}$ is the inverse of $\mu \mapsto \rho = \mu - 1/\mu$.

Proof: The equivalence of (i) and (ii) is a direct consequence of Theorem 3.6. The equivalence with (iii) is easily obtained since the transformation $\mu = A(\rho)$ makes the functions $s_{\alpha,\eta} \circ A$ linear. ■

To formulate the case $d = 3$ we introduce the abbreviation

$$D_{\text{inc}}(a, h, k; \mu) = a + \max\{\, Sh \cdot \begin{pmatrix} \mu_1 \\ \mu_2/\mu_1 \\ 1/\mu_2 \end{pmatrix} + Sk \cdot \begin{pmatrix} 1/\mu_1 \\ \mu_1/\mu_2 \\ \mu_2 \end{pmatrix} \mid S \in \mathcal{S}_3 \,\},$$

where $a \in \mathbb{R}$, $h, k \in \mathbb{R}^3$ and $\mu \in \widehat{\mathcal{I}}_3$. As in the case $d = 2$ we obtain the following characterization as a direct consequence of Theorem 3.6.

Theorem 5.2 *For $\widehat{\varphi} : \widehat{\mathcal{I}}_3 \to \mathbb{R}_\infty$ define $\varphi_{\rm inc} : \mathcal{I}_3 \to \mathbb{R}_\infty; (\mu_1, \mu_2/\mu_1, 1/\mu_2)^{\mathsf{T}} \mapsto \widehat{\varphi}(\mu)$, then Φ defined in* (5.1) *is singular–value polyconvex if and only if*

$$\widehat{\varphi}(\mu) = \sup\{\, D_{\rm inc}(a,h,k;\mu) \mid D_{\rm inc}(a,h,k;\cdot) \leq \widehat{\varphi}\,\},$$

i.e., $\widehat{\varphi}$ is a pointwise supremum of functions in $\mathbb{F}_3^{\rm inc} := \{\, D_{\rm inc}(a,h,k;\cdot)\,|\, a \in \mathbb{R},\ h,k \in \mathbb{R}^3\,\}$.

Unfortunately the set $\mathbb{F}_3^{\rm inc}$ is hard to analyze. Unlike in the two–dimensional case, we again have functions which are piecewise defined. For instance, the function $D_{\rm inc}(h,k;\cdot)$ with $h = (7,2,4)^{\mathsf{T}}$ and $k = (8,2,6)^{\mathsf{T}}$ is a piecewise rational function of μ on three different domains, viz., near $\mu = (1,1)$, near $\mu = (\rho,\rho^2)$ and near (ρ^2,ρ) for $\rho \gg 1$. We conjecture that each function $D_{\rm inc}(h,k;\cdot)$ is piecewise rational on at most three connected subdomains of $\widehat{\mathcal{I}}_3$

Proposition 5.3 *(a) Any function $D_{\rm inc}(a,h,k;\cdot)$ is nondecreasing in each μ_j.*
(b) Any nondecreasing, convex function $\widehat{\varphi} : \widehat{\mathcal{I}}_3 \to \mathbb{R}_\infty$ generates a singular–value polyconvex function Φ.
(c) If Φ in the form (5.1) *is singular–value polyconvex, then the associated function $\widehat{\varphi} : \widehat{\mathcal{I}}_3 \to \mathbb{R}_\infty$ is nondecreasing in each μ_j.*

Proof: For part (a) choose $\mu \in \widehat{\mathcal{I}}_3$ and assume $D_{\rm inc}(a,h,k;\mu) = a + Sh\cdot(\mu_1, \mu_2/\mu_1, 1/\mu_2)^{\mathsf{T}} + Sk\cdot(1/\mu_1, \mu_1/\mu_2, \mu_2)^{\mathsf{T}}$ for some $S \in \mathcal{S}_3$. Without loss of generality we may assume $S = \mathbb{I}$ (replace (h,k) by (Sh, Sk)). Since $D_{\rm inc}$ is defined as a maximum, we have results which are not larger for any other S. Choosing $S = \widehat{S}_\pm$ with $\widehat{S}_\pm e_1 = \pm e_2$, $\widehat{S}_\pm e_2 = \pm e_1$, and $\widehat{S}_\pm e_3 = e_3$ we obtain the two estimates

$$(h_1 - h_2)\tfrac{\mu_1^2 - \mu_2}{\mu_1} + (k_2 - k_1)\tfrac{\mu_1^2 - \mu_2}{\mu_1\mu_2} \geq 0, \quad (h_1 + h_2)\tfrac{\mu_1^2 + \mu_2}{\mu_1} + (k_2 + k_1)\tfrac{\mu_1^2 + \mu_2}{\mu_1\mu_2} \geq 0.$$

This implies the estimate $\mu_2 h_1 + k_2 \geq |\mu_2 h_2 + k_1| \geq 0$. With

$$\begin{aligned} \partial_{\mu_1} D_{\rm inc}(h,k;\mu) &= h_1 + k_2/\mu_2 - (k_1 + h_2\mu_2)/\mu_1^2 \\ &\geq (\mu_2 h_1 + k_2)/\mu_2 - (\mu_2 h_1 + k_2)/\mu_1^2 \;=\; \tfrac{\mu_1^2 - \mu_2}{\mu_1^2 \mu_2}(\mu_2 h_1 + k_2) \end{aligned}$$

we conclude $\partial_{\mu_1} D_{\rm inc}(h,k;\mu) \geq 0$ as desired. By symmetry we also have $\partial_{\mu_2} D_{\rm inc}(h,k;\mu) \geq 0$ and part (a) is proved.

Any nondecreasing, convex function $\widehat{\varphi}$ is the pointwise limit of functions of the form $\mu \mapsto a + g\cdot\mu$ with $g_1, g_2 \geq 0$. Thus, the assertion follows, if all these functions are contained in $\mathbb{F}_3^{\rm inc}$. However, it is easily seen that $h = (g_1, 0, 0)^{\mathsf{T}}$ and $k = (0, 0, g_2)^{\mathsf{T}}$ gives $D_{\rm inc}(a,h,k;\mu) = a + g\cdot\mu$. This proves part (b).

Part (c) is a simple consequence of part (a) and Theorem 5.2, since a pointwise supremum of monotone functions is again montone. ∎

The result in part (b) is well–known and we added it here just for completeness. The following examples shows that in general singular–value polyconvexity of Φ does not imply convexity of $\widehat{\varphi}$.

Example 5.4 For $\kappa > 1$ consider the function

$$\widehat{\varphi}_\kappa(\mu) = \max\{\kappa\mu_1 - 1/\mu_1, \mu_2 - \kappa/\mu_2\} = D_{\rm inc}((\kappa,0,0)^\mathsf{T}, (-1,0,0)^\mathsf{T}; \mu),$$

which defines a singular–value polyconvex function Φ via (5.1).

Clearly, $\widehat{\varphi}_\kappa$ is nonconvex, in fact, $\mathrm{D}^2\widehat{\varphi}_\kappa$ is negative semi–definite. Out of this example we may construct a smooth nonconvex function by choosing a function $\chi \in \mathrm{C}^k(\mathbb{R})$ such that $\mathrm{supp}\chi \in [2,4]$, $\chi \geq 0$, $\int_\mathbb{R} \chi(\kappa)\,\mathrm{d}\kappa = 1$ and $\int_\mathbb{R} \kappa\chi(\kappa)\,\mathrm{d}\kappa = 3$. Then, the function

$$\widehat{\varphi}(\mu) = \int_\mathbb{R} \widehat{\varphi}_\kappa(\mu)\,\mathrm{d}\kappa = X(\mu_2/\mu_1)(\mu_2+1/\mu_1) - G(\mu_2/\mu_1)(\mu_1+1/\mu_2) + 3\mu_1 - 1/\mu_1,$$

with $X(t) = \int_0^t \chi(\kappa)\,\mathrm{d}\kappa$ and $G(t) = \int_0^t \kappa\chi(\kappa)\,\mathrm{d}\kappa$, again defines a smooth, nonconvex singular–value polyconvex function $\Phi : \mathcal{V}_3 \to \mathbb{R}_\infty$.

A Co–positivity and positive semi–definiteness

The aim of this appendix is to prove the following result.

Lemma A.1 *Let $H \in \mathbb{R}^{3\times 3}$ be symmetric and positive semi–definite. Let $r^{(\pm 1)} \in \mathbb{R}^3$ be two vectors with $r_j^{(-1)} \leq r_j^{(1)}$ and define $\mathcal{R} = [r^{(-1)}, r^{(1)}]$. Then, the assertions (i) and (ii) are equivalent.*
(i) For all $\varepsilon \in \{-1,1\}^3$ the symmetric matrix $\mathrm{diag}\,\varepsilon\,[H - B(\widehat{r}^\varepsilon)]\,\mathrm{diag}\,\varepsilon$ is co–positive, where

$$B(r) = \begin{pmatrix} 0 & r_3 & r_2 \\ r_3 & 0 & r_1 \\ r_2 & r_1 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{r}^\varepsilon = (r_1^{(-\varepsilon_2\varepsilon_3)}, r_2^{(-\varepsilon_1\varepsilon_3)}, r_3^{(-\varepsilon_1\varepsilon_2)})^\mathsf{T}.$$

(ii) There exists $r \in \mathcal{R}$ such that $H - B(r)$ is positive semi–definite.

Proof: By continuity arguments it is sufficient to prove the result for matrices H which are strictly positive definite. By shifting r by $(H_{32}, H_{13}, H_{12})^\mathsf{T}$ we may assume that H is a diagonal matrix. Replacing r_j by $r_j\sqrt{H_{j+1,j+1}H_{j+2,j+2}}$ we may further assume that H is the identity matrix $\mathbb{I}$. The two sets

$$\mathcal{T} := \{\, r \in \mathbb{R}^3 \mid \mathbb{I} - B(r) \text{ positive semi–definite} \,\},\ \mathcal{C} := \{\, r \in \mathbb{R}^3 \mid \mathbb{I} - B(r) \text{ co–positive } \}$$

have the explicit characterizations

$$\begin{aligned} \mathcal{T} &= \{\, r \in \mathbb{R}^3 \mid |r_j| \leq 1,\ |r|^2 + 2r_1r_2r_3 \leq 1 \,\}, \\ \mathcal{C} &= \{\, r \in \mathbb{R}^3 \mid r_1, r_2, r_3 \leq 1 \text{ and } \big(r_1+r_2+r_3 \leq 1 \text{ or } r \in \mathcal{T}\big) \,\}. \end{aligned}$$

From this it is clear that (ii) implies (i).

To show that (i) implies (ii), we first note that $\mathcal{T}$ is invariant under the rotations by the angle π around each of the three coordinate axes. Moreover, $\mathcal{C}$ consists exactly of those points r which satisfy $r_j \leq r_j^\mathcal{T}$ for some $r^\mathcal{T} \in \mathcal{T}$. Condition (i) reads

$$r^{(-1)} = \begin{pmatrix} r_1^{(-1)} \\ r_2^{(-1)} \\ r_3^{(-1)} \end{pmatrix}, \begin{pmatrix} r_1^{(-1)} \\ -r_2^{(1)} \\ -r_3^{(1)} \end{pmatrix}, \begin{pmatrix} -r_1^{(1)} \\ r_2^{(-1)} \\ -r_3^{(1)} \end{pmatrix}, \begin{pmatrix} -r_1^{(1)} \\ -r_2^{(1)} \\ r_3^{(-1)} \end{pmatrix} \in \mathcal{C}. \tag{A.1}$$

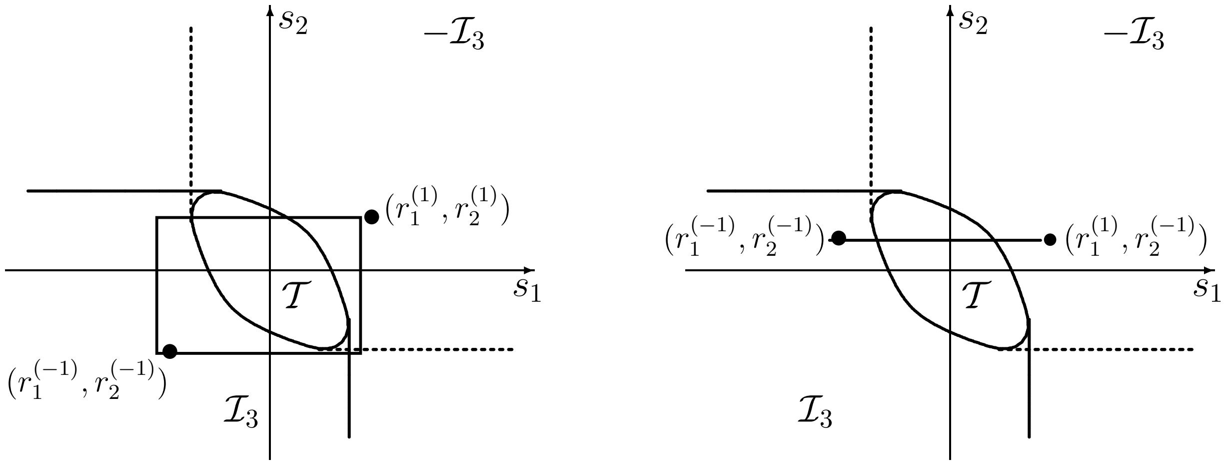

Figure 2: Intersections of the box $\mathcal{R}$ and $\mathcal{T}$: (a) the case $r_1^{(-1)} \leq r_2^{(-1)} \leq -1 < r_3^{(-1)} \leq 1$, (b) the case $r_1^{(-1)} \leq -1 < r_2^{(-1)} \leq r_3^{(-1)} \leq 1$

Since $\mathcal{C}$ lies in $\{\, r \mid r_j \leq 1 \,\}$ we conclude $r_j^{(1)} \geq -1$.

We have to show that $\mathcal{R} \cap \mathcal{T}$ is nonempty. To construct $r \in \mathcal{R} \cap \mathcal{T}$ we first observe that we may renumber the coordinates such that $r_1^{(-1)} \leq r_2^{(-1)} \leq r_3^{(-1)} \leq 1$ by using the symmetry under permutations. We now consider three cases.

Case $r_3^{(-1)} \leq -1$:
Since we always have $r_j^{(1)} \geq -1$ we immediately conclude $(-1,-1,-1)^\mathsf{T} \in \mathcal{R} \cap \mathcal{T}$.

Case $r_2^{(-1)} \leq -1 < r_3^{(-1)} \leq 1$:
Define the set $\mathcal{I}_3 = \{\, (s_1, s_2) \in \mathbb{R}^2 \mid (s_1, s_2, r_3^{(-1)})^\mathsf{T} \in \mathcal{C} \,\} \subset (-\infty, 1]^2$. Then, (A.1) implies $(r_1^{(1)}, r_2^{(1)}) \in -\mathcal{I}_3$. Clearly, rectangle $[(r_1^{(-1)}, r_2^{(-1)}), (r_1^{(1)}, r_2^{(1)})]$, which corresponds to the face of the box $\mathcal{R}$ with $r_3 = r_3^{(-1)}$, intersects $\mathcal{T} \cap \{r_3 = r_3^{(-1)}\}$, see Figure 2(a). Thus, $\mathcal{R} \cap \mathcal{T} \neq \emptyset$.

Case $-1 < r_2^{(-1)} \leq r_3^{(-1)} \leq 1$:
As in the above case we have $(r_1^{(-1)}, r_2^{(-1)}) \in \mathcal{I}_3$ and $(r_1^{(1)}, r_2^{(1)}) \in -\mathcal{I}_3$. Now the line segment connecting $(r_1^{(-1)}, r_2^{(-1)})$ and $(r_1^{(1)}, r_2^{(-1)})$, which corresponds to an edge of the box $\mathcal{R}$, intersects $\mathcal{T}$. Note therefore, that the part of the boundary of $-\mathcal{I}_3$ which lies on $s_1 = -1$ meets the boundary of $\mathcal{T}$ (the ellipse) at $s_2 = r_3^{(-1)} \geq r_2^{(-1)}$, see Figure 2(b). ∎

B Polyconvexity for Example 4.2

In this appendix we prove that the function $\phi : \mathcal{V}_2 \to \mathbb{R}$ defined via

$$\phi(\nu) = \begin{cases} \frac{2}{\alpha}\sqrt{\nu_1^\alpha(\nu_2^\alpha + b)} & \text{for } \nu_1 \geq \widehat{\nu}_2, \\ \frac{1}{\alpha}(\nu_1^\alpha + \nu_2^\alpha + b) & \text{for } \nu_1 \in [\nu_2, \widehat{\nu}_2], \end{cases} \qquad \text{where } \widehat{\nu}_2 = (\nu_2^\alpha + b)^{1/\alpha},$$

is singular–value polyconvex for all $\alpha \geq 2$.

According to Theorem 4.1 we need to specify $c = c(\gamma)$ such that (4.1) holds. We first consider the case $\gamma_1 \geq \widehat{\gamma}_2$. We define the function

$$\widehat{\phi}(\nu) = \tfrac{2}{\alpha}\sqrt{\nu_1^\alpha(\nu_2^\alpha{+}b)} = \tfrac{2}{\alpha}(\nu_1\widehat{\nu}_2)^{\alpha/2} \quad \text{and} \quad \Psi(\nu_1,\delta) = \tfrac{2}{\alpha}\sqrt{\delta^\alpha{+}b\nu_1^\alpha}.$$

Then, for $\nu \in \mathcal{V}_2$ we have $\phi(\nu) \geq \widehat{\phi}(\nu)$ and for $\nu_1 \geq \widehat{\nu}_2$ we have $\phi(\nu) = \widehat{\phi}(\nu) = \Psi(\nu_1, \nu_1\nu_2)$. Since for $\alpha \geq 2$ Ψ is convex, we find for all ν (with $\delta_\gamma = \gamma_1\gamma_2$ and $\delta_\nu = \nu_1\nu_2$)

$$\begin{aligned}\phi(\nu) \geq \widehat{\phi}(\nu) \;&=\; \Psi(\nu_1,\delta_\nu)\\ &\geq \Psi(\gamma_1,\delta_\gamma) + \mathrm{D}\Psi(\gamma_1,\delta_\gamma)\cdot\binom{\nu_1-\gamma_1}{\delta_\nu-\delta_\gamma}\\ &= \phi(\gamma) + \mathrm{D}\phi(\gamma)\cdot(\nu-\gamma) + \widehat{c}\,(\nu_1-\gamma_1)(\nu_2-\gamma_2),\end{aligned}$$

where $\widehat{c} = \gamma_1^{\alpha/2-1}\gamma_2^{\alpha-1}/\widehat{\gamma}_2^{\alpha/2}$. This proves the estimate in (4.1), and the restriction on c is easily obtained with

$$f = \mathrm{D}\phi(\gamma) = (\gamma_1^{\alpha/2-1}\widehat{\gamma}_2^{\alpha/2}, \gamma_1^{\alpha/2}\gamma_2^{\alpha-1}/\widehat{\gamma}_2^{\alpha/2})^\mathsf{T}$$

and $\gamma_2 \leq \widehat{\gamma}_2$.

The case $\gamma \in \mathcal{H} := \{\, \gamma \in \mathcal{V}_2 \mid \gamma_2 \leq \gamma_1 \leq \widehat{\gamma}_2 \,\}$ is more involved. Consider $\rho(t) = t^\alpha/\alpha$, then $\rho(t) \geq \rho(s) + \rho'(s)(t{-}s) + \frac{\alpha-1}{\alpha}s^{\alpha-2}(t{-}s)^2$ for all $t,s \geq 0$. Hence, for $\nu \in \mathcal{H}$ we find

$$\begin{aligned}\phi(\nu) - \phi(\gamma) - \mathrm{D}\phi(\gamma)\cdot(\nu{-}\gamma) &\geq \tfrac{\alpha-1}{\alpha}\big(\gamma_1^{\alpha-2}(\nu_1{-}\gamma_1)^2 + \gamma_2^{\alpha-2}(\nu_2{-}\gamma_2)^2\big)\\ &\geq 2\tfrac{\alpha-1}{\alpha}(\gamma_1\gamma_2)^{\alpha/2-1}\big|(\nu_1{-}\gamma_1)(\nu_2{-}\gamma_2)\big|.\end{aligned}$$

Hence, for $\nu \in \mathcal{H}$ we can choose any c with $|c| \leq c_\mathcal{H} = 2\frac{\alpha-1}{\alpha}(\gamma_1\gamma_2)^{\alpha/2-1}$. We show now that the choice $c = \gamma_2^{\alpha-1}/\gamma_1 \leq c_\mathcal{H}$ fulfills (4.1). With $f = \mathrm{D}\phi(\gamma) = (\gamma_1^{\alpha-1}, \gamma_2^{\alpha-1})^\mathsf{T}$, the conditions on c in (4.1) are easily checked.

It remains to establish the estimate (4.1) for $\nu \notin \mathcal{H}$. For this purpose define

$$w_\gamma(\nu) = \widehat{\phi}(\nu) - \phi(\gamma) - \mathrm{D}\phi(\gamma)\cdot(\nu{-}\gamma) - \tfrac{\gamma_2^{\alpha-1}}{\gamma_1}(\nu_1{-}\gamma_1)(\nu_2{-}\gamma_2)$$

We have to show $\min\{\, w_\gamma(\nu) \mid \nu \in \mathcal{V}_2 \setminus \mathcal{H} \,\} \geq 0$. By continuity it is sufficient to study the case $\alpha > 2$.

We first show that there is no ν with $\mathrm{D}w_\gamma(\nu) = 0$. From

$$0 = \partial_{\nu_2} w_\gamma(\nu) = \nu_1^{\alpha/2}\frac{\nu_2^{\alpha-1}}{\widehat{\nu}_2^{\alpha/2}} - \nu_1\frac{\gamma_2^{\alpha-1}}{\gamma_1}$$

we obtain the curve $\nu_1 = N(\nu_2)$ defined via $\nu_1^{\alpha/2-1} = \frac{\widehat{\nu}_2^{\alpha/2}}{\nu_2^{\alpha-1}}\frac{\gamma_2^{\alpha-1}}{\gamma_1}$. Next consider

$$0 = \partial_{\nu_2} w_\gamma(N(\nu_2),\nu_2) = \tfrac{\gamma_2^{\alpha-1}}{\gamma_1}\Big(\tfrac{b}{\nu_2^{\alpha-1}} - \gamma_2\big[(\gamma_1/\gamma_2)^\alpha - 1\big]\Big),$$

which shows that $\mathrm{D}w_\gamma(\nu) = 0$ has the unique solution

$$\nu^* = (N(\nu_2^*), \nu_2^*) \quad \text{with} \quad \nu_2^* = \gamma_2\Big(\tfrac{b}{\gamma_1^\alpha - \gamma_2^\alpha}\Big)^{1/(\alpha-1)} \geq \gamma_2.$$

To see that $\nu^* \notin \mathcal{V}_2 \setminus \mathcal{H}$ we note

$$\partial_{\nu_2} w_\gamma(\widehat{\nu}_2^*, \nu_2^*) = (\nu_2^*)^{\alpha-1} - \tfrac{\gamma_2^{\alpha-1}}{\gamma_1}\widehat{\nu}_2^* = \gamma_2^{\alpha-1}\Big(\tfrac{b}{\gamma_1^\alpha-\gamma_2^\alpha} - \tfrac{\gamma_2}{\gamma_1}\Big(\tfrac{b}{\gamma_1^\alpha-\gamma_2^\alpha}\Big)^{1/(\alpha-1)}\Big) \geq 0.$$

The curve $\partial_{\nu_2} w_\gamma(\nu) = 0$ intersects the boundary of $\mathcal{H}$ only once at $(\widehat{\nu}_2^\circ, \nu_2^\circ)$ and $\partial_{\nu_2} w_\gamma(\widehat{\nu}_2, \nu_2)$ is positive for large ν_2. Hence, $\nu_2^\circ \le \nu_2^*$ and with $N'(\nu_2) < 0$ we conclude $\nu_1^* = N(\nu_2^*) \le \widehat{\nu}_2^*$, as desired.

Since we have established that w_γ doesn't have a critical point in the interior the minimum must be attained at the boundary. For $|\nu| \to \infty$ we find $w_\gamma(\nu) \to +\infty$ since we have a lower bound $c_1|\nu|^{\alpha/2} - C_2$. At $\nu_2 = 0$ we have $\partial_{\nu_2} w_\gamma(\nu_1, 0) < 0$ which implies that the minimum is attained along the curve $\{\, (\widehat{\nu}_2, \nu_2) \mid \nu_2 > 0 \,\}$. However, this curve is part of the boundary of $\mathcal{H}$ and the desired estimate $w_\gamma(\nu) \ge 0$ for $\nu \in \mathcal{H}$ was already established above.

This concludes the proof that the function ϕ is singular–value polyconvex.

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