Energetic formulation of multiplicative elasto–plasticity using dissipation distances*

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Abstract

We introduce a new energetic formulation for the inelastic behavior of standard generalized materials. This formulation is solely based on the classical elastic energy–storage potential $\widehat{\psi}$ and a dissipation potential $\widehat{\Delta}$, and it replaces the classical variational inequalities which describe the flow rules for the inelastic variables like the plastic deformation and the hardening parameters.

The energetic formulation has the major advantage that it is defined for a larger class of processes since it does not involve any derivatives of the strains or the internal variables, thus allowing for an analysis of processes involving sharp interfaces, localization or microstructure. Two new quantities are derived from $\widehat{\psi}$ and $\widehat{\Delta}$. First, this is the global dissipation distance $\widehat{D}$ on the manifold of internal states. Second, the reduced stored–energy density Ψ^{red} contains the comprised information of the elastic and plastic material properties via minimization of $\widehat{\psi}+\widehat{D}$ over the new internal variable. Several stability concept are derived and used to analyze failure mechanism. Finally, a natural incremental method is proposed which reduces to a minimization problem and can be solved efficiently using Ψ^{red}.

1 Introduction

The theory of elasto–plasticity can be seen as a special case of the general theory of inelastic material behavior. In this work we will restrict ourselves to its simplest setting which is the framework of "standard generalized materials" (cf. [Ric71, HN75, Mor83, GNS83, ZW87, Hac97]). For this class we derive a new energetical formulation which allows for an easier modeling and simulation of rate–independent evolution of elastic and inelastic material behavior in the context of full boundary value problems. Our approach is based on previous work for phase transformation problems in [MT99, MTL02] and for elasto–plasticity in [Mie02a, CHM02].

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The theory of elasto–plasticity at finite strain has undergone a rapid development over the last few decades, see for instance [Lee69, Man73, Lub73, TS76, SO85, Sim88, MS92, Mau92, Mie96, OS99, MSS99, HH01, Nef00, Nef01]. All this work was developed to describe sufficiently smooth processes where rates of the internal variables are well defined. However, recently also processes are studied where discontinuities in space and time are of interest. In particular, this is the case when localization in shear bands occur, when microstructure develops or when phase transformations are relevant. To describe such processes it is useful to find a formulation of the process which avoids derivatives in space and time and takes care of discontinuities automatically.

Our framework is based entirely on two energetic functionals, one being the total elastic plus potential energy and one being the dissipation distance between two internal states. The solution processes are then characterized by a stability criterion (S) with respect to perturbations of the deformation and the internal variables and by the energy inequality (E). A first advantage of the energetic formulation (S) & (E) is that it is totally derivative free: neither the constitutive functions for the solution (strains and internal variables) need to be differentiated. Another advantage arises from the fact that everything is based on functionals defined via integration over the body; hence we can use the highly developed mathematical tools in the calculus of variations such as homogenization and relaxation techniques. Texture and microstructure evolution can be understood much better in this setting, cf. the follower paper [Mie02c]. A further advantage is the natural appearance of incremental problems which have the form of minimization problems. Such variational update algorithms were developed independently in [OR99, MSL01, ML01] and proved to be very successful. We compare our incremental algorithm with this one in the conclusions.

The theory is based on the elastic potential ψ and the dissipation potential Δ as the underlying constitutive functions (cf. [ZW87]):

$$\psi = \widehat{\psi}(\mathbf{F}, \mathbf{P}, p) \text{ and } \Delta = \widehat{\Delta}(\mathbf{P}, p, \dot{\mathbf{P}}, \dot{p}) \geq 0$$

where $\mathbf{F} = \mathrm{D}\boldsymbol{\varphi} = \mathbf{F}_{\mathrm{el}}\mathbf{F}_{\mathrm{pl}}$ is the deformation gradient, $\mathbf{P} = \mathbf{F}_{\mathrm{pl}}^{-1}$ the plastic tensor and $p \in \mathbb{R}^m$ contains the hardening parameters. The plastic tensor $\mathbf{P}$ is usually assumed to have determinant 1, i.e. $\mathbf{P}$ is element of the special linear group $\mathrm{SL}(d) = \{ \mathbf{P} \in \mathbb{R}^{d\times d} \mid \det \mathbf{P} = 1 \}$. (In most of this work we will use this assumption, however the theory work for any subgroup of $\mathrm{GL}_+(d)$, see Section 3.) Consequently, $\widehat{\Delta}$ is defined on the tangent bundle of the manifold $\mathrm{SL}(d)\times\mathbb{R}^m$. The multiplicative decomposition or equivalently the axiom of plastic indifference means

$$\widehat{\psi}(\mathbf{F}, \mathbf{P}, p) = \widetilde{\psi}(\mathbf{F}\mathbf{P}, p), \qquad \widehat{\Delta}(\mathbf{P}, p, \dot{\mathbf{P}}, p) = \widetilde{\Delta}(p, \mathbf{P}^{-1}\dot{\mathbf{P}}, \dot{p}).$$

This implies that the underlying mathematical structure is that of the Lie group $\mathrm{SL}(d)$; however, we will not emphasize this aspect here and refer the reader to [Mie02a, HMM02] for more details. Rate–independency is expressed in the fact, that $\widehat{\Delta}$ is homogeneous of degree 1 in the rate $(\dot{\mathbf{P}}, \dot{p})$, see (A3) in Section 3.1.

The function $\widehat{\Delta}$ can be understood as an infinitesimal metric on $\mathrm{SL}(d)\times\mathbb{R}^m$ which defines a (global) distance $\widehat{D}$, called dissipation distance in the sequel:

$$\begin{aligned}\widehat{D}((\mathbf{P}_0,p_0),(\mathbf{P}_1,p_1)) = \inf\{ \textstyle\int_0^1 \widehat{\Delta}(\mathbf{P}(s),p(s),\dot{\mathbf{P}}(s),\dot{p}(s))\,\mathrm{d}s \mid \\ (\mathbf{P},p)\in \mathrm{C}^1([0,1],\mathrm{SL}(d)\times\mathbb{R}^m), \\ (\mathbf{P}(j),p(j))=(\mathbf{P}_j,p_j) \text{ for } j=0,1\,\}\end{aligned}$$

Consider now a body $\Omega\subset\mathbb{R}^d$, a deformation $\boldsymbol{\varphi}:\Omega\mapsto\mathbb{R}^d$ as well as internal states $(\mathbf{P}_j,p_j):\Omega\mapsto\mathrm{SL}(d)\times\mathbb{R}^n$, then integration over Ω gives the total energies

$$\begin{aligned}\mathcal{E}(t,\boldsymbol{\varphi},\mathbf{P},p) &= \textstyle\int_\Omega \widehat{\psi}(\mathrm{D}\boldsymbol{\varphi}(x),\mathbf{P}(x),p(x))\,\mathrm{d}x - \langle \ell(t),\boldsymbol{\varphi}\rangle,\\ \mathcal{D}((\mathbf{P}_0,p_0),(\mathbf{P}_1,p_1)) &= \textstyle\int_\Omega \widehat{D}((\mathbf{P}_0(x),p_0(x)),(\mathbf{P}_1(x),p_1(x)))\,\mathrm{d}x,\end{aligned}$$

where $\ell(t)$ denotes the external loading depending on the process time $t\in[0,T]$. Here, $\mathcal{E}(t,\boldsymbol{\varphi},\mathbf{P},p)$ is the Gibbs' energy at time t associated with the state $(\boldsymbol{\varphi},\mathbf{P},p):\Omega\to\mathbb{R}^d\times\mathrm{SL}(d)\times\mathbb{R}^n$, and $\mathcal{D}((\mathbf{P}_0,p_0),(\mathbf{P}_1,p_1))$ is the minimal amount of dissipation occuring when the internal state $(\mathbf{P}_0,p_0)$ is changed continuously into $(\mathbf{P}_1,p_1)$.

A triple $(\boldsymbol{\varphi},\mathbf{P},p):[0,T]\times\Omega\mapsto\mathbb{R}^d\times\mathrm{SL}(d)\times\mathbb{R}^m$ is called a **solution process** if it satisfies the following stability condition (S) and the energy inequality (E):

(S) [**Stability**] For all $t\in[0,T]$ and all comparison states $(\widetilde{\boldsymbol{\varphi}},\widetilde{\mathbf{P}},\widetilde{p})$ we have

$$\mathcal{E}(t,\boldsymbol{\varphi}(t),\mathbf{P}(t),p(t)) \le \mathcal{E}(t,\widetilde{\boldsymbol{\varphi}},\widetilde{\mathbf{P}},\widetilde{p}) + \mathcal{D}((\mathbf{P}(t),p(t)),(\widetilde{\mathbf{P}},\widetilde{p})).$$

(E) [**Energy inequality**] For all $0\le t_1<t_2\le T$ we have

$$\begin{aligned}&\mathcal{E}(t_2,\boldsymbol{\varphi}(t_2),\mathbf{P}(t_2),p(t_2)) + \mathrm{Diss}((\mathbf{P},p);[t_1,t_2])\\ &\le \mathcal{E}(t_1,\boldsymbol{\varphi}(t_1),\mathbf{P}(t_1),p(t_1)) - \textstyle\int_{t_1}^{t_2}\langle\dot{\ell}(s),\boldsymbol{\varphi}(s)\rangle\,\mathrm{d}s.\end{aligned}$$

Here the dissipation $\mathrm{Diss}((\mathbf{P},p);[t_1,t_2])$ is defined via $\mathcal{D}$ for any process $(\mathbf{P},p):[0,T]\times\Omega\mapsto\mathrm{SL}(d)\times\mathbb{R}^m$ and coincides with $\int_{t_1}^{t_2}\int_\Omega\widehat{\Delta}(\mathbf{P},p,\dot{\mathbf{P}},\dot{p})\,\mathrm{d}x\,\mathrm{d}s$ for smooth processes.

Note that (S) & (E) characterize the process completely and that the formulation does neither involve derivatives of $\mathbf{F}=\mathrm{D}\boldsymbol{\varphi},\mathbf{P}$ and p with respect to t or x nor derivatives of the constitutive functions $\widehat{\psi}$ and $\widehat{\Delta}$. It is shown in Section 3.2 that the present formulation is consistent with the classical elasto–plastic flow rules if the solution of (S) & (E) is sufficiently smooth. In fact, using the Legendre transform there is a one–to–one correspondence between $\widehat{\Delta}$ and an associative flow rule for a suitable yield surface.

A central rôle in this formulation is played by the reduced energy density

$$\Psi^{\mathrm{red}}((\mathbf{P}_{\mathrm{old}},p_{\mathrm{old}});\mathbf{F}) = \min_{(\mathbf{P}_{\mathrm{new}},p_{\mathrm{new}})} \Psi(\mathbf{F},\mathbf{P}_{\mathrm{new}},p_{\mathrm{new}}) + \widehat{D}((\mathbf{P}_{\mathrm{old}},p_{\mathrm{old}}),(\mathbf{P}_{\mathrm{new}},p_{\mathrm{new}})),$$

which contains the condensed information on the interplay of energy storage via $\widehat{\psi}$ and energy dissipation via $\widehat{\Delta}$. For instance, for any solution process the deformation $\boldsymbol{\varphi}(t):\Omega\mapsto\mathbb{R}^d$ must be a minimizer of the reduced functional

$$\boldsymbol{\varphi}\mapsto\mathcal{E}^{\mathrm{red}}((\mathbf{P}(t),p(t));t,\boldsymbol{\varphi}) := \textstyle\int_\Omega \Psi^{\mathrm{red}}((\mathbf{P}(t,x),p(t,x));\mathrm{D}\boldsymbol{\varphi}(x))\,\mathrm{d}x - \langle\ell(t),\boldsymbol{\varphi}\rangle.$$

Hence, Ψ^{red} contains significant information on the possibility of formation of microstructure (via loss of quasiconvexity [CHM02, HH01]) or failure via fracture or localization, see Section 5.

Another important aspect is that the energetic formulation leads to a very natural incremental problem in variational form. Consider a time discretization $0 = t_0 < t_1 < \ldots < t_N = T$ and an initial state $(\mathbf{P}_0, p_0) : \Omega \mapsto \mathrm{SL}(d) \times \mathbb{R}^m$.

(IP) Incremental Problem: For $k = 1, 2, \ldots, N$ find

$$(\boldsymbol{\varphi}_k, \mathbf{P}_k, p_k) \in \underset{(\boldsymbol{\varphi}, \mathbf{P}, p)}{\operatorname{argmin}} \; \mathcal{E}(t_k, \boldsymbol{\varphi}, \mathbf{P}, p) + \mathcal{D}((\mathbf{P}_{k-1}, p_{k-1}), (\mathbf{P}, p)),$$

where $\operatorname{argmin} \mathcal{J}$ denotes the set of minimizers of $\mathcal{J}$. The suitability of (IP) is manifested in Theorem 2.3 where it is shown that solutions of (IP) are always stable and satisfy a discretized version of the energy inequality (E). Here, it is essential that (IP) contains the (nonlinear) dissipation distance $\mathcal{D}$ and not its linear approximation via $\widehat{\Delta}$. Moreover, the reduced energy density Ψ^{red} can be used to solve (IP) efficiently, since $\boldsymbol{\varphi}_k$ must be a minimizer of $\mathcal{E}^{\mathrm{red}}((\mathbf{P}_{k-1}, p_{k-1}); t_k, \cdot)$.

Since our theory is based strongly on the dissipation distance $\widehat{D}$ we give a few examples in Section 3.4. We discuss the isotropic von Mises plasticity with hardening using the results without hardening from [Mie02a]. Moreover, we show that the model for single–crystal plasticity in [OR99, Gur00] can be put into this form. A one–dimensional example is treated in more detail in Section 5 to highlight some of the effects of the theory. In particular, we work out the differences between the local flow rule and the energetic formulation when discontinuous solution occur. In Section 6 we discuss the issue of global versus local minimization in the incremental problem (IP).

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2 Basic modeling of rate–independent processes

In this section we consider general materials which have elastic as well as inelastic behavior, the latter one being described by an internal variable z. In Section 3 we will specialize the situation to elasto–plasticity by choosing suitable z. It will turn out that our model captures exactly the class of generalized standard media, cf. [HN75, Hac97].

2.1 Elastic behavior

We consider a body $\Omega \subset \mathbb{R}^d$ which undergoes a deformation $\boldsymbol{\varphi} : \Omega \mapsto \mathbb{R}^d$ such that the deformation gradient $\mathbf{F}(x) := \mathrm{D}\boldsymbol{\varphi}(x) \in \mathbb{R}^{d \times d}$ lies in $\mathrm{GL}_+(d) = \{\, \mathbf{F} \in \mathbb{R}^{d \times d} \mid \det \mathbf{F} > 0 \,\}$. Additionally there are internal variables $z = (z^1, \ldots, z^n) \in Z \subset \mathbb{R}^n$ where Z denotes a manifold like, for instance, $\mathrm{SL}(d) \times \mathbb{R}^m \subset \mathbb{R}^{d \times d} \times \mathbb{R}^m$ in elasto–plasticity. The elastic properties are described via the elastic potential ψ which is a function of the material point $x \in \Omega$ and $(\mathbf{F}, z)$:

$$\psi = \widehat{\psi}(x, \mathbf{F}, z).$$

For fixed (x, z) we assume coercivity of $\widehat{\psi}(x, \cdot, z)$:

$$\widehat{\psi}(x, \mathbf{F}, z) \to \infty \quad \text{for } (\det \mathbf{F})^{-1} + |\mathbf{F}| \to \infty \quad \text{and } \widehat{\psi}(x, \mathbf{F}, z) = \infty \text{ for } \det \mathbf{F} \leq 0. \tag{2.1}$$

In addition, we describe boundary conditions and external loadings as follows. Assume that the boundary $\partial\Omega$ of Ω decomposes into a Dirichlet and a Neumann part: $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}}$. For simplicity, we consider time–independent Dirichlet data $\boldsymbol{\varphi}_{\mathrm{Dir}} : \Gamma_{\mathrm{Dir}} \mapsto \mathbb{R}^d$. The volume forces $\mathbf{f}_{\mathrm{vol}} : [0, T] \times \Omega \mapsto \mathbb{R}^d$ and the boundary tractions $\mathbf{f}_{\mathrm{tract}} : [0, T] \times \Gamma_{\mathrm{Neu}} \mapsto \mathbb{R}^d$ are assumed to be continuous. We define the total elastic and potential energy of a given state $(\boldsymbol{\varphi}, z) : \Omega \mapsto \mathbb{R}^d \times Z$ at time t is the Gibbs' energy

$$\mathcal{E}(t, \boldsymbol{\varphi}, z) := \int_\Omega \widehat{\psi}(x, \mathrm{D}\boldsymbol{\varphi}(x), z(x))\, \mathrm{d}x - \langle \ell(t), \boldsymbol{\varphi} \rangle$$

with $\langle \ell(t), \boldsymbol{\varphi} \rangle = \int_\Omega \mathbf{f}_{\mathrm{vol}}(t, x) \cdot \boldsymbol{\varphi}(x)\, \mathrm{d}x + \int_{\Gamma_{\mathrm{Neu}}} \mathbf{f}_{\mathrm{tract}}(t, y) \cdot \boldsymbol{\varphi}(y)\, \mathrm{d}a(y)$.

A basic assumption of our rate–independent model will be that a solution process has to be in a stable elastic equilibrium for all $t \in [0, T]$, which implies that $\boldsymbol{\varphi}(t, \cdot) : \Omega \mapsto \mathbb{R}^d$ is a (global) minimizer of $\mathcal{E}(t, \cdot, z(t, \cdot))$ on the set $\mathcal{F}$ of admissible deformations

$$\mathcal{F} := \{\, \boldsymbol{\varphi} : \Omega \mapsto \mathbb{R}^d \mid \boldsymbol{\varphi}|_{\Gamma_{\mathrm{Dir}}} = \boldsymbol{\varphi}_{\mathrm{Dir}}, \mathrm{D}\boldsymbol{\varphi}(x) \in \mathrm{GL}_+(d) \text{ on } \Omega \,\}.$$

The set of internal states z is denoted by

$$\mathcal{Z} := \{\, z : \Omega \mapsto Z \mid z \text{ measurable } \}.$$

Throughout this work we try to use calligraphic letters like $\mathcal{E}, \mathcal{F}, \mathcal{Z}$, and $\mathcal{D}$ to denote objects which involve functions defined over the body Ω.

2.2 Inelastic internal variables and dissipation

Changes of the internal variables during a slow loading or unloading process will give rise to internal friction which dissipates energy via the rate $\dot{z}$:

$$\Delta = \widehat{\Delta}(x, z, \dot{z}) \geq 0 \text{ where } \dot{z}(t, x) = \tfrac{\partial}{\partial t} z(t, x).$$

The function $\widehat{\Delta} : \Omega \times \mathrm{T}Z \mapsto [0, \infty]$ is called a dissipation potential (cf. [ZW87]), since its derivative with respect to the rate $\dot{z}$ gives the internal frictional forces arising from changing z. We will call $\widehat{\Delta}$ also the (infinitesimal) dissipation metric, as it has the mathematical meaning of a metric on the manifold Z. Rate–independency is modeled by the assumption of homogeneity in $\dot{z}$ of degree 1, namely

$$\widehat{\Delta}(x, z, \alpha\dot{z}) = \alpha \widehat{\Delta}(x, z, \dot{z}) \text{ for } \alpha \geq 0.$$

Furthermore, we assume

$$\widehat{\Delta}(x, z, \cdot) : \mathrm{T}_z Z \mapsto [0, \infty] \text{ is convex and } \widehat{\Delta}(x, z, v) \geq c|v| \text{ for some } c > 0. \tag{2.2}$$

Considering a smooth process $z : [0,T]\times\Omega \mapsto Z$ the dissipation on an interval $[t_0, t_1]$ is

$$\mathrm{Diss}(z; [t_0, t_1]) = \int_{t_0}^{t_1} \int_\Omega \widehat{\Delta}(x, z(t,x), \dot{z}(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

For each $x \in \Omega$ the dissipation $\widehat{\Delta}(x, \cdot, \cdot)$ defines a distance metric on Z via

$$\widehat{D}(x; z_0, z_1) = \inf\{\, \int_0^1 \widehat{\Delta}(x, z(s), \dot{z}(s))\,\mathrm{d}s \mid z \in \mathrm{C}^1([0,1], Z), z(0) = z_0, z(1) = z_1 \,\}.$$

This defines also a metric on the set of internal states $\mathcal{Z}$ by setting, for $z_0, z_1 \in \mathcal{Z}$,

$$\mathcal{D}(z_0, z_1) = \int_\Omega \widehat{D}(x; z_0(x), z_1(x))\,\mathrm{d}x.$$

The distance $\mathcal{D}$ is called the dissipation distance between the internal states z_0 and z_1 in $\mathcal{Z}$. By definition the functions $\widehat{D}(x, \cdot, \cdot)$ and hence the function $\mathcal{D}$ satisfy the triangle inequality

$$\mathcal{D}(z_0, z_2) \le \mathcal{D}(z_0, z_1) + \mathcal{D}(z_1, z_2) \quad \text{for } z_0, z_1, z_2 \in \mathcal{Z}. \tag{2.3}$$

Condition (2.2) implies $\mathcal{D}(z_0, z_1) > 0$ for $z_0 \neq z_1$. However, we note that we never assume that $\widehat{\Delta}(x, z, \cdot)$ is symmetric (which would mean $\widehat{\Delta}(x, z, -v) = \widehat{\Delta}(x, z, v)$). Hence the order will matter in $\mathcal{D}$, i.e., in general $\mathcal{D}(z_0, z_1) \neq \mathcal{D}(z_1, z_0)$. For instance, such an unsymmetry is needed to describe damage or hardening in plasticity.

The dissipation along a curve $z : [0,T] \mapsto \mathcal{Z}$ can be expressed without any rate via

$$\mathrm{Diss}(z, [t_0, t_1]) = \sup \sum_{i=1}^{M} \mathcal{D}(z(\tau_{i-1}), z(\tau_i))$$

where the supremum is taken over all $M \in \mathbb{N}$ and all discretizations $t_0 \le \tau_0 < \tau_1 < \ldots < \tau_M \le t_1$. Note that $\mathcal{D}$ and $\mathrm{Diss}(z, [t_0, t_1])$ have the physical meaning of energies whereas $\int_\Omega \widehat{\Delta}(x, z(t,x), \dot{z}(t,x))\,\mathrm{d}x$ denotes a power.

2.3 Global energetic formulation of the model

Recall that our model is completely described by the two constitutive relations $\psi = \widehat{\psi}(x, \mathbf{F}, z)$ and $\Delta = \widehat{\Delta}(x, z, \dot{z})$ and by the loadings $\boldsymbol{\varphi}_{\mathrm{Dir}}$, $\mathbf{f}_{\mathrm{vol}}$ and $\mathbf{f}_{\mathrm{tract}}$.

Definition 2.1 *A process $(\boldsymbol{\varphi}, z) : [0,T] \mapsto \mathcal{F}\times\mathcal{Z}$ is called a solution of the rate–independent problem associated to $(\mathcal{F}\times\mathcal{Z}, \mathcal{E}, \mathcal{D})$, if (S) and (E) hold:*

(S) [Stability] *For all $t \in [0,T]$ we have*

$$\mathcal{E}(t, \boldsymbol{\varphi}(t), z(t)) \le \mathcal{E}(t, \widehat{\boldsymbol{\varphi}}, \widehat{z}) + \mathcal{D}(z(t), \widehat{z}) \text{ for all } (\widehat{\varphi}, \widehat{z}) \in \mathcal{F}\times\mathcal{Z}.$$

(E) [Energy inequality] *For all t_0, t_1 with $0 \le t_0 < t_1 \le T$ we have*

$$\mathcal{E}(t_1, \boldsymbol{\varphi}(t_1), z(t_1)) + \mathrm{Diss}(z, [t_0, t_1]) \le \mathcal{E}(t_0, \boldsymbol{\varphi}(t_0), z(t_0)) - \int_{t_0}^{t_1} \langle \dot{\ell}(t), \boldsymbol{\varphi}(t) \rangle\,\mathrm{d}t.$$

The stability condition (S) has a clear mechanical interpretation. Letting $\widehat{z} = z(t)$ we have $\mathcal{D}(z(t), \widehat{z}) = 0$ and the condition implies that $\boldsymbol{\varphi}(t)$ is a global minimizer of $\mathcal{E}(t, \cdot, z(t))$ on $\mathcal{F}$, which give the elastic equilibrium. However, varying $\widehat{z}$ we find that changing the internal variable must dissipate at least as much energy as the associated gain in the elastic energy would be. Thus, (S) is equivalent to a principle of maximal dissipation. The internal variable will move (and dissipate energy) as soon as the energy release in the elastic energy is large enough to compensate for the dissipation.

The energy inequality (E) has an obvious interpretation, since the modified work of the external forces is given by the last term [true work of external forces $:= \int_0^t \langle \ell(\tau), \dot{\boldsymbol{\varphi}}(\tau) \rangle \,\mathrm{d}\tau = \langle \ell(t), \dot{\boldsymbol{\varphi}}(t) \rangle - \langle \ell(0), \dot{\boldsymbol{\varphi}}(0) \rangle - \int_0^t \langle \dot{\ell}(\tau), \boldsymbol{\varphi}(\tau) \rangle \,\mathrm{d}\tau$]. Under rather general assumptions, see [MT01], it can be shown that (E) and (S) together imply that (E) holds with equality.

Closely related stability and energetic considerations were done in [FE97, FE89]. However, the characterization of the full rate–independent problem with (S) and (E) seems to be first observed in [MTL98, MT99].

This abstract setting has applications in many rate–independent continuum models. We refer to [MTL98, KMS99, GMH02, MTL02, Rou02a] for applications to phase transformations in shape–memory alloys. Applications to ferromagnetics are given in [Rou02b, RK02b, RK02a]. In [FM93, FM98, KMR02] fracture, damage and delamination is modeled on similar grounds using the crack size as an internal variable. The treatment of elasto–plasticity is the purpose of this work.

2.4 Local formulation and flow rules

Finally we want to show that (S) and (E) lead to a flow rule for the internal variable and thus gives the more common way of formulating rate–independent problems. However, we insist that the energetic formulation using (S) and (E) is much more flexible and gives additional insight which is not obvious from the (local) flow formulation. Hence, we will return to (S) and (E) soon. For establishing the equivalence of the energetic formulation and the classical flow formulation we need more smoothness of the solutions and the constitutive relations. Under these smoothness assumptions the energetic formulation gives the classical associative flow rules for generalized standard materials.

For the local formulation we define the thermodynamically conjugate variables

$$\mathbf{T} := \tfrac{\partial}{\partial \mathbf{F}} \widehat{\psi}(x, \mathbf{F}, z) \in \mathbb{R}^{d\times d} \text{ and } q := -\tfrac{\partial}{\partial z} \widehat{\psi}(x, \mathbf{F}, z) \in \mathrm{T}_z^* Z.$$

Here $\mathbf{T}$ is the first Piola–Kirchhoff stress tensor, which we shortly call stress tensor later on. The definition of q involves a minus sign as is common usage in this context and $\mathrm{T}_z^* Z$ is the dual space of $\mathrm{T}_z Z$.

Doing the local variations $\widehat{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \varepsilon \mathbf{u}$ and $\widehat{z} = z(t) + \varepsilon v$ in (S) and extracting the terms of order ε we find $\int_\Omega [\mathbf{T}{:}\mathrm{D}\mathbf{u} - q{\cdot}v + \widehat{\Delta}(z, v)] \,\mathrm{d}x - \langle \ell(t), \mathbf{u} \rangle \geq 0$. This gives the local stability criterion

$$(\mathrm{S})_{\mathrm{loc}} \qquad \begin{cases} \int_\Omega \mathbf{T} : \mathrm{D}\mathbf{u} \,\mathrm{d}x = \langle \ell(t), \mathbf{u} \rangle \text{ for all } \mathbf{u} \text{ with } \mathbf{u}|_{\Gamma_{\mathrm{Dir}}} = 0; \\ q(t,x){\cdot}v \leq \widehat{\Delta}(x, z(t,x), v) \text{ for all } v \in \mathrm{T}_{z(t,x)} Z. \end{cases}$$

The first equation is the weak form of the elastic equilibrium while the second says that the thermodynamical driving force $q(x)$ is not strong enough to overcome the threshold of dissipational friction.

Differentiating (E) with respect to time and using $(\mathrm{S})_{\mathrm{loc}}$ with $\mathbf{u} = \dot{\boldsymbol{\varphi}}$ we are left with

$$(\mathrm{E})_{\mathrm{loc}} \qquad \int_\Omega \widehat{\Delta}(x, z(t,x), \dot{z}(t,x))\,\mathrm{d}x \leq \int_\Omega q(t,x)\cdot\dot{z}(t,x)\,\mathrm{d}x.$$

Defining the set–valued subdifferential $\partial_v\widehat{\Delta}(x,z,\cdot) \subset \mathrm{T}_z^*Z$ of the convex function $\widehat{\Delta}(x,z,\cdot) : \mathrm{T}_z Z \mapsto [0,\infty]$ via

$$\partial_v\widehat{\Delta}(x,z,v) = \{\, \eta \in \mathrm{T}_z^*Z \mid \widehat{\Delta}(x,z,v{+}\widetilde{v}) \geq \widehat{\Delta}(x,z,v) + \eta\cdot\widetilde{v} \text{ for all } \widetilde{v} \in \mathrm{T}_z Z \,\},$$

we see that $(\mathrm{E})_{\mathrm{loc}}$ together with the second condition in $(\mathrm{S})_{\mathrm{loc}}$ is equivalent to

$$q(t,x) \in \partial_v\widehat{\Delta}(x, z(t,x), \dot{z}(t,x)). \tag{2.4}$$

Finally we define the elastic region (whose boundary is the yield surface) via

$$\mathbb{Q}(x,z) = \partial_v\widehat{\Delta}(x,z,0) \subset \mathrm{T}_z^*Z.$$

Using the Legendre transform, it is shown that (2.4) is equivalent to the flow rule

$$\dot{z}(t,x) \in \partial\mathcal{X}_{\mathbb{Q}(x,z(t,x))}(q(t,x)) = \mathrm{N}_{q(t,x)}\mathbb{Q}(x,z(t,x)); \tag{2.5}$$

we refer to [Mie02a, MT01] for more details. We continue to use $\mathcal{X}_A$ to denote the indicator function (of convex analysis) of a set A which satisfies $\mathcal{X}_A(a) = 0$ for $a \in A$ and ∞ else. Thus, (2.5) is the general mathematical form of flow rules which involves the (outward) normal cone $\mathrm{N}_q\mathbb{Q}$ of the elastic region. This formulation holds even in the case of multisurface plasticity.

The associative flow rule (2.5) can also be reformulated by as the equivalent variational inequality

$$\langle \dot{z}(t,x), q(t,x){-}\widehat{q}\rangle \geq 0 \quad \text{ for all } \widehat{q} \in \mathbb{Q}(x,z(t,x)).$$

If the elastic regions has a smooth boundary and is given by $\Phi(x,z,q) \leq 0$, then for $\Phi(x,z,q) = 0$ (in the yield surface) we have $\mathrm{N}_q\mathbb{Q}(x,z) = \{\, \lambda\mathrm{D}_q\Phi(x,z,q) \mid \lambda \geq 0 \,\}$ and arrive at the classical Karush–Kuhn–Tucker formulation:

$$\dot{z} = \lambda\mathrm{D}_q\Phi(z,q), \quad \lambda \geq 0,\ \Phi(z,q) \leq 0,\ \lambda\Phi(z,q) = 0.$$

Remark 2.2 *It might seem as a severe restriction that the dissipation potential Δ takes the form $\Delta = \widehat{\Delta}(x,z,\dot{z})$ and thus doesn't depend on $\mathbf{F}$ or $\dot{\mathbf{F}}$. This restriction is essential for our approach, since we need the dissipation distance $\widehat{D}$ on Z and the integrated dissipation distance $\mathcal{D}$ on $\mathcal{Z}$. However, it turns out that our assumption is equivalent to the usual assumptions for generalized standard materials ([HN75, Mau92, Hac97]) where the yield function Φ is assumed to depend only on (x,z,q) and q is the thermo–dynamically conjugate force to z. For a given yield function in the form $\Phi = \widehat{\Phi}(x,z,q)$ we can define the elastic region $\mathbb{Q}(x,z) = \{\, q \in \mathrm{T}_z^*Z \mid \widehat{\Phi}(x,z,q) \leq 0 \,\}$ and obtain $\widehat{\Delta}(x,z,\cdot)$ as inverse Legendre transform of $\mathcal{X}_{\mathbb{Q}(x,z)}$, or more directly as $\widehat{\Delta}(x,z,v) = \max_{\eta\in\mathbb{Q}(x,z)}\langle \eta, v\rangle$.*

2.5 Deformation paths minimizing the elastic work

We want to connect the above conditions (S) and (E) to the elastic work done along a deformation path. It is sufficient here to consider one material point, cf. [OR99]. For a path $(\mathbf{F}, z) : [0, T] \mapsto \mathrm{GL}_+(d) \times Z$ the elastic work is

$$\begin{aligned} W(\mathbf{F}(\cdot), z(\cdot)) &:= \int_0^T \mathbf{T}{:}\dot{\mathbf{F}}\,\mathrm{d}t = \int_0^T [\dot{\psi} + q\cdot\dot{z}]\,\mathrm{d}t = \widehat{\psi}(\mathbf{F}, z)|_0^T + \int_0^T q(t)\cdot\dot{z}(t)\,\mathrm{d}t \\ &\geq \widehat{\psi}(\mathbf{F}, z)|_0^T + \int_0^T \widehat{\Delta}(z(t), \dot{z}(t))\,\mathrm{d}t \geq \widehat{\psi}(\mathbf{F}, z)|_0^T + \widehat{D}(z(0), z(T)). \end{aligned}$$

The first equality follows from the chain rule and the definitions of $\mathbf{T}$ and q, viz., $\dot{\psi} = (\mathrm{d}/\mathrm{d}t)\widehat{\psi}(\mathbf{F}(t), z(t)) = \mathbf{T}{:}\dot{\mathbf{F}} - q\cdot\dot{z}$. The first estimate uses the energetic threshold condition $q\cdot\dot{z} \geq \widehat{\Delta}(z, \dot{z})$ from $(\mathrm{E})_{\mathrm{loc}}$ and the second estimate uses the definition of the distance $\widehat{D}$ on Z, see Section 2.2.

This lower bound on the elastic work motivates that the minimization of the elastic work with given initial data $\mathbf{F}_0$ and z_0 and given end value $\mathbf{F}_1$ can be achieved as follows. Define the reduced energy density

$$\Psi^{\mathrm{red}}(z_0; x, \mathbf{F}) = \min_{z \in Z} [\widehat{\psi}(x, \mathbf{F}, z) + \widehat{D}(x, z_0, z)] \tag{2.6}$$

and choose $z = z_1$ as a minimizer in the definition of $\Psi^{\mathrm{red}}(z_0; \mathbf{F}_1)$. Now find a curve $z : [0, T] \mapsto Z$ such that

$$\widehat{D}(z_0, z(T)) = \int_0^T \widehat{\Delta}(z(t), \dot{z}(t))\,\mathrm{d}t \quad \text{and } z(0) = z_0,\ z(T) = z_1.$$

(Such a curve is called a curve of minimal dissipation length or simply a geodesic curve with respect to the metric $\widehat{\Delta}(x, \cdot)$.) Finally $\mathbf{F}(t)$ is chosen such that $q(t)\cdot\dot{z} = \widehat{\Delta}(z(t), \dot{z}(t))$ where $q = -\mathrm{D}_z\widehat{\psi}(\mathbf{F}, z)$. This construction yields a deformation paths which minimizes the elastic work. Such paths are called *minimizing deformation paths* in [OR99].

2.6 An incremental time–stepping algorithm

One of the major advantages of the energetic formulation (S) & (E) is that it immediately gives rise to a natural incremental algorithm where each step is realized as a variational minimization problem.

We discretized the time interval $[0, T]$ via $0 = t_0 < t_1 < \ldots < t_N = T$ and give a stable initial condition $(\boldsymbol{\varphi}_0, z_0) \in \mathcal{F} \times \mathcal{Z}$:

> **Incremental Problem (IP):** For $k = 1, \ldots, N$ find iteratively $(\boldsymbol{\varphi}_k, z_k) \in \mathcal{F} \times \mathcal{Z}$ such that it minimizes the functional
>
> $$(\boldsymbol{\varphi}, z) \mapsto \mathcal{E}(t_k, \boldsymbol{\varphi}, z) + \mathcal{D}(z_{k-1}, z)$$
>
> on the set $\mathcal{F} \times \mathcal{Z}$.

Note that this is the same as minimizing the incremental energy

$$\mathcal{E}(t_k, \boldsymbol{\varphi}, z) - \mathcal{E}(t_{k-1}, \boldsymbol{\varphi}_{k-1}, z_{k-1}) + \mathcal{D}(z_{k-1}, z),$$

and thus it is the global analog of minimizing the work along a deformation path.

The fact that this incremental problem is very useful is manifested through the following result which states that incremental solutions are always stable and satisfy a discretized version of the energy inequality. In our incremental problem we have not approximated the dissipation distance $\mathcal{D}$ and that is why we obtain a rather nice stability result for the solutions of (IP). In practical applications one needs to approximate $\mathcal{D}$, and then one has to control the error on the stability condition.

Theorem 2.3 *Let $(\boldsymbol{\varphi}_k, z_k)_{k=1,\dots,N}$ be a solution of the incremental problem. Then, for $k = 1, \dots, N$ we have*
(i) stability of $(\boldsymbol{\varphi}_k, z_k)$ at time t_k, i.e.

$$\mathcal{E}(t_k, \boldsymbol{\varphi}_k, z_k) \leq \mathcal{E}(t_k, \widehat{\boldsymbol{\varphi}}, \widehat{z}) + \mathcal{D}(z_k, \widehat{z}) \text{ for all } (\widehat{\boldsymbol{\varphi}}, \widehat{z}) \in \mathcal{F}\times\mathcal{Z};$$

(ii) and the two–sided discretized energy estimate

$$\begin{aligned} \mathcal{E}(t_k, \boldsymbol{\varphi}_k, z_k) - \mathcal{E}(t_{k-1}, \boldsymbol{\varphi}_k, z_k) &= \textstyle\int_{t_{k-1}}^{t_k} \frac{\partial}{\partial t}\mathcal{E}(s, \boldsymbol{\varphi}_k, z_k)\,\mathrm{d}s \\ &\leq \mathcal{E}(t_k, \boldsymbol{\varphi}_k, z_k) - \mathcal{E}(t_{k-1}, \boldsymbol{\varphi}_{k-1}, z_{k-1}) + \mathcal{D}(z_{k-1}, z_k) \\ &\leq \textstyle\int_{t_{k-1}}^{t_k} \frac{\partial}{\partial t}\mathcal{E}(s, \boldsymbol{\varphi}_{k-1}, z_{k-1})\,\mathrm{d}s = \mathcal{E}(t_k, \boldsymbol{\varphi}_{k-1}, z_{k-1}) - \mathcal{E}(t_{k-1}, \boldsymbol{\varphi}_{k-1}, z_{k-1}). \end{aligned}$$

Proof: We give the simple arguments of the proof as they indicate the main features of the theory. To simplify the notation we introduce the shorthand $y_k := (\boldsymbol{\varphi}_k, z_k) \in \mathcal{F}\times\mathcal{Z} =: \mathcal{Y}$. The stability (i) is obtained as follows. For $\widehat{y} \in \mathcal{Y}$ we have

$$\begin{aligned} \mathcal{E}(t_k, \widehat{y}) + \mathcal{D}(z_k, \widehat{z}) &= \mathcal{E}(t_k, \widehat{y}) + \mathcal{D}(z_{k-1}, \widehat{z}) + \mathcal{D}(z_k, \widehat{z}) - \mathcal{D}(z_{k-1}, \widehat{z}) \\ &\geq \mathcal{E}(t_k, y_k) + \mathcal{D}(z_{k-1}, z_k) + \mathcal{D}(z_k, \widehat{z}) - \mathcal{D}(z_{z-k}, \widehat{z}) \geq \mathcal{E}(t_k, y_k). \end{aligned}$$

Here the first estimate uses that y_k is a global minimizer (as it solves the incremental problem) and the second estimate is the triangle inequality for the metric $\mathcal{D}$.

The lower estimate in (ii) is a consequence of the stability of y_{k-1} at time t_{k-1}:

$$\begin{aligned} \mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_k) &= \mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{E}(t_{k-1}, y_{k-1}) - \mathcal{E}(t_{k-1}, y_k) \\ &\leq \mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(z_{k-1}, z_k). \end{aligned}$$

The upper estimate in (ii) follows similarly as y_k is a minimizer at time t_k:

$$\mathcal{E}(t_k, y_k) + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, y_{k-1}) \leq \mathcal{E}(t_k, y_{k-1}) - \mathcal{E}(t_{k-1}, y_{k-1}).$$

■

Sofar we didn't make any statement on the solvability of the incremental problem (IP). In fact, for elasto–plasticity there are many occasions where the solvability of (IP) fails due to formation of microstructure in infimizing sequence, see [OR99, CHM02]. In this work we will not pursue this question further but refer to the follower paper [Mie02c]. However, we mention here already that the properties (i) & (ii) derived in the theorems are the cornerstone in establishing the convergence of the solutions of (IP) to a solution $(\boldsymbol{\varphi}, z) : [0, T] \mapsto \mathcal{F}\times\mathcal{Z}$ of (S) & (E), when the fineness $\max_{k=1,\dots N} |t_k^N - t_{k-1}^N|$ of the partitions tends to 0. In a very abstract setting such convergence results are given in [MT01]. In models for phase transformations in shape–memory alloys convergence results are given in [KM00, MTL02]. A numerical implementation and sample calculations of such a two–dimensional problem are provided in [CP01].

Finally we emphasize that the local nature of z (i.e. no gradients appear) can be used to work out the minimization in z pointwise. Recall that (IP) amounts in minimizing

$$\mathcal{E}(t_k,\boldsymbol{\varphi},z)+\mathcal{D}(t_{k-1},z)=\int_\Omega[\widehat{\psi}(x,\mathrm{D}\boldsymbol{\varphi}(x),z(x))+\widehat{D}(x,z_{k-1}(x),z(x))]\,\mathrm{d}x-\langle\ell(t_k),\boldsymbol{\varphi}\rangle.$$

This tells us that the minimization with respect to $z\in\mathcal{Z}$ can be done first and pointwise in $x\in\Omega$. To this end we use the reduced energy density Ψ^{red} as defined in (2.6):

$$\begin{array}{rcl}\mathcal{E}^{\mathrm{red}}(z_{k-1};t_k,\boldsymbol{\varphi}) & := & \min_{z\in\mathcal{Z}}[\mathcal{E}(t_k,\boldsymbol{\varphi},z)+\mathcal{D}(z_{k-1},z)] \\ & = & \int_\Omega \Psi^{\mathrm{red}}(z_{k-1}(x);x,\mathrm{D}\boldsymbol{\varphi}(x))\,\mathrm{d}x-\langle\ell(t_k),\boldsymbol{\varphi}\rangle.\end{array}$$

The new constitutive function $\Psi^{\mathrm{red}}(z;x,\mathbf{F})$ (see (2.6)) is uniquely defined by $\widehat{\psi}$ and $\widehat{\Delta}$; and it turns out that it contains all necessary information on the combined elastic and plastic behavior of the material. Moreover, there exists a mapping $\widehat{Z}_{\mathrm{new}}:\Omega\times\mathrm{GL}_+(d)\times Z\mapsto Z$ which selects a minimizer in the definition of Ψ^{red}, i.e.

$$\Psi^{\mathrm{red}}(z_{\mathrm{old}};x,\mathbf{F})=\widehat{\psi}(x,\mathbf{F},z_{\mathrm{new}})+\widehat{D}(z_{\mathrm{old}},z_{\mathrm{new}})\text{ where }z_{\mathrm{new}}=\widehat{Z}_{\mathrm{new}}(x,\mathbf{F},z_{\mathrm{old}}).\tag{2.7}$$

Thus the kth step of (IP) is solved if we find a minimizer of $\mathcal{E}^{\mathrm{red}}(z_{k-1};t_k,\,\cdot\,)$ on $\mathcal{F}$. If $\boldsymbol{\varphi}_k$ is such a minimizer, then $(\boldsymbol{\varphi}_k,z_k)$ with $z_k(x)=\widehat{Z}_{\mathrm{new}}(x,\mathrm{D}\boldsymbol{\varphi}_k(x),z_{k-1}(x))$ is one of the desired minimizer in (IP).

The minimization problem for $\mathcal{E}^{\mathrm{red}}_{z_{k-1}}(t_k,\cdot)$ has the standard form of a problem of nonlinear elasticity, where $\Psi^{\mathrm{red}}(z_{k-1}(x);x,\cdot)$ plays the role of the elastic potential. By construction we have frame indifference, i.e. $\Psi^{\mathrm{red}}(z;x,\mathbf{F})=\Psi^{\mathrm{red}}(z;x,\mathbf{RF})$ for all $\mathbf{R}\in\mathrm{SO}(d)$. However, other properties, like the coercivity in (2.1), are not so obvious; we will discuss this below. Similarly, quasi– and rank–one convexity may no longer hold for Ψ^{red}.

3 Elasto–Plasticity with hardening parameters

In this section we specify the general theory of Section 2 to the special case of elasto–plasticity within the framework of the multiplicative decomposition of the deformation gradient $\mathbf{F}=\mathrm{D}\boldsymbol{\varphi}$. We follow here the recent papers [OR99, Gur00] and [Mie02a], where the latter one introduced the dissipation metric in the context of ideal plasticity (i.e. without hardening).

3.1 Constitutive laws

Multiplicative elasto–plasticity uses the splitting

$$\mathbf{F}=\mathrm{D}\boldsymbol{\varphi}=\mathbf{F}_{\mathrm{el}}\mathbf{F}_{\mathrm{pl}},$$

where $\mathbf{F}_{\mathrm{pl}}$ is an internal variable which is assumed to be generated by movements of dislocations and is such that it maps the crystallographic lattice onto itself, in the sense that $\mathbf{F}_{\mathrm{pl}}(x)$ maps the tangent space $\mathrm{T}_x\Omega$ onto itself (which makes sense in the isotropic case as well). Only the remainder $\mathbf{F}_{\mathrm{el}} = \mathbf{F}\mathbf{F}_{\mathrm{pl}}^{-1}$ is the part which accounts for elastic energy and stresses. To simplify notation we use the plastic transformation $\mathbf{P} = \mathbf{F}_{\mathrm{pl}}^{-1}$ as an internal variable together with suitable hardening parameters $p \in \mathbb{R}^m$, i.e.,

$$z = (\mathbf{P}, p) \in \mathfrak{G}\times\mathbb{R}^m, \qquad \text{with } \mathfrak{G} \subset \mathrm{GL}_+(d).$$

Here $\mathfrak{G}$ is a Lie group (which is simply a matrix subgroup of $\mathrm{GL}_+(d)$) which might be different from model to model. Typically one chooses $\mathfrak{G} = \mathrm{SL}(d) := \{\, \mathbf{F} \mid \det \mathbf{F} = 1 \,\}$; however, other cases are possible, e.g., $\mathfrak{G} = \mathrm{GL}_+(d)$ if volumetric changes are allowed or, if only one slip system with $|\mathbf{n}| = |\mathbf{d}| = 1$ and $\mathbf{n}\cdot\mathbf{d} = 0$ is active, $\mathfrak{G} = \{\, \mathbf{1} + \alpha\mathbf{d}\otimes\mathbf{n} \,|\, \alpha \in \mathbb{R} \}$. The hardening parameters can include isotropic, kinematic or slip–strain hardening, see below.

For the constitutive function $\widehat{\psi}$ and $\widehat{\Delta}$ we now specify the associated symmetry conditions. They involve the material symmetry group $\mathfrak{S} \subset \mathrm{O}(\mathbb{R}^d) \cap \mathfrak{G}$ and they are supposed to hold for all $(x, \mathbf{F}, \mathbf{P}, p) \in \Omega\times\mathrm{GL}_+(d)\times\mathfrak{G}\times\mathbb{R}^m$:

(Sy1) **Objectivity (frame indifference):**

$$\widehat{\psi}(x, \mathbf{R}\mathbf{F}, \mathbf{P}, p) = \widehat{\psi}(x, \mathbf{F}, \mathbf{P}, p) \text{ for all } \mathbf{R} \in \mathrm{SO}(3);$$

(Sy2) **Plastic indifference:**

$$\begin{aligned} &\widehat{\psi}(x, \mathbf{F}\mathbf{G}^{-1}, \mathbf{G}\mathbf{P}, p) = \widehat{\psi}(x, \mathbf{F}, \mathbf{P}, p) \text{ and} \\ &\widehat{\Delta}(x, \mathbf{G}\mathbf{P}, p, \mathbf{G}\dot{\mathbf{P}}, \dot{p}) = \widehat{\Delta}(x, \mathbf{P}, p, \dot{\mathbf{P}}, \dot{p}) \text{ for all } \mathbf{G} \in \mathfrak{G}; \end{aligned}$$

(Sy3) **Rate independency:**

$$\widehat{\Delta}(x, \mathbf{P}, p, \alpha\dot{\mathbf{P}}, \alpha\dot{p}) = \alpha\widehat{\Delta}(x, \mathbf{P}, p, \dot{\mathbf{P}}, \dot{p}) \text{ for } \alpha \geq 0;$$

(Sy4) **Material symmetry:**

$$\begin{aligned} \widehat{\psi}(x, \mathbf{F}, \mathbf{P}\mathbf{S}, \tau_{\mathbf{S}} p) &= \widehat{\psi}(x, \mathbf{F}, \mathbf{P}, p) \text{ and} \\ \widehat{\Delta}(x, \mathbf{P}\mathbf{S}, \tau_{\mathbf{S}} p, \dot{\mathbf{P}}\mathbf{S}, \tau_{\mathbf{S}}\dot{p}) &= \widehat{\Delta}(x, \mathbf{P}, p, \dot{\mathbf{P}}, \dot{p}) \text{ for all } \mathbf{S} \in \mathfrak{S}. \end{aligned}$$

Here $\tau_{\mathbf{S}} \in \mathbb{R}^{m\times m}$ denotes a linear representation of the material symmetry group $\mathfrak{S}$ on the hardening parameters in $\mathbb{R}^m$. It satisfies $\tau_{(\mathbf{S}_1\mathbf{S}_2)} = \tau_{\mathbf{S}_2}\tau_{\mathbf{S}_1}$.

The special assumption for elasto–plasticity is the "plastic indifference" (Sy2) which leads to the multiplicative split $\mathbf{F} = \mathbf{F}_{\mathrm{el}}\mathbf{F}_{\mathrm{pl}}$ (cf. [Lee69]) as well as to the "plastically invariant" time rate $\mathbf{P}^{-1}\dot{\mathbf{P}}$ in the flow rules. We find

$$\widehat{\psi}(x, \mathbf{F}, \mathbf{P}, p) = \widetilde{\psi}(x, \mathbf{F}\mathbf{P}, p), \qquad \widehat{\Delta}(x, \mathbf{P}, p, \dot{\mathbf{P}}, p) = \widetilde{\Delta}(x, p, \mathbf{P}^{-1}\dot{\mathbf{P}}, \dot{p}). \tag{3.1}$$

In [Ber98] condition (Sy2) is called plastic *material isomorphism* between different elastic ranges. The consequences of the other symmetries will be studied along with the examples treated below.

To simplify the subsequent notation we suppress the explicit dependence of the material laws $\widehat{\psi}$ and $\widehat{\Delta}$ on the material point $x \in \Omega$ from now on. Of course, everything below is valid for general heterogeneous media.

3.2 Associative flow rules

In this section we show that the abstract flow rules of Section 2.4 lead in this context to the well–known associative flow rules of finite–strain elasto–plasticity. Note, however, that the aim of this work is not to derive such flow rules but rather to replace them by the energetic formulation (S) & (E) and the incremental problem (IP). Nevertheless, we derive the flow rules for consistency and for finding suitable dissipation potentials $\widehat{\Delta}$ for some well–established models.

We define the thermodynamically conjugate variables

$$\begin{aligned}\mathbf{Q} &= -\tfrac{\partial}{\partial \mathbf{P}}\widehat{\psi}(\mathbf{F},\mathbf{P},p) = -\mathbf{F}^{\mathsf{T}}\tfrac{\partial}{\partial \mathbf{F}_{\mathrm{el}}}\widetilde{\psi}(\mathbf{F}\mathbf{P},p) \in \mathrm{T}^*_{\mathbf{P}}\mathfrak{G} = \mathbf{P}^{-\mathsf{T}}\mathfrak{g}^* \subset \mathbb{R}^{d\times d},\\ \mathbf{q} &= -\tfrac{\partial}{\partial p}\widehat{\psi}(\mathbf{F},\mathbf{P},p) = -\tfrac{\partial}{\partial p}\widetilde{\psi}(\mathbf{F}\mathbf{P},p) \in \mathbb{R}^m_*.\end{aligned}$$

Here $\mathfrak{g}^*$ denotes the dual Lie algebra which is the set of linear mappings from $\mathfrak{g}$ into $\mathbb{R}$ and similarly $\mathbb{R}^m_*$ denotes the dual space of $\mathbb{R}^m$. For $\mathfrak{G} = \mathrm{SO}(d)$ the Lie algebra is $\mathfrak{g} = \mathfrak{sl}(R^d) = \{\, \boldsymbol{\xi} \in \mathbb{R}^{d\times d} \mid \mathrm{tr}\,\boldsymbol{\xi} = 0 \,\}$. This notation makes the distinction between the primal internal variables $(\mathbf{P},p) \in \mathfrak{G}\times\mathbb{R}^m$ and the dual (thermo–mechanically conjugate) variables $(\mathbf{P}^{\mathsf{T}}\mathbf{Q},q) \in \mathfrak{g}^*\times\mathbb{R}^m_*$ more transparent.

The elastic domain $\mathbb{Q}(\mathbf{P},p)$ associated with $\widehat{\Delta}(\mathbf{P},\mathbf{p},\cdot,\cdot)$ is the set of all thermodynamic forces $(\mathbf{Q},q)$ which are not strong enough to overcome the dissipational friction:

$$\begin{aligned}\mathbb{Q}(\mathbf{P},p) &= [\partial_{(\dot{\mathbf{P}},\dot{p})}\widehat{\Delta}(\mathbf{P},p,\cdot,\cdot)](\mathbf{0},0)\\ &= \{\, (\mathbf{Q},q) \mid \mathbf{Q}{:}\mathbf{V}{+}q{\cdot}v \leq \widehat{\Delta}(\mathbf{P},p,\mathbf{V},v) \text{ for all } (\mathbf{V},v) \in \mathrm{T}_{\mathbf{P}}\mathfrak{G}\times\mathbb{R}^m \,\}.\end{aligned}$$

Using (Sy2) and the Lie group structure, which implies $\mathrm{T}_{\mathbf{P}}\mathfrak{G} = \mathbf{P}\mathfrak{g}$, leads to

$$\begin{aligned}\mathbb{Q}(\mathbf{P},p) &= \{\, (\mathbf{Q},q) \mid \mathbf{Q}{:}(\mathbf{P}\boldsymbol{\xi}){+}q{\cdot}v \leq \widetilde{\Delta}(p,\boldsymbol{\xi},v) \text{ for all } (\boldsymbol{\xi},v) \in \mathfrak{g}\times\mathbb{R}^m \,\}\\ &= \{\, (\mathbf{Q},q) \mid (\mathbf{P}^{\mathsf{T}}\mathbf{Q},q) \in [\partial_{(\boldsymbol{\xi},\dot{p})}\widetilde{\Delta}(p,\cdot)](\mathbf{0},0) \,\} \subset \mathrm{T}^*_{\mathbf{P}}\mathfrak{G}\times\mathbb{R}^m_*.\end{aligned}$$

Defining $\widetilde{\mathbb{Q}}(p) = [\partial_{(\boldsymbol{\xi},\dot{p})}\widetilde{\Delta}(p,\cdot,\cdot)](\mathbf{0},0) \subset \mathfrak{g}^*\times\mathbb{R}^m_*$ which is equal to $\mathbb{Q}(\mathbf{1},p)$ we find

$$(\mathbf{Q},q) \in \mathbb{Q}(\mathbf{P},p) \iff (\mathbf{P}^{\mathsf{T}}\mathbf{Q},q) \in \widetilde{\mathbb{Q}}(p). \tag{3.2}$$

The flow rule (2.4) now takes the form $(\mathbf{Q},q) \in [\partial_{(\dot{\mathbf{P}},\dot{p})}\widehat{\Delta}(\mathbf{P},p,\cdot,\cdot)](\dot{\mathbf{P}},\dot{p})$. Using the above transformation by $\mathbf{P}$ this is equivalent to

$$(\mathbf{P}^{\mathsf{T}}\mathbf{Q},q) \in [\partial_{(\boldsymbol{\xi},\dot{p})}\widetilde{\Delta}(p,\cdot,\cdot)](\mathbf{P}^{-1}\dot{\mathbf{P}},\dot{p}) \subset \mathfrak{g}^*\times\mathbb{R}^m_*.$$

This is the flow rule in the thermodynamically conjugated space. Using the Legendre transform we obtain the formulation in the internal variable space which corresponds to (2.5):

$$(\mathbf{P}^{-1}\dot{\mathbf{P}},\dot{p}) \in \partial\mathcal{X}_{\widetilde{\mathbb{Q}}(p)}(\mathbf{P}^{\mathsf{T}}\mathbf{Q},q) = \mathrm{N}_{(\mathbf{P}^{\mathsf{T}}\mathbf{Q},q)}\widetilde{\mathbb{Q}}(p) \subset \mathfrak{g}\times\mathbb{R}^m.$$

this is the well–known associative flow rule of multiplicative elasto–plasticity. It contains the “plastically indifferent” plastic rate $\mathbf{P}^{-1}\dot{\mathbf{P}}$ as well as the “plastically indifferent” conjugate force $\mathbf{P}^{\mathsf{T}}\mathbf{Q} = -\mathbf{F}_{\mathrm{el}}^{\mathsf{T}}\frac{\partial}{\partial \mathbf{F}_{\mathrm{el}}}\widetilde{\psi}(\mathbf{F}_{\mathrm{el}},p)$.

3.3 Linearized elasto–plasticity

For completeness we shortly mention that linearized, rate–independent elasto–plasticity also takes the form of the abstract theory of Section 2. In contrast to plasticity with finite strains the internal plastic variable is now in a linear space. The linear model of elasto–plasticity involves a geometric linearization (infinitesimally small strains) as well as linear constitutive laws. However, the full problem is nonlinear, because of the plastic flow rule.

The linearized strain $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathrm{D}\mathbf{u}+(\mathrm{D}\mathbf{u})^\mathsf{T})$, where $\mathbf{u} : \Omega \to \mathbb{R}^d$ is the displacement, is decomposed into an elastic and a plastic part. The energy density function is quadratic:

$$\widehat{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\pi}, p) = \tfrac{1}{2}\mathbb{C}[\boldsymbol{\varepsilon}_{\mathrm{el}}]{:}\boldsymbol{\varepsilon}_{\mathrm{el}} + h(p) \quad \text{with } \boldsymbol{\varepsilon}_{\mathrm{el}} = \boldsymbol{\varepsilon}{-}\boldsymbol{\pi}.$$

Here $\mathbb{C} \in \mathrm{Lin}(\mathbb{R}^{d\times d})$ is the elasticity tensor. The internal variables are $z = (\boldsymbol{\pi}, p) \in \Sigma{\times}\mathbb{R}^m$, where $\Sigma = \{\, \boldsymbol{\pi} \in \mathbb{R}^{d\times d} \mid \boldsymbol{\pi} = \boldsymbol{\pi}^\mathsf{T},\ \mathrm{tr}\,\boldsymbol{\pi} = 0 \,\}$ and p denotes hardening parameters.

This special form gives the conjugate forces

$$\mathbf{T} := \frac{\partial\widehat{\psi}}{\partial\boldsymbol{\varepsilon}} = \mathbb{C}[\boldsymbol{\varepsilon}_{\mathrm{el}}] \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ \ \mathbf{Q} := -\frac{\partial\widehat{\psi}}{\partial\boldsymbol{\pi}} = \mathrm{dev}\,\mathbb{C}[\boldsymbol{\varepsilon}_{\mathrm{el}}] \in \Sigma^*,\ \ q := \frac{\partial\widehat{\psi}}{\partial p} = -\mathrm{D}h(p),$$

which gives $\mathbf{Q} = \mathrm{dev}\,\mathbf{T}$, where $\mathrm{dev}\,\boldsymbol{\xi} = \boldsymbol{\xi} - \frac{1}{d}(\mathrm{tr}\,\boldsymbol{\xi})\mathbf{1}$.

The simplest case with hardening has $p \in \mathbb{R}$ and $h(p) = p^2/2$ and the elastic domain is given by

$$\mathbb{Q} = \{\, (\mathbf{Q}, q) \in \Sigma^*{\times}\mathbb{R}_* \mid q \leq 0,\ \|\mathbf{Q}\|_*{+}s_1 q \leq s_2 \,\}$$

where $\|\cdot\|_* : \Sigma^* \mapsto [0,\infty)$ is some (possibly nonsmooth) norm on Σ^* and $s_1, s_2 > 0$ are constants. Denoting by $\|\cdot\| : \Sigma \mapsto [0,\infty)$ the norm associated with $\|\cdot\|_*$ (i.e., $\|\boldsymbol{\varepsilon}\| = \max\{\, \boldsymbol{\varepsilon}{:}\mathbf{Q} \mid \|\mathbf{Q}\|_* \leq 1 \,\}$) a simple calculation gives the dissipation metric

$$\widehat{\Delta}(\dot{\boldsymbol{\pi}}, \dot{p}) = \begin{cases} s_2\|\dot{\boldsymbol{\pi}}\| & \text{if } \dot{p} \geq s_1\|\dot{\boldsymbol{\pi}}\|, \\ \infty & \text{else.} \end{cases}$$

For this example the calculation of the dissipation distance is trivial as we may connect the states $(\boldsymbol{\pi}_0, p_0)$ and $(\boldsymbol{\pi}_1, p_1)$ by a straight line in $\Sigma{\times}\mathbb{R}$. We find

$$\widehat{D}((\boldsymbol{\pi}_0, p_0), (\boldsymbol{\pi}_1, p_1)) = \begin{cases} s_2\|\boldsymbol{\pi}_1{-}\boldsymbol{\pi}_0\| & \text{if } p_1 \geq p_0{+}s_1\|\boldsymbol{\pi}_1{-}\boldsymbol{\pi}_0\|, \\ \infty & \text{else.} \end{cases}$$

3.4 Examples of dissipation distances

Before writing the elasto–plastic problem out in full detail we present a few concrete examples which enlighten the correspondence between the dissipation metric $\widehat{\Delta}$ and the associated elastic domain $\mathbb{Q}(\mathbf{P}, p)$. By (3.2) it suffices to study the plastically indifferent objects $\widetilde{\Delta}(p,\cdot,\cdot) : \mathfrak{g}{\times}\mathbb{R}^m \mapsto [0,\infty]$ and $\widetilde{\mathbb{Q}}(p) \subset \mathfrak{g}^*{\times}\mathbb{R}^m_*$ which are connected via $\widetilde{\mathbb{Q}}(p) = [\partial_{(\boldsymbol{\xi},\dot{p})}\widetilde{\Delta}(p,\cdot,\cdot)](\mathbf{0}, 0)$. More precisely, for given convex $\widetilde{\mathbb{Q}}(p)$ the function $\widetilde{\Delta}(p,\cdot)$ is obtained by Legendre transformation of $\mathcal{X}_{\widetilde{\mathbb{Q}}(p)}$, i.e.

$$\widetilde{\Delta}(p, \boldsymbol{\xi}, v) = \sup_{(\boldsymbol{\eta},q)\in\widetilde{\mathbb{Q}}(p)} (\boldsymbol{\eta}{:}\boldsymbol{\xi}{+}v{\cdot}q) = \sup_{(\boldsymbol{\eta},q)\in\mathfrak{g}^*\times\mathbb{R}^m_*} [\boldsymbol{\eta}{:}\boldsymbol{\xi}{+}v{\cdot}q{-}\mathcal{X}_{\widetilde{\mathbb{Q}}(p)}(\boldsymbol{\eta}, q)]. \tag{3.3}$$

3.4.1 A one–dimensional model with hardening

We consider the case $d = 1$ with $\mathfrak{G} = \mathrm{GL}_+(1) = (0,\infty)$ and isotropic hardening with the scalar parameter $p \in \mathbb{R}$. We let

$$\widetilde{\mathbb{Q}}(p) = \{\ (\boldsymbol{\eta}, q) \in \mathbb{R}^2 \mid |\boldsymbol{\eta}|+a(p,q) \leq 0\ \}$$

where $a(p,\cdot) : \mathbb{R} \mapsto \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is the convex function given by

$$a(p,q) = \begin{cases} -\alpha-\beta(p)+q & \text{for } q \leq \beta(p), \\ \infty & \text{for } q > \beta(p). \end{cases}$$

where $\alpha \geq 0$ and $\beta : \mathbb{R} \mapsto [0,\infty)$ is a continuous function. This $\mathbb{Q}(p)$ is of the form (A.2) and Appendix A gives

$$\widetilde{\Delta}(p,\boldsymbol{\xi},v) = \begin{cases} \infty & \text{for } v < |\boldsymbol{\xi}|, \\ \alpha|\boldsymbol{\xi}| + \beta(p)v & \text{for } v \geq |\boldsymbol{\xi}|. \end{cases}$$

Thus, $\widetilde{D}(\mathbf{P}_0^{-1}\mathbf{P}_1, p_0, p_1) = \widehat{D}((\mathbf{P}_0,p_0),(\mathbf{P}_1,p_1))$ can be calculated by minimizing the length $\int_0^1 [\alpha|\dot{\mathbf{P}}(t)/\mathbf{P}(t)|+\beta(p(t))\dot{p}(t)]\,\mathrm{d}t$ under the constraint $\dot{p} \geq |\mathbf{P}^{-1}\dot{\mathbf{P}}|$. We obtain

$$\widetilde{D}(\mathbf{P},p_0,p_1) = \begin{cases} \alpha|\log\mathbf{P}|+B(p_1)-B(p_0) & \text{for } p_1 \geq p_0+\alpha|\log\mathbf{P}|, \\ \infty & \text{else}; \end{cases} \tag{3.4}$$

where $B(p) = \int_0^p \beta(s)\,\mathrm{d}s$.

3.4.2 Dissipation distance without hardening

To facilitate the subsequent discussion we first consider the case without hardening, then $\Delta = \widehat{\Delta}(\mathbf{P},\dot{\mathbf{P}}) = \widetilde{\Delta}(\mathbf{P}^{-1}\dot{\mathbf{P}})$ is defined via a norm–like function $\widetilde{\Delta} : \mathfrak{g} \mapsto [0,\infty]$. This situation is discussed in [Mie02a], here we just give some results.

The dissipation distance satisfies $\widehat{D}(\mathbf{P}_0,\mathbf{P}_1) = \widetilde{D}(\mathbf{P}_0^{-1}\mathbf{P}_1)$, where $\widetilde{D} : \mathfrak{G} \mapsto [0,\infty]$ satisfies $\widetilde{D}(\mathbf{1}+\varepsilon\boldsymbol{\xi}) = \varepsilon\widetilde{\Delta}(\boldsymbol{\xi})+O(\varepsilon^2)$ for $\varepsilon \to 0$. If $\mathbf{P} = \mathrm{e}^{\boldsymbol{\xi}_1}\cdot\ldots\cdot\mathrm{e}^{\boldsymbol{\xi}_k}$ then $\widetilde{D}(\mathbf{P}) \leq \sum_{j=1}^k \widetilde{\Delta}(\boldsymbol{\xi}_j)$, which follows with the triangle inequality and $\widetilde{D}(\mathrm{e}^{\boldsymbol{\xi}}) \leq \widetilde{\Delta}(\boldsymbol{\xi})$. It is important to observe that, in general, $\widetilde{D}(\mathrm{e}^{\boldsymbol{\xi}}) < \widetilde{\Delta}(\boldsymbol{\xi})$ which indicates that generally the matrix exponential curves $t \mapsto \mathrm{e}^{t\boldsymbol{\xi}}$ are **not** the paths of minimal dissipation, see also (3.7).

Example 3.1 [Large shear] Large plastic shearing in the form $\mathbf{P} = \mathbf{1} + \gamma\,\mathbf{s}\otimes\mathbf{n}$ with $\gamma \gg 1$ ($|\mathbf{s}| = |\mathbf{n}| = 1$ and $\mathbf{s}\cdot\mathbf{n} = 0$) can be connected a linear shear $\mathbf{P}(t) = \mathbf{1} + t\gamma\,\mathbf{s}\otimes\mathbf{n}$ to the identity matrix $\mathbf{1}$. This curve has the dissipation length $\gamma\widetilde{\Delta}(\mathbf{s}\otimes\mathbf{n})$. However, the dissipation distance of $\widehat{D}(\mathbf{1},\mathbf{P})$ is much smaller. Consider the curve

$$\mathbf{P}(t) = \begin{cases} \begin{pmatrix} \mathrm{e}^{3t\beta} & 0 \\ 0 & \mathrm{e}^{-3t\beta} \end{pmatrix} & \text{for } t \in [0,1/3], \\ \begin{pmatrix} \mathrm{e}^{\beta} & (3t-1)\gamma\mathrm{e}^{-\beta} \\ 0 & \mathrm{e}^{-\beta} \end{pmatrix} & \text{for } t \in [1/3,2/3], \\ \begin{pmatrix} \mathrm{e}^{3(1-t)\beta} & \gamma\mathrm{e}^{3(t-1)\beta} \\ 0 & \mathrm{e}^{3(t-1)\beta} \end{pmatrix} & \text{for } t \in [2/3,1], \end{cases} \tag{3.5}$$

which has the length $\int_0^1 \widetilde{\Delta}(\mathbf{P}^{-1}(t)\dot{\mathbf{P}}(t))\,\mathrm{d}t = \beta\delta_1 + \gamma\mathrm{e}^{-2\beta}\delta_2 + \beta\delta_3$ where $\delta_1 = \widetilde{\Delta}(\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right))$, $\delta_2 = \widetilde{\Delta}(\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right))$ and $\delta_3 = \widetilde{\Delta}(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right))$. Choosing $\beta = \frac{1}{2}\log\gamma$ we see that the dissipation distance $\widetilde{D}(\left(\begin{smallmatrix}1 & \gamma\\ 0 & 1\end{smallmatrix}\right))$ grows at most like $\log\gamma$ for $\gamma \to \infty$.

It is shown in [Mie02a] that curves with corners like $\mathbf{P}$ in (3.5) actually occur as shortest paths if the boundary of the unit ball $\{\, \boldsymbol{\xi} \in \mathfrak{g} \mid \widetilde{\Delta}(\boldsymbol{\xi}) \leq 1 \,\}$ for the norm $\widetilde{\Delta}$ is not strictly convex. This corresponds to the case of single crystal plasticity where $\widetilde{\Delta}$ is piecewise linear and the unit ball is a polyhedron, see [Mie02a] and Section 3.4.4.

Up to now only one nontrivial case is known where the dissipation distance $\widetilde{D}$ can be calculated. It is given in the following example.

Example 3.2 [Isotropic elasticity] This case relates to isotropic plasticity of Prandtl–Reuss type using the von Mises flow rule. The plastic transformation $\mathbf{P}$ lies in $\mathfrak{G} = \mathrm{SL}(d)$ and the material symmetry–group is $\mathfrak{S} = \mathrm{O}(d)$, see (Sy4).

$$\widetilde{\Delta}(\boldsymbol{\xi}) = (\alpha|\boldsymbol{\xi}_{\mathrm{sym}}|^2 + \beta|\boldsymbol{\xi}_{\mathrm{anti}}|^2)^{1/2} \text{ with } \boldsymbol{\xi}_{\mathrm{sym}} = \tfrac{1}{2}(\boldsymbol{\xi}+\boldsymbol{\xi}^{\mathsf{T}}) \text{ and } \boldsymbol{\xi}_{\mathrm{anti}} = \tfrac{1}{2}(\boldsymbol{\xi}-\boldsymbol{\xi}^{\mathsf{T}}). \tag{3.6}$$

It is shown in [Mie02a] that a curve is a shortest path if and only if it has the form

$$\mathbf{P}(t) = \mathbf{P}(0)M_{\beta/\alpha}(t\boldsymbol{\xi}) \text{ where } M_\delta(\boldsymbol{\xi}) = \mathrm{e}^{\boldsymbol{\xi}_{\mathrm{sym}} - \delta\boldsymbol{\xi}_{\mathrm{anti}}}\mathrm{e}^{(1+\delta)\boldsymbol{\xi}_{\mathrm{anti}}}. \tag{3.7}$$

As a consequence, the dissipation distance associated with $\widetilde{\Delta}$ from (3.6) reads

$$\widetilde{D}(\mathbf{P}) = \min\{\, \widetilde{\Delta}(\boldsymbol{\xi}) \mid \mathbf{P} = M_{\alpha/\beta}(\boldsymbol{\xi}) \,\}.$$

For symmetric, positive definite matrices $\mathbf{P} = \mathbf{P}^{\mathsf{T}} > 0$ we find

$$\widetilde{D}(\mathbf{P}) = \sqrt{\alpha}\,|\log\mathbf{P}| = \sqrt{\alpha}\ (\mathbf{P}{:}\mathbf{P})^{1/2}\,.$$

Using the polar decomposition $\mathbf{P} = \mathbf{R}\mathbf{U}$ with $\mathbf{R} \in \mathrm{SO}(d)$ and $\mathbf{U} = \mathbf{U}^{\mathsf{T}} > 0$ together with the triangle inequality (2.3) we arrive at the explicit estimate

$$\min\{\widetilde{D}(\mathbf{R}), \widetilde{D}(\mathbf{U})\} \leq \widetilde{D}(\mathbf{P}) \leq \widetilde{D}(\mathbf{R}) + \widetilde{D}(\mathbf{U}), \tag{3.8}$$

where $\widetilde{D}(\mathbf{U}) = \frac{1}{2}\sqrt{\alpha}\,|\log(\mathbf{P}^{\mathsf{T}}\mathbf{P})|$. Again we see that large shears $\mathbf{1} + \gamma\mathbf{d}\otimes\mathbf{n}$ have a dissipation distance to $\mathbf{1}$ which grows at most like $\log|\gamma|$ for $\gamma \to \infty$.

An important rôle in isotropic plasticity plays the case of zero plastic spin. This is realized by $\alpha = 1$ and $\beta = \infty$ in (3.6), or more precisely, on $\mathfrak{sl}(d) = \mathrm{T}_{\mathbf{1}}(\mathrm{SL}(d))$ we set

$$\widetilde{\Delta}_{\mathrm{no\,spin}}(\boldsymbol{\xi}) = \begin{cases} |\boldsymbol{\xi}_{\mathrm{sym}}| & \text{if } \boldsymbol{\xi}_{\mathrm{anti}} = \mathbf{0}, \\ \infty & \text{else.} \end{cases}$$

The associated geodesic curves are $\mathbf{P}(t) = \mathbf{P}(0)\mathrm{e}^{t(\boldsymbol{\sigma}-\boldsymbol{\omega})}\mathrm{e}^{t\boldsymbol{\omega}}$ with $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$ and $\boldsymbol{\omega} = -\boldsymbol{\omega}^{\mathsf{T}}$. Note that $\mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t) = \mathrm{e}^{-t\boldsymbol{\omega}}\boldsymbol{\sigma}\mathrm{e}^{t\boldsymbol{\omega}}$ is symmetric but not constant; nevertheless it gives a constant and finite value $\Delta = \widetilde{\Delta}(\mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t)) = |\boldsymbol{\sigma}|$. In particular, $\mathbf{P}(0)^{-1}\mathbf{P}(t)$ can reach every matrix in $\mathrm{SL}(d)$, not just symmetric ones. We find

$$\widetilde{D}_{\mathrm{no\,spin}}(\mathbf{P}) = \min\{\, |\boldsymbol{\sigma}| \mid \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}} \text{ and there exists } \boldsymbol{\omega} = -\boldsymbol{\omega}^{\mathsf{T}} \text{ with } \mathbf{P} = \mathrm{e}^{\boldsymbol{\sigma}-\boldsymbol{\omega}}\mathrm{e}^{\boldsymbol{\omega}} \,\}. \tag{3.9}$$

In [HMM02] it is shown that for $d=2$ the dissipation distances $\widehat{D}_{\text{no spin}}$ can be calculated more explicitly. For $\mathbf{P}\in\mathrm{SL}(2)$ we have

$$\widetilde{D}_{\text{no spin}}(\mathbf{P})=\min\{\,\rho\geq 0\mid \text{there exists } \gamma\in\mathbb{R} \text{ with } N(\rho,\gamma)=\mathbf{P}{:}\mathbf{P} \text{ and } T(\rho,\nu)=\operatorname{tr}\mathbf{P}\,\}$$

where the functions N and T are defined via

$$C(t)=\begin{cases}\cosh\sqrt{t} & \text{for } t\geq 0,\\ \cos\sqrt{-t} & \text{for } t\leq 0,\end{cases}\qquad\text{and}\qquad S(t)=\begin{cases}\frac{\sinh\sqrt{t}}{\sqrt{t}} & \text{for } t>0\\ 1 & \text{for } t=0\\ \frac{\sin\sqrt{-t}}{\sqrt{-t}} & \text{for } t<0\end{cases}$$

as follows

$$N(\rho,\gamma)=2(1+2\rho^2[S(\rho^2{-}\gamma^2)]^2),\quad T(\rho,\gamma)=2[C(\rho^2{-}\gamma^2)\cos\gamma+\gamma S(\rho^2{-}\gamma^2)\sin\gamma].$$

(In fact, it suffices to consider $\gamma\in[0,\sqrt{\pi^2{+}\rho^2}\,]$ in the minimum defining $\widetilde{D}_{\text{no spin}}$.) For instance, we have $\widehat{D}(\mathbf{1},-\mathbf{1})=\widetilde{D}(-\mathbf{1})=\sqrt{3}\,\pi$ which is realized with $\rho=\sqrt{3}\,\pi$ and $\gamma=2\pi$ and the geodesic curve $\mathbf{P}(t)=\mathrm{e}^{t(\boldsymbol{\sigma}-\boldsymbol{\omega})}\mathrm{e}^{t\boldsymbol{\omega}}$ with $\boldsymbol{\sigma}=\begin{pmatrix}\rho & 0\\ 0 & -\rho\end{pmatrix}$ and $\boldsymbol{\omega}=\begin{pmatrix}0 & \gamma\\ -\gamma & 0\end{pmatrix}$.

Example 3.3 [Four slip systems in $d=2$] Another interesting case is studied in [HMM02] which corresponds to $d=2$ and single–crystal plasticity with four slip systems, see Section 3.4.4. Here we just mention a few results to indicate the importance of geodesic curves and to show that generally we have to expect that geodesic curves have corners. We consider $d=2$ and

$$\widetilde{\Delta}\Big(\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}\Big)=\begin{cases}|\beta|+|\gamma| & \text{if } \alpha=0,\\ \infty & \text{else;}\end{cases}$$

which corresponds to the four slip systems $\{\pm\mathbf{S}^1,\pm\mathbf{S}^2\}$ with $\mathbf{S}^1=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $\mathbf{S}^2=\begin{pmatrix}0&0\\1&0\end{pmatrix}$. It can be shown that all geodesic curves $t\mapsto\mathbf{P}(t)$ satisfy

$$\boldsymbol{\xi}(t)=\mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t)\in\{\pm\mathbf{S}^1,\pm\mathbf{S}^2,\pm\tfrac{1}{2}(\mathbf{S}^1{+}\mathbf{S}^2)\}.$$

Moreover, any $\mathbf{P}\in\mathrm{SL}(2)$ can be connected to $\mathbf{1}$ by a geodesic curve such that $\boldsymbol{\xi}:[0,T]\to\mathfrak{sl}(2)$ jumps at most 5 times.

3.4.3 Isotropic and kinematic hardening

For isotropic elasto–plasticity the von Mises yield criterion involving the L^2–norm of the deviator of the symmetric stress tensor is often used. The radius of the yield surface corresponds to a scalar hardening variable. We combine this with kinematic hardening where a tensor–valued variable allows for a shift of the center of the yield surface.

Before going into more detail we recall the well–known fact that the plastic stress $\mathbf{P}^{\mathsf{T}}\mathbf{Q}\in\mathfrak{sl}(d)$ is symmetric, if $\widehat{\psi}$ is isotropic (i.e. the material symmetry group $\mathfrak{S}$ equals $\mathrm{SO}(d)$, see (Sy4) in Section 3.1). We have $\psi=\widetilde{\psi}(\mathbf{F}_{\text{el}},p)=\overline{\psi}(\mathbf{C}_{\text{el}},p)$ with $\mathbf{C}_{\text{el}}=\mathbf{F}_{\text{el}}^{\mathsf{T}}\mathbf{F}_{\text{el}}$ and obtain $\frac{\partial}{\partial\mathbf{F}_{\text{el}}}\widetilde{\psi}(\mathbf{F}_{\text{el}},p)=2\mathbf{F}_{\text{el}}\frac{\partial}{\partial\mathbf{C}_{\text{el}}}\overline{\psi}(\mathbf{C}_{\text{el}},p)$. Thus, we have

$$\mathbf{P}^{\mathsf{T}}\mathbf{Q}=-\mathbf{P}^{\mathsf{T}}\tfrac{\partial}{\partial\mathbf{P}}\widehat{\psi}(\mathbf{F},\mathbf{P},p)=-\mathbf{P}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}}\tfrac{\partial}{\partial\mathbf{F}_{\text{el}}}\widetilde{\psi}(\mathbf{F}_{\text{el}},p)=-2\mathbf{C}_{\text{el}}\tfrac{\partial}{\partial\mathbf{C}_{\text{el}}}\overline{\psi}(\mathbf{C}_{\text{el}},p),$$

which is symmetric under suitable additional assumptions on $\overline{\psi}$ (like $\overline{\psi}(\mathbf{C}_{\rm el}, p) = \psi_1(\mathbf{C}_{\rm el}) + \psi_2(p)$). However, the plastically indifferent stress $\mathbf{P}^{\mathsf{T}}\mathbf{Q}$ need not to be symmetric, it should be considered as an element of $\mathfrak{sl}(d)$ (set of trace–free matrices) and hence the addition of $\alpha\mathbf{1}$ is irrelevant. To indicate this and to agree with standard notation we use the deviator $\operatorname{dev}\boldsymbol{\xi} := \boldsymbol{\xi} - \frac{1}{d}(\operatorname{tr}\boldsymbol{\xi})\mathbf{1}$ where "tr" denotes the trace of a tensor.

The hardening variable $p \in \mathbb{R}^m$ is now taken in the form $p = (r, \mathbf{k})$ with $\mathbf{k} \in [\mathfrak{sl}(d)]_{\rm sym}$ such that $m = d(d{+}1)/2 - 1$. The elastic domain $\widetilde{\mathbb{Q}}(r, \mathbf{k})$ is given in the form

$$\widetilde{\mathbb{Q}}(p) = \{\ (\boldsymbol{\eta}, \rho, \boldsymbol{\kappa}) \in \mathfrak{sl}(d)^* {\times} \mathbb{R}_* {\times} [\mathfrak{sl}(d)]^*_{\rm sym}\ \mid\ \rho \leq 0,\ |\boldsymbol{\eta}_{\rm sym} {-} \alpha\boldsymbol{\kappa}| {-} s_1 {+} s_2\rho \leq 0\ \}$$

where $s_1, s_2 > 0$, $\boldsymbol{\eta} = \operatorname{dev}\mathbf{P}^{\mathsf{T}}\mathbf{Q}$, $\rho = -\partial_r\widehat{\psi}$, $\boldsymbol{\kappa} = -\partial_{\mathbf{k}}\widehat{\psi}$, and $\alpha \in [0, 1]$ is a parameter interpolating between the case without kinematic hardening ($\alpha = 0$) and the case with full kinematic hardening ($\alpha = 1$). The flow rule reads

$$\begin{pmatrix} \mathbf{P}^{-1}\dot{\mathbf{P}} \\ \dot{r} \\ \dot{\mathbf{k}} \end{pmatrix} \in \mathrm{N}_{(\boldsymbol{\eta},\rho,\boldsymbol{\kappa})}\widetilde{\mathbb{Q}}(r, \mathbf{k}) \quad \text{or} \quad \begin{pmatrix} \mathbf{P}^{-1}\dot{\mathbf{P}} \\ \dot{r} \\ \dot{\mathbf{k}} \end{pmatrix} = \lambda \begin{pmatrix} \boldsymbol{\eta}_{\rm sym} {-} \alpha\boldsymbol{\kappa} \\ |\boldsymbol{\eta}_{\rm sym} {-} \alpha\boldsymbol{\kappa}| \\ \alpha(\alpha\boldsymbol{\kappa} {-} \boldsymbol{\eta}_{\rm sym}) \end{pmatrix}$$

where $\lambda = 0$ in the interior of $\widetilde{\mathbb{Q}}(r, \mathbf{k})$ and $\lambda \geq 0$ at the boundary.

The associated dissipation metric can be calculated via the methods of Appendix A, cf. (A.1) and Proposition A.1. We find

$$\widetilde{\Delta}(\boldsymbol{\xi}, \dot{r}, \dot{\mathbf{k}}) = \begin{cases} s_1|\boldsymbol{\xi}| & \text{if } \dot{r} \geq s_2|\boldsymbol{\xi}|,\ \boldsymbol{\xi}_{\rm anti} = \mathbf{0} \text{ and } \dot{\mathbf{k}} = \alpha\boldsymbol{\xi}, \\ \infty & \text{else.} \end{cases} \tag{3.10}$$

The model with $\alpha > 0$ is only useful if $\mathbf{k}$ and $\boldsymbol{\kappa}$ are related suitably. This is most easily reached by letting

$$\widetilde{\psi}(\mathbf{FP}, r, \mathbf{k}) = \widetilde{W}(\mathbf{FP}) + h(r) + \tfrac{1}{2\alpha}|\mathbf{k}|^2,$$

which implies $\boldsymbol{\kappa} = -\frac{1}{\alpha}\mathbf{k}$. Thus, the desired flow rule for kinematic hardening is obtained: $\dot{\boldsymbol{\kappa}} = \lambda(\boldsymbol{\eta}_{\rm sym} {-} \alpha\boldsymbol{\kappa})$ with $\lambda \geq 0$.

The behavior of $\mathbf{k}$ may be oscillatory such that the latent hardening energy $\frac{1}{2\alpha}|\mathbf{k}|^2$ can have nonmonotone behavior. Nevertheless the total plastic power

$$h'(r)\dot{r} + \tfrac{1}{\alpha}\mathbf{k}{\cdot}\dot{\mathbf{k}} + \widetilde{\Delta}(\boldsymbol{\xi}, \dot{r}, \dot{\mathbf{k}})$$

will be nonnegative under suitable constitutive assumptions.

Finally, we restrict our view to the case without kinematic hardening ($\alpha = 0$) where $\mathbf{k}$ and $\boldsymbol{\kappa}$ can be omitted completely. From the dissipation metric $\widetilde{\Delta}$ in (3.10) and Section 3.4.2 we find the dissipation distance

$$\widehat{D}((\mathbf{P}_0, r_0), (\mathbf{P}_1, r_1)) = \begin{cases} s_1\widetilde{D}_{\rm no\,spin}(\mathbf{P}_0^{-1}\mathbf{P}_1) & \text{if } r_1 \geq r_0 {+} s_2\widetilde{D}_{\rm no\,spin}(\mathbf{P}_0^{-1}\mathbf{P}_1), \\ \infty & \text{else;} \end{cases}$$

where $\widetilde{D}_{\rm no\,spin}$ is given in (3.9).

3.4.4 Single–crystal plasticity

In single–crystal plasticity the plastic flow occurs through plastic slips realized by movements of dislocations. Let $\mathbf{S}^\alpha = \mathbf{d}^\alpha \otimes \mathbf{n}^\alpha$, $\alpha = 1, \ldots, m$, be the m slip systems where $\mathbf{n}^\alpha$ is the unit normal to the αth slip plane and $\mathbf{d}^\alpha$ is the slip direction with $|\mathbf{d}^\alpha| = 1$ and $\mathbf{d}^\alpha \cdot \mathbf{n}^\alpha = 0$. All plastic flow has the form

$$\dot{\mathbf{P}} = \mathbf{P} \textstyle\sum_{\alpha=1}^m \nu_\alpha \mathbf{S}^\alpha$$

where the slip rates ν_α are taken to be positive. This means we distinguish the slip systems $\mathbf{S}^\alpha$ and $-\mathbf{S}^\alpha$ as mechanically the slip strains in these directions must be distinguished, see [OR99, Gur00].

The crystal symmetry group $\mathfrak{S} \subset \mathrm{O}(d)$ is discrete and associates a permutation $\pi_{\mathbf{R}} \in \mathrm{Perm}(m)$ to each $\mathbf{R} \in \mathfrak{S}$ such that $\pi_{\mathbf{R}\widehat{\mathbf{R}}} = \pi_{\mathbf{R}} \circ \pi_{\widehat{\mathbf{R}}}$ (composition of permutations) and

$$(\mathbf{R}\mathbf{d}^\alpha, \mathbf{R}\mathbf{n}^\alpha) = (\mathbf{d}^{\pi_{\mathbf{R}}(\alpha)}, \mathbf{n}^{\pi_{\mathbf{R}}(\alpha)}) \quad \Longleftrightarrow \quad \mathbf{S}^{\pi_{\mathbf{R}}(\alpha)} = \mathbf{R}\mathbf{S}^\alpha \mathbf{R}^\mathsf{T}.$$

The set of all slip systems $\{\, \mathbf{S}^\alpha \mid \alpha = 1, \ldots, m \,\}$ determines the associated Lie algebra $\mathfrak{g}$ (and hence the Lie group $\mathfrak{G} \subset \mathrm{GL}_+(d)$) as the smallest Lie algebra containing all slip systems:

$$S := \mathrm{span}\{\mathbf{S}^\alpha : \alpha = 1, \ldots, m\} \subset \mathfrak{g} = \mathrm{T}_{\mathbf{1}}\mathfrak{G}.$$

Here $\mathfrak{g}$ may be strictly bigger than S, as is seen in Example 3.3 where $\dim S = 2 < \dim \mathfrak{g} = 3$. From $\mathbf{d}^\alpha \cdot \mathbf{n}^\alpha = 0$ we know $\mathrm{tr}(\mathbf{S}^\alpha) = 0$ and hence $\mathfrak{g} \subset \mathfrak{sl}(d)$ and $\mathfrak{G} \subset \mathrm{SL}(d)$.

As additional internal variables we introduce the vector of accumulated slip strains

$$p = \gamma = (\gamma_1, \ldots, \gamma_m)^\mathsf{T} \text{ where } \gamma_\alpha(t,x) = \textstyle\int_0^t \nu_\alpha(s,x)\,\mathrm{d}s.$$

We now postulate the dissipation metric and then show that it gives rise to the classical single–crystal flow rule for the shear stresses τ_α in each slip system. With $\boldsymbol{\xi} = \mathbf{P}^{-1}\dot{\mathbf{P}}$ the dissipation metric is

$$\widetilde{\Delta}(\boldsymbol{\xi}, \dot{\gamma}) = \begin{cases} \sum_{\alpha=1}^m \kappa_\alpha \dot{\gamma}_\alpha & \text{if all } \dot{\gamma}_\alpha \geq 0 \text{ and } \boldsymbol{\xi} = \sum_{\alpha=1}^m \dot{\gamma}_\alpha \mathbf{S}^\alpha, \\ \infty & \text{else,} \end{cases}$$

where $\kappa_\alpha > 0$ are threshold parameters, see [OR99, Gur00].

The associated elastic domain $\widehat{\mathbb{Q}}(\mathbf{P}, \gamma)$ is formulated in the thermo–mechanically conjugated variables $\mathbf{Q} = -\partial_{\mathbf{P}}\widehat{\psi}$ and $q = (q_1, \ldots, q_m) = -\partial_\gamma \widehat{\psi}$. The invariant form $\widetilde{\mathbb{Q}}(\gamma)$ using $\boldsymbol{\eta} = \mathrm{dev}\, \mathbf{P}^\mathsf{T}\mathbf{Q}$ is given by

$$\widetilde{\mathbb{Q}}(\gamma) = \{\, (\boldsymbol{\eta}, q) \in \mathfrak{sl}(d)^* \times \mathbb{R}^m_* \mid q_\alpha - \kappa_\alpha + \mathbf{S}^\alpha : \boldsymbol{\eta} \leq 0 \text{ for } \alpha = 1, \ldots, m \,\}.$$

Hence the elastic domain is characterized by one yield condition for each slip systems $\mathbf{S}^\alpha$. Denoting by $\tau_\alpha = \mathbf{S}^\alpha : \boldsymbol{\eta}$ the resolved shear stress and using the plastic backstress q_α, the slip system $\mathbf{S}^\alpha$ becomes active ($\dot{\gamma}_\alpha > 0$) only if $q_\alpha + \tau_\alpha = \kappa_\alpha > 0$.

In [OR99] the quantity $g^\alpha = -q_\alpha$ is used and the slip criterion is $\tau_\alpha - g^\alpha = 0$ while $\tau_\alpha - g^\alpha \leq 0$ is the elastic domain. There, the threshold value $\kappa_\alpha > 0$ is incorporated into the energy density $\widehat{\psi}(\mathbf{F},\mathbf{P},\gamma)=\widetilde{W}(\mathbf{FP})+\psi_{\rm hard}(\gamma)$. By adding $\sum_{\alpha=1}^m \kappa_\alpha\gamma_\alpha$ to $\psi_{\rm hard}(\gamma)$ we shift $g^\alpha=-q_\alpha$ exactly by the amount κ_α.

An interesting feature of single–crystal plasticity is the occurrence of latent hardening which can be implemented by choosing suitable $\psi_{\rm hard}$. In particular, the influence of the accumulated slip strain in one slip system on the hardening in other systems plays a crucial role in the formation of microstructure, see [OR99, ORS00].

4 The reduced energy density

In Section 2.5 we have defined the reduced energy density

$$\Psi^{\rm red}((\mathbf{P}_{\rm old},p_{\rm old});\mathbf{F})=\min_{(\mathbf{P}_{\rm new},p_{\rm new})}\widehat{\psi}(\mathbf{F},\mathbf{P}_{\rm new},p_{\rm new})+\widehat{D}((\mathbf{P}_{\rm old},p_{\rm old}),(\mathbf{P}_{\rm new},p_{\rm new})) \tag{4.1}$$

for characterizing minimization properties of stable solutions as well as for solving the incremental time–stepping algorithm. Here we study this function further as it is an in excellent indicator for the joint effect of elasticity and plasticity through its definition involving $\widehat{\psi}$ and $\widehat{D}$. In particular, we will see how formation of microstructure, localization and failure can be read off from $\Psi^{\rm red}$. Moreover, the global stability condition (S) is intrinsically linked with $\Psi^{\rm red}$.

For general situations it is very difficult to calculate $\Psi^{\rm red}$ as it is already difficult to calculate the dissipation distance. However, sometimes not all of $\widehat{D}$ is needed, and in Section 4.4 we provide an example where $\Psi^{\rm red}$ can be calculated despite the fact that $\widehat{D}$ is not known explicitly. Another advantage of the reduced energy density $\Psi^{\rm red}$ is that the ambiguity of splitting the plastic energy into dissipation $\widehat{\Delta}$ and into hardening energy $h=\widehat{\psi}_{\rm hard}$ disappears completely.

4.1 General properties

A simple consequence of plastic indifference (Sy2) is

$$\Psi^{\rm red}((\mathbf{P}_{\rm old},p_{\rm old});\mathbf{F})=\widetilde{\Psi}^{\rm red}(p_{\rm old};\mathbf{FP}_{\rm old})\quad\text{where } \widetilde{\Psi}^{\rm red}(p;\mathbf{F}):=\Psi^{\rm red}((\mathbf{1},p);\mathbf{F}). \tag{4.2}$$

Thus, it suffices to calculate $\Psi^{\rm red}$ for $\mathbf{P}_{\rm old}=\mathbf{1}$, only. Additionally, we have defined a function $\widehat{Z}_{\rm new}$ which associates to $\mathbf{F}$ and $(\mathbf{P}_{\rm old},p_{\rm old})$ one minimizer in the definition of $\Psi^{\rm red}$, i.e.

$$\Psi^{\rm red}((\mathbf{P}_{\rm old},p_{\rm old});\mathbf{F})=\Big(\widehat{\psi}(\mathbf{F},\mathbf{P},p)+\widehat{D}((\mathbf{P}_{\rm old},p_{\rm old}),(\mathbf{P},p))\Big)\Big|_{(\mathbf{P},p)=\widehat{Z}_{\rm new}(\mathbf{F},(\mathbf{P}_{\rm old},p_{\rm old}))}.$$

Again by plastic indifference we obtain

$$\widehat{Z}_{\rm new}(\mathbf{F},(\mathbf{P}_{\rm old},p_{\rm old}))={\rm diag}(\mathbf{P}_{\rm old},\mathbf{1})\widetilde{Z}_{\rm new}(\mathbf{F},p_{\rm old})\text{ with }\widetilde{Z}_{\rm new}(\mathbf{F},p_{\rm old})=\widehat{Z}_{\rm new}(\mathbf{F},(\mathbf{1},p_{\rm old})).$$

Abbreviating $z=(\mathbf{P},p)$ and using the triangle inequality for $\widehat{D}$ we find the inequality

$$\Psi^{\rm red}(z_1;\mathbf{F})\le\Psi^{\rm red}(z_2;\mathbf{F})+\widehat{D}(z_1,z_2).$$

We may even have equality in certain cases, namely if

$$z := \widehat{Z}_{\mathrm{new}}(\mathbf{F}, z_1) = \widehat{Z}_{\mathrm{new}}(\mathbf{F}, z_2) \text{ and } \widehat{D}(z_1, z) = \widehat{D}(z_1, z_2) + \widehat{D}(z_2, z).$$

For computational purposes it is also important to have the pseudo stresses $\mathrm{D}_{\mathbf{F}}\Psi^{\mathrm{red}}$ and the pseudo tangent moduli $\mathrm{D}^2_{\mathbf{F}}\Psi^{\mathrm{red}}$.

Proposition 4.1 *For Ψ^{red} as constructed above we have*

$$\mathrm{D}_{\mathbf{F}}\Psi^{\mathrm{red}}(z_{\mathrm{old}}; \mathbf{F}) = \mathrm{D}_{\mathbf{F}}\widehat{\psi}(\mathbf{F}, \widehat{Z}_{\mathrm{new}}(\mathbf{F}, z_{\mathrm{old}})).$$

If $\widehat{Z}_{\mathrm{new}}(\cdot, z_{\mathrm{old}})$ is differentiable at $\mathbf{F}$ and $\widehat{D}(z_{\mathrm{old}}, \cdot)$ is twice differentiable at $z = \widehat{Z}_{\mathrm{new}}(\mathbf{F}, z_{\mathrm{old}})$, then

$$\mathrm{D}^2_{\mathbf{F}}\Psi^{\mathrm{red}}(z_{\mathrm{old}}; \mathbf{F}) = \mathbb{A} - \mathbb{B}\,\mathbb{D}^{-1}\mathbb{B}^{\mathsf{T}}, \tag{4.3}$$

where $\mathbb{A}, \mathbb{B}$, and $\mathbb{D}$ are given via $\left(\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B}^{\mathsf{T}} & \mathbb{D} \end{smallmatrix}\right) := \mathrm{D}^2_{\mathbf{F},z}(\widehat{\psi}(\cdot,\cdot) + \widehat{D}(z_{\mathrm{old}}, \cdot))(\mathbf{F}, \widehat{Z}_{\mathrm{new}}(\mathbf{F}, z_{\mathrm{old}}))$.

Obviously, the minimization with respect to z_{new} always gives rise to a softening via the term $-\mathbb{B}\mathbb{D}^{-1}\mathbb{B}^{\mathsf{T}}$. Note that (4.3) is useless in points where $\widehat{D}$ is not smooth. In particular, inside the elastic regime we have $\widehat{Z}_{\mathrm{new}}(\mathbf{F}, z_{\mathrm{old}}) = z_{\mathrm{old}}$ and $\Psi^{\mathrm{red}}(z_{\mathrm{old}}; \mathbf{F}) = \widehat{\psi}(\mathbf{F}, z_{\mathrm{old}})$ which shows no softening.

Proof: The argument works for each fixed z_{old}, so we suppress it in the subsequent calculation and let $W(\mathbf{F}, z) = \widehat{\psi}(\mathbf{F}, z) + \widehat{D}(z_{\mathrm{old}}, z)$ and $Z(\mathbf{F}) = \widehat{Z}_{\mathrm{new}}(\mathbf{F}, z_{\mathrm{old}})$. By the definition of the reduced energy density we have $\Psi^{\mathrm{red}}(\mathbf{F}) = \min_z W(\mathbf{F}, z) = W(\mathbf{F}, Z(\mathbf{F}))$. This implies

$$\mathrm{D}_z W(\mathbf{F}, Z(\mathbf{F})) = 0 \quad \text{and} \quad \mathbb{D} = \mathrm{D}^2_z W(\mathbf{F}, Z(\mathbf{F})) \geq 0, \tag{4.4}$$

since $z = Z(\mathbf{F})$ is a minimizer.

By the chain rule, we obtain the directional derivatives ($\mathbf{H}$ arbitrary)

$$\mathrm{D}\Psi^{\mathrm{red}}(\mathbf{F})[\mathbf{H}] = \mathrm{D}_{\mathbf{F}}W(\mathbf{F}, Z(\mathbf{F}))[\mathbf{H}] + \mathrm{D}_z W(\mathbf{F}, Z(\mathbf{F}))[\mathrm{D}Z(\mathbf{F})[\mathbf{H}]],$$

but the second term vanishes because of (4.4). Differentiating again, we find

$$\begin{aligned} \mathrm{D}^2\Psi^{\mathrm{red}}(\mathbf{F})[\mathbf{H}, \mathbf{H}] &= \mathrm{D}^2_{\mathbf{F}}W(\mathbf{F}, Z(\mathbf{F}))[\mathbf{H}, \mathbf{H}] + \mathrm{D}_{\mathbf{F}}\mathrm{D}_z W(\mathbf{F}, Z(\mathbf{F}))[\mathbf{H}, \mathrm{D}Z(\mathbf{F})[\mathbf{H}]] \\ &= \mathbb{A}[\mathbf{H}]{:}\mathbf{H} + \mathbb{B}[\mathrm{D}Z(\mathbf{F})[\mathbf{H}]]{:}\mathbf{H}. \end{aligned} \tag{4.5}$$

Differentiating the first equation in (4.4) we obtain $\mathbb{B}^{\mathsf{T}}{:}\mathbf{H} + \mathbb{D}[\mathrm{D}Z(\mathbf{F})[\mathbf{H}]] = 0$. Solving this for $\mathrm{D}Z(\mathbf{F})[\mathbf{H}]$ and inserting the result into (4.5) the result follows. ■

4.2 Spatially localized stability and quasiconvexity

We distinguish the notion **local stability** meaning stability with respect to small perturbation throughout Ω and **spatially localized stability** meaning stability with respect to arbitrary perturbations which are localized in a small neighborhood of a point $x_0 \in \Omega$. For sufficiently small $\varepsilon > 0$ we define test functions $(\widehat{\boldsymbol{\varphi}}, \widehat{\mathbf{P}}, \widehat{p})$ via

$$(\widehat{\boldsymbol{\varphi}}_\varepsilon(x), \widehat{\mathbf{P}}_\varepsilon(x), \widehat{p}_\varepsilon(x)) = \big(\boldsymbol{\varphi}(x) + \varepsilon\overline{\boldsymbol{\varphi}}(\tfrac{x-x_0}{\varepsilon}), \mathbf{P}(x)\overline{\mathbf{P}}(\tfrac{x-x_0}{\varepsilon}), p(x) + \overline{p}(\tfrac{x-x_0}{\varepsilon})\big) \tag{4.6}$$

where $(\overline{\boldsymbol{\varphi}}(y), \overline{\mathbf{P}}(y), \overline{p}(y)) = (\mathbf{0}, \mathbf{1}, 0)$ for $y \notin B = \{\, y \in \mathbb{R}^d \mid |y| < 1 \,\}$. Hence, $(\overline{\boldsymbol{\varphi}}, \overline{\mathbf{P}}, \overline{p})$ can be interpreted as microscopic fluctuation on the macroscopic field $(\boldsymbol{\varphi}, \mathbf{P}, p)$.

Proposition 4.2 *If $(\boldsymbol{\varphi},\mathbf{P},p)$ is a stable state in the sense of (S), then for each point $x_0\in\Omega$ where $(\boldsymbol{\varphi},\mathbf{P},p)$ is continuous we have*

$$\tfrac{1}{\mathrm{vol}B}\int_{y\in B}\Psi^{\mathrm{red}}\left((\mathbf{P}_0,p_0);\mathbf{F}_0{+}\mathrm{D}\overline{\boldsymbol{\varphi}}(y)\right)\mathrm{d}y\geq\widehat{\psi}(\mathbf{F}_0,\mathbf{P}_0,p_0) \tag{4.7}$$

for all $\overline{\boldsymbol{\varphi}}\in\mathrm{C}^1(B,\mathbb{R}^d)$ with $\overline{\boldsymbol{\varphi}}|_{\partial B}=\mathbf{0}$, where $(\mathbf{F}_0,\mathbf{P}_0,p_0)=(\mathbf{F}(x_0),\mathbf{P}(x_0),p(x_0))$.

Proof: We insert (4.6) into (S), subtract the term of order ε^0, divide by ε^d and take the limit $\varepsilon\to 0$ to arrive at

$$\begin{aligned}&\int_B\left[\widehat{\psi}(\mathbf{F}_0{+}\mathrm{D}\overline{\varphi}(y),\mathbf{P}_0\overline{\mathbf{P}}(y),p_0{+}\overline{p}(y)){+}\widehat{D}((\mathbf{P}_0,p_0),(\mathbf{P}_0\overline{\mathbf{P}}(y),p_0{+}\overline{p}(y)))\right]\mathrm{d}y\\&\geq\mathrm{vol}(B)\,\widehat{\psi}(\mathbf{F}_0,\mathbf{P}_0,p_0)\end{aligned}$$

for all $(\overline{\boldsymbol{\varphi}},\overline{\mathbf{P}},\overline{p})$. Minimizing under the integral with respect to $(\overline{\mathbf{P}}(y),\overline{p}(y))$ yields the assertion. ■

Using the definition of the quasiconvex hull QC in (D.3) and the trivial estimates

$$[\mathsf{QC}\Psi^{\mathrm{red}}((\mathbf{P},p);\,\cdot\,)](\mathbf{F})\leq\Psi^{\mathrm{red}}((\mathbf{P},p);\mathbf{F})\leq\widehat{\psi}(\mathbf{F},\mathbf{P},p)$$

we obtain the following equivalent form of Proposition 4.2.

Corollary 4.3 *Under the assumptions of Proposition 4.2 we have*

$$\Psi^{\mathrm{red}}((\mathbf{P}_0,p_0);\mathbf{F}_0)=\widehat{\psi}(\mathbf{F}_0,\mathbf{P}_0,p_0); \tag{a}$$

$$\begin{aligned}&\Psi^{\mathrm{red}}((\mathbf{P}_0,p_0);\cdot)\ \textit{is quasiconvex at}\ \mathbf{F}_0\\&(\textit{i.e.},\ [\mathsf{QC}\Psi^{\mathrm{red}}((\mathbf{P}_0,p_0);\,\cdot\,)](\mathbf{F}_0)=\Psi^{\mathrm{red}}((\mathbf{P}_0,p_0);\mathbf{F}_0)\,).\end{aligned} \tag{b}$$

This indicates two different failure mechanisms. Assume a small material probe is loaded such that the mesoscopic strain $\mathbf{F}(x_0)$ is enforced. However, Ψ^{red} is such that either (a) or (b) does not hold. If (a) is violated then the energy can be lowered by keeping $\mathbf{F}(x_0)$ fixed and by changing the plastic parameters from $(\mathbf{P}(x_0),p(x_0))$ into some $(\mathbf{P},p)$. This may either lead to localization effects or to phenomena which are like phase transformations (sudden plastic slip). If (b) is violated then the missing quasiconvexity means that the energy can be reduced by introducing microscopic fluctuations while keeping the mesoscopic strain fixed. In such situation we should expect formation of plastic microstructure, cf. [OR99, CHM02].

A third failure criterion can be obtained from pure traction loading, which in elasto–plastic processes is mainly of theoretical interest. Assume that the body is under boundary traction $\mathbf{f}_{\mathrm{tract}}=\mathbf{T}_{\mathrm{ext}}\boldsymbol{\nu}$ where $\boldsymbol{\nu}$ is the outer unit normal vector. We say that the material can sustain the stress $\mathbf{T}_{\mathrm{ext}}$ if

$$Y(\mathbf{T}_{\mathrm{ext}}):=\inf_{(\mathbf{F},\mathbf{P},p)}\left[\widehat{\psi}(\mathbf{F},\mathbf{P},p)+\widehat{D}((\mathbf{1},0),(p,p))-\mathbf{F}{:}\mathbf{T}_{\mathrm{ext}}\right]>-\infty.$$

(Instead of $(\mathbf{1},0)$ we could have used any $(\mathbf{P}_0,p_0)$ without changing the condition $Y(\mathbf{T}_{\mathrm{ext}})>-\infty$.) Clearly, $Y(\mathbf{T}_{\mathrm{ext}})=-\infty$ means that a probe would stretch out to infinity to lower the energy, which simply means fracture. Again, the reduced density is helpful since

$$Y(\mathbf{T}_{\mathrm{ext}})=\inf_{\mathbf{F}}\left[\Psi^{\mathrm{red}}((\mathbf{1},0);\mathbf{F})-\mathbf{F}{:}\mathbf{T}_{\mathrm{ext}}\right].$$

Thus, the sustainable stresses lead to linear growth conditions for $\Psi^{\rm red}$ in $\mathbf{F}$. A rough estimate of the growth of $\Psi^{\rm red}$ can be obtained in the following way. Assume that energy density ψ has the form $\widehat{\psi}(\mathbf{F},\mathbf{P},p) = W(\mathbf{FP})+h(p)$ with $W(\mathbf{F}_{\rm el}) \sim \|\mathbf{F}_{\rm el}\|^\alpha$ and $h(p) \sim {\rm e}^{\beta p}$. Moreover, assume that the dissipation distance satisfies $\widetilde{D}(\mathbf{P},p_0,p_1) \sim \|\log \mathbf{P}^{\mathsf{T}}\mathbf{P}\|$ for $p_1 \geq p_0+\|\log \mathbf{P}^{\mathsf{T}}\mathbf{P}\|$. Then, the reduced energy density $\widetilde{\Psi}^{\rm red}((\mathbf{1},p);\mathbf{F})$ grows at most like $\|\mathbf{F}\|^{\alpha\beta/(\alpha+\beta)}$.

Global minimization over $\mathbf{F} \in {\rm GL}_+(d)$ seems unnatural here and one would like to consider local minimizers here. This important issue will be discussed in Section 6.

4.3 A one–dimensional example

The present example serve illustrational purposes. However the one–dimensional case can be analyzed in much more detail and so we can learn about the implications of our theory. In particular, in Section 5 we will be able to study the evolution of such a model. In the case $d = 1$ we write $\mathbf{F}_1^{-1}\mathbf{F}_2$ as $\mathbf{F}_2/\mathbf{F}_1$, since the matrices reduce to real scalars.

As in Section 3.4.1 we let $\mathbf{F} = {\rm D}\boldsymbol{\varphi} \in \mathfrak{G} = {\rm GL}_+(1) = (0,\infty)$,

$$\widehat{\psi}(\mathbf{F},\mathbf{P},p) = W(\mathbf{FP})+h(p), \text{ and } \widetilde{D}(\mathbf{P},p_0,p_1)) = \begin{cases} |\log \mathbf{P}| & \text{for } p_1 \geq p_0+|\log \mathbf{P}|, \\ \infty & \text{else.} \end{cases}$$

Minimizing in the definition (4.1) for $\Psi^{\rm red}$ with respect to $p_{\rm new}$ we find $p_{\rm new} = p_{\rm old}+ |\log(\mathbf{P}_{\rm new}/\mathbf{P}_{\rm old})|$ and hence $\Psi^{\rm red}((\mathbf{P},p);\mathbf{F}) = \widetilde{\Psi}^{\rm red}(p;\mathbf{FP})$ takes the form

$$\widetilde{\Psi}^{\rm red}(p;\mathbf{F}) = \min_{\mathbf{P}_{\rm new}} \Big[W(\mathbf{FP}_{\rm new}) + h(p+|\log \mathbf{P}_{\rm new}|) + |\log \mathbf{P}_{\rm new}| \Big]. \tag{4.8}$$

To allow for explicit calculations we restrict ourselves to the case

$$W(\mathbf{F}) = \tfrac{1}{\alpha}(\mathbf{F}^\alpha+\mathbf{F}^{-\alpha}-2) \text{ and } h(p) = \begin{cases} 1+(\alpha-1)p & \text{for } p \leq 0, \\ {\rm e}^{\alpha p} - p & \text{for } p \geq 0; \end{cases} \tag{4.9}$$

with $\alpha \geq 1$ (making W convex). Hence $p < 0$ corresponds to no hardening whereas $p > 0$ gives a suitable amount of hardening.

The reduced energy density has the additional symmetry $\widetilde{\Psi}^{\rm red}(p,1/\mathbf{F}) = \widetilde{\Psi}^{\rm red}(p,\mathbf{F})$ due to $W(1/\mathbf{F}) = W(\mathbf{F})$. Hence, it suffices to give $\Psi^{\rm red}$ for $\mathbf{F} \geq 1$ only. Using the formula (C.4) we find for this case

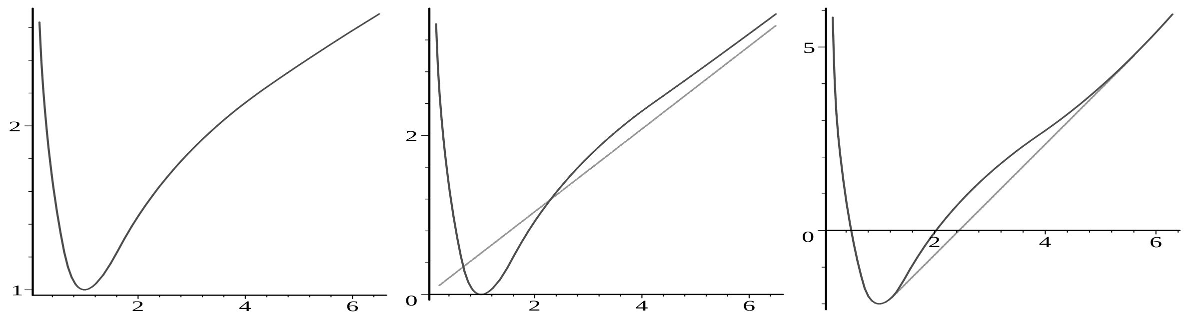
$$\widetilde{\Psi}^{\rm red}(p;\mathbf{F}) = \begin{cases} W(\mathbf{F}) + h(p) & \text{for } 1 \leq \mathbf{F}^\alpha \leq \mathbf{F}^{-\alpha} + b_p s_p, \\ W(\mathbf{F}_*) + \alpha \log(\mathbf{F}/\mathbf{F}_*) + h(p) & \text{for } \mathbf{F}_*^\alpha \leq \mathbf{F}^\alpha \leq s_p \mathbf{F}_*^\alpha, \\ \frac{2}{\alpha}(\sqrt{1+b_p\mathbf{F}^\alpha} - 1) - p & \text{for } \mathbf{F}^\alpha \geq s_p^2(\mathbf{F}^{-\alpha}+b_p); \end{cases} \tag{4.10}$$

where $b_p = \alpha {\rm e}^{\alpha p}$, $s_p = \max\{1, {\rm e}^{-\alpha p}\}$ and $\mathbf{F}_* > 1$ is the unique solution of $\mathbf{F}_*^\alpha = \mathbf{F}_*^{-\alpha} + \alpha$. For $p \geq 0$ we have $s_p = 1$ and hence the second, nonconvex regime, which involves the logarithm, is not present. We have convexity of $\widetilde{\Psi}^{\rm red}$ if and only if $\alpha \geq 2$ and $p \geq 0$.

We see a major difference between extension, $\mathbf{F} > 1$, and compression $\mathbf{F} < 1$. For $\mathbf{F} < 1$ and $p < 0$ the logarithmic term takes the form $\widetilde{\Psi}^{\rm red}(p,\mathbf{F}) = \widetilde{\Psi}^{\rm red}(p,1/\mathbf{F}) =$

$W(\mathbf{F}_*) - \alpha \log(\mathbf{F}\mathbf{F}_*) + h(p)$ which is convex whereas the corresponding part for $\mathbf{F} > 1$ is concave. Such a difference cannot be seen in a linearized theory.

For $1 \leq \alpha < 2$ the third regime shows concave behavior as $\widetilde{\Psi}^{\mathrm{red}}(p;\mathbf{F}) \sim \mathbf{F}^{\alpha/2}$ for $\mathbf{F} \gg 1$. As a consequence the (quasi–) convexification of $\widetilde{\Psi}^{\mathrm{red}}$ is given by $\widetilde{\Psi}^{\mathrm{red}}(p,\mathbf{F})$ for $\mathbf{F} \leq 1$ and it equals $\widetilde{\Psi}^{\mathrm{red}}(p,1)$ for all $\mathbf{F} \geq 1$. In the case $\alpha = 2$ the function $\widetilde{\Psi}^{\mathrm{red}}(p,\cdot)$ has a linear asymptote for $\mathbf{F} \to \infty$, see Figure 4.1.



(a) $\alpha = 1$ (b) $\alpha = 2$ (c) $\alpha = 4$

Figure 4.1: The reduced energy density $\Psi^{\mathrm{red}}(p;\cdot)$ for $p{=}{-1}$ and three different α. For α=2 the asymptote is given, for α=4 the convexification is shown. [The horizontal axis shows $\mathbf{F}$ and the vertical axis shows $\psi = \Psi^{\mathrm{red}}(p;\mathbf{F})$.]

4.4 Isotropic hardening

We consider the case of isotropic elasto–plasticity, where the elastic stored–energy density $\widehat{\psi}$ as well as the dissipation metric $\widehat{\Delta}$ are isotropic, i.e., the material symmetry group $\mathfrak{S}$ in (Sy4) equals $\mathrm{SO}(d)$, cf. Section 3.1. The associated plastic Lie group is $\mathfrak{G} = \mathrm{SL}(d)$. Plastic hardening is described by the scalar hardening parameter $p \in \mathbb{R}$. We assume

$$\widehat{\psi}(\mathbf{F},\mathbf{P},p) = \widetilde{W}(\mathbf{F}\mathbf{P}) + h(p), \quad \widetilde{\Delta}(p,\boldsymbol{\xi},\dot{p}) = \begin{cases} |\boldsymbol{\xi}| & \text{for } \boldsymbol{\xi} \in \Sigma \text{ and } \dot{p} \geq |\boldsymbol{\xi}|, \\ \infty & \text{else;} \end{cases}$$

with $h'(p) \geq 0$ for $p \geq p_0$ and $|\boldsymbol{\xi}| = (\boldsymbol{\xi}{:}\boldsymbol{\xi})^{1/2}$. Here $\Sigma = \mathfrak{sl}(d)_{\mathrm{sym}} := \{\, \boldsymbol{\sigma} \in \mathbb{R}^{d\times d} \mid \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}, \operatorname{tr}\boldsymbol{\sigma} = 0 \,\}$, which implies that the dissipation metric does not allow for infinitesimal plastic spin (nevertheless $\mathbf{P}$ will be unsymmetric in general).

This example was already studied in [CHM02] where it was shown that the reduced density is not rank–one convex. Here we work out the example in the context of dissipation distances and give the reduced energy density in more detail.

As indicated in Section 3.4.3 it is not easy to calculate the dissipation distance which has the form

$$\widetilde{D}(\mathbf{P},p_0,p_1) = \begin{cases} \widetilde{D}_{\mathrm{no\,spin}}(\mathbf{P}) & \text{if } p_1 \geq p_0 + \widetilde{D}_{\mathrm{no\,spin}}(\mathbf{P}) \\ \infty & \text{else} \end{cases}$$

where $\widetilde{D}_{\mathrm{no\,spin}}$ is defined in (3.9) and satisfies

$$|\boldsymbol{\sigma}| \leq \widetilde{D}_{\mathrm{no\,spin}}(\mathrm{e}^{\boldsymbol{\sigma}}\mathbf{R}) \leq |\boldsymbol{\sigma}| + \widetilde{D}_{\mathrm{no\,spin}}(\mathbf{R}) \quad \text{for } \mathbf{R} \in \mathrm{SO}(d) \text{ and } \boldsymbol{\sigma} \in \Sigma, \tag{4.11}$$

see (3.8) and [Mie02a] for more details. Surprisingly, this information is enough to characterize $\Psi^{\rm red}$ in the isotropic case.

Proposition 4.4 *If $d \leq 3$, $\widetilde{W} : \mathrm{GL}_+(d) \mapsto \mathbb{R}_\infty$ is polyconvex, and $\widetilde{D}$ and h are given as above, then*

$$\widetilde{\Psi}^{\rm red}(p_{\rm old}; \mathbf{F}) = \min_{\mathbf{s}\in S_0} \Big(\widetilde{W}(\mathrm{diag}(\mathbf{f})\mathrm{e}^{\mathrm{diag}(\mathbf{s})}) + h(p_{\rm old}{+}|\mathbf{s}|) + |\mathbf{s}| \Big),$$

where $\mathbf{f} = (f_1, \ldots, f_d) \in (0,\infty)^d$ is the vector of singular values of $\mathbf{F}$ (eigenvalues of $(\mathbf{F}^\mathsf{T}\mathbf{F})^{1/2}$), $S_0 = \{\, \mathbf{s} \in \mathbb{R}^d \mid \mathbf{s}{\cdot}(1,\ldots,1)^\mathsf{T} = 0 \,\}$, and $|\mathbf{s}| = (\sum_1^d s_i^2)^{1/2}$.

Proof: By isotropy of $\widetilde{W}$ we find

$$\begin{aligned}\widetilde{\Psi}^{\rm red}(p_{\rm old}; \mathbf{F}) &= \min_{(\mathbf{P}_{\rm new}, p_{\rm new})} \Big(\widetilde{W}(\mathbf{F}\mathbf{P}_{\rm new}) + h(p_{\rm new}) + \widetilde{D}(\mathbf{P}_{\rm new}, p_{\rm old}, p_{\rm new}) \Big) \\ &= \min_{\boldsymbol{\sigma}\in\Sigma, \mathbf{R}\in \mathrm{SO}(d)} \Big(\widetilde{W}(\mathbf{F}\mathrm{e}^{\boldsymbol{\sigma}}\mathbf{R}) + h(p{+}d(\boldsymbol{\sigma},\mathbf{R})) + d(\boldsymbol{\sigma},\mathbf{R}) \Big),\end{aligned}$$

where we have used $d(\boldsymbol{\sigma},\mathbf{R}) := \widetilde{D}_{\rm no\,spin}(\mathrm{e}^{\boldsymbol{\sigma}}\mathbf{R})$ and that h is nondecreasing. Minimizing with respect to $\mathbf{R} \in \mathrm{SO}(d)$, and using isotropy and (4.11) we find

$$\widetilde{\Psi}^{\rm red}(p_{\rm old}; \mathbf{F}) = \min_{\boldsymbol{\sigma}\in\Sigma} \Big(\widetilde{W}(\mathbf{F}\mathrm{e}^{\boldsymbol{\sigma}}) + h(p_{\rm old}{+}|\boldsymbol{\sigma}|) + |\boldsymbol{\sigma}| \Big). \tag{4.12}$$

Let $\boldsymbol{\sigma} = \mathbf{R}_1^\mathsf{T}\,\mathrm{diag}(\mathbf{s})\mathbf{R}_1$ and $\mathbf{F} = \mathbf{R}_2\,\mathrm{diag}(\mathbf{f})\mathbf{R}_3$ such that $\mathbf{s} \in \mathbb{R}^d$ contains the eigenvalues of $\boldsymbol{\sigma}$ and $\mathbf{R}_j \in \mathrm{SO}(d)$. With $|\boldsymbol{\sigma}| = |\,\mathrm{diag}(\mathbf{s})| = |\mathbf{s}|$ and

$$\widetilde{W}(\mathbf{F}\mathrm{e}^{\boldsymbol{\sigma}}) = \widetilde{W}(\mathbf{R}_2\,\mathrm{diag}(\mathbf{f})\mathbf{R}_3\mathbf{R}_1^\mathsf{T}\mathrm{e}^{\mathrm{diag}(\mathbf{s})}\mathbf{R}_1) \tag{4.13}$$

we see that the outside rotations $\mathbf{R}_2$ and $\mathbf{R}_1$ can be omitted. In [Mie02b] it is shown that the minimum in (4.13) over $\mathbf{R}_1 \in \mathrm{SO}(d)$ is attained for $\mathbf{R}_1 = \mathbf{\Pi}\mathbf{R}_3$ for some permutation $\mathbf{\Pi}$. (Here we use $d \leq 3$). This gives the result. ∎

Similar results in this direction are given in [Šil01b, Šil01a]. They relate to the case without hardening and involve the concept of logarithmic convexity which fits well to the logarithmic nature of our dissipation distances.

In a specific example with $d = 2$ we can analyze $\Psi^{\rm red}$ further. Denote by $\mathbf{f} = (f_1, f_2)^\mathsf{T} \in (0,\infty)^2$ the vector of singular values of $\mathbf{F} \in \mathrm{GL}_+(2)$ and let $V : \mathbb{R} \mapsto \mathbb{R}_\infty$ be a convex, lower semicontinuous function with $V(\delta) = \infty$ for $\delta \leq 0$. For instance, the case $V(\delta) = +\infty$ for $\delta \neq 1$ and $V(1) = 0$ gives incompressible materials. Now let $\alpha \geq 1$ and

$$\widetilde{W}(\mathbf{F}) = V(\det \mathbf{F}) + \tfrac{1}{\alpha}(f_1^\alpha{+}f_2^\alpha) \quad \text{ where } \det \mathbf{F} = f_1 f_2,$$

then $\widetilde{W}$ is polyconvex and hence quasiconvex.

In Proposition 4.4 the minimum is taken over $\mathbf{s} \in S_0 = \{\, (s,-s) \mid s \in \mathbb{R} \,\}$ if $d = 2$. Hence $\det \mathrm{e}^{\mathrm{diag}(\mathbf{s})} = 1$ and the volumetric part $V(\det \mathbf{F})$ does not involve $\mathbf{s} \in S$. In order to do explicit calculations we choose the same hardening function h as in (4.9). The reduced energy density takes the form

$$\widetilde{\Psi}^{\rm red}(p_{\rm old};\mathbf{F}) = V(\det \mathbf{F}) + R(p_{\rm old};\mathbf{f}) - p_{\rm old}$$

where R is calculated using (C.5). With the abbreviation $b_p = \alpha e^{\alpha p/\sqrt{2}}$ and for $p \geq 0$ we find

$$R(p;\mathbf{f}) = \begin{cases} \frac{2}{\alpha}\sqrt{f_1^\alpha(f_2^\alpha + b_p)} & \text{for } f_2^\alpha - f_1^\alpha \leq -b_p, \\ \frac{1}{\alpha}(f_1^\alpha + f_2^\alpha + b_p) & \text{for } |f_2^\alpha - f_1^\alpha| \leq b_p, \\ \frac{2}{\alpha}\sqrt{f_2^\alpha(f_1^\alpha + b_p)} & \text{for } f_2^\alpha - f_1^\alpha \geq b_p. \end{cases} \tag{4.14}$$

We note that R satisfies the lower bound $R(p;\mathbf{f}) \geq c|\mathbf{F}|^{\alpha/2}$ as $|\mathbf{F}|^2 = f_1^2 + f_2^2$, which implies superlinear growth for $\alpha > 2$. Moreover, in [Mie02b] it is shown that for $\alpha \geq 2$ and $p_{\rm old} \geq 0$ the stored–energy density $\widetilde{\Psi}^{\rm red}(p_{\rm old};\cdot)$ is polyconvex. Thus, we conclude that the hardening function $h : p \mapsto e^{\alpha p} - p$ is strong enough to prevent failure. In [Mie03] it is shown that for this two–dimensional model with $\alpha > 6$ the incremental problem (IP) of Section 2.6 always has solutions in suitable function spaces.

A model which is not rank–one convex (and hence not quasiconvex) is obtained when $p < 0$ is assumed. Using (C.5) we find

$$R(p;\mathbf{f}) = \begin{cases} (f_1^\alpha + f_2^\alpha)/\alpha + 1 + \frac{\alpha p}{\sqrt{2}} & \text{for } f_2^\alpha \in [f_1^\alpha, \leq f_1^\alpha + \alpha], \\ \Gamma_{\mathbf{f}} + \log[\alpha(\Gamma_{\mathbf{f}} - 1)/(2 f_1^\alpha)] + 1 + \frac{\alpha p}{\sqrt{2}} & \text{for } f_2^\alpha \in [f_1^\alpha + \alpha, e^{-\sqrt{2}\alpha p} f_1^\alpha + \alpha e^{-\alpha p/\sqrt{2}}], \\ \frac{2}{\alpha}\sqrt{f_2^\alpha(f_1^\alpha + b_p)} & \text{for } f_2^\alpha \geq e^{-\sqrt{2}\alpha p} f_1^\alpha + \alpha e^{-\alpha p/\sqrt{2}}, \\ R(p; f_2, f_1) & \text{for } f_2 \in (0, f_1]; \end{cases}$$

where $\Gamma_{\mathbf{f}} = \sqrt{1 + 4 f_1^\alpha f_2^\alpha/\alpha^2}$. The limit $f_1 \to 0$ gives

$$R(p;0,f_2) = \begin{cases} f_2^\alpha/\alpha + 1 + \alpha p/\sqrt{2} & \text{for } 0 \leq f_2^\alpha \leq \alpha, \\ 1 + \log(f_2^\alpha/\alpha) + 1 + \alpha p/\sqrt{2} & \text{for } \alpha \leq f_2^\alpha \leq \alpha^2/b_p, \\ 2\sqrt{b_p} f_2^{\alpha/2}/\alpha & \text{for } f_2^\alpha \geq \alpha^2/b_p; \end{cases}$$

which clearly shows nonconvexity for the intermediate regime. Hence, the function R cannot be rank–one convex (since $R_{,22}(p, f_1, f_2) < 0$ for suitable f_2 and $0 < f_1 \ll 1$). Certainly there are volumetric functions $V : (0,\infty) \to \mathbb{R}_\infty$ such that the reduced energy density $\Psi^{\rm red}(p_{\rm old};\cdot) = V(\det\,\cdot\,) + R(p_{\rm old};\cdot) - p_{\rm old}$ is still not rank–one convex. However, for situations which are close to incompressibility quasiconvexity of $\Psi^{\rm red}$ could still hold, if the volumetric function V is chosen appropriately.

4.5 Formation of microstructure

We shortly comment on the possibility of formation of microstructure and refer to [OR99, ORS00, CHM02, MSL01, ML01, Mie02c] for more details.

If the the reduced energy density $\Psi^{\rm red}(p_{\rm old};\cdot)$ is not quasiconvex, we cannot guarantee the existence of minimizers for the incremental problem (IP), since any minimizer $\boldsymbol{\varphi}$ leads to macroscopic gradients $\mathbf{F}$ such that $\Psi^{\rm red}((\mathbf{P}(x_0), p(x_0));\cdot) : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ must be quasiconvex at $\mathbf{F}(x_0)$, see Theorem 2.3 and Corollary 4.3. In such a situation the incremental problem will have infimizing sequences $(\boldsymbol{\varphi}^j, z^j) \in \mathcal{F}\times\mathcal{Z}$ such that

$$\mathcal{E}(t_k, \boldsymbol{\varphi}^j, z^j)+\mathcal{D}(z_{k-1}, z^j) \to \alpha_k := \inf_{(\boldsymbol{\varphi},z)\in\mathcal{F}\times\mathcal{Z}} \mathcal{E}(t_k, \boldsymbol{\varphi}, z)+\mathcal{D}(z_{k-1}, z).$$

The major problem arises from the fact that the infimum α_k is not attained and only approximated through finer and finer microstructure in the functions $(\boldsymbol{\varphi}^j, z^j)$. Thus, microscopic fluctuations are needed to make the global incremental energy as small as possible. To handle such situations new methods are needed which keep track of these microscopic fluctuations as well as the overall macroscopic behavior. Such methods are suggested in [OR99, ORS00, HH01, MSL01, ML01] and are discussed theoretically in [Mie02c].

In Section 5 we also deal with a one–dimensional situation where the reduced energy density is not quasiconvex. There the formation of microstructure can be avoided as the one–dimensional case is very degenerate.

4.6 Localization

Localization means that the system prefers very localized strains or plastic changes over distributed ones. An indication for such a behavior is given when Ψ^{red} has linear behavior at infinity. Here we only discuss the simple one–dimensional model of Section 4.3 and leave higher–dimensional cases for future research.

As localization leads to failure or fracture the model really breaks down in such situations. Thus, the present discussion is just meant to discuss the question under what conditions localization occurs. If our understanding of this process is good enough, we can implement methods for preventing the undesired behavior.

Recall the model of Section 4.3 where we now restrict to the special case $\alpha = 2$:

$$W(\mathbf{F}) = \tfrac{1}{2}(\mathbf{F}^2+\mathbf{F}^{-2}-2), \qquad h(p) = \begin{cases} 1+p & \text{for } p \le 0, \\ \mathrm{e}^{2p} - p & \text{for } p \ge 0. \end{cases}$$

For $p \le 0$ the reduced energy density $\Psi^{\mathrm{red}}(p;\cdot)$ has the asymptote $A_p : \mathbf{F} \mapsto -1-p+\sqrt{2}\mathrm{e}^p\mathbf{F}$. There are three cases: (i) for $p \in (p_{\mathrm{t}}, 0]$ the asymptote lies strictly below $\widetilde{\Psi}^{\mathrm{red}}$, (ii) for $p = p_{\mathrm{t}} \approx -0.528$, the asymptote touches the graph of $\widetilde{\Psi}^{\mathrm{red}}$ in a point $F_{\mathrm{t}}(1, F_*)$, and (iii) for $p < p_{\mathrm{t}}$ the asymptote intersects the graph of $\widetilde{\Psi}^{\mathrm{red}}$.

In case (i) the convexification of $\widetilde{\Psi}^{\mathrm{red}}$ is obtained as in Section 4.3, i.e. by a tangent touching twice and having a slope smaller than the asymptote. In case (ii) the asymptote is the limit of the tangents when $p \to p_{\mathrm{t}}$ from above; and the asymptote forms the convexification. In case (iii) the convexification is given by the tangent which is parallel to the asymptote but lies below of it, see Figure 4.2.

To see the effect of the asymptotes we now perform a pure displacement experiment on the domain $\Omega = (0, 1) \subset \mathbb{R}^1$. We prescribe displacement conditions $\boldsymbol{\varphi}(t, 0) = 0$ and $\boldsymbol{\varphi}(t, 1) = l(t) = t+1$ and the initial conditions $(\mathbf{P}(0, x), p(0, x)) = (\mathbf{1}, p_0)$.

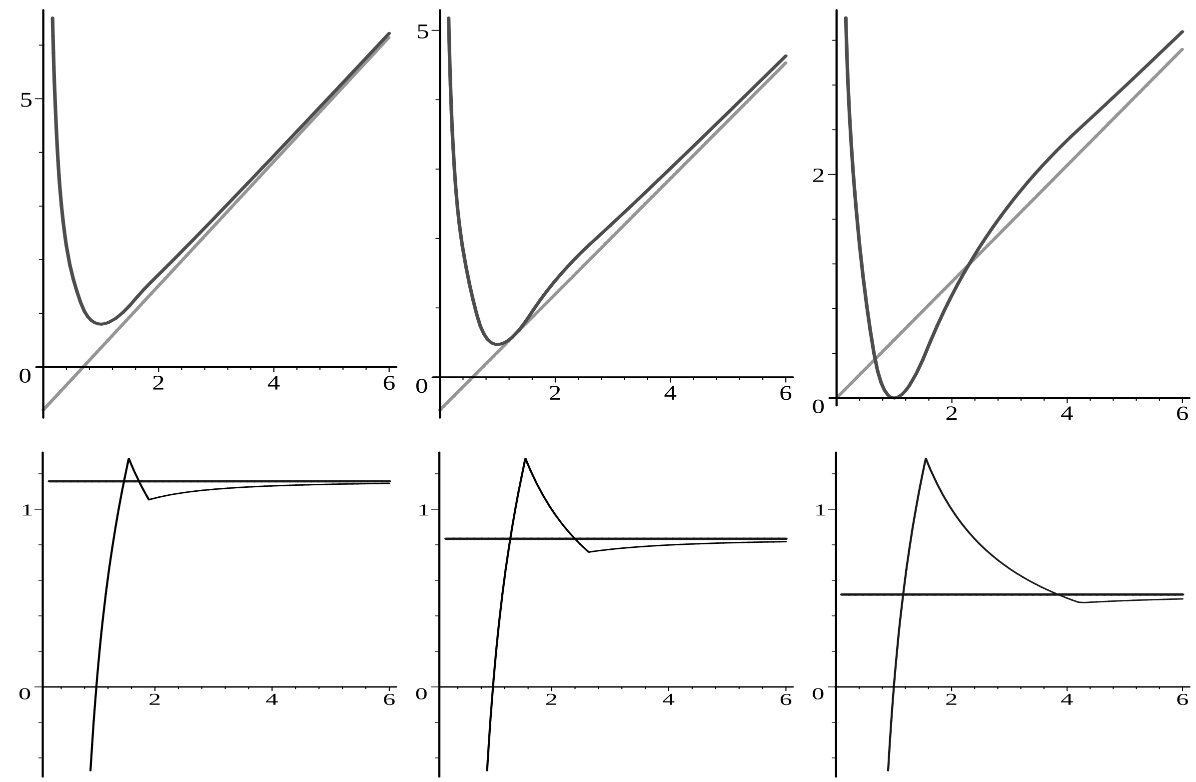


Figure 4.2: $\Psi^{\rm red}(p;\cdot)$ with its asymptote (above) and the stress–strain relation (below) for (i) $p=-0.2$, (ii) $p=p_{\rm t}\approx -0.528$, and (iii) $p=-1$. [The horizontal axis shows $\mathbf{F}$ and the vertical axis shows $\psi=\Psi^{\rm red}(p;\mathbf{F})$ (above) and $\mathbf{T}=\partial_{\mathbf{F}}\Psi^{\rm red}(p;\mathbf{F})$ (below).]

First consider the case $p_0\geq 0$. Then, $\Psi^{\rm red}(p;\cdot)$ is convex and we obtain a unique solution in the form $\boldsymbol{\varphi}(t,x)=l(t)x$ and $(\mathbf{P}(t,x),p(t,x))=(\mathrm{e}^{\pi(t)},\pi(t))$. The second case $p_0\in(p_{\rm t},0)$ we omit here and leave it for Section 5.

In the third case $p\leq p_{\rm t}$ we denote by $\mathbf{F}_0>1$ the unique point where the convexification changes from $\Psi^{\rm red}(p_0;\cdot)$ to the left to the tangent, which is parallel to the asymptote, on the right. Then, the solution is homogeneous for $t\leq t_0:=\mathbf{F}_0{-}1$. For $t>t_0$ there cannot exist a homogeneous solution, since for a stable solution $\Psi^{\rm red}$ must coincide with its convexification and since for $\mathbf{F}>\mathbf{F}_0$ no such points exist. It is easy to check that up to t_0 the solution behaves purely elastic (i.e., $\mathbf{F}_0<\mathbf{F}_*$). For $t>t_0$ the solution must be inhomogeneous with average strain $\int_0^1\mathbf{F}(x)\,\mathrm{d}x=l(t)=t{+}1$. However, the energetic minimum cannot be achieved with a classical deformation. To see this note that the stress in the one–dimensional setting must be constant and the only possible value is the slope of the asymptote. Yet this slope is only achieved at $\mathbf{F}_0$ and at infinity. Thus a generalized solution is obtained by choosing $\mathbf{F}(x)=\mathrm{D}\boldsymbol{\varphi}(t,x)$ such that it is of the form

$$\mathbf{F}(t,x)=\mathbf{F}_0+\textstyle\sum_{j=1}^N\alpha_j(t)\delta_{x_j}(x)\quad\text{with } \dot\alpha_j\geq 0 \text{ and } l(t)=\mathbf{F}_0{+}\textstyle\sum_{j=1}^N\alpha_j(t),$$

where $x_j\in\Omega$ can be understood as fracture points. Thus, we have localized points where the deformation is extreme whereas in other areas the deformation remains bounded.

(A good way to understand this phenomenon is to study infimizing sequences for the incremental problem, where the infimum is not attained in the set of classical functions.)

A natural setting in this context is the space of functions of bounded variations, $\mathrm{BV}((0,1))$, which contains those functions whose distributional derivative is a signed measure. We refer to [FF95, AFF97, MT01] and the references therein for the rich literature in this context.

5 One–dimensional models

To illustrate the above theory we study the somewhat unphysical one–dimensional situation. However, this model shows typical features of finite–strain elasto–plasticity and has the big advantage that it can be solved analytically.

Again we write $\mathbf{F}_2/\mathbf{F}_1$ instead of $\mathbf{F}_1^{-1}\mathbf{F}_2$, since the matrices reduce to real scalars.

5.1 The model

We let $\Omega = (0,1) \subset \mathbb{R}^1$ and

$$\widehat{\psi}(\mathbf{F},\mathbf{P},p) = \widetilde{\psi}(\mathbf{F}_{\mathrm{el}},p) = W(\mathbf{F}_{\mathrm{el}}) + h(p),$$

where $W : \mathbb{R} \mapsto \mathbb{R}$ is convex with $W(1) = 0$, $W'(0) = 0$ and $W(\mathbf{F}) \to \infty$ for $\mathbf{F} \searrow 0$ or $\mathbf{F} \nearrow \infty$. The hardening energy will satisfy $h'(p), h''(p) \geq 0$ for $p \in \mathbb{R}$. We will see below that $h'' > 0$ corresponds to hardening whereas $h''(p) < 0$ indicates regions of softening.

In addition we prescribe the external driving of the system by the displacement conditions $\boldsymbol{\varphi}(t,0) = 0$ and $\boldsymbol{\varphi}(t,1) = \ell(t)$. For simplicity we consider the simple pure tension test $\ell(t) = t{+}1$. For the internal variables we take the initial data $(\mathbf{P}(0,x), p(0,x)) = (\mathbf{1}, 0)$.

The thermodynamically conjugated forces are

$$\mathbf{T} = \tfrac{\partial}{\partial \mathbf{F}}\widehat{\psi} = \mathbf{P}W'(\mathbf{F}\mathbf{P}), \quad \mathbf{Q} = -\tfrac{\partial}{\partial \mathbf{P}}\widehat{\psi} = -\mathbf{F}W'(\mathbf{F}\mathbf{P}), \quad q = -\tfrac{\partial}{\partial p}\widehat{\psi} = -h'(p).$$

The elastic domain is $\mathbb{Q}(\mathbf{P},p) = \mathrm{diag}(\mathbf{P}^{-\mathsf{T}}, 1)\widetilde{\mathbb{Q}}$ where $\widetilde{\mathbb{Q}}$ is assumed to have the form $\widetilde{\mathbb{Q}} = \{\, (\boldsymbol{\eta}, q) \in \mathbb{R}^2 \mid |\boldsymbol{\eta}| \leq 1{-}q,\ q < 0 \,\}$. With Section 3.4.1 we obtain

$$\widehat{\Delta}(\mathbf{P},p,\dot{\mathbf{P}},\dot{p}) = \widetilde{\Delta}(\dot{\mathbf{P}}/\mathbf{P},\dot{p}) = \begin{cases} |\dot{\mathbf{P}}/\mathbf{P}| & \text{for } \dot{p} \geq |\dot{\mathbf{P}}/\mathbf{P}|, \\ \infty & \text{else.} \end{cases} \tag{5.1}$$

According to Sections 3.4.1 and 4.3 the dissipation distance reads $\widehat{D}((\mathbf{P}_0,p_0),(\mathbf{P}_1,p_1)) = \widetilde{D}(\mathbf{P}_1/\mathbf{P}_0,p_0,p_1) = |\log(\mathbf{P}_1/\mathbf{P}_0)|$ for $p_1 \geq p_0 + |\log(\mathbf{P}_1/\mathbf{P}_0)|$ and ∞ else. Moreover, the reduced energy density takes the form

$$\begin{aligned} \Psi^{\mathrm{red}}((\mathbf{P}_{\mathrm{old}},p_{\mathrm{old}});\mathbf{F}) &= \min_{(\mathbf{P},p)} \Psi(\mathbf{F},\mathbf{P},p) + \widetilde{D}(\mathbf{P}/\mathbf{P}_{\mathrm{old}},p_{\mathrm{old}},p) \\ &= \min_{\mathbf{P}>0} W(\mathbf{F}\mathbf{P}) + h(p_{\mathrm{old}}{+}|\log(\mathbf{P}/\mathbf{P}_{\mathrm{old}})|) + |\log(\mathbf{P}/\mathbf{P}_{\mathrm{old}})|, \end{aligned}$$

where we have used $h' \geq 0$.

The main assumptions for the following analysis are the standard properties $W'(\mathbf{F}) > 0$ and $W''(\mathbf{F}) > 0$ for $\mathbf{F} > 1$ and the assumptions $h'(p), h''(p) \geq 0$ which forbids softening, see (5.5). To obtain nontrivial results we make the additional assumption that $\Psi^{\rm red}((\mathbf{P}_{\rm old}, p_{\rm old}); \cdot)$ is nonconvex in the region $\mathbf{F} > 1$. It is important to note that this does not contradict the other assumptions. A first example for this fact was given in Section 4.3.

Another example, which is more explicit, is the following. It is in some sense singular as h_* is nonsmooth, but it satisfies the assumptions in a generalized sense:

$$h_*(p) = 0 \text{ for } p \leq \log 2 \quad \text{and} \quad h_*(p) = \infty \text{ for } p > \log 2.$$

Hence, there is no hardening for $p \leq \log 2$ while there is total hardening at $p = \log 2$ (no plasticity at all). For the energy density we assume $W_*(\mathbf{F}) = \frac{1}{4}(\mathbf{F}-1)^2$ for $\mathbf{F} \geq 1$. (Throughout the sub– or superscript $*$ indicates that the quantity associates to this particular example.)

For $\mathbf{P} \in [1/2, 1]$ and $\mathbf{F} \geq 1$ we find

$$\Psi_*^{\rm red}((\mathbf{P}, \log(1/\mathbf{P})); \mathbf{F}) = \begin{cases} W_*(\mathbf{FP}) & \text{for } 1 \leq \mathbf{FP} \leq 2, \\ W_*(2) + \log(\mathbf{FP}/2) & \text{for } 2 \leq \mathbf{FP} \leq 4\mathbf{P}, \\ W_*(\mathbf{F}/2) + \log(2\mathbf{P}) & \text{for } \mathbf{F} \geq 4. \end{cases} \tag{5.2}$$

Observe that for $\mathbf{P} \in (1/2, 1]$ this function is nonconvex in $\mathbf{F} \geq 1$ and that it has a continuous derivative.

In particular, the convexification of $\Psi_*^{\rm red}((\mathbf{1}, 0); \cdot)$ is given by replacing $\Psi_*^{\rm red}((\mathbf{1}, 0); \cdot)$ on the interval $(\mathbf{F}_1^*, \mathbf{F}_2^*)$ by the linear segment which is formed by the coinciding tangents in $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$. A calculation gives $\mathbf{F}_1^* = (\sqrt{1+12\log 2}+2)/3 \approx 1.68$ and $\mathbf{F}_2^* = 4\mathbf{F}_1^*-2 \approx 4.74$, and the tangent has the slope $(\mathbf{F}_1^*-1)/2 \approx 0.342$.

5.2 Solutions of the local flow rule

The local problem is given by $(\mathrm{S})_{\rm loc}$ and $(\mathrm{E})_{\rm loc}$. According to Section 2.4 the local stability condition $(\mathrm{S})_{\rm loc}$ consists of the elastic equilibrium (here: constant stress in Ω) and the local flow rule

$$\mathbf{QV} + qv \leq \widehat{\Delta}(\mathbf{P}, p, \mathbf{V}, v) = \widetilde{\Delta}(\mathbf{V}/\mathbf{P}, v) \text{ for all } (\mathbf{V}, v) \in \mathbb{R}^2.$$

Letting $\mathbf{V} = \mathbf{P}\boldsymbol{\xi}$ and using (5.1) the second condition takes the form $\mathbf{PQ}\boldsymbol{\xi} + qv \leq |\boldsymbol{\xi}|$ for all $(\boldsymbol{\xi}, v)$ with $|\boldsymbol{\xi}| \leq v$. With $(\mathbf{Q}, q) = -(\boldsymbol{\varphi}'W'(\boldsymbol{\varphi}'\mathbf{P}), h(p))$ this leads to

$$\begin{aligned} &(\mathrm{S})_{\rm loc} && \begin{cases} \mathbf{P}W'(\boldsymbol{\varphi}'\mathbf{P}) = c(t) \text{ (const. on } \Omega), \\ |\boldsymbol{\varphi}'\mathbf{P}W'(\boldsymbol{\varphi}'\mathbf{P})| \leq 1+h'(p), \ h'(p) \geq 0; \end{cases} \\ &(\mathrm{E})_{\rm loc} && |\dot{\mathbf{P}}/\mathbf{P}| \leq -\boldsymbol{\varphi}'W'(\boldsymbol{\varphi}'\mathbf{P})\dot{\mathbf{P}} - h'(p)\dot{p}. \end{aligned} \tag{5.3}$$

From $(\mathrm{S})_{\rm loc}$ and $(\mathrm{E})_{\rm loc}$ we deduce $\dot{p} = |\dot{\mathbf{P}}/\mathbf{P}|$ which shows that the hardening parameter was chosen such that it grows together with $|\dot{\mathbf{P}}/\mathbf{P}|$. Other choices would lead to equivalent models if the hardening energy h is adjusted accordingly.

The problem (5.3) has a unique homogeneous solution of the form $\boldsymbol{\varphi}(t,x) = \ell(t)x = (t{+}1)x$, $(\mathbf{P}(t,x), p(t,x)) \equiv (\mathbf{P}(t), p(t))$. From $\dot{\mathbf{F}} = \dot{\ell} = 1$ we easily find $\dot{\mathbf{P}} \leq 0$, i.e., $\mathbf{F}_{\rm pl} = 1/\mathbf{P} \geq 1$ grows. Hence, $\dot{p} = -\dot{\mathbf{P}}/\mathbf{P}$ and we can eliminate p via $p(t) = |\log \mathbf{P}(t)| = \log(1/\mathbf{P}(t))$. With this (5.3) reduces to

$$|\ell \mathbf{P} W'(\ell\mathbf{P})| \leq 1{+}g(\mathbf{P}) \text{ and } \ell\mathbf{P}W'(\ell\mathbf{P})\dot{\mathbf{P}} = (1{+}g(\mathbf{P}))\dot{\mathbf{P}} \tag{5.4}$$

where $g(\mathbf{P}) = h'(\log(1/\mathbf{P}))$. From this we find $\mathbf{P}(t) = 1$ for $t \in [0, \ell_{\rm y}{-}1]$ where the yield strain $\ell_{\rm y} > 1$ is defined via $\ell_{\rm y}W'(\ell_{\rm y}) = 1{+}g(0)$. From then on $\mathbf{P}(t)$ is defined implicitly via

$$G(\ell, \mathbf{P}) := \ell\mathbf{P}W'(\ell\mathbf{P}) - 1 - g(\mathbf{P}) = 0.$$

Taking the time derivative and using $\dot{\ell} = 1$ we find the ODE

$$\dot{\mathbf{P}}/\mathbf{P} = -\tfrac{\boldsymbol{\Sigma}(\ell\mathbf{P})}{\ell\boldsymbol{\Sigma}(\ell\mathbf{P}) - g'(\mathbf{P})} \text{ with } \boldsymbol{\Sigma}(\mathbf{F}) = W'(\mathbf{F}) + \mathbf{F}W''(\mathbf{F}),$$

which is certainly well–posed as long as $g'(\mathbf{P}) = -h''(\log(1/\mathbf{P}))/\mathbf{P} \leq 0$ (i.e. no softening occurs), since $\boldsymbol{\Sigma} > 0$ under tension. In fact, we could allow for a moderate softening without losing existence of solutions.

For the elastic strain $\mathbf{F}_{\rm el} = \mathbf{F}\mathbf{P} = \ell\mathbf{P}$ we have

$$\dot{\mathbf{F}}_{\rm el} = \mathbf{P} + \ell\dot{\mathbf{P}} = -\mathbf{P}g'(\mathbf{P})/(\ell\boldsymbol{\Sigma}(\ell\mathbf{P}){-}g'(\mathbf{P})), \tag{5.5}$$

which manifests hardening and softening for $\operatorname{sign}(-g'(\mathbf{P})) = \operatorname{sign}(h''(\mathbf{P}))$ positive and negative, respectively.

For the special example with W_* and h_* we cannot apply the ODEs but there is still a simple solution of (5.4) if we add the condition $\dot{\mathbf{P}} = 0$ if $\mathbf{P} = 1/2$. We have $\boldsymbol{\varphi}_{\rm loc}(t,x) = \ell(t)x$, the yield strain is $\ell_{\rm y}^* = 2$ and the plastic flow starts at $t_{\rm y}^* = 1$ and stops at $t = 3$:

$$(\mathbf{P}_{\rm loc}(t), \mathbf{F}_{\rm el,loc}(t)) = \begin{cases} (\quad 1 \quad, \quad t{+}1 \quad) & \text{for } t \in [0,1], \\ (\,2/(t{+}1)\,, \quad 2 \quad) & \text{for } t \in [1,3], \\ (\quad 1/2 \quad, (t{+}1)/2\,) & \text{for } t \geq 3. \end{cases} \tag{5.6}$$

The stress–strain relation in the $(\ell, \mathbf{T})$–plane is depicted in Figure 5.1 as a full line.

5.3 Solution of the global problem (S) and (E)

To study the global problem we first give a characterization of stability.

Proposition 5.1 *A state $(\boldsymbol{\varphi}, \mathbf{P}, p)$ satisfies (S) if and only if there exists a constant $k \in \mathbb{R}$ such that*

$$\Psi^{\rm red}((\mathbf{P}(x), p(x)); \mathbf{F}) \geq \widehat{\psi}(\boldsymbol{\varphi}'(x), \mathbf{P}(x), p(x)) + k(\mathbf{F}{-}\boldsymbol{\varphi}'(x)) \tag{5.7}$$

for almost all $x \in \Omega$ and all $\mathbf{F} > 0$.

Proof: Minimizing with respect to $\widehat{z} = (\widehat{\mathbf{P}}, \widehat{p})$ in the definition of stability we find that $(\boldsymbol{\varphi}, \mathbf{P}, p) : \Omega \mapsto \mathbb{R}^d$ is stable if and only if

$$\textstyle\int_0^1 \widehat{\psi}(\boldsymbol{\varphi}'(x), \mathbf{P}(x), p(x))\,\mathrm{d}x \leq \int_0^1 \Psi^{\rm red}((\mathbf{P}(x), p(x)); \widehat{\boldsymbol{\varphi}}'(x))\,\mathrm{d}x \tag{5.8}$$

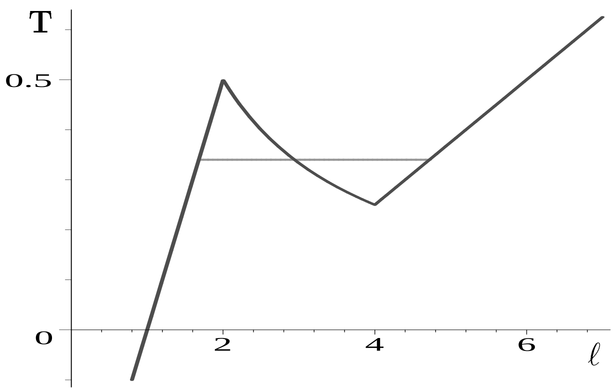


Figure 5.1: The stress–strain relation of the solution of the local problem (full line) and for the solution of the global problem (grey horizontal line).

for all $\widehat{\boldsymbol{\varphi}} : \Omega \mapsto \mathbb{R}$ with $\widehat{\boldsymbol{\varphi}}(0) = 0$ and $\widehat{\boldsymbol{\varphi}}(1) = \boldsymbol{\varphi}(1)$.

Assuming that (5.7) holds we easily obtain (5.8) by letting $\mathbf{F}(x) = \widehat{\boldsymbol{\varphi}}'(x)$ and integrating over Ω. Since the term involving k cancels due to the boundary conditions we find the desired stability.

For the opposite conclusion we start from stability in the form (5.8) and denote the right–hand side by $\mathcal{J}(\widehat{\boldsymbol{\varphi}})$. By construction of the reduced energy density we have $\widehat{\psi}(\boldsymbol{\varphi}', \mathbf{P}, p) \geq \Psi^{\mathrm{red}}((\mathbf{P}, p); \widehat{\boldsymbol{\varphi}}')$ and stability in the form of Proposition 4.2 implies in fact equality. Hence, (5.8) implies that $\boldsymbol{\varphi}$ is a global minimizer of $\mathcal{J}$. By Weierstraß' necessary conditions for strong (local) minimizers we find that $\partial_{\mathbf{F}}\Psi^{\mathrm{red}}((\mathbf{P}(x), p(x)); \boldsymbol{\varphi}'(x))$ must be equal to a constant k and that $\Psi^{\mathrm{red}}((\mathbf{P}(x), p(x)); \cdot)$ must lie above its tangent in $\mathbf{F} = \boldsymbol{\varphi}'(x)$. This gives (5.7). ■

With this result we are now able to construct solutions of the global problem (S) & (E). For simplicity we do this only for the special case defined via W_* and h_*. To this end we first check whether the local solution satisfies the global stability condition. By construction the energy inequality (E) is satisfied already.

For $t \in [0, 1]$ we have $(\mathbf{F}, \mathbf{P}, p) = (t{+}1, 1, 0)$. Proposition 5.1 gives stability as long as $\mathbf{F}(t) = t{+}1$ remains in the convexity region of $\Psi_*^{\mathrm{red}}((1, 0); \cdot)$; and this is the case for $t \in [0, t_1^*]$ with $t_1^* = \mathbf{F}_1^*{-}1 \approx 0.684$. For $t \in (t_1^*, 3)$ we cannot have global stability, since $(\mathbf{F}(t), \mathbf{P}(t), p(t)) = (t{+}1, 2/(t{+}1), \log([t{+}1]/2))$ and Ψ_*^{red} does not lie above its tangent in this point. For $t \geq 3$ we have $(\mathbf{F}(t), \mathbf{P}(t), p(t)) = (t{+}1, 1/2, \log 2)$ and global stability holds since Ψ_*^{red} is now convex.

The above stability check implies that the solution of the global problem (S) & (E) coincides with that of the local problem $(\mathrm{S})_{\mathrm{loc}}$ & $(\mathrm{E})_{\mathrm{loc}}$ on the time interval $[0, t_1^*]$. We set $t_2^* = \mathbf{F}_2{-}1 \approx 3.74$ and define a solution of (S) & (E) as follows. Choose any function $\mathcal{X} : [t_1^*, t_2^*]{\times}\Omega \to \{0, 1\}$ with the following properties:

$$\textstyle\int_\Omega \mathcal{X}(t, x)\,\mathrm{d}x = \frac{t - t_1^*}{t_2^* - t_1^*} \quad \text{and for each } x \in \Omega,\ t_1^* \leq r < s \leq t_2^* :\ \mathcal{X}(r, x) \leq \mathcal{X}(s, x),$$

and define $(\boldsymbol{\varphi}_{\mathrm{glob}}, \mathbf{P}_{\mathrm{glob}}, p_{\mathrm{glob}})$ on $[t_1, t_2]$ via $\boldsymbol{\varphi}_{\mathrm{glob}}(t, x) = \mathbf{F}_1^* x + \int_0^x \mathcal{X}(t, y)\,\mathrm{d}y\,(\mathbf{F}_2^*{-}\mathbf{F}_1^*)$ and

$$(\mathbf{P}_{\mathrm{glob}}(t, x), p_{\mathrm{glob}}(t, x)) = \begin{cases} (1, 0) & \text{for } \mathcal{X}(t, x) = 0, \\ (1/2, \log 2) & \text{for } \mathcal{X}(t, x) = 1. \end{cases} \tag{5.9}$$

For $t \geq t_2^*$ we continue the solution simply by the (homogeneous) solution of the local problem. Altogether we have found a big variety of solutions of the global problem. However, the elastic stress $\mathbf{T}$ is the same for all the different solutions, namely $\mathbf{T}(t) = W_*'(\mathbf{F}_1^*) = \frac{1}{2}W_*'(\mathbf{F}_2^*/2)$. In Figure 5.1 the stress–strain relation for the solution of the global problem is indicated by the dashed line.

5.4 Local stability and continuity

In this section we want to compare the solution $(\boldsymbol{\varphi}_{\rm glob}, z_{\rm glob})$ of the global problem and the solution $(\boldsymbol{\varphi}_{\rm loc}, z_{\rm loc})$ of local problem, see (5.9) and (5.6). To shorten notation we use $z = (\mathbf{P}, p)$.

To this end we introduce a topology in the space $\mathcal{F}\times\mathcal{Z}$ to measure distances and continuities of solution paths. The denote by $\|(\boldsymbol{\varphi}, z)\|_r$ the norm of $\mathcal{Y}^r := \mathrm{W}^{1,r}(\Omega)\times[\mathrm{L}^r(\Omega)]^2$ where r lies in $[1,\infty]$. As a first result we see that the functions $t \mapsto (\boldsymbol{\varphi}_{\rm loc}(t), z_{\rm loc}(t)) \in \mathcal{Y}^r$ is Lipschitz continuous for all $r \in [1,\infty]$. In contrast, the function $t \mapsto (\boldsymbol{\varphi}_{\rm glob}(t), z_{\rm glob}(t)) \in \mathcal{Y}^r$ satisfies only the Hölder continuity

$$\|(\boldsymbol{\varphi}_{\rm glob}(t), z_{\rm glob}(t)) - (\boldsymbol{\varphi}_{\rm glob}(s), z_{\rm glob}(s))\|_r \leq C_r|t-s|^{1/r},$$

where the problem arises for $s, t \in [t_1^*, t_2^*]$. For $r = \infty$ we don't have any continuity.

Another question arises with respect to the stability properties of $(\boldsymbol{\varphi}_{\rm loc}, z_{\rm loc})$. We may define local stability in terms of the r–norm $\|\cdot\|_r$ as follows. The state $(\boldsymbol{\varphi}_{\rm loc}(t), z_{\rm loc}(t))$ is **locally stable** if there exists a $\delta > 0$ such that

$$\begin{aligned}&\mathcal{E}(t, \boldsymbol{\varphi}_{\rm loc}(t), z_{\rm loc}(t)) \leq \mathcal{E}(t, \widehat{\boldsymbol{\varphi}}, \widehat{z}) + \mathcal{D}(z_{\rm loc}(t), \widehat{z})\\ &\text{for all } (\widehat{\varphi}, \widehat{z}) \in \mathcal{F}\times\mathcal{Z} \text{ with } \|(\boldsymbol{\varphi}_{\rm loc}(t), z_{\rm loc}(t)) - (\widehat{\boldsymbol{\varphi}}, \widehat{z})\|_r \leq \delta.\end{aligned} \tag{5.10}$$

We obtain the following result.

Proposition 5.2 *In the case $r \in [1,\infty)$ the solution $(\boldsymbol{\varphi}_{\rm loc}(t), z_{\rm loc}(t))$ is locally stable in the sense of (5.10) only for $t \notin (t_1^*, 3)$ with $t_1^* \approx 0.684$.*

In the case $r = \infty$ the solution $(\boldsymbol{\varphi}_{\rm loc}(t), z_{\rm loc}(t))$ is locally stable for all $t \notin [1, 3)$.

Proof: For the case $r < \infty$ we use the fact that local stability in $\mathcal{Y}^r$ implies spatially localized stability in the sense of Section 4.2. This is easily seen by introducing perturbations of the form (4.6) and using the fact that, for fixed $(\overline{\boldsymbol{\varphi}}, \overline{\mathbf{P}}, \overline{p})$ we have $\|(\boldsymbol{\varphi}, z) - (\widehat{\boldsymbol{\varphi}}_\varepsilon, \widehat{z}_\varepsilon)\|_r = O(\varepsilon^{1/r})$ for $\varepsilon \to 0$. Hence, local stability implies via (4.7) that $\Psi_*^{\rm red}(z_{\rm loc}; \cdot)$ must be convex at $\mathbf{F}_{\rm loc}(t) = \ell(t)$ (i.e., it lies above its tangent in this point).

For $t \notin (t_1^*, 3)$ we have global and hence local stability in $\mathcal{Y}^r$. For $t \in (t_1^*, 3)$ we consider the reduced energy density $\Phi_t := \Psi_*^{\rm red}((\mathbf{P}_{\rm loc}(t), \log[1/\mathbf{P}_{\rm loc}(t)]); \cdot)$ as given in (5.2). Since it is not convex at $\ell(t)$, the desired local instability is established.

For the case $r = \infty$ we proceed differently. Local stability for $t \notin (t_1^*, 3)$ follows from that for finite r, since $\|\cdot\|_r \leq \|\cdot\|_\infty$. Instead of $\Psi_*^{\rm red}$ we use the locally reduced density

$$\Psi^\delta(z_{\rm old}; \mathbf{F}) = \min_{|z - z_{\rm old}| \leq \delta} \widehat{\psi}(\mathbf{F}, z) + \widehat{D}(z_{\rm old}, z).$$

Clearly, local stability of $(\boldsymbol{\varphi}_{\rm loc}(t), z_{\rm loc}(t))$ holds if and only if $\Psi^\delta(z_{\rm loc}(t);\cdot)$ lies above its tangent in a neighborhood of $\mathbf{F}_{\rm loc}(t) = \ell(t)$ (note that the gradients are also restricted via the norm in $\mathcal{Y}^\infty$). This condition is clearly satisfied for $t < 1$ since in this regime we have $(\mathbf{P}_{\rm loc}(t) = \mathbf{1}$ and $\Psi^\delta(z_{\rm loc}(t), \mathbf{F}) = W_*(\mathbf{F})$. For $t \in [1, 3]$ we are in the plastic flow regime with $\mathbf{F}_{\rm el}(t) = \ell(t)\mathbf{P}_{\rm loc}(t) = 2$. Considering the definition of Ψ^δ and $\Psi^{\rm red}_*$ we find that these two functions coincide in a neighborhood of $(z_{\rm loc}(t), \mathbf{F}_{\rm loc}(t))$. Moreover, $\Psi^{\rm red}_*(z_{\rm loc}(t), \cdot)$ is not convex in $\mathbf{F}_{\rm loc}(t)$ because of the logarithmic term to the right of $\mathbf{F}_{\rm loc}(t)$, cf. (5.2). This proves local instability in $\mathcal{Y}^\infty$. ■

Remark 5.3 *In our special case defined through W_* and h_* we found that the solution $(\boldsymbol{\varphi}_{\rm loc}, z_{\rm loc})$ was never locally stable in the case when plastic flow occurs. This is due to the fact that the system has no hardening, viz., $h'_*(p) = 0$ for $p < 2$. Treating the general one–dimensional system of Section 5.1 with real hardening we obtain local stability in $\mathcal{Y}^\infty$, if the second derivative (from the right) of $\Psi^{\rm red}(z_{\rm loc}(t);\cdot)$ at $\mathbf{F}_{\rm loc}(t)$ is nonnegative.*

Denoting by $\mathbf{F}^{\rm y}_{\rm el}(p)$ the elastic yield strain associated with the hardening parameter p and using the formula from Proposition 4.1 we obtain, for the invariant form $\widetilde{\Psi}^{\rm red}$ of the reduced energy density, the formula

$$\lim_{\mathbf{F}\searrow\mathbf{F}^{\rm y}_{\rm el}(p)} \mathrm{D}^2_{\mathbf{F}}\widetilde{\Psi}^{\rm red}(p;\mathbf{F}) = \Big[W''(\mathbf{F}) - \frac{[W''(\mathbf{F})\mathbf{F}+W'(\mathbf{F})]^2}{W''(\mathbf{F})+h'(p)+h''(p)}\Big]_{\mathbf{F}=\mathbf{F}^{\rm y}_{\rm el}(p)}.$$

This indicates that a certain amount of hardening in the form of $h'(p)+h''(p)$ is needed to have this second derivative nonnegative. Only then the classical solutions can be stable during plastic yielding. Obviously this condition does not hold in the above simple model.

6 Discussion

In this work we have established a mathematical basis for variational incremental methods used in finite–strain elasto–plasticity in [OS99, OR99, ORS00, HH01, ML01, MSL01]. The main advantages of the energetic formulation of elasto–plasticity via the properties (S) & (E) are the following:

(i) No derivatives of $\mathbf{F} = \mathrm{D}\boldsymbol{\varphi}$, $\mathbf{P}$ and p with respect to space x and time t appear; hence very weak solutions can be considered (even Young measure solutions, cf. [Mie02c]).

(ii) No derivatives of the constitutive functions $\widehat{\psi}$ and $\widehat{\Delta}$ appear; hence general constitutive laws like incompressibility of multisurface plasticity can be treated easily.

(iii) The reduced energy density $\Psi^{\rm red}$ appears naturally and displays many important features of the interplay between elastic and plastic properties. Thus, it provides a new theoretical tool for evaluating complex material laws.

(iv) The most natural incremental problem (IP) is a variational problem leading to variational updates having symmetric algorithmic tangent moduli. Moreover, homogenization and relaxation techniques can be applied to the functionals $\mathcal{E}$ and $\mathcal{D}$, cf. [Mie02c]).

Of course, the present model is extreme in two respects. First, it is strictly rate independent. Only in the full rate–independent limit (not just quasistatic) the system is always completely relaxed, since the relaxation time scales for the elastic deformation and the internal dynamics are much smaller than the time scale of the external loading. Only in this situation the stability concept (S) is useful. Second, we impose global stability and not just local stability. This requirement is essential in the formulation of the incremental problem (IP) where we ask for the global minimizer not just for a local one. In particular, an analog of the basic Theorem 2.3 would not hold if just local minimization is done. In [MTL02], Sect. 3.1, there is a discussion the modeling of rate–independent hysteresis using local minimization versus global minimization using an additional dissipation to penalize the jumping out of a local minimum.

Nevertheless, it seems more natural to replace the assumption of global minimization by just asking for local minimization. This leads to the following localized incremental problem. It seems to be more appropriate to describe nature as well as numerical codes, since in both cases only nearby local minima are relevant. For mathematicians there is the problem, that there doesn't exists a good existence theory for local minimizers.

(IP)$_{\rm loc}$ Localized Incremental Problem: For $k = 1, 2, \ldots, N$ find

$$(\boldsymbol{\varphi}_k, z_k) \in \underset{\substack{(\boldsymbol{\varphi}, z) \in \mathcal{F}\times\mathcal{Z} \\ \mathcal{N}((\boldsymbol{\varphi}, z), (\boldsymbol{\varphi}_k, z_k)) \leq \delta}}{\operatorname{argmin}} \mathcal{E}(t_k, \boldsymbol{\varphi}, z) + \mathcal{D}(z_{k-1}, z).$$

We refer to [EM03] for a first analysis of a simplified version of such a system.

There are two questions concerning this localized version. First we have to decide about the new distance measure $\mathcal{N}$ on $\mathcal{F}\times\mathcal{Z}$ which is used to decide what local really means. Second we have to decide about the smallness parameter δ which needs to be related to the size of the time steps $\tau = \max_{k=1,\ldots,N} t_k - t_{k-1}$. A suitable choice would be $\delta = \sqrt{\tau}$ to allow for fast reactions like jumps.

The question about the topology related to the distance $\mathcal{N}$ is a delicate one. We have investigated the stability properties of the local solution $\boldsymbol{\varphi}_{\rm loc}, z_{\rm loc})$ in the one–dimensional model in Section 5.4. We found significant differences in the stability properties with respect to the norms in $\mathcal{Y}^r$, if $r < \infty$ and $r = \infty$ are considered. Only in the case $r = \infty$ one obtains stability up to the point $\ell(t) = 2$ which might be considered to be the point where the local failure happens. However, in continuum mechanics it is not reasonable to make restriction in the L^∞–norm. One should rather use integral quantities. In particular, in localization phenomena or in phase separations (formation of twins) the gradients will jump at least in a small subset (nucleus) to a distant point which going along a continuous path, at least in the rate–independent limit.

One possible way out of this dilemma would be to use the dissipation itself as the distance measure $\mathcal{N}$ in (IP)$_{\rm loc}$. Another way would be to ask that $(\boldsymbol{\varphi}_k, z_k)$ has to lie in the same potential well as $(\boldsymbol{\varphi}_{k-1}, z_{k-1})$ with possible switches if they are really necessary. At this point we cannot resolve this question and we refer to future research.

There is an easy way to check whether numerically obtained solutions $(\boldsymbol{\varphi}_k, z_k)_{k=1,\ldots,N}$ are in fact global or local minimizers. The point is that true solutions of (S) & (E) (which are global minimizers) do in fact satisfy an *energy equality* and not just an inequality, see

Lemma 3.7 in [MT01] and also our lower estimate in Theorem 2.3(ii). As a consequence one obtains that the sum $\mathcal{E}(t,\boldsymbol{\varphi}(t),z(t))+\mathrm{Diss}(z,[0,t])$ is continuous even in situations when both terms jump. Hence, for the incremental problem one should calculate the discrete values

$$\begin{aligned} e_k^{(1)} &= \textstyle\sum_{j=1}^k \mathcal{E}(t_j,\boldsymbol{\varphi}_j,z_j)-\mathcal{E}(t_{j-1},\boldsymbol{\varphi}_j,z_j) = \sum_{j=1}^k\langle \ell(t_{j-1})-\ell(t_j),\boldsymbol{\varphi}_j\rangle,\\ e_k^{(2)} &= \mathcal{E}(t_k,\boldsymbol{\varphi}_k,z_k)+\textstyle\sum_{j=1}^k \mathcal{D}(z_{j-1},z_j),\\ e_k^{(3)} &= \textstyle\sum_{j=1}^k \mathcal{E}(t_j,\boldsymbol{\varphi}_{j-1},z_{j-1})-\mathcal{E}(t_{j-1},\boldsymbol{\varphi}_{j-1},z_{j-1}) = \sum_{j=1}^k\langle \ell(t_{j-1})-\ell(t_{j-1}),\boldsymbol{\varphi}_j\rangle, \end{aligned}$$

which by Theorem 2.3(ii) satisfy $e_k^{(1)}\leq e_k^{(2)}\leq e_k^{(3)}$. The first and the third term depend nicely on k since $\dot{\ell}$ is bounded. However, if $(\boldsymbol{\varphi}_k,z_k)$ drop out of a local minimum, then there will be a large energy release which is not accompanied by a dissipation of the same size. Thus, we are likely to find that $e_k^{(2)}$ lies significantly below of $e_k^{(1)}$. We make this argument clearer by the following toy model.

Example 6.1 *We consider $z\in\mathcal{Z}=\mathbb{R}$, the dissipation distance $\mathcal{D}(z_0,z_1)=|z_0-z_1|$, the energy $\mathcal{E}(t,z)=\frac{1}{2}\max\{(z+^2)^2,(z-2)^2\}-t\,z$, and the initial condition $z(0)=-3$. We describe the time–continuous solutions z_{glob} of the global problem (S) & (E) as well as the solution z_{loc} of (S)$_{\mathrm{loc}}$ & (E)$_{\mathrm{loc}}$. The associated solutions of the incremental problem (IP) and (IP)$_{\mathrm{loc}}$ are simply obtained by evaluating the time-continuous solutions at the discrete time steps.*

The solution z_{glob} of the global problem is given by $z_{\mathrm{glob}}(t)=t-3+4\,\mathrm{H}(t-1)$ where H denotes the Heaviside function with $\mathrm{H}(t)=0$ for $t<0$ and $\mathrm{H}(t)=1$ for $t\geq 0$. Note that z_{glob} jumps at the time $t=1$ from the value $z=-2$ to $z=+2$. The dissipation increases by 4 and the energy drops by 4. Thus, we have continuity of $\mathcal{E}(t,z(t))+\mathrm{Diss}(z,[0,t])$ and the energy equality holds.

The local solution is $z_{\mathrm{loc}}(t)=t-3+4\mathrm{H}(t-3)$, where the jump occurs only at time $t=3$. The delayed jump from $z{=}0$ to $z=+4$ is due to the fact that the solution is still locally stable for $t\in[1,3)$. The dissipation again jumps by $+4$ but the energy jumps by -12. Hence, the energy inequality will be strict, since the missing difference of $8=4-12$ will not disappear in subsequent steps.

A very similar, but much more involved analysis can be made for the one–dimensional model of Section 5. There the local solution $(\boldsymbol{\varphi}_{\mathrm{loc}},\mathbf{P}_{\mathrm{loc}}):(t,x)\mapsto((t{+}1)x,\min\{1,2/(t{+}1)\})$ for $t\in[0,3]$ loses local stability for $t>1$. Hence it must be replaced by an inhomogeneous solution whose energy release will be bigger than the associated dissipation.

A Legendre transformation and polar functions

We provide a few formulae which are useful for applying duality arguments. Throughout we assume that V is a finite–dimensional vector space and V^* its dual and $\langle\sigma,v\rangle$ denotes the dual pairing for $(\sigma,v)\in V^*{\times}V$. For a convex function $f:V\mapsto\mathbb{R}_\infty$ the Legendre–Fenchel transform $\mathcal{L}$ is defined via

$$f^*(\sigma)=\mathcal{L}f(\sigma):=\sup_{v\in V}\,\langle\sigma,v\rangle-f(v),$$

and it is again a convex function. It is important to allow for the value $+\infty$ here, then $f : v \mapsto \|v\|$ leads to $f^* = \mathcal{X}_{B_1^*(0)}$, where $B_1^*(0) = \{\, \sigma \in V^* \,|\, \|\sigma\|_* \leq 1 \,\}$ and $\mathcal{X}_A$ denotes the indicator function of convex analysis, i.e. $\mathcal{X}_A(a) = 0$ for $a \in A$ and ∞ else. We collect some well–known facts about $\mathcal{L}$, where f is assumed to be convex and lower semicontinuous.

(i) If $f(\alpha v) = \alpha^\gamma f(v)$ for all $\alpha \geq 0$ with $\gamma > 1$ fixed, then $f^*(\alpha\sigma) = \alpha^{\gamma/(\gamma-1)} f^*(\sigma)$.
(ii) $f^{**} = f$.
(iii) $\eta \in \partial f(v) \iff v \in \partial f^*(\eta) \iff \langle \eta, v\rangle = f(v) + f^*(\eta)$.

If $K \subset V$ is a closed convex set, then $\mathrm{T}_v K = \overline{\{\, w \in V \,|\, v + \varepsilon w \in K \text{ for small } \varepsilon > 0 \,\}}$ is the (inner) tangent cone and $\mathrm{N}_v K = \{\, \sigma \in V^* \,|\, \langle \sigma, w\rangle \leq 0 \text{ for all } w \in \mathrm{T}_v K \,\}$ is the (outer) normal cone. We have the relation $\mathrm{N}_v K = \partial \mathcal{X}_K(v)$. For convex function $k : V \mapsto [0,\infty]$ which are also homogeneous of degree one we define the polar function via

$$k^\circ(\sigma) = \sup_{v \text{ with } k(v) \leq 1} \langle \sigma, v\rangle.$$

Clearly, the result is again a convex function $k^\circ : V^* \mapsto [0,\infty]$ which is homogeneous of degree one and $k^{\circ\circ} = k$. An example is any norm $\|\cdot\|$ on V, then $\|\cdot\|^\circ$ is the norm $\|\cdot\|_*$ on V_* given by $\|\sigma\|_* = \sup_{\|v\| \leq 1} \langle \sigma, v\rangle$.

The following formula is useful in problems with kinematic hardening. Let $k : V \mapsto [0,\infty]$ be convex and homogeneous of degree 1 and consider $k_2 : V \times V \mapsto [0,\infty]$, $(v,w) \mapsto k(v-w)$. Then, $(k_2)^\circ : V^* \times V^* \mapsto [0,\infty]$ takes the form

$$(k_2)^\circ(\sigma,\tau) = \begin{cases} \frac{1}{2} k^\circ(\sigma-\tau) & \text{for } \sigma+\tau = 0, \\ \infty & \text{for } \sigma+\tau \neq 0. \end{cases} \tag{A.1}$$

In applications in plasticity the vector space V decomposes into plastic strains $\boldsymbol{\xi} = \mathbf{P}^{-1}\dot{\mathbf{P}}$ and the hardening rates $w = \dot{p}$. Hence, we consider the case $V = X \times W$.

Proposition A.1 *Let $k : X^* \mapsto [0,\infty]$ be homogeneous of degree 1 and convex and let $a : W^* \mapsto \mathbb{R}_\infty$ be convex with $a(0) \leq 0$. Define the elastic domain $\mathbb{Q} = \{\, (\eta, q) \in X^* \times W^* \,|\, k(\eta) + a(q) \leq 0 \,\}$. Then $\Delta = \mathcal{L}\mathcal{X}_{\mathbb{Q}} : X \times W \mapsto [0,\infty]$ is given by*

$$\Delta(\xi, v) = B(k^0(\xi), v) \ \text{with}\ B(k,v) = \begin{cases} b^\infty(v) & \text{for} \quad k = 0, \\ k b(\frac{1}{k} v) & \text{for} \quad k \in (0,\infty), \\ \infty & \text{for} \quad k = \infty \end{cases}$$

where $b(w) = \sup_{a(q)\leq 0} \langle q, w\rangle - a(q)$ and $b^\infty(w) = \sup_{a(q)<0} \langle q, w\rangle$.

To prove this result consider $\Delta(\xi, v) = \sup_{k(\eta)+a(q)\leq 0} [\langle \eta, \xi\rangle + \langle q, v\rangle] = \sup_{q \in W} [\langle q, v\rangle + s(q,\xi)]$ where $s(q,\xi) = \sup_{k(\eta \leq -a(q)} \langle \eta, \xi\rangle$ for $a(q) \leq 0$ and $s(q,\xi) = -\infty$ for $a(q) > 0$. For $a(q) \leq 0$ we find $s(q,\xi) = -a(q) k^\circ(\xi)$ since k is homogeneous of degree one. This proves the result if $k^\circ(\xi) = 0$ or ∞. In the final case we have

$$\Delta(\xi, v) = \sup_{a(q)\leq 0} [\langle q, v\rangle - (k^\circ(\xi) a(q)] = k^\circ(\xi) \sup_{a(q)\leq 0} [\langle q, \frac{1}{k^\circ(\xi)} v\rangle - a(q)]$$

which is the desired result. A simple example is obtained for $X = W = \mathbb{R}$, $k(\eta) = k^\circ(\eta) = |\eta|$ and

$$a(q) = \begin{cases} -r - s(t-q) & \text{for} \quad q \leq t \\ \infty & \text{for} \quad q > t \end{cases} \tag{A.2}$$

where $r, s, t \geq 0$ are given parameters. We obtain

$$b(w) = \begin{cases} \infty & \text{for} \quad w < s \\ r{+}tw & \text{for} \quad w \geq s; \end{cases} \quad \text{which gives} \quad \Delta(\xi, v) = \begin{cases} \infty & \text{for} \quad v < s|\xi|, \\ r|\xi|{+}t|v| & \text{for} \quad v \geq s|\xi|. \end{cases}$$

Proposition A.2 *Let $A : X \mapsto Y$ be a linear continuous mapping and $J : Y \mapsto \mathbb{R}_\infty$ be convex and lower semicontinuous, then the function $\widetilde{J} : X \to \mathbb{R}_\infty;\ x \mapsto J(Ax)$, satisfies $\partial\widetilde{J}(x) = A^*\partial J(Ax)$ for all $x \in X$.*

Proposition A.3 *Let $A \in \mathcal{L}(X, W)$ and $J : X \mapsto \mathbb{R}_\infty$ be convex and lower semicontinuous. Then $\widetilde{J} : X{\times}W \mapsto \mathbb{R}_\infty;\ (x, w) \mapsto J(x{-}Aw)$ has the Legendre–Fenchel transform*

$$(\mathcal{L}\widetilde{J})(x^*, w^*) = \begin{cases} (\mathcal{L}J)(x^*) & \text{if } w^* = -A^*x, \\ \infty & \text{else.} \end{cases}$$

B Dissipation distances

Proposition B.1 *If $\widetilde{\mathbb{Q}}$ and hence $\widehat{\Delta}$ do not depend on p, then the dissipation distance $\widehat{D}$ satisfies $\widehat{D}((\mathbf{P}_1, p_1), (\mathbf{P}_2, p_2)) = \widetilde{D}(\mathbf{P}_1^{-1}\mathbf{P}_2, p_2{-}p_1)$.*

Proposition B.2 *Let $\mathbb{Q}, \Delta$ and B be as in Proposition A.1, then the dissipation distance satisfies $\widetilde{D}(\mathbf{P}, p) = B(\widetilde{D}_{k^0}(\mathbf{P}), p)$ with $\widetilde{D}_{k^0}(\mathbf{P}_1) = \inf \int_0^1 k^0(\mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t))\,\mathrm{d}t$.*

This result follows from the convexity of $B : [0, \infty){\times}\mathbb{R}^m \mapsto [0, \infty]$ and the monotonicity of $B(\cdot, v)$ as follows. For any $\mathbf{P} : [0, 1] \mapsto G$ let $\kappa(t) = k^0(\mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t))$ then $\int_0^1 \kappa(t)\,\mathrm{d}t \geq \widetilde{D}_{k^0}(\mathbf{P}_1)$ with equality for a shortest path. Thus, we have $\int_0^1 \widehat{\Delta}(\mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t), \dot{p}(t))\,\mathrm{d}t = \int_0^1 B(\kappa(t), v(t))\,\mathrm{d}t$ which may be minimized in $(\kappa, v) : [0, 1] \mapsto [0, \infty){\times}\mathbb{R}^m$ under the constraints $\int_0^1 \kappa\,\mathrm{d}t = \widetilde{D}_{k^0}(\mathbf{P}_1)$ and $\int_0^1 v\,\mathrm{d}t = p$. By convexity the constant function is a global minimizer.

C Some minimization problems

For fixed $\beta \in [0, 1]$ we define the continuous and convex function

$$\widehat{H}_\beta(q) = \begin{cases} 1 + \beta q & \text{for } q \leq 0, \\ \mathrm{e}^q & \text{for } p \geq 0. \end{cases}$$

For $q \in \mathbb{R}$ and $a, b > 0$ we want to calculate the minimum

$$r_\beta(q; a, b) = \min_{t>0} \Big(at + b/t + \widehat{H}_\beta(q{+}|\log t|) \Big). \tag{C.1}$$

For $q \geq 0$ the result is independent of β and reads

$$r(q; a, b) = \begin{cases} 2\sqrt{a(b{+}\mathrm{e}^q)} & \text{for } b - a \leq -\mathrm{e}^q, \\ a + b + \mathrm{e}^q & \text{for } |b - a| \leq \mathrm{e}^q, \\ 2\sqrt{b(a{+}\mathrm{e}^q)} & \text{for } b - a \geq \mathrm{e}^q. \end{cases} \tag{C.2}$$

The result for $q<0$ is more involved. To simplify the presentation we note $r_\beta(q;a,b)=r_\beta(q;b,a)$ since in the minimization process the variable t can be replaced by $1/t$. Thus we give the results for $0<a\leq b$ only:

$$r_\beta(q;a,b)=\left\{\begin{array}{cl} a+b+1+\beta q & \text{for } a\leq b\leq a+\beta,\\ \Gamma{+}1{+}\beta q{+}\beta\log[(\Gamma{-}\beta)/(2a)] & \text{for } a{+}\beta\leq b\leq \mathrm{e}^{-2q}a{+}\mathrm{e}^{-q}\beta,\\ a\mathrm{e}^{-q}+b\mathrm{e}^{q}+1 & \text{for } \mathrm{e}^{-q}\beta\leq b{-}\mathrm{e}^{-2q}a\leq \mathrm{e}^{-q},\\ 2\sqrt{b(a{+}\mathrm{e}^q)} & \text{for } b\geq \mathrm{e}^{-2q}a{+}\mathrm{e}^{-q}, \end{array}\right. \tag{C.3}$$

where $\Gamma=\sqrt{\beta^2+4ab}$. The minimum is attained at $t=\widehat{T}_\beta(q;a,b)$ with

$$\widehat{T}_\beta(q;a,b)=\left\{\begin{array}{cl} 1 & \text{for } a\leq b\leq a+\beta,\\ (\Gamma{-}\beta)/(2a) & \text{for } a+\beta\leq b\leq \mathrm{e}^{-2q}a+\mathrm{e}^{-q}\beta,\\ \mathrm{e}^{-q} & \text{for } \mathrm{e}^{-2q}a{+}\mathrm{e}^{-q}\beta\leq b\leq \mathrm{e}^{-2q}a{+}\mathrm{e}^{-q},\\ \sqrt{b/(a{+}\mathrm{e}^q)} & \text{for } b\geq \mathrm{e}^{-2q}a{+}\mathrm{e}^{-q}, \end{array}\right.$$

We may apply this to calculate the reduced energy density of Section 4.3 letting

$$W(F)=(F^\alpha{+}F^{-\alpha}{-}2)/\alpha \text{ and } H(p)=\widehat{H}_1(\alpha p) \text{ with } \beta=1.$$

Then, the reduced energy density satisfies

$$\begin{aligned}\Psi^{\mathrm{red}}((1,p);F) &= \min_{P>0}\Big(\tfrac{F^\alpha}{\alpha}P^\alpha{+}\tfrac{F^{-\alpha}}{\alpha}P^{-\alpha}{+}\widehat{H}_1(\alpha p{+}\alpha|\log P|)\Big)-2/\alpha-p\\ &= r_1(\alpha p;F^\alpha/\alpha,F^{-\alpha}/\alpha)-2/\alpha-p,\end{aligned} \tag{C.4}$$

since we may substitute $t=P^\alpha$. This proves the formula given in (4.10). For the two–dimensional model of Section 4.4 we have to calculate

$$R(p;f_1,f_2)=\min_{s\in\mathbb{R}}\Big(\tfrac{1}{\alpha}(f_1\mathrm{e}^s)^\alpha{+}\tfrac{1}{\alpha}(f_2\mathrm{e}^{-s})^\alpha\Big)+\widehat{H}_1(\tfrac{\alpha}{\sqrt{2}}p{+}\alpha|s|). \tag{C.5}$$

Substituting $t=\mathrm{e}^{\alpha s}$ in (C.1) we find $R(p,f_1,f_2)=r_1(\alpha p/\sqrt{2};f_1^\alpha/\alpha,f_2^\alpha/\alpha)$.

D Convexity properties

We consider functions $\widehat{\psi}:\mathbb{R}^{d\times d}\mapsto\mathbb{R}_\infty$, where we always assume that ψ is lower semi–continuous, i.e., if $\mathbf{F}_k\to\mathbf{F}_\infty$, then $\widehat{\psi}(\mathbf{F}_\infty)\leq\liminf_{k\to\infty}\widehat{\psi}(\mathbf{F}_k)$. A typical example is constituted by an incompressible neo–Hookean material with

$$\widehat{\psi}(\mathbf{F})=\left\{\begin{array}{cl}\frac{\alpha}{2}|\mathbf{F}|^2 & \text{for } \det\mathbf{F}=1,\\ \infty & \text{else.}\end{array}\right.$$

The function is called **convex**, if for all $\mathbf{F}$ and $\mathbf{G}$ the function

$$\mathbb{R}\ni t\mapsto\widehat{\psi}(\mathbf{F}{+}t\mathbf{G})\in\mathbb{R}_\infty \tag{D.1}$$

is convex. Moreover, $\widehat{\psi}$ is called **rank–one convex**, if (D.1) holds for all $\mathbf{G}$ with $\operatorname{rank}\mathbf{G}=1$. The function $\widehat{\psi}$ is called **polyconvex**, if there exists a lower semi–continuous convex function $g:\mathbb{R}^{s(d)}\mapsto\mathbb{R}_\infty$ such that $\widehat{\psi}(\mathbf{F})=g(M(\mathbf{F}))$ where $s(d)=\sum_{j=1}^d\binom{d}{j}^2$ (i.e. $s(1)=1$, $s(2)=5$,

$s(3) = 19$) and $M : \mathbb{R}^{d\times d} \mapsto \mathbb{R}^{s(d)}$ associates to $\mathbf{F}$ the set of all its minors (determinants of quadratic submatrices). The function $\widehat{\psi}$ is called **quasiconvex at $\mathbf{F}_0$** if for all $\overline{\boldsymbol{\varphi}} \in \mathrm{C}^1(B,\mathbb{R}^d)$ with $\overline{\boldsymbol{\varphi}}|_{\partial B} = \mathbf{0}$ we have

$$\frac{1}{\operatorname{vol} B} \int_{y\in B} \widehat{\psi}(\mathbf{F}_0 + \mathrm{D}\overline{\boldsymbol{\varphi}}(y))\,\mathrm{d}y \geq \widehat{\psi}(\mathbf{F}_0), \tag{D.2}$$

where $B = \{\, y \in \mathbb{R}^d \mid |y| < 1 \,\}$. The function $\widehat{\psi}$ is called **quasiconvex** if (D.2) holds for all $\mathbf{F}_0 \in \mathbb{R}^{d\times d}$ where $\widehat{\psi}(\mathbf{F}_0) < \infty$. It is a well–known fact that

$$\widehat{\psi} \text{ convex } \Rightarrow \widehat{\psi} \text{ polyconvex } \Rightarrow \widehat{\psi} \text{ quasiconvex } \Rightarrow \widehat{\psi} \text{ rank–one convex},$$

where the first two implications are strict for $d \geq 2$. For the last implication it is known that it is strict for $d \geq 3$ the case $d = 2$ remaining open. ("Strict" means that the implication cannot be reverted.) For $d = 1$ rank–one convexity and convexity (and hence all other convexity notions) coincide. The quasiconvex hull $\mathsf{QC}\widehat{\psi}$ of a function $\widehat{\psi}$ is defined as

$$[\mathsf{QC}\widehat{\psi}](\mathbf{F}) = \inf_{\overline{\boldsymbol{\varphi}}} \frac{1}{\operatorname{vol} B} \int_{y\in B} \widehat{\psi}(\mathbf{F}+\mathrm{D}\overline{\boldsymbol{\varphi}}(y))\,\mathrm{d}y \tag{D.3}$$

where the set of admissible $\overline{\boldsymbol{\varphi}}$ is as in (D.2). The quasiconvex hull $\mathsf{QC}\widehat{\psi}$ is the largest quasiconvex function lying below of $\widehat{\psi}$.

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