Universität Stuttgart
Mathematisches Institut A
1. Lehrstuhl

Angewandte Analysis
Pfaffenwaldring 57
D–70569 Stuttgart

http://www.mathematik.uni-stuttgart.de/mathA/lst1/

On interfaces between incompatible wells and elastic deformations of infinite cylinders

Boris Buffoni*
Département de mathématiques
Ecole Polytechnique Fédérale, 1015 Lausanne, Suisse
Boris.Buffoni@epfl.ch

Alexander Mielke
Mathematisches Institut A, Universität Stuttgart
Pfaffenwaldring 57, 70569 Stuttgart, Germany
mielke@mathematik.uni-stuttgart.de

7 March 2000

Abstract

We consider the minimization problem for the functional $\mathcal{E}(u) = \int_\Omega E(\nabla u(z))\,\mathrm{d}z$ where $\Omega = \mathbb{R} \times \Sigma$ is an infinitely long cylinder. The density is polyconvex and assumed to be 0 on a set $K \subset \mathbb{R}^{M\times N}$ of wells and positive elsewhere. We show that the gradients of solutions with finite energy have to approach one component K_+ for $x \to \infty$ and one component K_- for $x \to -\infty$, if the number of components is finite (among other conditions). Moreover, for certain pairs of distinct components we construct nontrivial minimizers within the class of solutions approaching the given components.

We follow ideas developed in the variational study of heteroclinic connections for Lagrangian systems and we put special emphasis on multiplicity of such interface solutions.

We discuss an application in the theory of nonlinear elasticity, where such solutions are called *semi-necks*. When a two-dimensional infinite hyperelastic strip is stretched along its infinite direction it may occur that for a given tensile load many homogeneous deformations are possible. In such a case we show by infimizing the energy functional the existence of configurations that tend asymptotically to two different homogeneous deformations.

Mathematics Subject Classification: 73C50, 35J50, 34C37.

*Supported by a grant of the Swiss National Science Foundation.

1 Introduction

The motivation of this work arises from elasticity theory. We consider elastostatic deformations of an infinitely long cylindrical body with an undeformed cross–section $\Sigma \subset \mathbb{R}^{N-1}$. With respect to the reference configuration $\Omega = \mathbb{R} \times \Sigma$, the configuration of a hyperelastic body is described by a deformation $u : \Omega \to \mathbb{R}^N$ such that $\det \nabla u > 0$ a.e. The elastic properties of the material is described by a stored–energy density $W = W(\nabla u)$ which depends only upon the deformation gradient. The associated weak form of the Euler-Lagrange equation is

$$\textstyle\int_\Omega W'(\nabla u) : \nabla v \,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{for all } v \in W^{1,\infty}(\Omega) \text{ with } \operatorname{sppt} v \subset [-l, l] \times \overline{\Sigma} \text{ for some } l > 0. \tag{1.1}$$

Here $(W'(F))_{i,j} = \partial_{F_{i,j}} W(F)$ and $A : B = \operatorname{trace}(A^T B)$.

Our aim is to construct solutions u which behave for $x \to \pm\infty$ like two different straight cylinders under extension, namely $u(x, y) - U_\pm(x, y) \to 0$ for $x \to \pm\infty$. Such solutions are also called semi-necking solutions. Here $U_\pm$ are the straight solutions

$$U_\pm(x, y) = a_\pm + F_\pm \begin{pmatrix} x \\ y \end{pmatrix},$$

where $F_\pm$ are matrices. Two such solutions can only be connected by a nontrivial solution if they correspond to the same resultant force $\mathcal{F}^W$ and the same Hamiltonian energy $\mathcal{H}^W$ per cross–section:

$$\begin{aligned} \mathcal{F}^W(u) &= \textstyle\int_\Sigma W'(\nabla u(x, y))e_1 \,\mathrm{d}y \in \mathbb{R}^N, \\ \mathcal{H}^W(u) &= \textstyle\int_\Sigma W'(\nabla u(x, y))e_1 \cdot \partial_x u(x, y) - W(\nabla u(x, y)) \,\mathrm{d}y \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where e_1 denotes the first vector of the canonical basis of $\mathbb{R}^N$. The resultant force $\mathcal{F}^W$ is independent of $x \in \mathbb{R}$ for sufficiently smooth solutions of (1.1) by Noether's theorem applied to the translational invariance $u \mapsto u + \gamma$, $\gamma \in \mathbb{R}^N$. The Hamiltonian energy $\mathcal{H}^W$ is associated similarly with the translation invariance $x \mapsto x + \xi$, $\xi \in \mathbb{R}$.

The elasticity problem is treated as a special case of the general problem to make the functional

$$\mathcal{E}(u) = \int_\Omega E(z, \nabla u(z)) \,\mathrm{d}z, \quad z = (x, y),$$

minimal in the class of functions $u : \Omega \to \mathbb{R}^M$ satisfying $\nabla u(x, y) - F_\pm \rightharpoonup 0$ for $x \to \pm\infty$ in a suitable sense to be defined below. Our existence result (see Theorems 2.5 and 4.4) will be in the sense of minimizers for $\mathcal{E}$ in suitable sets of functions, in particular such minimizers u satisfy

$$\mathcal{E}(u) \le \mathcal{E}(u+v) \text{ for all } v \in W^{1,p}(\Omega). \tag{1.3}$$

If E and u are sufficiently smooth, this implies that the weak form of the Euler–Lagrange equation holds, cf. (1.1).

Here $E : \mathbb{R}^{M \times N} \to [0, \infty]$ is assumed to be polyconvex, that is it can be written as a convex function of all the minors of F. Moreover, the limit gradients $F_\pm$ satisfy

$E(F_\pm) = 0 = \min\{E(F) : F \in \mathbb{R}^{M\times N}\}$. Under the additional coercivity assumption $E(F) \geq -C + C|F|^p$ for all F with $p > N$, the direct method combined with ideas from compactness via concentration (cf. [Lio84]) allows us to obtain the desired minimizers if E vanishes only at the two points $F_\pm$. We then say that F_- connects to F_+.

A simple example for a suitable energy density $E : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is given by

$$E(F) = \frac{1}{4}(F_{11}^4 + F_{12}^4 + F_{21}^4 + F_{22}^4) + F_{12}^2 + F_{21}^2 - \det F + 1/2. \tag{1.4}$$

It decouples into $E(F) = e_1(F_{11}, F_{22}) + e_2(F_{12}, F_{21})$, where e_2 is strictly convex; hence $F_{12} = F_{21} = 0$ on any critical point of E. A simple calculation gives the three critical points $-I$, 0 and I, with energies 0, $1/2$ and 0. Thus we have two wells.

If E has a finite set of minimizers $\{F^{(i)} : 1 \leq i \leq I\}$ (i.e., $E(F^{(i)}) = 0$) with $I \geq 3$, then one cannot expect connecting solutions between each pair $(F^{(i)}, F^{(j)})$. However, for each pair there exists a finite sequence $i = i_0, i_1, \ldots, i_\nu = j$ such that $F^{(i_{l-1})}$ connects to $F^{(i_l)}$ for $l = 1, \ldots, \nu$.

However, in general, we should expect E to have a set K of minimizers that have components not consisting of isolated points. In elasticity, this is the case when $N \geq 3$ because of frame indifference. This more general setting is dealt with in the same way, with the difference that the asymptotic convergence is now toward components. Our method is inspired by those used in the study of heteroclinic orbits of natural Lagrangian systems with space-periodic potentials (see e.g. [Koz85, Rab89, Rab97]).

For densities E which are sufficiently smooth and satisfy suitable bounds for $|E(F)|$ and $|E'(F)|$, it can be shown that the minimizers are actually weak solutions of the Euler–Lagrange equation (1.1) (see e.g. section I.5 in [Gia83]). If E does not depend on x, the associated integrals

$$\mathcal{F}^E(u) = \int_\Sigma E'(y, \nabla u)e_1 \,\mathrm{d}y \in \mathbb{R}^M, \;\; \mathcal{H}^E(u) = \int_\Sigma [E'(y, \nabla u)e_1 \cdot \nabla u\, e_1 - E(y, \nabla u)] \,\mathrm{d}y \in \mathbb{R},$$

are equal to 0 for a.e. $x \in \mathbb{R}$. However, our theory is based on general polyconvex densities $E : \mathbb{R}^{M\times N} \to [0, \infty]$ which may not be continuous. In particular, we allow E to attain the value $+\infty$ which is important for nonlinear elasticity where $W(F) = \infty$ for $\det F \leq 0$ is physically reasonable to prevent local interpenetration of material.

In such a situation the cross–sectional resultants $\mathcal{F}^E$ and $\mathcal{H}^E$ in (1.2) do not make sense and in Section 3 we generalize the identities $\mathcal{F}^E \equiv 0$ and $\mathcal{H}^E \equiv 0$ using rank–one convexity. We define the convex functions

$$\begin{aligned} G_x^E(\gamma) &= \textstyle\int_\Sigma E(y, \nabla u(x, y) + \gamma \otimes e_1) \,\mathrm{d}y \quad \text{and} \\ B_x^E(\beta) &= \textstyle\int_\Sigma E(y, \nabla u(x, y)\mathrm{diag}(\beta, 1, \ldots, 1)) - \beta E(y, \nabla u(x, y)) \,\mathrm{d}y \end{aligned}$$

and their subdifferentials (where $x \in \mathbb{R}$ remains fixed). We find the conditions

$$0 \in \partial G_x^E(0) \subset \mathbb{R}^M \quad \text{and} \quad 0 \in \partial B_x^E(1) \subset \mathbb{R}. \tag{1.5}$$

The connection of the abstract approach with the elasticity problem is obtained by considering $M = N \in \{2, 3\}$ and the functional $\mathcal{E}(u) = \int_\Omega E_\mu(\nabla u) \,\mathrm{d}y \,\mathrm{d}x$ with

$$E_\mu(F) = W(F) - \mu \cdot (Fe_1) - w_{\min}(\mu) \quad \text{with } \mu \in \mathbb{R}^N, \tag{1.6}$$

where $w_{\min}(\mu) := \inf\{W(F) - \mu \cdot (Fe_1) : F \in \mathbb{R}^{N\times N}\}$ is supposed to be finite. The vector $\mu \in \mathbb{R}^N$ will be the resultant force per cross–section Σ, namely $\mathcal{F}^W(u)$ if u is a smooth minimizer. A major feature of nonlinear elasticity is frame indifference (i.e., $W(F) = W(RF)$ for all F and all $R \in \mathrm{SO}(N)$ – called also objectivity) as well as $W(F) \to +\infty$ for $\det F \to 0$. Moreover, we let $W(F) = \infty$ for $\det F \leq 0$.

In this situation our existence results for minimizers are still valid. However, until now it is an unsolved problem to show that these minimizers satisfy (1.1) (see [Ba77a, Gia83, Eva86, AcF87] for a discussion). In addition, it is not clear how the cross–sectional resultants $\mathcal{F}$ and $\mathcal{H}$ can be defined for such minimizers. (The general problem is to control $W'(\nabla u)$ in regions where $\det \nabla u$ is very small.)

We discuss nonlinearly elastic material laws satisfying all physical requirements as well as polyconvexity. This material law has the form

$$W(F) = \sum_{k=1}^{N} a(\nu_k) + b(\det F), \qquad (\ \det F = \prod_{k=1}^{N} \nu_k\ ),$$

where $\nu_k \geq 0$, $k = 1, \dots, N$, are the principal stretches (that is, the eigenvalues of $(F^TF)^{1/2}$), $a : [0, \infty) \to \mathbb{R}$ is nondecreasing, $b(z) = \infty$ for $z \leq 0$ and a and b are convex. This leads to an isotropic elastic constitutive law which is polyconvex (see Thm 5.1 in [Ba77b] or p. 255 in [Dac89]). Moreover, there exists a $\hat{\mu} > 0$ such that for $\mu = \hat{\mu} e_1 \in \mathbb{R}^N$ the density E_μ in (1.6) has two or more minimizing gradients $F^{(i)}$.

Together with the abstract theory this implies the existence of semi–neck solutions. Using the abstract definition of the resultants involving the subdifferentials in (1.5) it is possible to find the analogues of the classical resultants $\mathcal{F}^W(u) \in \mathbb{R}^N$ and $\mathcal{H}^W(u) \in \mathbb{R}$ for the general case, namely

$$\mu \in \partial G_x^W(0) \quad \text{ and } \quad -w_{\min} \in \partial B_x^W(1),$$

where $G_x^W(\gamma) = \int_{-1}^1 W(\nabla u(x,y) + \gamma \otimes e_1)\, \mathrm{d}y$ and $B_x^W(\beta) = \int_{-1}^1 [W(\nabla u(x,y) \mathrm{diag}(\beta, 1)) - \beta W(\nabla u(x,y))]\, \mathrm{d}y$.

2 The abstract existence theory

We consider a general energy density $E : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty]$ such that $E(x, y, F)$ is periodic in x with period 1 for all y in the bounded cross–section $\Sigma \subset \mathbb{R}^{N-1}$ and all $F \in \mathbb{R}^{M\times N}$. The difficulty in the analysis is that the cylinder Ω is unbounded and hence we cannot apply standard compactness arguments. Instead we shall rely on the important fact that the functional

$$\mathcal{E}(u) = \int_\Omega E(x, y, \nabla u(x, y))\, \mathrm{d}y\, \mathrm{d}x \in [0, \infty], u \in W^{1,p}_{\mathrm{loc}}(\Omega),$$

is invariant under translations of u by integer values along the x-axis, i.e. $T_\xi : u \mapsto u(\cdot - \xi, \cdot)$ with $\xi \in \mathbb{Z}$.

To obtain minimizers as suitable limits of infimizing sequences we need coercivity and weak lower semicontinuity of $\mathcal{E}$. In particular, we suppose that E satisfies the coercivity condition

$$E(z, F) \geq -C + C^{-1}|F|^p = -C + C^{-1}\Big(\sum_{i,j} |F_{ij}|^2\Big)^{p/2}, \ \forall\, (z, F) \in \Omega \times \mathbb{R}^{M\times N}, \qquad (2.1)$$

for some $0 < C < \infty$ and $1 < p < \infty$, and it is polyconvex, which means the following. Denote by $\mathbb{M}(F) \in \mathbb{R}^\tau$ the set of all minors (subdeterminants) of F where $\tau = \sum_{s=1}^{\min\{M,N\}} \binom{M}{s}\binom{N}{s}$. Then, we call $E : \Omega \times \mathbb{R}^{M\times N} \to (-\infty, \infty]$ polyconvex, if there exists a function $g : \Omega \times \mathbb{R}^\tau \to (-\infty, \infty]$ which satisfies the following:

$$
\begin{array}{ll}
\text{(i)} & E(z,F) = g(z, \mathbb{M}(F)) \text{ for all } z, F \in \Omega \times \mathbb{R}^{M\times N}, \\
\text{(ii)} & \text{for each } m \in \mathbb{R}^\tau \text{ the function } g(\cdot, m) \text{ is measurable,} \\
\text{(iii)} & \text{for each } z \in \Omega \text{ the function } g(z,\cdot) : \mathbb{R}^\tau \to (-\infty,\infty] \text{ is convex} \\
& \text{and lower semicontinuous.}
\end{array}
\tag{2.2}
$$

Clearly, polyconvex densities are also rank–one convex, as all minors are rank–one linear, that is, for all $t \in \mathbb{R}$ we have

$$\mathbb{M}(tF + (1-t)G) = t\mathbb{M}(F) + (1-t)\mathbb{M}(G) \quad \text{if } \operatorname{rank}(F-G) = 1.$$

Moreover, the following weak lower semicontinuity result is a direct consequence of polyconvexity, cf. [Ba77a, Dac89]:

$$u^n \rightharpoonup u \text{ weakly in } W^{1,p}(\mathcal{C}) \Rightarrow \left\{ \begin{array}{c} \int_{\mathcal{C}} E(z, \nabla u(z))\,\mathrm{d}z \leq \liminf_{n\to\infty} \int_{\mathcal{C}} E(z, \nabla u^n(z))\,\mathrm{d}z, \\ \text{for all bounded domains } \mathcal{C} \subset \Omega. \end{array} \right. \tag{2.3}$$

For continuous functions $E(z,\cdot) : \mathbb{R}^{M\times N} \to [0,\infty)$ (∞ not allowed) this condition is more or less equivalent to quasiconvexity of $E(z,\cdot)$ (see e.g. [AcF84, Dac89]).

For convenience we define the averaging operator $[\![\cdot]\!]$ and the normalization operator $\mathcal{N}$ via

$$[\![u]\!] = \frac{1}{\operatorname{vol}(\Omega_*)} \int_{\Omega_*} u(z)\,\mathrm{d}z \quad \text{and} \quad \mathcal{N} : \left\{ \begin{array}{rcl} W^{1,p}_{\mathrm{loc}}(\Omega) & \to & W^{1,p}_{\mathrm{loc}}(\Omega), \\ u & \mapsto & u - [\![u]\!], \end{array} \right.$$

where $\Omega_* = (-1/2, 1/2) \times \Sigma$.

To deal with the case that E depends non-trivially on z we set

$$\mathcal{L} = \{\, \mathcal{N}u|_{\Omega_*} : u \in W^{1,p}_{\mathrm{loc}}(\Omega), E(z, \nabla u(z)) = 0 \text{ a.e. in } \Omega \,\} \neq \emptyset.$$

Then $\mathcal{L}$ is compact in $L^1(\Omega_*)$, since the coercivity of E implies a bound on ∇u and the normalization bounds u. Here and throughout we use the fact that, for all bounded domains $\mathcal{C} \subset \mathbb{R}^N$, weak convergence in $W^{1,p}(\mathcal{C})$ implies strong convergence in $L^1(\mathcal{C})$.

The set $\mathcal{L}$ plays a crucial role in our further analysis. To obtain a feeling for this set we consider the case that E does not depend on z. Set $K = \{F \in \mathbb{R}^{M\times N} : E(F) = 0\}$ and $u_F(z) = Fz$. Then, under suitable assumptions (see e.g. Proposition 4.3), we have $\mathcal{L} = \{u_F|_{\Omega_*} : F \in K\}$ and clearly $\mathcal{L} \subset L^1(\Omega_*)$ has the same topological structure as $K \subset \mathbb{R}^{M\times N}$. However, if the set K contains a rank–one connection, i.e. $F_0 + \lambda a \otimes b \in K$ for $\lambda \in [0,1]$, then $\mathcal{L}$ will be much richer. Indeed, for every function $h : \mathbb{R} \to \mathbb{R}$ with $h'(t) \in [0,1]$ a.e. in $\mathbb{R}$ the mapping $u : z \mapsto F_0 z + h(b\cdot z)a$ lies in $W^{1,p}_{\mathrm{loc}}(\Omega)$ and satisfies $E(\nabla u(z)) = 0$ a.e. on Ω. In that situation $\mathcal{L}$ will be much bigger than $\{u_F|_{\Omega_*} : F \in K\}$.

We assume that $\mathcal{L}$ can be decomposed in subsets $\mathcal{L}_i$ as follows. There exists $\delta > 0$, $I \in \mathbb{N} \setminus \{1\}$ and $\mathcal{L}_i$ with $\mathcal{L} = \cup_{i=1}^I \mathcal{L}_i$ and $\mathcal{L}_i \neq \emptyset$ for all $i = 1, \ldots, I$ such that

$$\min\{\|u - v\|_{L^1(\Omega_*)} : u \in \mathcal{L}_i,\ v \in \mathcal{L}_j,\ i \neq j\} \geq \delta. \tag{2.4}$$

We call the sets $\mathcal{L}_i$ the I wells of E. In general, they are unions of connected components of $\mathcal{L}$. If $\mathcal{L}$ has only a finite number of components and $\delta > 0$ is small enough, then the $\mathcal{L}_i$ are exactly the components of $\mathcal{L}$.

The crucial assumption on the decomposition $\mathcal{L} = \cup_1^I \mathcal{L}_i$ is the following:

$$\begin{array}{l} \textit{If } \mathcal{E}(u) = 0 \textit{ and there exists } \xi \in \mathbb{Z} \textit{ such that } \mathcal{N}u|_{\Omega_*} \in \mathcal{L}_i \\ \textit{and } (\mathcal{N}T_\xi u)|_{\Omega_*} \in \mathcal{L}_j, \textit{ then } i = j. \end{array} \tag{2.5}$$

This assumption says that the different wells can not be connected by zero–energy solutions and it is crucial in the proof of Lemma 2.4.

From this the main idea of our work follows. We show that under this assumption the infimum of the energy over all solutions connecting any two given wells is positive and that this infimum is actually attained for some pairs of wells while it may not be attained for other pairs.

A simple example of this type is the one–dimensional scalar case with $M = N = 1$, but now the energy density E also depends on u, namely $E(u', u) = (u')^2 + (u+1)^2 u^2 (u-1)^2$. Here E has the three wells (for u rather than for u') $\{-1\}$, $\{0\}$ and $\{1\}$. Clearly the adjacent wells are connected by minimizers with equal energy $c_0 > 0$. The infimum for connections between $\{-1\}$ and $\{1\}$ is $2c_0$ and it is not attained, as infimizing sequences u_n are such that u_n' develops two bumps whose distance tends to ∞.

Before continuing the construction we comment further on the condition (2.5) for the case that E does not depend on $z \in \Omega$. Then $\mathcal{L}$ is characterized by the zero–energy gradients $K = \{F : E(F) = 0\}$. An important observation is that condition (2.5) implies that the components of K are not rank–one connected, that is

$$\Big(F_1 \in K_i,\ F_2 \in K_j,\ i \neq j \Big) \implies \operatorname{rank}(F_1 - F_2) \geq 2.$$

This follows easily form the fact that E is polyconvex, which implies rank–one convexity, and that E does not equal 0 in the line connecting F_1 and F_2. The question whether condition (2.5) is in fact equivalent to the nonexistence of rank–one connections is a delicate one, see [BaJ87, Sve92]. The answer is positive if $\mathcal{L}_i = \{\mathcal{N}u_{F_i}\}$ and $I \leq 3$, whereas the case $I \geq 4$ seems to be open in general, see [Mül98] for a survey. In our application on nonlinear elasticity with $M = N \in \{2, 3\}$ we can show condition (2.5) for any number I of wells, since the sets K_i take the form $\{\operatorname{diag}(\beta^{(i)}, \alpha^{(i)} Q) : Q \in \mathrm{SO}(N-1)\}$, see Proposition 4.3.

From now on we assume that $I \geq 2$ and we start with a few lemmas preparing the main existence theorem.

Lemma 2.1. *Let the sequence $(u^n) \subset W^{1,p}_{\mathrm{loc}}(\Omega)$ be such that $\sup_n \mathcal{E}(u^n) < \infty$ and $[\![u^n]\!] = 0$ for all $n \in \mathbb{N}$. Then, there exists a subsequence $(u^{n_m})_{m \in \mathbb{N}}$ and $\tilde{u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $u^{n_m} \rightharpoonup \tilde{u}$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ and*

$$\int_{\mathcal{C}} E(x, y, \nabla \tilde{u}(x, y))\, \mathrm{d}y\, \mathrm{d}x \leq \liminf_{m \to \infty} \int_{\mathcal{C}} E(x, y, \nabla u^{n_m}(x, y))\, \mathrm{d}y\, \mathrm{d}x$$

for all domains $\mathcal{C} \subset \Omega$ (not necessarily bounded).

Proof. For $\gamma \in \mathbb{N}$, let $\mathcal{B}(\gamma) := (-\gamma, \gamma) \times \Sigma$. The coercivity hypothesis (2.1) gives $\sup_n \|\nabla u^n\|^p_{L^p(\mathcal{B}(\gamma))} < \infty$. Together with $[\![u^n]\!] = 0$ we find $\sup_n \|u^n\|_{L^p(\mathcal{B}(\gamma))} < \infty$.

By a diagonal sequence argument, there exists a subsequence (u^{n_m}) that converges weakly in $W^{1,p}_{\mathrm{loc}}(\Omega)$ to some $\widetilde{u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$. For all bounded domains $\mathcal{C} \subset \Omega$ we obtain that

$$\int_{\mathcal{C}} E(x, y, \nabla\widetilde{u}(x, y))\,\mathrm{d}y\,\mathrm{d}x \leq \liminf_{m\to\infty} \int_{\mathcal{C}} E(x, y, \nabla u^{n_m}(x, y))\,\mathrm{d}x\,\mathrm{d}y$$

by condition (2.3) and this can be easily extended to unbounded domains $\mathcal{C}$. □

To compare the functions $T_\xi u$ in Ω_* with elements in $\mathcal{L}_i$ we define

$$d_i(\xi, u) = \min\left\{\|(\mathcal{N}T_\xi u)|_{\Omega_*} - v\|_{L^1(\Omega_*)} : v \in \mathcal{L}_i\right\} \text{ and } d(\xi, u) = \min\{d_i(\xi, u) : 1 \leq i \leq I\}$$

for all $\xi \in \mathbb{Z}$. Again using compactness we obtain the following useful result.

Lemma 2.2. *The functionals $d_i(\xi, \cdot)$ are weakly continuous from $W^{1,p}_{\mathrm{loc}}(\Omega)$ into $\mathbb{R}$, that is $u^n \rightharpoonup \widetilde{u}$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ implies that $d_i(\xi, u^n)$ converges to $d_i(\xi, \widetilde{u})$ for all $\xi \in \mathbb{Z}$. The same holds true for the functional $d(\xi, \cdot)$.*

We immediately find that whenever $d(\xi, u) < \delta/2$ then there exists a unique $i \in \{1, \ldots, I\}$ such that $d_i(\xi, u) < \delta/2$. Let us denote it by $i = \mathcal{I}(\xi, u)$. When $d(\xi, u) \geq \delta/2$, we set $\mathcal{I}(\xi, u) := 0$. The basic idea in studying the infimizing sequences of the functional $\mathcal{E}$ is to obtained from looking at the energy density integrated over parts of the cylinder:

$$e(\xi, u) = \int_{\Omega_*} E(z, T_\xi \nabla u(z))\,\mathrm{d}z \geq 0, \quad \xi \in \mathbb{Z}.$$

Then, $\mathcal{E}(u) = \sum_{\mathbb{Z}} e(\xi, u)$ and $\mathcal{E}(u) < \infty$ implies that $\mathcal{N}T_\xi u$ should converge somehow to the wells. Thus, we need to show that $d(\xi, u)$ converges to 0 if $e(\xi, u)$ does so. So far we only know that $e(\xi, u) = 0$ for all $\xi \in \mathbb{Z}$ implies implies $d(\xi, u) = 0$ for all ξ.

The convergence to exactly one well for $\xi \to \infty$ and one for $\xi \to -\infty$ is deduced using compactness and condition (2.5).

Lemma 2.3. *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies $\mathcal{E}(u) < \infty$, then there exist $1 \leq i_-, i_+ \leq I$ such that*

$$(a) \lim_{\xi\to-\infty} d_{i_-}(\xi, u) = 0, \qquad (b) \lim_{\xi\to+\infty} d_{i_+}(\xi, u) = 0. \tag{2.6}$$

We shall use the notation $\mathcal{I}(\pm\infty, u) := i_\pm$.

Proof. Let us prove (b), then (a) follows by changing ξ into $-\xi$. It suffices to show that $d(\xi, u) \to 0$ for $\xi \to \infty$. Then, $d(\xi, u) < \delta/2$ for $\xi > \xi_1$ which implies the existence of $i_+ \in \{1, \ldots, I\}$ with $d_{i_+}(\xi, u) = d(\xi, u)$ for $\xi > \xi_1$.

To obtain a contradiction assume that $d(\xi, u)$ does not converge to 0 for $\xi \to \infty$. Then, there exist a sequence $\xi_n \to \infty$ and an $\varepsilon > 0$ such that $\lim_{n\to\infty} d(\xi_n, u) \geq \varepsilon$.

Setting $u^n := \mathcal{N}T_{\xi_n} u$, we get $\int_{\mathcal{C}} E(z, \nabla u^n(z))\,\mathrm{d}z \to 0$ as $n \to \infty$ for all bounded domains $\mathcal{C} \subset \Omega$. Since $[\![u^n]\!] = 0$ for all $n \in \mathbb{N}$ and $\sup_n \mathcal{E}(u^n) = \mathcal{E}(u) < \infty$, Lemma 2.1 gives a subsequence (u^{n_m}) converging to some $\widetilde{u}$ weakly in $W^{1,p}_{\mathrm{loc}}(\Omega)$ with $\int_\Omega E(z, \nabla\widetilde{u}(z))\,\mathrm{d}z = 0$.

Hence $d(0, \widetilde{u}) = 0$. The weak continuity of d (Lemma 2.2) gives

$$d(0, \widetilde{u}) = \lim_{m\to\infty} d(0, u^{n_m}) = \lim_{m\to\infty} d(\xi_{n_m}, u) \geq \varepsilon.$$

This is a contradiction and the assertion is proved. □

Lemma 2.4. *We have*

$$c_* = \inf\{\, \mathcal{E}(u) : u \in W^{1,p}_{\mathrm{loc}}(\Omega) \text{ with } \mathcal{I}(0,u) \neq \mathcal{I}(\xi,u) \text{ for some } \xi \in \mathbb{Z}\,\} > 0.$$

Proof. As $\mathcal{I}(\xi,u) \in \{0,\dots,I\}$ and $\xi \in \mathbb{Z}$ take discrete values and as $\mathcal{E}$ is translation invariant, the infimum remains the same if we consider the case $\xi = 1$ only.

Now assume the infimum equals 0. Then there exists a sequence $(u^n) \subset W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $\mathcal{E}(u^n) \to 0$ and one of the following properties holds for all $n \in \mathbb{N}$:

(a) $d(0,u^n) \geq \delta/2$ or $d(1,u^n) \geq \delta/2$,

(b) $d(0,u^n), d(1,u^n) < \delta/2$ and $\mathcal{I}(0,u^n) \neq \mathcal{I}(1,u^n)$.

We can further assume that $[\![u^n]\!] = 0$ for all n and, by Lemma 2.1, that (u^n) converges weakly in $W^{1,p}_{\mathrm{loc}}(\Omega)$ to some $\tilde{u}$ with $\mathcal{E}(\tilde{u}) = 0$. Using the definitions of $\mathcal{L}$ and d as well as the weak continuity of d (cf. Lemma 2.2) we conclude $d(0,\tilde{u}) = d(1,\tilde{u}) = 0$ and $\mathcal{I}(0,\tilde{u}) = \mathcal{I}(1,\tilde{u})$ which contradicts (a). However, condition (2.5) gives $\mathcal{I}(0,\tilde{u}) = \mathcal{I}(1,\tilde{u})$ which contradicts (b), now employing the weak continuity of $d_{\mathcal{I}(\xi,\tilde{u})}(\xi,\cdot)$. □

Lemma 2.3 describes the asymptotic behavior as $x \to \pm\infty$ of $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $\mathcal{E}(u) < \infty$. We now focus our attention to subclasses of solutions which have prescribed behavior at infinity. For $1 \leq i_-, i_+ \leq I$ we let

$$\begin{aligned} \mathcal{A}(i_-,i_+) &= \{\, u \in W^{1,p}_{\mathrm{loc}}(\Omega) : d_{i_\pm}(\xi,u) \to 0 \text{ for } \xi \to \pm\infty \,\} \\ c(i_-,i_+) &= \inf\{\, \mathcal{E}(u) : u \in \mathcal{A}(i_-,i_+) \,\}. \end{aligned}$$

If $\mathcal{A}(i_-,i_+) = \emptyset$ then we let $c(i_-,i_+) = \infty$. From Lemma 2.4 we know $c(i_-,i_+) \geq c_* > 0$ for $i_- \neq i_+$.

It should be noted on the one hand that these sets are not weakly closed since for any $u \in \mathcal{A}(i_-,i_+)$ the translates $\mathcal{N}T_\xi u$ converge for $\xi \to \infty$ weakly in $W^{1,p}_{\mathrm{loc}}(\Omega)$ to a function in $\mathcal{A}(i_-,i_-)$. On the other hand we see that $u \in \mathcal{A}(i_-,i_+)$ and $v \in W^{1,p}(\Omega)$ (not the local form) implies that $u+v$ lies in $\mathcal{A}(i_-,i_+)$ again, as $T_\xi v|_{\Omega_*} \to 0$ in $W^{1,p}(\Omega_*)$ for $|\xi| \to \infty$. Thus, finding a minimizer of $\mathcal{E}$ in this set we conclude the minimization property given in (1.3).

The convergence conditions in the sets $\mathcal{A}(i_-,i_+)$ is rather weak, even in the case that the sets $\mathcal{L}_i$ consist of one linear function $u_{F^{(i)}} : z \mapsto F^{(i)}z$. We only ask for $\mathcal{N}T_\xi u$ to converge in $L^1(\Omega_*)$ to $u_{F^{(i)}}$. By the boundedness of $\mathcal{N}T_\xi u$ in $W^{1,p}(\Omega_*)$ this implies also weak convergence in $W^{1,p}(\Omega_*)$. If $p > N$ we may deduce uniform convergence on Ω_* of $\mathcal{N}T_\xi u$ to $u_{F^{(i)}}$ for $\xi \to \infty$. However, this convergence does not imply convergence to a linear function $u_{\mathrm{lin}} : z \mapsto u_0 + F^{(i)}z$ in the sense $T_\xi(u - u_{\mathrm{lin}})|_{\Omega_*} \to 0$ in $L^1(\Omega_*)$. The difference might be a function $w : \Omega \to \mathbb{R}^M$ which grows but still satisfies $\mathcal{N}T_\xi w|_{\Omega_*} \to 0$, like, for instance,

$$w(x,y) = \cos(x^\delta)x^\gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ with } \delta,\gamma \geq 0 \text{ and } 0 < \gamma + \delta < 1. \tag{2.7}$$

Other strange behaviors might occur if $\mathcal{L}_i$ consists of more than one point, cf. (4.5). These pathologies would disappear if we would be able to show that the convergence is fast enough, e.g. exponential in ξ.

We consider functions $u \in \mathcal{A}(i_-, i_+)$ and try to locate regions where $\mathcal{N}T_\xi u|_{\Omega_*}$ is close to one $\mathcal{L}_i$ or far from $\mathcal{L}$. The function $\mathcal{I}(\xi, u)$ is a good indicator for this. For $u \in \mathcal{A}(i_-, i_+)$ with $i_- \neq i_+$, define $h_0 := -\infty$ and, by induction, the integers $h_1 < \ldots < h_k$ with $k \geq 1$ by

$$h_l = \min\{\xi \in \mathbb{Z} : \xi > h_{l-1},\, \mathcal{I}(\xi, u) \neq \mathcal{I}(\xi{+}1, u)\}$$

if the minimum is taken over a nonempty set, otherwise $k := l - 1$. When required, we shall use the notations $h_l(u)$ and $k(u)$.

We are now ready to state and prove our existence and multiplicity result. As shown in the simple example above, it is not possible to show that for each set $\mathcal{A}(i_-, i_+)$ the infimum $c(i_-, i_+)$ of $\mathcal{E}$ is attained, since the infimizing sequences are generated by a finite chain of interfaces between different wells.

Theorem 2.5. *Assume* (2.1), (2.3), (2.4) *and* (2.5). *Then for each pair* (i_-, i_+) *with* $i_- \neq i_+$ *and* $c(i_-, i_+) < \infty$, *there is an* $r \in \mathbb{N}$ *and a sequence* $i_- = j_0, j_1, \ldots, j_r = i_+$ *of elements of* $\{1, \ldots, I\}$ *such that* $\mathcal{E}$ *restricted to* $\mathcal{A}(j_{q-1}, j_q)$ *has a global minimizer* u^q *for each* $q = 1, \ldots, r$. *Each* u^q *satisfies (1.3).*

Remark 2.6. In this statement we allow the sequence $j_0, \ldots, j_r$ to contain many copies of a same element of $\{1, \ldots I\}$ because this is more convenient for the proof. Of course we could state the theorem as sharp as possible, that is $j_q \neq j_p$ whenever $q \neq p$, by killing everything in between. From (2.8) in the proof we see that $c(i_-, i_+) \geq \sum_{q=1}^r c(j_{q-1}, j_q)$. We conjecture equality here, but so far we are unable to prove it.

Proof. Let $(u^n) \subset \mathcal{A}(i_-, i_+)$ be a infimizing sequence and set $k_n = k(u^n)$ and $h_{l,n} = h_l(u^n)$.

Taking a subsequence if necessary (by the diagonal procedure), we can assume that the sequence (k_n) is nondecreasing, that $h_{l,n} - h_{l-1,n}$ converge in $[1, \infty]$ as $n \to \infty$, and that the limit

$$\lim_{n\to\infty} \mathcal{N}T_{h_{l,n}} u^n := u_l$$

exists for $l \geq 1$, with convergence in $L^1_{\rm loc}(\Omega)$ and weak convergence in $W^{1,p}_{\rm loc}(\Omega)$.

Denote by $1 = l_1 < l_2 < \ldots$ the (possibly infinite and in fact finite – see below) sequence made of those $l \geq 1$ such that $h_{l,n} - h_{l-1,n} \to \infty$ as $n \to \infty$.

Let us prove that $\mathcal{I}(-\infty, u_{l_m}) = \mathcal{I}(+\infty, u_{l_{m-1}})$ for all $m \geq 2$. For contradiction, suppose that, for some fixed $m \geq 2$, $j_- := \mathcal{I}(\infty, u_{l_{m-1}}) \neq \mathcal{I}(-\infty, u_{l_m}) := j_+$. Then there exist $\xi_n^- < \xi_n^+ \in \mathbb{Z}$ such that $\xi_n^- - h_{l_{m-1},n} \to \infty$, $h_{l_m,n} - \xi_n^+ \to \infty$, $d_{j_-}(\xi_n^-, u^n) < \delta/2$ and $d_{j_+}(\xi_n^+, u^n) < \delta/2$ for large n. Hence there exists $l \in (l_{m-1}, l_m)$ such that $\xi_n^- < h_{l,n} < \xi_n^+$. Thus l_{m-1} and l_m cannot be consecutive in the sequence $l_1 < l_2 < \ldots$, which is a contradiction.

Also $\sum_{m\geq 1} \mathcal{E}(u_{l_m}) \leq c(i_-, i_+)$. Indeed

$$\begin{aligned}
\sum_{m\geq 1} \mathcal{E}(u_{l_m}) &= \lim_{M\to\infty} \sum_{m\leq M} \lim_{T\to\infty} \int_{|x|\leq T} \int_\Sigma E(z, \nabla u_{l_m}(z))\, \mathrm{d}x\, \mathrm{d}y \\
&\leq \lim_{M\to\infty} \lim_{T\to\infty} \liminf_{n\to\infty} \sum_{m\leq M} \int_{|x|\leq T} \int_\Sigma E(z, \nabla T_{h_{l_m,n}} u^n(z))\, \mathrm{d}x\, \mathrm{d}y \\
&\leq \lim_{M\to\infty} \lim_{T\to\infty} \liminf_{n\to\infty} \mathcal{E}(u^n) \qquad (2.8) \\
&= c(i_-, i_+).
\end{aligned}$$

Clearly $\mathcal{E}(u_{l_m}) \geq \inf\{\mathcal{E}(u) : \mathcal{I}(0,u) \neq \mathcal{I}(1,u)\} \geq c_* > 0$ by Lemma 2.4, which implies that the sequence $l_1 < l_2 < \dots$ is finite. Let $l_{\widetilde{m}}$ be it last element ($\widetilde{m} \geq 1$) and set $i_0 = i_-$ and $i_m := \mathcal{I}(+\infty, u_{l_m})$ for all $1 \leq m \leq \widetilde{m}$. Note that $i_{\widetilde{m}} = i_+$, $u_{l_m} \in \mathcal{A}(i_{m-1}, i_m)$ and $\mathcal{E}(u_{l_m}) < c(i_-, i_+)$ for all m if $\widetilde{m} \geq 2$.

For $\widetilde{m} = 1$ we conclude that u_{l_1} is the desired minimizer of $\mathcal{E}$ on the set $\mathcal{A}(i_-, i_+)$. The case $\widetilde{m} \geq 2$ is proved by induction. We enumerate the pairs $(i_-, i_+) \in \{1, \dots, I\}^2$ with $i_- \neq i_+$ in increasing order with respect to $c(i_-, i_+)$:

$$c(i_-(1), i_+(1)) \leq c(i_-(2), i_+(2)) \leq \dots \leq c(i_-(\kappa), i_+(\kappa)) \leq \dots \leq +\infty.$$

The argument is by induction on $\kappa = 1, \dots, I^2 - I$. In what precedes, set $i_- = i_-(\kappa)$ and $i_+ = i_+(\kappa)$. If $\kappa = 1$ we must have $\widetilde{m} = 1$ and we are finished.

Now suppose that $\widetilde{m} > 1$ and the theorem holds true for all $1 \leq \tilde{\kappa} < \kappa$. We can apply the induction hypothesis to all pairs (i_{m-1}, i_m) such that $i_{m-1} \neq i_m$, because $c(i_{m-1}, i_m) < c(i_-(\kappa), i_+(\kappa))$. If we denote by $j_0(\tilde{\kappa}), \dots, j_{r(\tilde{\kappa})}(\tilde{\kappa})$ the sequence given by the induction hypothesis applied to $(i_-(\tilde{\kappa}), i_+(\tilde{\kappa}))$ with $\tilde{\kappa} < \kappa$, then we choose for $j_0(\kappa), \dots, j_{r(\kappa)}(\kappa)$ the concatenation of the sequences corresponding to the couples (i_{m-1}, i_m) with $i_{m-1} \neq i_m$.
□

If we suppose that $c(i_-, i_+) < \infty$ for all $i_- \neq i_+$, then it follows from the previous theorem that there are at least as many locally minimizing semi-necks as the number of components of $\mathcal{L}$. If the set $\mathcal{L}$ in fact has infinite wells which can be separated by smaller and smaller $\delta > 0$ (cf. (2.4), then we can even conclude the existence of infinitely many nontrivial solutions connecting different wells.

3 Cross–sectional invariants

In the remaining sections we assume that E does not dependent on $x \in \mathbb{R}$. For notational simplicity we also drop the variable $y \in \Sigma$, but all the theory works with general y dependence.

Recall that polyconvexity implies rank-one convexity: for all $F \in \mathbb{R}^{M \times N}$, all $a \in \mathbb{R}^M$ and all $b \in \mathbb{R}^N$ the function

$$\mathbb{R} \ni t \mapsto E(F + t\, a \otimes b) \in [0, \infty] \quad \text{is convex.}$$

This implies that for all $F \in \mathbb{R}^{M \times N}$ and all $b \in \mathbb{R}^N$ the function

$$\mathbb{R}^M \ni \gamma \mapsto E(F + \gamma \otimes b) \in [0, \infty] \quad \text{is convex.}$$

Our variational problem is invariant under the symmetry $u \mapsto u + \gamma$, $\gamma \in \mathbb{R}^M$, as well as under the symmetry $x \mapsto x + \xi$, $\xi \in \mathbb{R}$. Thus, by Noether's theory there are associated integrals for the solutions of the Euler–Lagrange equations. Here we do not apply this theory directly since we want to treat cases where the minimizers are not necessarily (weak) solutions of the Euler–Lagrange equations. This problem can arise if the density $E : \mathbb{R}^{M \times N} \to \mathbb{R}_\infty$ is not smooth or not well–behaved for $|F| \to \infty$.

We now take a minimizer u for $\mathcal{E}$ on the set $\mathcal{A}(i_-, i_+)$ which exists according to the previous section for suitable $i_- \neq i_+$. Thus we have

$$\mathcal{E}(u) \leq \mathcal{E}(v) \quad \text{for all } v \in \mathcal{A}(i_-, i_+). \tag{3.1}$$

We now consider test functions v which associates to the symmetry $u \mapsto u + \gamma$:

$$v_{\chi,\gamma}(x,y) = u(x,y) + \chi(x)\gamma \quad \text{with } \gamma \in \mathbb{R}^M \text{ and } \chi \in W^{1,\infty}(\mathbb{R},\mathbb{R}).$$

We choose χ such that χ' has compact support, then $v \in \mathcal{A}(i_-, i_+)$.

Now let $\chi'(x) = 1$ for $x \in [x_1, x_2]$ and 0 else, then (3.1) gives

$$\int_{x_1}^{x_2} \int_\Sigma [E(\nabla u(x,y) + \gamma \otimes e_1) - E(\nabla u(x,y))] \,\mathrm{d}y \,\mathrm{d}x \geq 0,$$

for all $\gamma \in \mathbb{R}^N$ and all $x_1 < x_2$. By the rank–one convexity of E this integral is a convex function of $\gamma \in \mathbb{R}^M$ which attains its minimum at $\gamma = 0$. By varying x_1 and x_2 we arrive at part (a) of the following result.

Theorem 3.1. *Every minimizer u of $\mathcal{E}$ on $\mathcal{A}(i_-, i_+)$ satisfies for a.e. $x \in \mathbb{R}$, all $\gamma \in \mathbb{R}^M$, and all $\beta > 0$*

(a) $\int_\Sigma [E(\nabla u(x,y) + \gamma \otimes e_1) - E(\nabla u(x,y))] \,\mathrm{d}y \geq 0$;

(b) $\int_\Sigma [E(\nabla u(x,y) M_\beta) - \beta E(\nabla u(x,y))] \,\mathrm{d}y \geq 0$, *where* $M_\beta = \mathrm{diag}(\beta, 1, \ldots, 1) \in \mathbb{R}^{N \times N}$.

Proof. Part (b) follows in a similar spirit. We choose test functions $v \in \mathcal{A}(i_-, i_+)$ in the form $v(x,y) = u(\phi(x), y)$ which leads to

$$\nabla v(x,y) = \nabla u(\phi(x), y) \mathrm{diag}(\phi'(x), 1, \ldots, 1).$$

Choosing $\phi'(x) = \beta$ on $[x_1, x_1 + \eta/\beta]$ and 1 else gives the estimate

$$\begin{array}{rcl} 0 & \leq & \int_\mathbb{R} \int_\Sigma [E(\nabla u(\phi(x), y) M_\beta) - E(\nabla u(x,y))] \,\mathrm{d}y \,\mathrm{d}x \\ & = & \int_0^{\eta/\beta} \int_\Sigma E(\nabla u(x_1 + \beta z, y) M_\beta) \,\mathrm{d}y \,\mathrm{d}z - \int_{x_1}^{x_1+\eta} \int_\Sigma E(\nabla u(x,y)) \,\mathrm{d}y \,\mathrm{d}x \\ & = & \int_{x_1}^{x_1+\eta} \left[\frac{1}{\beta} E(\nabla u(x,y) M_\beta) - E(\nabla u(x,y)) \right] \mathrm{d}y \,\mathrm{d}x. \end{array}$$

Since x_1 and $x_2 = x_1 + \eta$ were arbitrary we obtain the desired result after multiplying with $\beta > 0$. □

The conditions in part (a) and (b) can also be written in terms of the subdifferential of convex functions. For a.e. $x \in \mathbb{R}$ the integral $\mathcal{E}_\Sigma(u)(x) = \int_\Sigma E(\nabla u(x,y)) \,\mathrm{d}y$ is well–defined. For these x define $G_x : \mathbb{R}^M \to [0, \infty]$ and $B_x : (0, \infty) \to [0, \infty]$ given by

$$G_x(\gamma) = \int_\Sigma E(\nabla u(x,y) + \gamma \otimes e_1) \,\mathrm{d}y \quad \text{and} \quad B_x(\beta) = \int_\Sigma [E(\nabla u(x,y) M_\beta) - \beta E(\nabla u(x,y))] \,\mathrm{d}y.$$

These functions are convex, and the conditions in Theorem 3.1 are equivalent to

$$\text{(a)' } \quad 0 \in \partial G_x(0) \subset \mathbb{R}^M \qquad\qquad \text{(b)' } \quad 0 \in \partial B_x(1) \subset \mathbb{R},$$

where ∂ denotes the subdifferential of convex functions.

Under suitable assumptions on the density E and on the solution u we derive the well–known resultant quantities for each cross–section $\{x\} \times \Sigma$ which do not depend on $x \in \mathbb{R}$. In the present setting all these quantities will be 0, however, in the elasticity application we will recover the appropriate continuum mechanical values $\mathcal{F}(u) \in \mathbb{R}^N$ and $\mathcal{H}(u) \in \mathbb{R}$. If G_x and B_x are differentiable at $\gamma = 0$ and $\beta = 1$, respectively, and the derivative can be interchanged with the integral over $y \in \Sigma$, then we obtain

$$\begin{array}{l} D_\gamma G_x(0) = \int_\Sigma E'(\nabla u(x,y)) e_1 \,\mathrm{d}y = 0 \in \mathbb{R}^M \quad \text{and} \\ D_\beta B_x(1) = \int_\Sigma E'(\nabla u(x,y)) e_1 \cdot \nabla u(x,y) e_1 - E(\nabla u(x,y)) \,\mathrm{d}y = 0 \in \mathbb{R} \quad \text{for a.e. } x \in \mathbb{R}. \end{array}$$

4 Application to nonlinear elasticity

4.1 Constitutive assumptions

In nonlinear elasticity theory we have $M = N$ with $N \in \{2,3\}$. It is important to choose the elastic energy density W as a function of $F = \nabla u \in \mathbb{R}^{N\times N}$ suitably. The major requirements are frame indifference (objectivity)

$$W(RF) = W(F) \quad \text{for all} \quad R \in \mathrm{SO}(N) \text{ and } F \in \mathbb{R}^{N\times N}.$$

Together with the obvious assumption that W does only depend on ∇u but not on u itself, this simply says that rigid body transformations don't change the elastic energy.

Moreover, from physical considerations one imposes the condition that very large stretch or volume compression cost a lot of energy:

$$W(F) \to \infty \quad \text{for} \quad |F| \to \infty \text{ or } \det F \to 0.$$

In fact one imposes $W(F) = \infty$ whenever $\det F \leq 0$. The determinant conditions immediately show that such a function W cannot be convex, since the convex hull of $\{F \in \mathbb{R}^{N\times N} : \det F > 0\}$ is $\mathbb{R}^{N\times N}$, see [Cia88]. Here polyconvexity is the right generalization of convexity. For $N = 2$ we may write $W(F) = g(F, \det F)$ with a convex function $g : \mathbb{R}^5 \to [0,\infty]$. In particular we may assume $g(F,f) = \infty$ for $f \leq 0$ as well as $g(F,f) \to \infty$ for $f \to 0$. This gives the desired energy blowup for volume compression without violating polyconvexity.

We may restrict the energy functional further by assuming isotropy:

$$W(FR) = W(F) \quad \text{for all} \quad R \in \mathrm{SO}(N) \text{ and } F \in \mathbb{R}^{N\times N}.$$

This means that the material behaves the same in all directions. In this situation W only depends on the singular values ν_k of F, i.e., the square roots of the eigenvalues of F^TF. It also implies that $W(RFQ) = W(F)$ for all $R, Q \in O(N)$ such that $\det R = \det Q$ and all $F \in \mathbb{R}^{N\times N}$. General constitutive laws of the form $W(F) = \Psi(\nu_1, \dots, \nu_n)$ were considered in [Ba77a].

Fix $\hat\mu > 0$. The first result relates the minimizers of $E_{\hat\mu}(F) := W(F) - \hat\mu F_{11} - w_{\min}(\hat\mu)$ with those of $\Psi_{\hat\mu} : (\nu_1, \dots, \nu_N) \mapsto \Psi(\nu_1, \dots, \nu_N) - \hat\mu\nu_1$.

Lemma 4.1. *The following equality holds true:*

$$\min\{W(F) - \hat\mu F_{11} : \det F > 0\} = \min\{\Psi(\nu_1, \dots, \nu_N) - \hat\mu\nu_1 : \nu_k > 0\}. \tag{4.1}$$

Moreover, if the minimizers of $\Psi_{\hat\mu}$ are given by $\{(\beta^{(i)}, \alpha_2^{(i)}, \dots, \alpha_N^{(i)}) : 1 \leq i \leq I\}$ where $\alpha_k^{(i)} < \beta^{(i)}$ for all k and i, and $I < \infty$, then the minimizers of $E_{\hat\mu}$ are of the form $\mathrm{diag}(\beta^{(i)}, \tilde R\,\mathrm{diag}(\alpha_2^{(i)}, \dots, \alpha_N^{(i)})\tilde Q)$ with $\tilde R, \tilde Q \in \mathrm{SO}(N-1)$ and $1 \leq i \leq I$.

Proof. We denote by m_F and m_ν the minima on the left–hand and on the right–hand side in (4.1), respectively. Clearly we have $m_F \leq m_\nu$, since a minimizer $(\nu_1, \dots, \nu_N)$ on the right–hand side leads to the admissible $F = \mathrm{diag}(\nu_1, \dots, \nu_N)$ on the left–hand side.

For the opposite estimate take any minimizer F for m_F. Then $\det F > 0$ and by the polar decomposition we have $F = RDQ$ with $R, Q \in \mathrm{SO}(N)$ and $D = \mathrm{diag}(\nu_1, \dots, \nu_N)$ with $\nu_1 \geq \dots \geq \nu_N$. This leads to

$$W(F) - \hat{\mu}F_{11} = \Psi(\nu_1, \dots, \nu_N) - \hat{\mu}e_1 \cdot RDQe_1.$$

Since F is a minimizer and $\hat{\mu} > 0$ the term $R^T e_1 \cdot DQe_1$ must be maximal, namely ν_1. We conclude $m_F \geq m_\nu$.

The further assertion of the minimizers for m_F follows from the above, since $F = R\,\mathrm{diag}(\beta^{(i)}, \alpha_1^{(i)}, \dots, \alpha_{N-1}^{(i)})Q$ can be minimizer if and only if $Re_1 = Qe_1 = R^T e_1 = Q^T e_1 = e_1$ (using the fact that R and Q are rotation matrices). $\square$

Here it is sufficient to consider the special case

$$W(F) = \Psi(\nu_1, \dots, \nu_N) = \sum_{k=1}^{N} a(\nu_k) + b(\det F), \quad \nu_k = \nu_k(F), \tag{4.2}$$

which is polyconvex if $a : [0, \infty) \to \mathbb{R}$ is nondecreasing and a and b are convex. We ask a to be strictly increasing. The coercivity assumption is easily obtained by imposing

$$\lim_{z\to\infty} a(z) \geq -C + C^{-1}z^p, \quad \lim_{z\to 0^+} b(z) = +\infty, \quad \inf_{z>0} b(z) > -\infty,$$

(and $b(z) = \infty$ for $z \leq 0$). Note that $|F|^2 = F : F = \mathrm{trace}(F^T F) = \sum_{k=1}^n \nu_k(F)^2$, which implies $W(F) \geq -nC + \tilde{c}|F|^p$ as desired. The particular form implies that the minimizers of $E_{\hat{\mu}} : F \mapsto W(F) - \hat{\mu}F_{11} - w_{\min}(\hat{\mu})$ are special.

Lemma 4.2. *Assume that W takes the form (4.2) and satisfies the assumptions thereafter. Then, the set of all minimizers of $E_{\hat{\mu}}$ takes the form*

$$K = \cup_{i=1}^I K_i \quad \text{with } K_i = \{\mathrm{diag}(\beta^{(i)}, \alpha^{(i)}Q) \in \mathbb{R}^{N\times N} : \ Q \in \mathrm{SO}(N-1)\,\}.$$

Two sets K_i and K_j are not rank–one connected if and only if $\beta^{(j)} \neq \beta^{(i)}$ and $\alpha^{(j)} \neq \alpha^{(i)}$ for the case $N = 2$ and $\alpha^{(j)} \neq \alpha^{(i)}$ for the case $N \geq 3$.

Proof. With Lemma 4.1 it is sufficient to find the minimizers of $\Psi_{\hat{\mu}}$. Take any minimizer $(\beta^{(i)}, \alpha_2^{(i)}, \dots, \alpha_N^{(i)}) \in (0, \infty)^2$ of $\Psi_{\hat{\mu}}$ as given in that lemma. For notational simplicity we omit the superscript $^{(i)}$.

Let $\hat{\alpha} = \left(\Pi_2^N \alpha_j\right)^{1/(N-1)}$. Then it is well–know that $\sum_2^N \alpha_j \geq (N-1)\hat{\alpha}$ holds true with equality if and only if all α_j are equal to $\hat{\alpha}$. Using the convexity and then monotonicity of a we conclude

$$\begin{aligned}\Psi_{\hat{\mu}}(\beta, \alpha_2, \dots, \alpha_N) &= b(\beta\,\Pi_2^N \alpha_j) + a(\beta) + \textstyle\sum_2^N a(\alpha_j) \\ &\geq b(\beta\hat{\alpha}^{N-1}) + a(\beta) + \textstyle\sum_2^N \Big(a(\hat{\alpha}) + a'(\hat{\alpha})[\alpha_j - \hat{\alpha}]\Big) \\ &\geq b(\beta\hat{\alpha}^{N-1}) + a(\beta) + \textstyle\sum_2^N a(\hat{\alpha}) \ = \ \Psi_{\hat{\mu}}(\beta, \hat{\alpha}, \dots, \hat{\alpha}).\end{aligned}$$

Since $a' > 0$ we conclude that equality can only hold if $\alpha_j = \hat{\alpha}$.

By definition, two sets K_i and K_j are not rank–one connected if $\mathrm{rank}(F - G) \geq 2$ for all $F \in K_i$ and $G \in K_j$. If $N = 2$ then both sets consist of one diagonal matrix and the assertion is clear. For $N \geq 3$ it is clear that $\alpha^{(i)} = \alpha^{(j)}$ implies a rank–one connection, e.g., between $\mathrm{diag}(\beta^{(i)}, \alpha^{(i)}I)$ and $\mathrm{diag}(\beta^{(j)}, \alpha^{(j)}I)$. For the opposite conclusion it suffices to show $\mathrm{rank}(\alpha^{(i)}R - \alpha^{(j)}Q) \geq 2$ for all $R, Q \in \mathrm{SO}(N-1)$. Letting $\lambda = \alpha^{(i)}/\alpha^{(j)} \neq 1$, $\widetilde{Q} = R^{-1}Q$ and multiplying with $\frac{1}{\alpha^{(j)}}R^{-1}$ shows that we have to check $\mathrm{rank}(\lambda I - \widetilde{Q}) \geq 2$. As $\widetilde{Q} \in \mathrm{SO}(N-1)$ and $|\lambda| \neq 1$ we conclude that the matrix $\lambda I - \widetilde{Q}$ is invertible, and hence it has full rank $N-1 \geq 2$. $\square$

Although the above form for W seems quite restrictive, the functions a and b and the load parameter $\hat{\mu} \geq 0$ can be chosen such that $E_{\hat{\mu}}(F)$ has several global minimizers. First examples of this kind were constructed by Mielke ([Mie91, Ch. 10]) in the framework of center manifold theory, by Owen [Owe87] for one-dimensional models and by Ball [Ba77b] for other problems in nonlinear elasticity. Below we provide a different example, which is based on the form (4.2) and which may have an arbitrary finite number of components of minimizers.

The special form of the set K given in the above proposition allows us to show that the corresponding set $\mathcal{L}$ satisfies the fundamental condition (2.5).

Proposition 4.3. *Let $N \in \{2,3\}$ and let $K = \{\, F \in \mathbb{R}^{M\times N} : E(F) = 0 \,\} = \cup_{i=1}^{I} K_i$ be as in Lemma 4.2. Assume that none of the wells K_i are pairwise rank–one connected. Define the sets $\mathcal{L}, \mathcal{L}_i \subset L^1(\Omega_*)$ as*

$$\mathcal{L} = \cup_{i=1}^{I} \mathcal{L}_i \quad \textit{and } \mathcal{L}_i = \{\, \mathcal{N}u_F|_{\Omega_*} \in L^1(\Omega_*) :\ u_F : z \mapsto Fz,\ \ F \in K_i \,\}.$$

Then the conditions (2.4) and (2.5) hold.

Proof. The condition (2.4) follows easily as the mapping $F \mapsto \mathcal{N}u|_{\Omega_*}$ is continuous.

To show the second condition consider any function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $\mathcal{E}(u) = 0$, then $\nabla u(z) \in K$ for a.e. $z \in \Omega$. Our aim is to show that all such functions have the form $u(z) = u(0) + Fz$ for one $F \in K$. This certainly implies condition (2.5). Write $u(x,y) = (u_1(x,y), \ldots, u_N(x,y))^T$. Then, the special structure of the components K_i and the condition $\nabla u(z) \in K$ imply that $\partial_y u_1 = 0$ and $\partial_x u_k = 0$ a.e. $z \in \Omega$ for all $k \geq 2$. For a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ this implies $u_1(x,y) = u_1(x)$ and $u_k(x,y) = u_k(y)$.

Now we use the fact that there are no rank–one connections. In the case $N = 2$ define the functions $\chi_i : \Omega \to \mathbb{R}$ with $\chi_i(x,y) = 1$ if $\nabla u(x,y) \in K_i$ and 0 else. Since all $\beta^{(i)}$ are different and u_1 depends only on x we find $\partial_y \chi_i = 0$ in the sense of distributions. Similarly we obtain $\partial_x \chi_i = 0$ as all $\alpha^{(j)}$ are different. Thus, one χ_i equals 1 and all others equal 0 a.e. in Ω.

In the case $N = 3$ we solely consider the mapping $\widetilde{u} : y \mapsto (u_2(y), u_3(y))^T$ with $\widetilde{u} \in W^{1,p}(\Sigma)$ and $\Sigma \subset \mathbb{R}^2$. We know $\nabla_y \widetilde{u} \in \cup_{i=1}^{I} \widetilde{K_i}$ with $\widetilde{K_i} = \alpha^{(i)}\mathrm{SO}(2)$. Note that all the matrices $\widetilde{F}$ in $\widetilde{K_i}$ satisfy the relations $\widetilde{F}_{11} = \widetilde{F}_{22}$ and $\widetilde{F}_{12} = -\widetilde{F}_{21}$. Thus, $\widetilde{u}$ is a solution of the Cauchy–Riemann equations in Σ. We conclude that $\widetilde{u}$ is holomorphic and the continuity of $\nabla_y \widetilde{u}$ then gives $\nabla_y \widetilde{u}(y) = \widetilde{F}_0 = \alpha^{(i_0)}Q_0$ for some $i_0 \in \{1, \ldots, I\}$ and $Q_0 \in \mathrm{SO}(2)$, see [Sve93] for more details. Together with $\nabla u \in K$, we obtain $\nabla u(z) = F_0$ with $F_0 = \mathrm{diag}(\beta^{(i_0)}, \widetilde{F}_0) \in K_{i_0}$ which gives the desired result. $\square$

4.2 An example

We construct specific examples of smooth functions a and b such that the associated density $W(F) = \Psi(\nu_1(F), \ldots, \nu_n(F)) - \hat{\mu}F_{11}$ has several minima.

Critical points of $\Psi(\nu_1, \ldots, \nu_n) - \hat{\mu}\nu_1$ satisfy

$$\begin{cases} a'(\nu_1) + b'(\delta)\delta/\nu_1 - \hat{\mu} = 0, \\ a'(\nu_k) + b'(\delta)\delta/\nu_k = 0 \text{ for } k = 2, \ldots, n\,, \end{cases} \qquad \text{where } \delta = \prod_1^N \nu_k. \tag{4.3}$$

From Lemma 4.2 we know that it suffices to search for minima in the form $\nu_1 = \beta$ and $\nu_k = \alpha$ for $k = 2, \ldots, N$. This means we have to minimize

$$w(\beta, \alpha) = a(\beta) + (N{-}1)a(\alpha) + b(\beta\alpha^{N-1}) - \hat{\mu}\beta.$$

Assuming $a(\nu) = c\nu$ with $c > \hat{\mu}$, critical points of w satisfy

$$c + b'(\beta\alpha^{N-1})\beta\alpha^{N-2} = 0 \text{ and } c + b'(\beta\alpha^{N-1})\alpha^{N-1} - \hat{\mu} = 0 \tag{4.4}$$

and we conclude $\alpha = \rho\beta$ with $\rho = (c{-}\hat{\mu})/c \in (0, 1)$.

Global minimizers of w are then given by the global minimizers of

$$v : \beta \mapsto v(\beta) = w(\beta, \rho\beta) = (c{-}\hat{\mu})N\beta + b(\rho^{N-1}\beta^N).$$

Setting $z = \rho^{N-1}\beta^N$, we have to find the global minimizers of

$$z \mapsto b(z) + \eta\, z^{1/N} \quad \text{with } \eta = (c{-}\hat{\mu})N\rho^{(1-N)/N} = nc\rho^{1/N} \in (0, Nc).$$

Since $z \to z^{1/N}$ is strictly concave it is easy to show that there exists a strictly convex function b (i.e., $b'' > 0$) such that the function $\beta \mapsto v(\beta)$ assumes its minimal value at several points $\beta_1, \ldots, \beta_I$. In particular, we may choose b such that $b(z) \to \infty$ and $b'(z) \to -\infty$ for $z \to 0+$.

Then we can redefine $a(\nu)$ for large z such that a remains convex and satisfies $a(\nu) \geq C\,\nu^p$ for $z \gg 1$ with $C > 0$ and $p > N$. Clearly, $w : [0, \infty)^2 \to \mathbb{R}$ will have no new global minimizers since $a(\nu) \geq c\nu$ for all $\nu \geq 0$.

4.3 Existence of semi–neck solutions

We now apply our abstract existence results to the special case of nonlinear elasticity, where $E(F) = W(F) - \hat{\mu}F_{11} - w_{\min}(\hat{\mu})$ with W as in (4.2) and Section 4.2.

First we treat the two–dimensional case $\Omega = \mathbb{R} \times (-1, 1)$, where the cross–section Σ is chosen to be the interval $(-1, 1)$. The above example shows that there are suitable polyconvex stored–energy densities $W(F)$ and an axial loading $\mu = \hat{\mu}e_1$ such that $F \mapsto W(F) - \hat{\mu}F_{11}$ attains its minimal value at a finite number of points. Using the isotropy we have a reflection symmetry which amounts to the fact that $u \in \mathcal{A}(i_-, i_+)$ is equivalent to $u_R \in \mathcal{A}(i_+, i_-)$, when the reflected deformation u_R is defined by $u_R(x, y) = \mathrm{diag}(-1, 1)u(-x, y)$.

In particular, we may choose W (i.e., a and b) and $\hat{\mu}$ such that E has exactly I minimal points, that is

$$K = \cup_{i=1}^I K_i \quad \text{with } K_i = \{\, \mathrm{diag}(\beta^{(i)}, \alpha^{(i)})\,\}.$$

Moreover, for $i \neq j$ we have $\beta^{(i)} \neq \beta^{(j)}$ and $\alpha^{(i)} \neq \alpha^{(j)}$. Then, Proposition 4.3 guarantees that the main existence theorem is applicable. In particular, in the case $I = 2$, we thus have at least one minimizer in $\mathcal{A}(1, 2)$ and at least one in $\mathcal{A}(2, 1)$. More generally, the following result holds.

Theorem 4.4. *Consider the case $N=2$ with W satisfying the above assumptions such that $\Psi_{\hat\mu}$ has a finite number of minima. Let $i_-, i_+ \in \{1,\dots,I\}$ with $i_+ \neq i_-$. Then, there exists $r\in\mathbb{N}$ with $r<I$ and indices i_j, $j=0,\dots,r$, with $i_j\neq i_k$ for $j\neq k$ and $i_0=i_-$ and $i_r=i_+$, such that for each $j=1,\dots,r$ the functionals $\mathcal{E}(u)=\int_\Omega W(\nabla u)-\hat\mu\partial_x u_1 - w_{\min}(\hat\mu)\,\mathrm{d}z$ attains its minimum on $\mathcal{A}(i_{j-1},i_j)$. This means there exist solutions $u^j\in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that*

$$\mathcal{E}(u^j)\leq \mathcal{E}(u^j+v) \text{ for all } v\in W^{1,p}(\Omega),$$
$$\left.\begin{array}{l} T_\xi \nabla u^j \rightharpoonup \mathrm{diag}(\beta^{(i_j)},\alpha^{(i_j)}), \\ T_{-\xi}\nabla u^j \rightharpoonup \mathrm{diag}(\beta^{(i_{j-1})},\alpha^{(i_{j-1})}) \end{array}\right\} \text{ for } \xi\to\infty$$

where the convergence is weak in $L^p(\Omega_*)$.

Recall that we do not claim convergence towards a linear solution $u^j_{\mathrm{lin}}(x,y)=u_0+(\beta^{(i_j)}x,\alpha^{(i_j)}y)$. In principle it is possible that $w=u^j-u^j_{\mathrm{lin}}$ grows with $x\to\infty$ but $\mathcal{N}T_\xi w\rightharpoonup 0$ weakly in $W^{1,p}(\Omega_*)$. An example of such a function is given in (2.7).

The application of the above theory to the three–dimensional case $\Omega=\mathbb{R}\times\Sigma$ with a bounded two–dimensional cross–section in $\mathbb{R}^2$ leads to a less precise description of the asymptotic behavior. The reason is that the connected components of K are given by circles

$$K_i=\{\,\mathrm{diag}(\beta^{(i)},\alpha^{(i)}Q) : Q=\begin{pmatrix} c & -s\\ s & c\end{pmatrix} \text{ with } c^2+s^2=1\,\}.$$

Here the rotation $\mathrm{diag}(1,Q)\in\mathrm{SO}(3)$ is related to the rigid–body rotation of of the cylinder about its axis.

We obtain a similar statements of existence of solutions connecting different wells, however, we cannot guarantee convergence of the gradient towards one matrix but only to the set K_{i_+}. This means there exists $Q_\xi\in\mathrm{SO}(2)$ such that

$$T_\xi\nabla u-\mathrm{diag}(\beta^{(i_+)},\alpha^{(i_+)}Q_\xi)\rightharpoonup 0 \text{ weakly in } L^p(\Omega_*). \tag{4.5}$$

With the present methods we are not able to exclude cases like $Q=\begin{pmatrix}\cos\xi^\gamma & -\sin\xi^\gamma\\ \sin\xi^\gamma & \cos\xi^\gamma\end{pmatrix}$ with $\gamma\in(0,1)$.

However, if the cross–section has at least one symmetry axis, then a slight modification of the above theory allows us to proof existence of semi–neck solutions converging asymptotically to homogeneous deformations. Assume that the cylinder $\Omega=\mathbb{R}\times\Sigma$ is invariant under the reflection $(x,y_1,y_2)\mapsto R(x,y_1,y_2)=(x,y_1,-y_2)$. Then we may restrict our view to deformations of the cylinder which have the same symmetry, that is $u(x,y)=R\,u(R(x,y))$, which implies for instance $u_3(x,y_1,-y_2)=-u_3(x,y_1,y_2)$. Defining the symmetric subspace $X_R=\{\,u\in W^{1,p}_{\mathrm{loc}}(\Omega): u=R\,u\circ R\,\}$ and using $\mathcal{E}(R\,u\circ R)=\mathcal{E}(u)$ we can restrict the minimization to the set $X\cap\mathcal{A}(i_-,i_+)$. Each minimizer u^R in this set has to converge weakly for $\xi\to\pm\infty$ to the intersection, $K^R_{i_\pm}$, of $K_{i_\pm}$ with the symmetry R, where

$$K^R_i=\{\,F\in K_i : F=RFR\,\}=\{\,F^\varepsilon_i=\mathrm{diag}(\beta^{(i)},\varepsilon\alpha^{(i)},\varepsilon\alpha^{(i)}):\varepsilon\in\{-1,1\}\,\}.$$

Since these sets are discrete the minimizer u^R satisfies $T_\xi\nabla u^R\rightharpoonup F^\varepsilon_{i_+}$ weakly in $L^p(\Omega_*)$ for $\xi\to\infty$.

It is interesting to look at the cross–sectional invariants. For this we recall the definition $E(F) = W(F) - \hat{\mu}F_{11} - w_{\min}(\hat{\mu})$, where $w_{\min}(\hat{\mu}) = \min\{W(F) - \hat{\mu}F_{11} : \det F > 0\}$. We insert this form into the results of Theorem 3.1 and the subsequent calculations. For a given minimizer define the convex functions

$$\begin{aligned} G_x^W(\gamma) &= \textstyle\fint_{-1}^{1} W(\nabla u(x,y) + \gamma \otimes e_1)\,\mathrm{d}y \quad \text{and} \\ B_x^W(\beta) &= \textstyle\fint_{-1}^{1} [W(\nabla u(x,y)M_\beta) - \beta W(\nabla u(x,y))]\,\mathrm{d}y. \end{aligned}$$

The superscript ^{W} indicates that only W is used instead of $E(F) = W(F) - \mu \cdot Fe_1 - w_{\min}(\mu)$. The relations $\mathbb{R}^2 \ni 0 \in \partial G_x(0)$ and $\mathbb{R} \ni 0 \in \partial B_x(1)$ for the subdifferentials translate into

$$\mu \in \partial G_x^W(0) \quad \text{and} \quad -w_{\min} \in \partial B_x^W(1).$$

Assuming sufficient smoothness for W and u we obtain the classical relations

$$\begin{aligned} \mu &= \textstyle\fint_\Sigma W'(\nabla u(x,y))e_1\,\mathrm{d}y \quad \text{and} \\ -w_{\min} &= \textstyle\fint_\Sigma [W'(\nabla u(x,y))e_1 \cdot \nabla u(x,y)e_1 - W(\nabla u(x,y))]\,\mathrm{d}y \end{aligned}$$

for all $x \in \mathbb{R}$, see [Eri77, Mie88, Mie91].

References

[AcF84] E. Acerbi, N. Fusco, *Semicontinuity problems in the calculus of variations*, Archive Rational Mechanics Analysis **86** (1984) 125–145.

[AcF87] E. Acerbi, N. Fusco, *A regularity theorem for minimizers of quasiconvex integrals*, Archive Rational Mechanics Analysis **99** (1987) 261–281.

[Ba77a] J.M. Ball, Constitutive inequalities and existence theorems in nonlinear elastostatics. In "*Nonlinear Analysis and Mechanics: Heriot–Watt Symposium*, Vol. I, R.J. Knops (ed), Pitman Research Notes Math. Vol. 17; 1977" pages 187–241.

[Ba77b] J.M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity.* Archive Rational Mechanics Analysis **63** (1977) 337–403.

[BaJ87] J.M. Ball, R.D. James, *Fine phase mixtures as minimizers of energy.* Archive Rational Mechanics Analysis **100** (1987) 13–52.

[Cia88] P.G. Ciarlet, *Mathematical Elasticity, Volume I: Three–Dimensional Elasticity* North–Holland, 1988.

[Dac89] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.

[Eri77] J. Ericksen, *On the formulation of Saint–Venants problem*, In "R.J. Knops (ed), Nonlinear Analysis and Mechanics: Heriot–Watt Symposium Vol. I, London: Pitman 1977, pages 158–186.

[Eva86] L.C. Evans, *Quasiconvexity and partial regularity in the calculus of variations*, Archive Rational Mechanics Analysis **95** (1986) 227–252.

[Gia83] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Princeton University Press, Princeton, 1983.

[Koz85] V. V. Kozlov, *Calculus of variations in the large and classical mechanics*, Russ. Math. Surveys **40(2)** (1985) 37–71.

[Mül98] S. Müller, *Variational models for microstructure and phase transitions*, Preprint MPI Leipzig (1998) 127 pages.

[Lio84] P. L. Lions, *The concentration-compactness principle in the calculus of variations, Parts 1 and 2*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984) 109–145 and 223–283.

[Mie88] A. Mielke, *Saint-Venant's problem and semi-inverse solutions in nonlinear elasticity*, Archive Rational Mechanics and Analysis **102** (1988) 205–229 (with Corrigendum in **110** (1990) 351–352.)

[Mie91] A. Mielke, *Hamiltonian and Lagrangian Flows on Center Manifolds*, Lecture Notes in Mathematics, no. 1489, Springer-Verlag, Berlin, 1991.

[Owe87] N. Owen, *Existence and stability of necking deformations for nonlinearly elastic rods*, Archive Rational Mechanics Analysis **98** (1987) 357–384.

[Rab89] P. H. Rabinowitz, *Periodic and heteroclinic orbits for a periodic Hamiltonian system*, Anal. Inst. Henri Poincaré Analyse Nonlinéaire **6(5)** (1989) 331–346.

[Rab97] P. H. Rabinowitz, *Heteroclinics for a Hamiltonian system of double pendulum type*, Topological Methods in Nonlinear Analysis, Journal of the Julius Schauder Center **9** (1997) 41–76.

[Sve92] V. Šverák. *New examples of quasiconvex functions.* Archive Rational Mechanics Analysis **119** (1992) 293–300.

[Sve93] V. Šverák. *On the problem of two wells.* In "Microstructure and phase transitions, D. Kinderlehrer, R.D. James, M. Luskin, J. Ericksen (eds), IMA Vol. Applied Math. **54**, Springer–Verlag 1993," 183–189.