Multi–pulse solutions to the Navier–Stokes problem between parallel plates*

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Abstract

We consider the three-dimensional Poiseuille problem of a viscous incompressible fluid flow between parallel plates. The flows under investigation are assumed to be traveling waves in streamwise direction with spatial periodicity $2\pi/\alpha$. In spanwise direction they are assumed to be uniformly close to the basic flow which enables us to use the spatial center–manifold reduction, where the spanwise variable takes the role of the time. For Reynolds numbers close to criticality the problem is reduced to a four-dimensional ODE whose lowest order terms coincide with the steady complex Ginzburg–Landau equation. Using perturbation arguments we relate reversible n–pulse solutions of this equation to n–pulse solutions of the problem on a center manifold. Thus, we obtain multi–pulse solutions of the Navier–Stokes problem for parameters slightly below criticality. These solutions are localised in spanwise direction but periodic in streamwise direction.

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1 Introduction

Let us consider viscous incompressible fluid flow between parallel plates under the action of the constant force in the domain $P = \mathbb{R} \times (-1, 1) \times \mathbb{R}$ with coordinates (x, y, z). This is three–dimensional Poiseuille problem which consists of the Navier–Stokes system together with the non–slip boundary conditions. The trivial steady state is the Poiseuille flow which reads in the nondimensional form

$$V(y) = (U(y), 0, 0)^T \quad \text{with } U(y) = 1 - y^2.$$

If $v = V + u$ is the total velocity, then u satisfies

$$\left.\begin{array}{rcl} \dfrac{\partial}{\partial t}u + (V \cdot \nabla)u + (u \cdot \nabla)V + (u \cdot \nabla)u + \nabla p & = & \dfrac{1}{R}\Delta u, \\ \nabla \cdot u & = & 0, \end{array}\right\} \quad \text{in} \quad P; \qquad \begin{array}{rcl} u & = & 0 \end{array} \quad \text{on} \quad \partial P. \tag{1.1}$$

Near the instability threshold a big variety of bounded solutions of (1.1) are known to exist, see e.g., [AnI72, ChJ73, Bri89, AfV91]. All of these solutions are downstream traveling waves which are periodic in x–direction and are either independent of z or periodic in z.

Here we follow the ideas developed in [AfM95] and consider solutions which have no prescribed behavior in z–direction. Only in the downstream direction x we assume a traveling wave form together with periodicity. In particular this means

$$u(t, x, y, z) = \widetilde{u}(x - ct, y, z) \quad \text{with } \widetilde{u}(\xi, y, z) = \widetilde{u}(\xi + 2\pi/\alpha, y, z). \tag{1.2}$$

Thus, problem (1.1) – (1.2) is an elliptic PDE which depends on the three real parameters R, c and α, see (2.2) for the explicit form. The pulse structure will be with respect to the spanwise direction z, that is our single and multi–pulse solutions will decay to zero for $|z| \to \infty$ and have several bumps inbetween. The amplitudes of the bumps are proportional to $(R_{\rm crit}(\alpha) - R)^{1/2}$, where the critical Reynolds number $R_{\rm crit}(\alpha)$ is defined below.

The approach in this paper is based on the numerical calculation of the coefficients as functions of the downstream wave length α, but otherwise it uses rigorous analytical methods. It is appropriate to mention here that all numerical information on the relevant coefficients needed to deduce the existence of single and multi–pulses bifurcating from Poiseuille flow was practically at hand in early 1970s, see [AnI72, ChJ73]. However, there is one major difference in contrast to Hopf bifurcation (see [Sat73] and references therein) since the shape of the first term in the expansion for the single pulse solution (see Theorems 1.1 and 2.4) is not determined by the linear theory.

In fact, the methods developed here can be used to construct time periodic solutions of the Poiseuille problem that have a complicated spatial structure in z direction. Therefore one needs to combine the arguments in [MiHO92] and [AfM98a] with the discussion of the Poiseuille problem here. Then, it is possible to speak about **spatial chaos** that appeares in the Poiseuille problem near the instability threshold. In our opinion, complicated spatial structure of the fluid flow may play a crucial role in understanding the transition to turbulence. In such a picture the single and multi–pulses constructed here play the role of metastable steady states, and thus constitute the simplest building blocks for coherent structures in fluids in the sense of [HLB96].

Of course the present analysis is not restricted to the Poiseuille problem, but is applicable to a large variety of pattern–forming systems as long as the same spatial symmetries are present, namely an SO(2) symmetry and a reflection symmetry in the unbounded axial direction. Moreover, the threshold of instability has to be reached at zero spanwise wave number. However, there are two cases depending on the fact whether the SO(2) action commutes with the reflection or not, see [AfM98b]. In the Poiseuille problem we only meet the commuting case.

To be more specific we return to the Poiseuille problem. Under the assumption (1.2) the spatial domain of the problem is $Q = \Sigma \times \mathbb{R}$ with $\Sigma = \mathbb{S}^1 \times (-1,1)$ and $\mathbb{S}^1 = \mathbb{R}/_{(2\pi/\alpha)\mathbb{Z}}$. From the mass conservation ($\operatorname{div} u = 0$) follows that the spanwise mass flux $\delta(z) = \int_\Sigma \widetilde{u}_3 \,\mathrm{d}\widetilde{x}\,\mathrm{d}y$ is a constant. We restrict the analysis to the case $\delta = 0$ in order to preserve the reflection symmetry in z direction. Moreover the problem admits two continuous symmetries: translation invariance along ξ and z direction. This means that all mappings in the right–hand side of (1.1) commute with

$$\begin{array}{rcccl} \tau_\alpha &:& (\widetilde{u}(\xi,y,z)) &\to& (\widetilde{u}(\xi+\alpha,y,z)),\\ \sigma_\beta &:& (\widetilde{u}(\xi,y,z)) &\to& (\widetilde{u}(\xi,y,z+\beta)),\\ \rho &:& (\widetilde{u}(\xi,y,z)) &\to& (\widehat{D}\widetilde{u}(\xi,y,-z)), \end{array} \tag{1.3}$$

where $\widehat{D} = \operatorname{diag}(1,1,-1)$. Obviously, $\rho^2 = 1$, $\rho\tau_\alpha = \tau_\alpha\rho$, $\sigma_\beta\rho = \rho\sigma_{-\beta}$, and $\sigma_\beta\tau_\alpha = \tau_\alpha\sigma_\beta$.

We consider problem (1.1)—(1.2) as a spatial dynamical system with z playing the role of a time–like variable. Then invariance under σ_β means that the system is autonomous. The reflection ρ gives a reversibility (later on denoted by $\mathcal{R}$), and σ_α gives a SO(2)–symmetry for this spatial dynamical system (later on denoted by T_α).

Our analysis is based on a spatial center–manifold reduction which originates in [Kir82, Mie88, IoM91] and was established in [AfM95] for the present version of the Poiseuille problem. Thus, our problem can be reduced to an ODE whose form relies strongly on the spectral properties of the linearization of (1.1) at $u = 0$. Having fixed the downstream periodicity to $2\pi/\alpha$ the critical parameters $(R_{\mathrm{crit}}(\alpha), c_{\mathrm{crit}}(\alpha))$ of the instability threshold are determined from the Orr-Sommerfeld spectral problem

$$\mathrm{i}\alpha R_{\mathrm{crit}}((U - c_{\mathrm{crit}})l_\alpha\varphi - U''\varphi) = l_\alpha^2\varphi \ \text{ for } \ y \in (-1,1), \quad \varphi = \varphi' = 0 \ \text{ at } \ y = \pm 1. \tag{1.4}$$

where $l_\alpha = (\mathrm{d}/\mathrm{d}y)^2 - \alpha^2$. From [Bri89, AfM95] it is known that the neutral curve of stability of the three-dimensional problem coincides with the 2D neutral curve for $\alpha \geq \alpha_4 \approx 0.98787$, but lies to the left for $\alpha < \alpha_4$, see Figure 1. The points T_1 and T_2 corresponds to the "top" and the "nose" of the neutral curve.

Our analysis will be restricted to the interval $\alpha \in (\alpha_4, \alpha_1)$, since we rely on the fact that the most unstable modes are perturbations of large wavelength. In that regime the

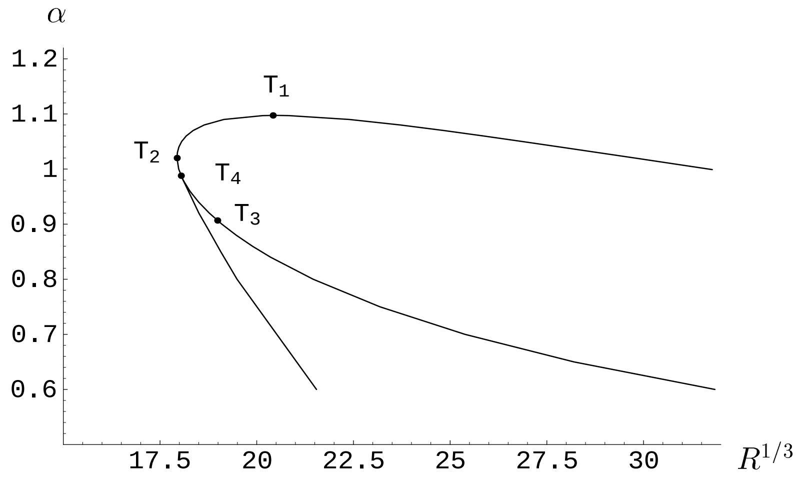

Figure 1: Neutral curve for the 2D and 3D Poiseuille flow
$T_1 = (R_1, \alpha_1) \approx (8591.4,\ 1.09732)$, $T_2 = (R_2, \alpha_2) \approx (5772.2,\ 1.02056)$,
$T_3 = (R_3, \alpha_3) \approx (6842.2,\ 0.90667)$, $T_4 = (R_4, \alpha_4) \approx (5874.7,\ 0.98787)$

spatial center–manifold reduction and a suitable scaling leads to the ODE

$$\frac{\mathrm{d}^2 a}{\mathrm{d}\zeta^2} - (1+\mathrm{i}\omega)^2 a + (1+\mathrm{i}\Omega)(2+\mathrm{i}\Omega)|a|^2 a + \eta^2 G_0\left(\omega, \Omega, \eta, a, \overline{a}, \frac{\mathrm{d}a}{\mathrm{d}\zeta}, \frac{\mathrm{d}\overline{a}}{\mathrm{d}\zeta}\right) = 0$$

where ω, Ω and η are functions of the physical parameters R, c and α such that $\eta^2 = \mathcal{O}(|R - R_{\mathrm{crit}}(\alpha)| + |c - c_{\mathrm{crit}}(\alpha)|)$. For $\eta = 0$ and $\omega = \Omega$ this equation has the single–pulse solution

$$a_{\mathrm{pulse}}(\zeta) = (\cosh\zeta)^{-(1+\mathrm{i}\omega)},$$

and using ODE methods, see Section 3, it is possible to prove persistence of this single–pulse solution under variations of η for almost all ω. The exceptional values for ω give rise to an exceptional set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\} \subset (\alpha_4, \alpha_1)$ for the Poiseuille problem, with $a_1 \approx 1.053$, $a_2 \approx 1.038$. Unfortunately, the numerics in [AfM98a] is not yet fine enough to calculate the whole set $\mathcal{A}$ and to guarantee $\alpha_2 \notin \mathcal{A}$.

We arrive at the following result concerning the existence of single–pulse solutions for the Poiseuille problem.

Theorem 1.1
For $\alpha_* \in (\alpha_4, \alpha_1)\backslash\mathcal{A}$ there exists $\varepsilon > 0$ and a function $c^{(1)}(\cdot, \alpha_*) : (R_{\mathrm{crit}}(\alpha_*) - \varepsilon, R_{\mathrm{crit}}(\alpha_*)] \to \mathbb{R}$ such that for any $R_* \in (R_{\mathrm{crit}}(\alpha_*) - \varepsilon, R_{\mathrm{crit}}(\alpha_*))$ and $c_* = c^{(1)}(\alpha_*, R_*)$ problem (1.1) – (1.2) with parameters $(R, c, \alpha) = (R_*, c_*, \alpha_*)$ has a single–pulse solution $\widetilde{u}^{(1)}$ in the form

$$\widetilde{u}^{(1)}(\xi, y, z) = \frac{\eta}{\rho}\,\mathrm{Re}\left([\cosh(\eta z)]^{-(1+\mathrm{i}\widehat{\Omega})}\Psi_0(\xi, y)\right) + \mathcal{O}(\eta^2 \mathrm{e}^{-|\eta z|}),$$

where $\Psi_0(\xi, y) = \mathrm{e}^{\mathrm{i}\alpha\xi}(\mathrm{i}\varphi'(y), \alpha\varphi(y), 0)^T$, $\eta = (\mu(R_{\mathrm{crit}}(\alpha_*) - R_*))^{1/2}$ and μ, ρ and $\widehat{\Omega}$ are positive constants depending only on α_*.

Note that single–pulse solutions in the supercritical regime $R > R_{\text{crit}}(\alpha)$ could not be stable, as they converge for $|z| \to \infty$ to the basic Poiseuille flow which is already unstable. The solution from the previous theorem exist in the subcritical regime $R < R_{\text{crit}}(\alpha)$; hence, there is some chance that they are stable, however there are no rigorous results in this direction.

Theorem 1.2
Let $\alpha_*, R_*, c_* = c^{(1)}(R_*, \alpha_*)$ and $\widetilde{u}^{(1)}$ be as in the previous theorem. Then, for each integer $n \geq 2$ and each small $r > 0$ there exists a parameter triple $(R^{(n)}, c^{(n)}, \alpha^{(n)})$ with $|(R_*, c_*, \alpha_*) - (R^{(n)}, c^{(n)}, \alpha^{(n)})| \leq r$ such that for $(R, c, \alpha) = (R_j^{(n)}, c_j^{(n)}, \alpha_j^{(n)})$ problem (1.1) – (1.2) has an n–pulse solution $\widetilde{u}^{(n)}$, that is, for $k = 1, \dots, n$ there exist $z_k^{(n)} \in \mathbb{R}$ and $\beta_k^{(n)} \in \mathbb{S}^1$ such that

$$\sup\{ \|\widetilde{u}_j^{(n)}(\cdot,\cdot,z) - \textstyle\sum_{k=1}^n \widetilde{u}^{(1)}(\cdot - \beta_k^{(n)}, \cdot, z - z_k^{(n)})\|_{H^2(\Sigma)} \ : \ z \in \mathbb{R} \} \leq r.$$

We have chosen an additive representation of the n–pulse solution which makes sense since the shifts $|z_k^{(n)} - z_{k+1}^{(n)}|$ between the single–pulse solutions tend to infinity for $r \to 0$. Thus, the n–pulse solution $\widetilde{u}^{(n)}$ looks like n displaced copies of the single–pulse solution $\widetilde{u}^{(1)}$. In addition to the displacements $z_k^{(n)}$ in spanwise direction z there are also displacements $\beta_k^{(n)} \in \mathbb{S}^1$ in downstream direction ξ.

For more precise statements of the above two theorems we refer to Theorem 2.4 and 2.5. The second result could be refined in such a way that the parameter triple $(R^{(n)}, c^{(n)}, \alpha^{(n)})$ could be chosen such that $R^{(n)} = R_*$ and $\alpha^{(n)} = \alpha_*$. In fact, the analysis in Section 3 shows that there are sequences of two–dimensional surfaces in the three–dimensional parameter space for (R, c, α) on which n–pulse solutions exists and which converge to the surface where the single–pulse solution exists. Yet, we stay with the weaker statement as it simplifies the notations and arguments considerably and thus displays the main ideas more clearly.

2 Results for Navier–Stokes equations

The Poiseuille problem (1.1) will be treated by the use of a spatial center–manifold reduction following the work in [AfM95]. The spanwise direction $z \in \mathbb{R}$ plays the role of the time–like variable which is used to construct the spatial center manifold. To formulate the precise statements we introduce the following notations.

We restrict our attention to special solutions $u(t, x, y, z)$ that travel downstream with speed c and have a permanent form which is periodic with period $2\pi/\alpha$ in the downstream direction, that is $u(t, x, y, z) = \widetilde{u}(\xi, y, z)$ with $\xi = x - ct$ and $\widetilde{u}(\xi + 2\pi/\alpha, y, z) = \widetilde{u}(\xi, y, z)$. It is essential that we do not prescribe any particular behavior in the z direction. Moreover, we assume that the mass flow in spanwise direction has average zero:

$$\delta \stackrel{\text{def}}{=} \int_{\xi=0}^{2\pi/\alpha} \int_{y=-1}^{1} \widetilde{u}_3(\xi, y, z) \, \mathrm{d}y \, \mathrm{d}\xi = 0 \quad \text{for all } z \in \mathbb{R}. \tag{2.1}$$

Using that $\widetilde{u}$ is divergence free it is immediate that this condition holds whenever it holds for one $z_0 \in \mathbb{R}$.

After dropping the tildes this leads to the elliptic system

$$\left.\begin{array}{rcl} c\frac{\partial}{\partial\xi}u + \frac{1}{R}\Delta u - (V\cdot\nabla)u - (u\cdot\nabla)V - (u\cdot\nabla)u - \nabla p &=& 0, \\ \nabla\cdot u &=& 0, \end{array}\right\} \text{ in } Q; \tag{2.2}$$
$$u(\xi,-1,z) = u(\xi,1,z) = 0, \quad u(\xi{+}2\pi/\alpha,y,z) = u(\xi,y,z) \text{ for } (\xi,y,z)\in Q.$$

Problem (2.2) depends on the three real parameters R, c and α. According to the linear stability analysis using the Orr–Sommerfeld equation (1.4) there exist functions $R_{\mathrm{crit}}, c_{\mathrm{crit}} : (\alpha_4,\alpha_1) \to \mathbb{R}$ such that the linearization of (2.2) allows for solutions of the form $u(\xi,y,z) = \Psi_0(y)\mathrm{e}^{\mathrm{i}\alpha\xi}$ that corresponds to the instability threshold. As a shorthand we write $R_\alpha \stackrel{\text{def}}{=} R_{\mathrm{crit}}(\alpha)$ and $c_\alpha \stackrel{\text{def}}{=} c_{\mathrm{crit}}(\alpha)$. Using φ from (1.4) we define $\Psi_0, \Psi_1 \in H^2(\Sigma)$ via

$$\Psi_0(\xi,y) = \mathrm{e}^{\mathrm{i}\alpha\xi}(\mathrm{i}\varphi'(y), \alpha\varphi(y), 0)^T, \quad \Psi_1(\xi,y) = \mathrm{e}^{\mathrm{i}\alpha\xi}(0,0,R_\alpha\psi(y))^T$$

where ψ is the unique solution of the linear equation

$$l_\alpha\psi = \frac{1}{\alpha R_\alpha}l_\alpha\varphi' + \mathrm{i}(U'\varphi - (U{+}c_\alpha)\varphi') \text{ for } y\in(-1,1) \quad \text{and } \psi(-1)=\psi(1)=0.$$

Thus, our vectors Ψ_0 and Ψ_1 are just the first three components of the vectors Ψ_0 and Ψ_1 in [AfM95]. The reduction theorem 4.2 in [AfM95] can now be reformulated as follows.

Theorem 2.1
Take any $\alpha_0 \in (\alpha_4,\alpha_1)$ then there exists an $\varepsilon > 0$ such that all small bounded solutions u of (2.1) and (2.2), i.e., $\|u(\cdot,z)\|_{H^2(\Sigma)} < \varepsilon$, with parameters R, c, α such that $|(R,c,\alpha) - (R_{\alpha_0}, c_{\alpha_0}, \alpha_0)| < \varepsilon$ lie on a four–dimensional spatial center manifold of the form

$$u(\cdot,z) = \mathrm{Re}[A(z)\Psi_0 + B(z)\Psi_1] + \mathcal{H}(R,c,\alpha,A(z),B(z),\overline{A}(z),\overline{B}(z)) \tag{2.3}$$

where $\mathcal{H}(\ldots) = \mathcal{O}(|(A,B)|^2 + |(A,B)|\cdot|(R{-}R_\alpha, c{-}c_\alpha)|)$.

Moreover, for $j = 1,2$ there are functions $h_j(R,c,\alpha,A,B,\overline{A},\overline{B}) = \mathcal{O}(|(A,B)|^2 + |(A,B)|\cdot|(R{-}R_\alpha, c{-}c_\alpha)|)$ such that any solution $(A,B) : \mathbb{R} \to \mathbb{C}^2$ of the ODE

$$\left.\begin{array}{rcl} \mathrm{d}A/\,\mathrm{d}z &=& B + h_1(R,c,\alpha,A,B,\overline{A},\overline{B}), \\ \mathrm{d}B/\,\mathrm{d}z &=& h_2(R,c,\alpha,A,B,\overline{A},\overline{B}), \end{array}\right\} \tag{2.4}$$

defines via (2.3) a solution of (2.1) and (2.2).

Since the Poiseuille problem defines an analytical spatial dynamical systems the functions $\mathcal{H}$, h_1 and h_2 in the theorem can be chosen such that they are $\mathcal{C}^k$ for any $k \in \mathbb{N}$, see [Mie88].

Theorem 4.2 in [AfM95] is more general as it also allows for solutions with non–zero average flux δ in spanwise direction, see (2.1). Then, the solutions of Navier–Stokes lie on a six–dimensional center manifold. The six–dimensional ODE can be reduced to the present four–dimensional case by using that the average flux δ is a first integral and that the average of the pressure does not appear, see Section 4 of [AfM95].

The reduction result allows us to find solutions of the Navier–Stokes equation by solving the ODE (2.4). Solving the first equation of (2.4) for B and then inserting it into the second we arrive at an equivalent second order ODE

$$\frac{\mathrm{d}^2 A}{\mathrm{d}z^2} = g(R, c, \alpha, A, \overline{A}, \frac{\mathrm{d}A}{\mathrm{d}z}, \frac{\mathrm{d}\overline{A}}{\mathrm{d}z}). \tag{2.5}$$

This equation is the same as (4.6) in [AfM95] except that here we restricted ourselves to the case of zero spanwise flux ($\delta = 0$) and we write out the dependence on α explicitly.

The symmetries (1.3) of the original problem reduce to symmetries of (2.5) as follows. The translation invariance along the z direction makes (2.5) autonomous. The translations along the ξ direction acts, due to space periodicity in ξ, as an SO(2)–action and the reflection symmetry $z \to -z$ of the Navier–Stokes equation, which is still present since we consider zero spanwise flux, gives rise to a reversibility of (2.5). Thus, g satisfies

$$\begin{aligned} g(R, c, \alpha, A, \overline{A}, \tfrac{\mathrm{d}A}{\mathrm{d}z}, \tfrac{\mathrm{d}\overline{A}}{\mathrm{d}z}) &= g(R, c, \alpha, A, \overline{A}, -\tfrac{\mathrm{d}A}{\mathrm{d}z}, -\tfrac{\mathrm{d}\overline{A}}{\mathrm{d}z}) \\ &= g(R, c, \alpha, \mathrm{e}^{\mathrm{i}a} A, \mathrm{e}^{-\mathrm{i}a}\overline{A}, \mathrm{e}^{\mathrm{i}a}\tfrac{\mathrm{d}A}{\mathrm{d}z}, \mathrm{e}^{-\mathrm{i}a}\tfrac{\mathrm{d}\overline{A}}{\mathrm{d}z}) \quad \text{for all } a \in \mathbb{S}^1. \end{aligned}$$

From these symmetries we deduce the expansion

$$g(R, c, \alpha, A, \overline{A}, \frac{\mathrm{d}A}{\mathrm{d}z}, \frac{\mathrm{d}\overline{A}}{\mathrm{d}z}) = g_0(R, c, \alpha)A + \widehat{d}(R, c, \alpha)A|A|^2 + \widehat{G}(R, c, \alpha, A, \overline{A}, \frac{\mathrm{d}A}{\mathrm{d}z}, \frac{\mathrm{d}\overline{A}}{\mathrm{d}z})$$

where $\widehat{G}(R, c, \alpha, A, \overline{A}, \frac{\mathrm{d}A}{\mathrm{d}z}, \frac{\mathrm{d}\overline{A}}{\mathrm{d}z}) = \mathcal{O}\left(|A|(|\frac{\mathrm{d}A}{\mathrm{d}z}|^2 + |A|^4)\right)$.

It is possible to prove that g_0 and $\widehat{d}$ are real analytic in (R, c, α), but for our aim it is enough to know that they are $\mathcal{C}^2$ and have the expansions

$$\begin{aligned} g_0(R, c, \alpha) &= M(\alpha)(R - R_\alpha) + N(\alpha)(c - c_\alpha) + \mathcal{O}((R - R_\alpha)^2 + (c - c_\alpha)^2), \\ \widehat{d}(R, c, \alpha) &= d(\alpha) + \mathcal{O}(|R - R_\alpha| + |c - c_\alpha|). \end{aligned} \tag{2.6}$$

Here $N(\alpha)$ and $M(\alpha)$ can be given explicitly in terms of the eigenfunction φ of the Orr–Sommerfeld equation and the eigenfunction ζ of the adjoint problem. We suppose that they are normalized such that $\int_{-1}^{1} l_\alpha \varphi \overline{\zeta}\, \mathrm{d}y = 1$, where $l_\alpha = (\mathrm{d}/\mathrm{d}y)^2 - \alpha^2$. Using the notations from [AfM95] we have

$$\begin{aligned} N(\alpha) &= -\mathrm{i}\alpha / e_{02}(\alpha), \quad M(\alpha) \;=\; -E(\alpha)/e_{02}(\alpha) \\ \text{with } E(\alpha) &= -\tfrac{\mathrm{i}\alpha}{R_\alpha} \int_{-1}^{1} \Big((U - c_\alpha) l_\alpha \varphi - U'' \varphi\Big) \overline{\zeta}\, \mathrm{d}y, \\ H(\alpha) &= -\tfrac{1}{R_\alpha} \int_{-1}^{1} \Big(4\alpha l_\alpha \varphi + \mathrm{i}R_\alpha(U l_\alpha \varphi - U''\varphi) + 2\mathrm{i}\alpha^2 R_\alpha (c_\alpha - U)\varphi\Big) \overline{\zeta}\, \mathrm{d}y, \\ e_{02}(\alpha) &= \tfrac{H(\alpha)}{2\alpha} + \tfrac{\mathrm{i}c_\alpha}{2\alpha} - \tfrac{R_\alpha E(\alpha)}{2\alpha^2}. \end{aligned}$$

The analysis of the ODE (2.5) follows the ideas of [KaMP96] which were refined in [AfM98a]. To include our problem into the frame of this work we rescale the variables A

and z and choose another set of three parameters. The form for our further investigation is the perturbed complex Ginzburg–Landau equation

$$\frac{\mathrm{d}^2 a}{\mathrm{d}\zeta^2} - (1+\mathrm{i}\omega)^2 a + (1+\mathrm{i}\Omega)(2+\mathrm{i}\Omega)|a|^2 a + \eta^2 G_0\left(\omega, \Omega, \eta, a, \overline{a}, \frac{\mathrm{d}a}{\mathrm{d}\zeta}, \frac{\mathrm{d}\overline{a}}{\mathrm{d}\zeta}\right) = 0. \tag{2.7}$$

Here $\omega, \Omega \in \mathbb{R}$ and η is a small positive parameter. The choice of these parameters is motivated by the subsequent analysis. For $\eta = 0$ we obtain the steady complex Ginzburg–Landau equation which in the case $\omega = \Omega$ has the explicit single–pulse solution

$$a_{\text{pulse}}(\zeta) = (\cosh \zeta)^{-(1+\mathrm{i}\omega)}, \tag{2.8}$$

which first appeared in [HoS72]. All solutions constructed below are in the parameter regime $\omega \approx \Omega$ and $\eta \approx 0$.

The relation between the parameters (R, c, α) and (ω, Ω, η) is given via

$$g_0(R, c, \alpha) = \eta^2(1+\mathrm{i}\omega)^2 \quad \text{and} \quad d(\alpha) = -\rho^2(1+\mathrm{i}\Omega)(2+\mathrm{i}\Omega). \tag{2.9}$$

If these relations are satisfied with $\rho, \eta > 0$, then equation (2.5) transforms into (2.7) by the change of variables $A(z) = \rho^{-1}\eta\, a(\eta z)$. If we choose $\alpha_* \in (\alpha_4, \alpha_1)$ and assume the nondegeneracy conditions

$$\begin{aligned} &\operatorname{Im} d(\alpha_*) \neq 0, \quad \tfrac{\mathrm{d}}{\mathrm{d}\alpha}\left(\tfrac{\operatorname{Re} d(\alpha)}{\operatorname{Im} d(\alpha)}\right)\Big|_{\alpha=\alpha_*} \neq 0, \\ &L(\alpha_*) \stackrel{\text{def}}{=} \operatorname{Re} M(\alpha_*) \operatorname{Im} N(\alpha_*) - \operatorname{Im} M(\alpha_*) \operatorname{Re} N(\alpha_*) \neq 0, \end{aligned} \tag{2.10}$$

then the relations (2.9) can be inverted locally in the form

$$R = \widehat{R}(\omega, \Omega, \eta), \quad c = \widehat{c}(\omega, \Omega, \eta), \quad \alpha = \widehat{\alpha}(\Omega), \tag{2.11}$$

by the use of the implicit function theorem. For $\eta = 0$ the three functions are independent of ω, and $\alpha = \widehat{\alpha}(\Omega)$ can be inverted in the form $\Omega = \widehat{\Omega}(\alpha)$ because of the second nondegeneracy condition. This implies

$$\widehat{R}(\omega, \widehat{\Omega}(\alpha), 0) = R_\alpha \text{ and } \widehat{c}(\omega, \widehat{\Omega}(\alpha), 0) = c_\alpha.$$

We need one further function which determines the direction of the bifurcation, namely $K(\alpha)$ defined through $\widehat{R}(\widehat{\Omega}(\alpha), \widehat{\Omega}(\alpha), \eta) = R_\alpha + K(\alpha)\eta^2 + \mathcal{O}(\eta^4)$. A simple calculation gives

$$K(\alpha) = \frac{1}{L(\alpha)}\Big((1-\widehat{\Omega}(\alpha)^2)\operatorname{Im} N(\alpha) - 2\widehat{\Omega}(\alpha)\operatorname{Re} N(\alpha)\Big).$$

With these notations the function G_0 is given by $G_0(\omega, \Omega, \eta, a, \overline{a}, b, \overline{b}) =$

$$\frac{\rho}{\eta^5}\Big((\widehat{d}(R, c, \alpha) - d(\alpha))\frac{\eta^3}{\rho^3}|a|^2 a + \widehat{G}(R, c, \alpha, \frac{\eta}{\rho}a, \frac{\eta}{\rho}\overline{a}, \frac{\eta^2}{\rho}b, \frac{\eta^2}{\rho}\overline{b})\Big),$$

where $(R, c, \alpha) = (\widehat{R}, \widehat{c}, \widehat{\alpha})(\omega, \Omega, \eta)$; and obviously G_0 is of order $\mathcal{O}(1)$ for $\eta \to 0$.

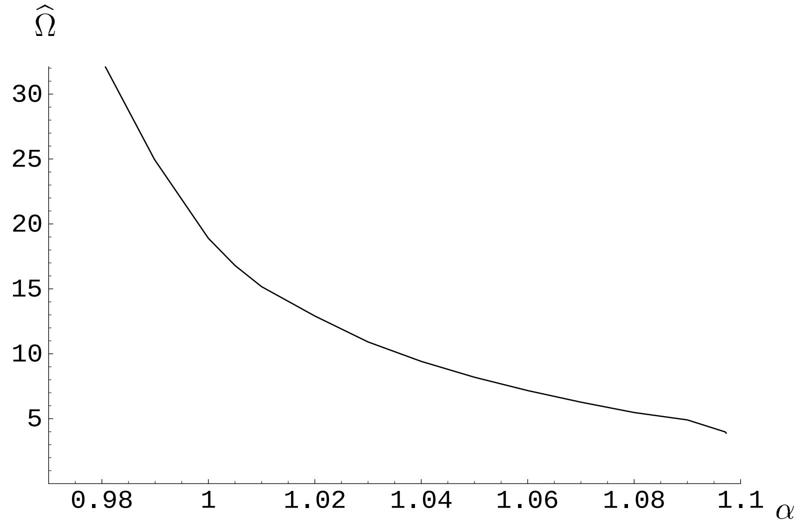

Figure 2: The function $\widehat{\Omega}(\alpha)$.

A further nondegeneracy condition arises from the fact that the single–pulse solution $a_{\rm pulse}$ in (2.8) has to be transversal in a way to be defined later, see Definition 3.6. This leads to exceptional values $\omega \in \mathcal{E} = \{0, \pm\omega_1, \pm\omega_2, \ldots\}$ where transversality fails, see [AfM98a]. Using the relation (2.11) for the critical values this defines an exceptional set of wave numbers

$$\mathcal{A} = \{\, \alpha \in (\alpha_4, \alpha_1) \;:\; \widehat{\Omega}(\alpha) \in \mathcal{E} \,\}.$$

It is now a numerical task to check the nondegeneracy conditions and to calculate $K(\alpha)$ and the exceptional set $\mathcal{A}$. We find that all three conditions in (2.10) hold along the whole interval (α_4, α_1). Thus, the function $\widehat{\Omega}$ exists globally and is monotone, see Figure 2.

Our calculations gives that $L(\alpha)$ is nonzero and $K(\alpha)$ is negative such that the bifurcation is subcritical. The exceptional set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ is finite since the set $\mathcal{E}$ does not have finite points of accumulation as it is derived from the zeros of the analytic function defined in [AfM98a]. From $\omega_1 \approx 8.03$ and $\omega_2 \approx 9.51$ we find the physically relevant values $a_1 \approx 1.053$ and $a_2 \approx 1.038$; but since the numerics in [AfM98a] is reliable only for $|\omega| \lessapprox 10$ we cannot exclude $\alpha_2 \in \mathcal{A}$ since $\widehat{\Omega}(\alpha_2) \approx 13$.

The two theorems in the introduction and their more precise version below will be direct consequences of the following two results concerning the perturbed complex Ginzburg–Landau equation. The proof of these two results will be given in the subsequent section. The first gives a persistence result for the single–pulse solution and the second states the existence of infinitely many n–pulse solutions for each $n \geq 2$.

Theorem 2.2
Choose $\Omega_* = \widehat{\Omega}(\alpha_*) \notin \mathcal{E}$. Then there exists an $\varepsilon > 0$ and a smooth function $\omega^{(1)} : [\Omega_* - \varepsilon, \Omega_* + \varepsilon] \times [0, \varepsilon] \to \mathbb{R}$ with $\omega^{(1)}(\Omega, \eta) = \Omega + \mathcal{O}(\eta^2)$ such that (2.7) has for $\omega = \omega^{(1)}(\Omega, \eta)$ a single–pulse solution $a(\zeta) = a^{(1)}(\Omega, \eta, \zeta)$ which is transversal with respect to perturba-

tions in Ω, is even in ζ and has the expansion

$$a^{(1)}(\Omega,\eta,\zeta) = \widehat{a}(\Omega,\eta)(\cosh\zeta)^{-(1+\mathrm{i}\omega^{(1)}(\Omega,\eta))} + \mathcal{O}(\eta^2 \mathrm{e}^{-2|\zeta|})$$

with $\widehat{a}(\Omega,\eta) = 1 + \mathcal{O}(\eta^2) \in \mathbb{R}$.

For notational simplicity we formulate the result on multi–pulse solutions as a one–parameter problem, that is, we fix ω and η and vary Ω. It would be possible to consider general curves $s \mapsto (\omega(s), \Omega(s), \eta(s))$, however, to keep the notations as simple as possible we restrict ourselves to this case.

Theorem 2.3
Choose $\Omega_* = \widehat{\Omega}(\alpha_*) \notin \mathcal{E}$. Then there exists an $\varepsilon > 0$ such that for all $(\Omega,\eta) \in [\Omega_*-\varepsilon, \Omega_*+\varepsilon] \times [0,\varepsilon]$ and all integers $n \geq 2$ there exists a sequence $\Omega_j^{(n)}$, $j \in \mathbb{N}$ with $\Omega_j^{(n)} \to \Omega$ such that (2.7) with parameters $(\omega^{(1)}(\Omega,\eta), \Omega_j^{(n)}, \eta)$ admits an n–pulse solution $a_j^{(n)}$, that is, there are $\zeta_{j,l}^{(n)} \in \mathbb{R}$, $\beta_{j,l}^{(n)} \in \mathbb{S}^1$ for $l = 1,\dots,n$ such that

$$\sup\{ \left| a_j^{(n)}(\zeta) - \sum_{l=1}^{n} \mathrm{e}^{\mathrm{i}\beta_{j,l}^{(n)}} a^{(1)}(\zeta - \zeta_{j,l}^{(n)}) \right| \,:\, \zeta \in \mathbb{R} \} \to 0 \quad \textit{for } j \to \infty, \tag{2.12}$$

where $a^{(1)}$ is the single–pulse solution of (2.7) with parameters $(\omega^{(1)}(\Omega,\eta), \Omega, \eta)$ from Theorem 2.2.

From the results for the perturbed complex Ginzburg–Landau equation we now easily derive the following results for the Poiseuille problem.

Theorem 2.4
Choose any $\alpha_* \in (\alpha_4, \alpha_1) \setminus \mathcal{A}$, then there exists an $\varepsilon > 0$ and a smooth function $c^{(1)} : \overline{S(\alpha_*,\varepsilon)} \to \mathbb{R}$, where

$$S(\alpha_*,\varepsilon) = \{ (R,\alpha) \,:\, |\alpha - \alpha_*| < \varepsilon,\ R \in (R_\alpha - \varepsilon, R_\alpha) \},$$

with $c^{(1)}(R_\alpha,\alpha) = c_\alpha$ such that the following holds: The Poiseuille problem (2.1) and (2.2) has for the parameters $(R, c^{(1)}(R,\alpha), \alpha)$ with $(\alpha, R) \in S(\alpha_*,\varepsilon)$ a single–pulse solution $u^{(1)}$ which satisfies the expansion

$$u^{(1)}(\xi,y,z) = \frac{\eta}{\rho} \operatorname{Re}\Big(\cosh(\eta z)^{-1-\mathrm{i}\omega} \Psi_0(\xi,y) \Big) + \mathcal{O}(\eta^2 \mathrm{e}^{-|\eta z|}),$$

where (ρ,η,ω) is related to (R,α) via

$$g_0(R, c^{(1)}(R,\alpha), \alpha) = \eta^2 (1+\mathrm{i}\omega)^2 \quad \textit{and} \quad d(\alpha) = -\rho^2 (1+\mathrm{i}\omega)(2+\mathrm{i}\omega).$$

Proof: The result follows from the center–manifold reduction in Theorem 2.1, the existence of the single–pulse solution for the complex Ginzburg–Landau equation and from the relations between the parameters (ω, Ω, η) and (R, c, α). For the latter note that the relations (2.11) map the surface $\omega = \omega^{(1)}(\Omega, \eta)$ obtained in Theorem 2.2 into the surface

$$R = \widetilde{R}(\Omega, \eta) \stackrel{\text{def}}{=} \widehat{R}(\omega^{(1)}(\Omega, \eta), \Omega, \eta), \;\; c = \widetilde{c}(\omega, \eta) \stackrel{\text{def}}{=} \widehat{c}(\omega^{(1)}(\Omega, \eta), \Omega, \eta), \;\; \alpha = \widehat{\alpha}(\Omega).$$

This surface can be written as a graph over the set $S(\alpha_*, \varepsilon)$ for $\varepsilon > 0$ sufficiently small, if $\partial_{(\eta^2)}\widetilde{R}(\widehat{\Omega}(\alpha_*), 0) < 0$. Now, using $\widehat{R}(\omega, \widehat{\Omega}(\alpha), 0) = R_\alpha$ we find easily $\partial_{(\eta^2)}\widetilde{R}(\widehat{\Omega}(\alpha_*), 0) = \partial_{(\eta^2)}\widehat{R}(\Omega_*, \Omega_*, 0) = K(\alpha_*) < 0$. QED

Theorem 2.5
Choose α_* and $\varepsilon > 0$ as in the previous theorem. Then for each integer $n \geq 2$ and each point $(R, \alpha) \in S(\alpha_*, \varepsilon)$ there exists a sequence of parameter triples $(R_j^{(n)}, c_j^{(n)}, \alpha_j^{(n)})$ with

$$(R_j^{(n)}, c_j^{(n)}, \alpha_j^{(n)}) \to (R, c^{(1)}(R, \alpha), \alpha) \quad \textit{for } j \to \infty$$

such that for these parameters the Poiseuille problem (2.1) *and* (2.2) *has an n–pulse solution $u_j^{(n)}$ which satisfies*

$$\sup\{\, \|u_j^{(n)}(\cdot, \cdot, z) - \sum_{k=1}^{n} u^{(1)}(\cdot - \beta_{j,k}^{(n)}, \cdot, z - z_{j,k}^{(n)})\|_{H^2(\Sigma)} \; : \; z \in \mathbb{R}\,\} \to 0 \quad \textit{for } j \to \infty$$

with suitable $z_{j,k}^{(n)} \in \mathbb{R}$ and $\beta_{j,k}^{(n)} \in \mathbb{S}^1$.

Proof: This result again follows directly from the center–manifold reduction in Theorem 2.1 and the existence result for n–pulse solutions of the complex Ginzburg–Landau equation in Theorem 2.3 together with the parameter mapping properties established in the proof of the previous theorem. QED

3 Multi–pulse solutions for the reduced equation

Our analysis follows closely to that in [AfM98a] which treats general reversible four–dimensional SO(2)–invariant vector fields with the saddle–focus constellation of eigenvalues of the linear problem. There, as a particular example, the unperturbed complex Ginzburg–Landau equation, that is (2.7) with $\eta = 0$ is studied. However, due to our special application we have certain complications which prevents us from using the results from there directly. The three major differences are the following. First, the analytical expression for a single–pulse solution is only known for parameters $\Omega = \omega$ and $\eta = 0$ where the inverse transformation (2.11) to physical parameters is singular. Second, because of the center manifold reduction we have to deal with $\mathcal{C}^k$ vector fields rather than analytical ones. Third, we have to characterise multi–pulse solutions for the Poiseuille problem, and for this purpose we establish the additive representation

$$a^{(n)}(x) = \sum_{l=1}^{n} \mathrm{e}^{\mathrm{i}\beta_l} a^{(1)}(x - x_l) + \text{h.o.t.}$$

of n–pulse solutions. In Section 3 we use, as in [AfM98a], the variable x for the independent time–like variable instead of ζ, which was used in Section 2 to avoid confusion with the notations from the Poisuille problem. Further on the dot ($\dot{}$) indicates $\mathrm{d}/\mathrm{d}x$.

The general strategy is the following. We know that (2.7) has a single–pulse solution for $\eta = 0$ and $\Omega = \omega$. Thus we can construct local Poincaré sections K_{in} and K_{out}, a local map $\widetilde{S}(\Omega, \eta, \cdot) : K_{\text{in}} \to K_{\text{out}}$ and a global map $\widetilde{G}(\Omega, \eta, \cdot) : K_{\text{out}} \to K_{\text{in}}$. These maps are studied after factorizing with respect to the SO(2) symmetry. Thus, it is possible to study the persistence of the explicit single–pulse solution for $(\Omega, \eta) = (\omega, 0)$ if the transversality condition holds. Using the same mappings it is then possible to deduce the result concerning the multi–pulse solutions.

The analysis of the ODE (2.7) is done for the equivalent first order equation obtained from

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = N^{-1} \begin{pmatrix} a \\ \dot{a} \end{pmatrix} \quad \text{where } N = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix} \text{ and } \lambda = 1 + \mathrm{i}\omega.$$

The complex Ginzburg–Landau equation formulated for $Y = (y_1, y_2) \in \mathbb{C}^2$ reads

$$\dot{Y} = JY + F(\varepsilon, \eta, Y, \overline{Y}), \quad Y = (y_1, y_2) \in \mathbb{C}^2, \quad F = (F_1, F_2) \in \mathbb{C}^2 \tag{3.1}$$

with the diagonal linear part $J = \operatorname{diag}(\lambda_1, \lambda_2) = \operatorname{diag}(\lambda, -\lambda)$ and $F_j(\varepsilon, \eta, Y, \overline{Y}) = \mathcal{O}(|Y|^3)$. The vector field (3.1) is $\mathcal{C}^k$ with $2 < k < \infty$. We have introduced

$$\varepsilon = \Omega - \omega,$$

which allows us to consider ε as a small bifurcation parameter whereas ω is a fixed parameter, which for brevity will be omitted further on in most expressions. The action of SO(2) and of the involution in the new coordinates is given by:

$$T_\alpha(y_1, y_2) = (\mathrm{e}^{\mathrm{i}\alpha} y_1, \mathrm{e}^{\mathrm{i}\alpha} y_2), \qquad \mathcal{R}(y_1, y_2) = (y_2, y_1).$$

The SO(2)–invariance of (3.1) means $F \circ T_\alpha = T_\alpha F$, and reversibility means $F \circ \mathcal{R} = -\mathcal{R} F$.

Remark 3.1 The analysis below is valid not only for the perturbed complex Ginzburg–Landau equation, but also for the general problems in the form (3.1) with the required symmetry, reversibility, eigenvalue constellation and primary homoclinic solution H_0 that exists for $\varepsilon = \eta = 0$.

Problem (3.1) is obviously reversible under the family $T_\alpha \mathcal{R}$ of transformations. But only two of them are involutions, namely those with $\alpha = 0, \alpha = \pi$. Thus, problem (3.1) must be treated as a bireversible system with respect to $\mathcal{R}$ and $-\mathcal{R}$. An important observation is that any orbit homoclinic to 0 is in fact reversible. Lemma 2.1 in [AfM98a] states that for every homoclinic orbit $H(x)$ of (3.1), there is a uniquely determined $\nu \in \{-1, 1\}$ and $\xi \in \mathbb{R}$ such that

$$H(x) = \nu \mathcal{R}\, H(\xi - x) \quad \text{for all } x \in \mathbb{R}. \tag{3.2}$$

The sign ν is called the parity of H and written as $\mathrm{par}(H) = \nu$.

For example, the function $H_0(x) = N^{-1}(a_{\mathrm{pulse}}(x), \dot{a}_{\mathrm{pulse}}(x))^T$ is a homoclinic solution of (3.1) with $\varepsilon = \eta = 0$. It has parity $\nu = 1$, i.e., $H_0(x) = \mathcal{R}H_0(-x)$.

Our goal is to study the existence of other homoclinic orbits for all small $\varepsilon \in \mathbb{R}, \eta > 0$. The last condition is important since the transformation (2.11) is singular at $\eta = 0$. Note that by SO(2)–invariance and reversibility the existence of reversible orbits is a phenomenon of codimension 1. Therefore it is natural to expect that homoclinic orbits exist on a disconnected set of curves in $(-\varepsilon_0, \varepsilon_0) \times (0, \eta_0)$.

3.1 The Poincaré maps

3.1.1 The local map

We are going to construct a Poincaré section and a suitable Poincaré return map which allows us to study the existence of homoclinic orbits which wind around the original homoclinic orbit n times. To this end we construct a local map $\widetilde{S}$ which takes care of the passage of orbits near the fixed point $Y = 0$ and a global map $\widetilde{G}$ which contains information about the flow near the homoclinic orbit.

Consider the $\mathcal{C}^\infty$ (or formal) ODE

$$\dot{Z} = JZ + \Psi(\varepsilon, \eta, Z, \overline{Z}), \qquad \Psi = (\psi_1, \psi_2) \quad \text{with } Z = (z_1, z_2) \in \mathbb{C}^2. \tag{3.3}$$

It is said to be in normal form if

$$\psi_j(\varepsilon, \eta, Z, \overline{Z}) = z_j \sum_{\langle \Lambda, Q \rangle = 0} g_{jQ}(\varepsilon, \eta)(Z, \overline{Z})^Q, \quad \Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}) \tag{3.4}$$

where $Q = (q_1, \ldots, q_4) \in \mathbb{Z}^4$, $q_j \geq -1$, $q_k \geq 0$ for $k \neq j$, and $(Z, \overline{Z})^Q = z_1^{q_1} z_2^{q_2} \overline{z}_1^{q_3} \overline{z}_2^{q_4}$. Monomials $z_j(Z, \overline{Z})^Q$ with $\langle \Lambda, Q \rangle := \lambda_1 q_1 + \cdots + \lambda_4 q_4 = 0$ are called resonant monomials.

Lemma 3.2
Let $E_0 = (-\varepsilon_0, \varepsilon_0)^2$ for sufficiently small ε_0. For any $k > 0$ there exists $n = n(k) \in \mathbb{N}$ such that if the nonlinearity F in (3.1) lies in $\mathcal{C}^n(E_0 \times \mathbb{C}^2, \mathbb{C}^2)$ and satisfies $F(\varepsilon, \eta, Y, \overline{Y}) = \mathcal{O}(|Y|^3)$, then there exist a neighborhood $U \subset \mathbb{C}^2$ of 0 and a near–identity transformation $M \in \mathcal{C}^k(E_0 \times U, \mathbb{C}^2)$ (that is $M(\varepsilon, \eta, Y, \overline{Y}) = Y + \mathcal{O}(|Y|^3)$ uniformly on E_0) which transforms (3.1) via $(z_1, z_2) = Z = M(\varepsilon, \eta, Y, \overline{Y})$ into the (polynomial) normal form

$$\dot{z}_1 = z_1[\lambda + \Phi(\varepsilon, \eta, |z_1 z_2|^2)], \quad \dot{z}_2 = -z_2[\lambda + \Phi(\varepsilon, \eta, |z_1 z_2|^2)], \tag{3.5}$$

where $\Phi(\varepsilon, \eta, \cdot)$ is a polynomial with $\Phi(\varepsilon, \eta, 0) = 0$. Moreover, the transformation M is SO(2)*– and $\mathcal{R}$–equivariant.*

Proof: Observe that the set of eigenvalues $\{\lambda_j\}, j = 1, \ldots, 4$ belongs to the Siegel domain (see [Arn83]). We write (3.1) as 4 complex equations with $W = (Y, \overline{Y}) \in \mathbb{C}^4$

by adding the complex conjugate equations, i.e., $\dot{W} = \operatorname{diag}(\Lambda)W + G(\varepsilon, \eta, W)$ where $G = (F_1, F_2, \overline{F}_1, \overline{F}_2)$. Of course, SO(2)–symmetry and reversibility are preserved.

We first give the formal normal form under the given symmetries. Since $\overline{\lambda}/\lambda \notin \mathbb{R}$ the resonance condition $\lambda q_1 - \lambda q_2 + \overline{\lambda} q_3 - \overline{\lambda} q_4 = 0$ yields $q_1 = q_2$ and $q_3 = q_4$. From SO(2)–invariance we conclude $q_1 + q_2 - q_3 - q_4 = 0$ which implies $q_1 = q_2 = q_3 = q_4$. Since normal–form transformations with no resonant terms preserve symmetries and reversibility (see [Brj71, Arn83]) we arrive at

$$\dot{w}_j = w_j[\lambda_j + \Phi_j(\varepsilon, \eta, w_1 w_2 w_3 w_4)] \quad \text{for } j = 1, \ldots, 4. \tag{3.6}$$

Using $(w_3, w_4) = (\overline{w}_1, \overline{w}_2)$ and reversibility we find $(\Phi_2, \Phi_3, \Phi_4) = (-\Phi_1, \overline{\Phi}_1, -\overline{\Phi}_1)$, and thus (3.5) is established with $\Phi = \Phi_1$.

For $\mathcal{C}^k$ vector fields this means that (3.1) can be transformed to (3.5) up to higher order terms with a polynomial transformation $\widetilde{M}$. It is left to use the Sternberg–Chen theorem in order to get the desired result. This theorem states that a vector field is $\mathcal{C}^k$–conjugate to its n–jet if $n = n(k)$ is sufficiently large. In fact, the Sternberg–Chen theorem (see [Har64], Thms 12.2, 12.3) deals with systems without parameters and no statement on the equivariance of the conjugating transformation was done. However, as the proof is constructive, it is sufficient to control the equivariance and the parameter dependence of the introduced constructions. Notice that the (polynomial) normal form (3.5) has the same symmetry and reversibility properties as the original problem. Therefore, if the cut–off function $\widetilde{\Phi}(t)$ used in [Har64], page 266, is chosen invariant under the action of the symmetry group, then the remainder $R^{(k)}$ is equivariant under the same action and hence the limit transformation is equivariant too. Of course, for the SO(2)–action and reflection $\mathcal{R}$ the existence of such a $\widetilde{\Phi}(t)$ is obvious. QED

Remark 3.3 Notice that a modern construction of [Sam82] for general parameter dependent vector fields, that also may include center variables, can also be generalized to obtain the equivariance of the conjugating diffeomorphism ([Sam98]).

We describe the local mapping in the normal form coordinates $(z_1, z_2) \in \mathbb{C}^2$. For a homoclinic solution $H_0(x)$ of (3.1) we know that $H_0(x) \in U$ for $|x| \geq x_0 \gg 1$. Hence, in Z–coordinates we have $\widetilde{H}_0(x) = M(\varepsilon, \eta, H_0(x))$ for $|x| \geq x_0$ with

$$\begin{aligned} Z(x) &= \widetilde{H}_0(x) = (c_+ \mathrm{e}^{-\lambda x}, 0) \quad \text{for } x \geq x_0, \\ Z(x) &= \widetilde{H}_0(x) = (0, \operatorname{par}(H_0) c_+ \mathrm{e}^{\lambda(\xi + x)}) \quad \text{for } x \leq -x_0, \end{aligned} \tag{3.7}$$

with $\xi \in \mathbb{R}$ and $c_+ \in \mathbb{C}$. To see this, we use the fact that the stable manifold of (3.5) is given by $z_1 \equiv 0$ while the unstable manifold is given by $z_2 \equiv 0$. The result on the parity is a consequence of (3.2).

We construct a local map $\widetilde{S}$ between the hypersurfaces

$$\begin{aligned} K_{\text{in}} &= \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq r, |z_2| = \delta\} \text{ and} \\ K_{\text{out}} &= \mathcal{R} K_{\text{in}} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = \delta, |z_2| \leq r\}, \end{aligned}$$

where δ and r are taken small enough. Clearly, $K_{\rm out}$ is transversal to the unstable manifold of $Z=0$ and $K_{\rm in}$ is transversal to the stable manifold. The local (Shilnikov) map induced by the flow of (3.5) is given by

$$\widetilde{S}(\varepsilon,\eta,\cdot):\begin{cases} K_{\rm in}\setminus\{(0,z_2):|z_2|=\delta\} & \to \quad K_{\rm out}, \\ (z_1,\delta\mathrm{e}^{\mathrm{i}\psi_2}) & \mapsto \quad (\delta\frac{z_1}{|z_1|}\mathrm{e}^{\mathrm{i}\Sigma(\varepsilon,\eta,|z_1|)},\ \mathrm{e}^{\mathrm{i}(\psi_2-\Sigma(\varepsilon,\eta,|z_1|))}|z_1|),\end{cases}$$

where

$$\Sigma(\varepsilon,\eta,|z_1|)=\frac{\omega+\operatorname{Im}\Phi(\varepsilon,\eta,\delta^2|z_1|^2)}{1+\operatorname{Re}\Phi(\varepsilon,\eta,\delta^2|z_1|^2)}\log\frac{\delta}{|z_1|}. \tag{3.8}$$

This special form of $\widetilde{S}$ follows easily from (3.5) since $|z_1z_2|$ is constant along solutions.

The reversibility of the problem is such that $\mathcal{R}$ maps $K_{\rm in}$ into $K_{\rm out}$ as well as $K_{\rm out}$ into $K_{\rm in}$. Moreover the mapping $\widetilde{S}(\varepsilon,\eta,\cdot)$ satisfies

$$\widetilde{S}(\varepsilon,\eta,\cdot)^{-1}=\mathcal{R}\widetilde{S}(\varepsilon,\eta,\mathcal{R}\,\cdot)\text{ and }\widetilde{S}(\varepsilon,\eta,T_\alpha\cdot)=T_\alpha\widetilde{S}(\varepsilon,\eta,\cdot), \tag{3.9}$$

for $\alpha\in\mathbb{S}^1=\mathbb{R}/_{2\pi\mathbb{Z}}$.

3.1.2 The global and the Poincaré map

The global map, which will take care of the flow along the homoclinic excursion, has to be constructed in the Y–coordinates since the validity of the Z–coordinates is restricted to a small neighborhood of $Z=0$. In Y–coordinates we define the Poincaré sections

$$\widehat{K}_{\rm in}=M^{-1}(\varepsilon,\eta,K_{\rm in})\subset\mathbb{C}^2\text{ and }\widehat{K}_{\rm out}=\mathcal{R}\,\widehat{K}_{\rm in}=M^{-1}(\varepsilon,\eta,K_{\rm out})\subset\mathbb{C}^2.$$

Since the homoclinic orbit $H_0(x)$ of (3.1) exists for $\varepsilon,\eta=0$ we conclude the existence of a smooth Poincaré mapping $\widehat{G}(\varepsilon,\eta,\cdot):\widehat{K}_{\rm out}\mapsto\widehat{K}_{\rm in}$, where as usual, $\widehat{G}$ may only be defined on a smaller set.

In order to be able to combine this map with the local map $\widetilde{S}$ we transform it back to Z–coordinates and let

$$\widetilde{G}(\varepsilon,\eta\cdot):\begin{cases} K_{\rm out} & \to \quad K_{\rm in}, \\ (\delta\mathrm{e}^{\mathrm{i}\psi_1},z_2) & \mapsto \quad M(\varepsilon,\eta,\widehat{G}(\varepsilon,\eta,M^{-1}(\varepsilon,\eta,(\delta\mathrm{e}^{\mathrm{i}\psi_1},z_2)))).\end{cases}$$

This map preserves the SO(2)–symmetry and the reversibility as follows:

$$\widetilde{G}(\varepsilon,\eta,\cdot)^{-1}=\mathcal{R}\widetilde{G}(\varepsilon,\eta,\mathcal{R}\,\cdot),\quad \widetilde{G}(\varepsilon,\eta,T_\alpha\cdot)=T_\alpha\widetilde{G}(\varepsilon,\eta,\cdot), \tag{3.10}$$

for $\alpha\in\mathbb{S}^1$. The assumption that (3.1) has a homoclinic orbit H_0 for $\varepsilon=\eta=0$ now means that

$$\widetilde{G}(0,0,(\delta\mathrm{e}^{\mathrm{i}\psi_1},0))=(0,0,(0,\nu_0\delta\mathrm{e}^{\mathrm{i}\psi_1}))\quad\text{with }\nu_0=\operatorname{par}(H_0). \tag{3.11}$$

This follows easily from (3.7) by shifting x to $x-\xi/2$.

We are now in the position to define the total Poincaré map

$$P(\varepsilon,\eta,\cdot)=\widetilde{G}(\varepsilon,\eta,\cdot)\circ\widetilde{S}(\varepsilon,\eta,\cdot):K_{\rm in}\to K_{\rm in}.$$

In fact, we can avoid the use of the two sections $K_{\rm in}$ and $K_{\rm out}$ by using the fact $\mathcal{R}K_{\rm out} = K_{\rm in}$ and defining the maps $S : K_{\rm in} \to K_{\rm in}$ and $G : K_{\rm in} \to K_{\rm in}$ via

$$S(\varepsilon, \eta, \cdot) = \mathcal{R}\widetilde{S}(\varepsilon, \eta, \cdot) \text{ and } G(\varepsilon, \eta, \cdot) = \widetilde{G}(\varepsilon, \eta, \mathcal{R}\,\cdot),$$

which implies $P = G \circ S$. As a consequence of the symmetries of $\widetilde{S}$ and $\widetilde{G}$ in (3.9) and (3.10), respectively, we have the following result.

Lemma 3.4
Both maps $S, G : K_{\rm in} \to K_{\rm in}$ are involutions on their domains of definition: $S \circ S = \mathrm{id}$ and $G \circ G = \mathrm{id}$. Moreover both maps are $\mathrm{SO}(2)$–equivariant, that is $S \circ T_\alpha = T_\alpha S$ and $G \circ T_\alpha = T_\alpha G$ for $\alpha \in \mathbb{S}^1$.

3.2 Local coordinates and factorization with respect to SO(2)

The next step is to use the $\mathrm{SO}(2)$–symmetry to reduce the dimension of the problem from 3 to 2. For this purpose we introduce in $K_{\rm in}$ local coordinates

$$\psi = \psi_2 \in \mathbb{S}^1 \text{ and } w = z_1 \mathrm{e}^{-\mathrm{i}\psi_2} \in K_r \stackrel{\rm def}{=} \{\, w \in \mathbb{C} \,:\, |w| \leq r \,\},$$

such that $(z_1, z_2) = (w\mathrm{e}^{\mathrm{i}\psi}, \delta \mathrm{e}^{\mathrm{i}\psi}) \in K_{\rm in}$. In these coordinates the $\mathrm{SO}(2)$–action is $T_\alpha(\psi, w) = (\psi + \alpha, w)$ and the mapppings S and G take the form

$$\begin{aligned} S(\varepsilon, \eta, \psi, w) &= (\psi + \sigma(\varepsilon, \eta, w), s(\varepsilon, \eta, w)) \text{ and} \\ G(\varepsilon, \eta, \psi, w) &= (\psi + \gamma(\varepsilon, \eta, w), g(\varepsilon, \eta, w)). \end{aligned} \tag{3.12}$$

For the local map we have the following explicit expressions

$$\sigma(\varepsilon, \eta, w) = \arg w + \Sigma(\varepsilon, \eta, |w|), \quad s(\varepsilon, \eta, w) = \mathrm{e}^{-2\mathrm{i}\Sigma(\varepsilon, \eta, |w|)}\overline{w} \tag{3.13}$$

The complex conjugation in $s(\varepsilon, \eta, \cdot)$ arises from $\mathcal{R}$ in the definition $S = \mathcal{R}\widetilde{S}$. Notice that for $w = 0$ there is the unique continuous extension $s(\varepsilon, \eta, 0) = 0$. In contrast, $\sigma(\varepsilon, \eta, w) : \mathbb{R}^2 \times K_r \setminus \{0\} \to \mathbb{R}$ cannot be extended continuously into $w = 0$.

The functions γ and g can be obtained from $\widetilde{G}$ via

$$(g(\varepsilon, \eta, w)\mathrm{e}^{\mathrm{i}\gamma(\varepsilon, \eta, w)}, \delta \mathrm{e}^{\mathrm{i}\gamma(\varepsilon, \eta, w)}) = \widetilde{G}(\varepsilon, \eta, (\delta, w)) = T_{-\psi}\widetilde{G}(\varepsilon, \eta, (\delta \mathrm{e}^{\mathrm{i}\psi}, \mathrm{e}^{\mathrm{i}\psi} w)). \tag{3.14}$$

The properties of the maps G and S in (ψ, w)–coordinates can be summarized as follows.

Lemma 3.5
Let $E_0 = (-\varepsilon_0, \varepsilon_0)^2$ for sufficiently small positive ε_0. Then, we have:

1. The functions $\gamma : E_0 \times K_r \to \mathbb{S}^1$ and $g : E_0 \times K_r \to \mathbb{C}$ are $\mathcal{C}^k$ and satisfy on their domain of definition

$$\gamma(\varepsilon, \eta, g(\varepsilon, \eta, w)) + \gamma(\varepsilon, \eta, w) = 0 \in \mathbb{S}^1 \text{ and } g \circ g = \mathrm{id}. \tag{3.15}$$

2. The functions $\sigma : E_0 \times K_r \setminus \{0\} \to \mathbb{S}^1$ and $s : E_0 \times K_r \setminus \{0\} \to K_r$ are $\mathcal{C}^k$ and $s(\varepsilon, \eta, \cdot)$ is a Lipschitz map on K_r. Moreover we have

$$\sigma(\varepsilon, \eta, s(\varepsilon, \eta, w)) + \sigma(\varepsilon, \eta, w) = 0 \in \mathbb{S}^1 \text{ and } s \circ s = \mathrm{id}. \tag{3.16}$$

3. Let $p(\varepsilon, \eta, w) = g(\varepsilon, \eta, s(\varepsilon, \eta, w))$ then $p^{-1} = s \circ p \circ s = g \circ p \circ g$.

The proof is straightforward from the above construction.

Finally the total Poincaré map $P = G \circ S$ is written in (ψ, w)–coordinates as

$$P(\varepsilon, \eta, \psi, w) = (\psi + \widetilde{\pi}(\varepsilon, \eta, w), p(\varepsilon, \eta, w))$$

where $\widetilde{\pi}(\varepsilon, \eta, w) = \gamma(\varepsilon, \eta, s(\varepsilon, \eta, w)) + \sigma(\varepsilon, \eta, w)$ and $p(\varepsilon, \eta, w) = g(\varepsilon, \eta, s(\varepsilon, \eta, w))$.

3.3 Single and multi–pulse solutions

The factorised global map g is a smooth involution with $g(0, 0, 0) = 0$ therefore we have the expansion

$$g(\varepsilon, \eta, w) = a(\varepsilon, \eta) + \Gamma(\varepsilon, \eta)w + \mathcal{O}(|w|^2) \quad \text{with } \Gamma^2 = I \text{ and } \Gamma a + a = 0. \tag{3.17}$$

The two identities for Γ and a easily follow from $g \circ g = \mathrm{id}$. This implies that eigenvalues of the matrix $\Gamma(\varepsilon, \eta)$ lie in $\{-1, 1\}$. To demonstrate that $\det \Gamma = -1$ notice that the Poincaré map $P = G \circ S$ from K_{in} into itself is orientation preserving and hence the same is true for the factorized map $p = g \circ s$, that is $\det D_w p(w) = 1$ whenever it is defined. Hence $\det D_w s(w) = -1$ implies $\det D_w g(w) = -1$ as desired.

Having $\Gamma^2 = I$ and $\det \Gamma = -1$ there is a rotation $w \mapsto \mathrm{e}^{\mathrm{i}\alpha} w$ such that $\mathrm{e}^{-\mathrm{i}\alpha} \Gamma \mathrm{e}^{\mathrm{i}\alpha} = \left(\begin{smallmatrix} 0 & b \\ 1/b & 0 \end{smallmatrix}\right)$ with $b > 0$. For $\Gamma = \left(\begin{smallmatrix} q & u \\ v & -q \end{smallmatrix}\right)$ take $\alpha = \frac{1}{2}\mathrm{arcctg}(-\frac{u+v}{q})$. Note that such a smooth rotation of $w \in K_r$ does change the local map s simply from Σ into $\Sigma + 2\alpha$. The global map in these coordinates is given by

$$g(\varepsilon, \eta, w) = h(\varepsilon, \eta) \begin{pmatrix} -b \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix} w + \mathcal{O}(|w|^2) \quad \text{with } h^2 = \frac{|a|^2}{1 + b^2}. \tag{3.18}$$

where the real coefficients $b(\varepsilon, \eta)$ and $h(\varepsilon, \eta)$ lie in $\mathcal{C}^k(E_0, \mathbb{R})$ since the rotation is smooth in ε and η. In order to get more information on homoclinic solutions we must impose a nondegeneracy condition.

Definition 3.6
Assume that $H_{\varepsilon_*, \eta_*}$ is a single–pulse solution, that is $g(\varepsilon_*, \eta_*, 0) = 0$. Then, $H_{\varepsilon_*, \eta_*}$ is called **transversal** *with respect to the perturbation in ε if*

$$c(\varepsilon_*, \eta_*) \stackrel{\text{def}}{=} \frac{\partial h(\varepsilon_*, \eta_*)}{\partial \varepsilon} \neq 0. \tag{3.19}$$

In [AfM98a] this transversality is shown to be the key feature to prove the existence of cascades of multi–pulse solutions. Moreover, it is essential to prove persistence of the single–pulse solutions in two–parameter families.

Here we can take advantage from the fact that for the case $\eta = 0$ we are dealing with the complex Ginzburg–Landau equation, where transversality of a_{pulse} in (2.8) was established numerically in [AfM98a] for all ω not lying in the exceptional set $\mathcal{E}$. Recall that $c(\varepsilon_0, \eta_0)$ in (3.19) also depends on the parameter ω. In the following proposition we introduce this dependence explicitly.

Proposition 3.7
If ω is chosen not in $\mathcal{E} = \{0, \pm\omega_1, \pm\omega_2, \ldots\}$, then there exists an $\varepsilon_0 > 0$ such that

$$c(\omega, \varepsilon, \eta) \neq 0 \quad \textit{for} \quad |\varepsilon| < \varepsilon_0 \quad \textit{and} \quad |\eta| < \varepsilon_0.$$

Proof: The definition of transversality is given with respect to a specific parameter in the equation which is different here from the definition in [AfM98a] and we have to show that these two definitions coincide. If this is the case then we know $c(\omega, 0, 0) \neq 0$ and the desired result follows from the continuity of c.

To avoid confusion we rename the parameter ε in [AfM98a] into $\widetilde{\varepsilon}$. There, the parameter dependence in the complex Ginzburg–Landau equation appears only via the term $d_{\widetilde{\varepsilon}}|a|^2a = (1+\mathrm{i}\widetilde{\varepsilon})(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)|a|^2a$. This case can be retransformed into our case with $\eta = 0$ where the parameter ε appears via $(1+\mathrm{i}\Omega)(2+\mathrm{i}\Omega)|a|^2a$ with $\Omega = \omega+\varepsilon$.

If we find ρ, Ω from $\rho^2(1+\mathrm{i}\Omega)(2+\mathrm{i}\Omega) = d_{\widetilde{\varepsilon}}$, then $\varepsilon = \Omega(\widetilde{\varepsilon})-\omega = S(\omega)\widetilde{\varepsilon}+\mathcal{O}(\widetilde{\varepsilon}^2)$, where $S(\omega) = 3(1+q^2)(3q+(8+9q^2)^{1/2})/(32+36q^2)^{1/2}$ with $q = (2-\omega^2)/3\omega$. From $S(\omega) \neq 0$ and $c(\omega, 0, 0)S(\omega) = \widetilde{c}(\omega)$, where $\widetilde{c}(\omega)$ is the associated transversality coefficient in [AfM98a], we conclude $c(\omega, 0, 0) \neq 0$ for $\omega \notin \mathcal{E}$ as desired. QED

With this proposition we are ready to prove Theorem 2.2 which is the persistence result for the single–pulse solution in the perturbed complex Ginzburg–Landau equation.

Proof of Theorem 2.2: We introduce the explicit dependence on ω. A single–pulse solution exists whenever $g(\omega, \varepsilon, \eta, 0) = 0 \in \mathbb{R}^2$. By the expansion (3.18) this is the case if and only if $h(\omega, \varepsilon, \eta) = 0 \in \mathbb{R}$. From Proposition 3.7 it follows that this scalar equation can be solved locally around $\Omega_* \notin \mathcal{E}$ in the form $\varepsilon = \varepsilon^{(1)}(\eta, \omega)$ with a smooth function $\varepsilon^{(1)}(\eta, \omega)$. Since $\varepsilon^{(1)}(0, \omega) = 0$, the relation $\Omega = \omega + \varepsilon^{(1)}(\eta, \omega)$ can be solved for $\omega = \omega^{(1)}(\Omega, \eta)$ and the result is estalished. QED

The result for the existence of infinite sequences of n–pulse solutions now follows directly from the analysis in [AfM98a] if we consider the single–pulse solution obtained above as the primary pulse, keep $\eta > 0$ fixed and vary ε in a neighborhood of $\varepsilon^{(1)}(\eta, \omega)$. Note that all the results in [AfM98a] concerning the n–pulse solutions do only involve continuity and transversality arguments, so that analyticity is not needed there. Thus, Theorem 2.3 is proved as well.

3.4 The additive representation of multi–pulses

We introduce an alternative characterisation of multi–pulse solutions in order to be able to define multi–pulse solutions for elliptic problems without appealing to the reduced ODE on the center manifold. According to [AfM98a] a homoclinic solution Y is called n–pulse solution if there are $\beta_1, \ldots, \beta_n \in \mathbb{S}^1$ such that Y winds in each of the tubular neighborhoods $T_{\beta_j}N_r$ of the homoclinic orbit $T_{\beta_j}H^{(1)}$ exactly once, where $H^{(1)}$ is the primary homoclinic orbit and $N_r = \{\, Y \in \mathbb{C}^2 \ : \ \exists x \in \mathbb{R} : |Y-H^{(1)}(x)| < r \,\}$. We denote an n–pulse solution of (3.1) located in $\cup_{j=1}^n T_{\beta_j}N_r$ by $H_r^{(n)}(\widehat{\varepsilon}, \widehat{\eta}, x)$.

Proposition 3.8
For each integer $n \geq 2$ there exist $C > 0$ and $r_0 > 0$ such that for all $r \in (0, r_0)$ and for all n–pulse solutions $H_r^{(n)}(\widehat{\varepsilon}, \widehat{\eta}, x)$ of (3.1) there exist $\beta_j, x_j \in \mathbb{R}$ with $j = 1, \cdots, n$ such that

$$\sup_{x\in\mathbb{R}} \left| H_r^{(n)}(\widehat{\varepsilon}, \widehat{\eta}, x) - \sum_{j=1}^{n} T_{\beta_j} H^{(1)}(\varepsilon^{(1)}(\widehat{\eta}), \widehat{\eta}, x-x_j) \right| < C\sqrt{r}. \tag{3.20}$$

Proof: We denote the local coordinates introduced on $K_{\text{in}} \setminus \{(0, z_2) : |z_2| = \delta\}$ in Section 3.2 as $(\psi^{\text{in}}, w^{\text{in}})$ and on $K_{\text{out}} \setminus \{(z_1, 0) : |z_1| = \delta\}$ as $(\psi^{\text{out}}, w^{\text{out}})$. According to our definition we have $Z^{\text{in}} = (w^{\text{in}}\mathrm{e}^{\mathrm{i}\psi^{\text{in}}}, \delta\mathrm{e}^{\mathrm{i}\psi^{\text{in}}})$ and $Z^{\text{out}} = (\delta\mathrm{e}^{\mathrm{i}\psi^{\text{out}}}, w^{\text{out}}\mathrm{e}^{\mathrm{i}\psi^{\text{out}}})$.

From the construction of the Poincaré map we conclude that the n–homoclinic orbit $H_r^{(n)}$ intersects $\widehat{K}_{\text{out}}$ in n different points $Y_k^{\text{out}} = H_r^{(n)}(\widehat{\varepsilon}, \widehat{\eta}, x_k^{\text{out}}) \in \widehat{K}_{\text{out}}$, $k = 1, \ldots, n$ with $Z_k^{\text{out}} \stackrel{\text{def}}{=} M(\widehat{\varepsilon}, \widehat{\eta}, Y_k^{\text{out}})$ with local coordinates $(\psi_k^{\text{out}}, w_k^{\text{out}}) \in K_{\text{out}}$, $k = 1, \ldots, n$. In a same way $H_r^{(n)}$ intersects $\widehat{K}_{\text{in}}$ in points $Y_k^{\text{in}} = H_0^{(n)}(\widehat{\varepsilon}, \widehat{\eta}, x_k^{\text{in}}) \in \widehat{K}_{\text{in}}$, $k = 1, \ldots, n$ with $Z_k^{\text{in}} \stackrel{\text{def}}{=} M(\widehat{\varepsilon}, \widehat{\eta}, Y_k^{\text{in}})$ with local coordinates $(\psi_k^{\text{in}}, w_k^{\text{in}}) \in K_{\text{in}}$, $k = 1, \ldots, n$.

The intersection times are in the order $x_1^{\text{out}} < x_1^{\text{in}} < \ldots < x_j^{\text{out}} < x_j^{\text{in}} < \ldots < x_n^{\text{out}} < x_n^{\text{in}}$. Moreover, we have $w_1^{\text{out}} = w_n^{\text{in}} = 0$ and $\psi_n^{\text{in}} - \psi_1^{\text{out}} = \arg(\nu)$ where $\nu = \operatorname{par}(H_r^{(n)}(\widehat{\varepsilon}, \widehat{\eta}, \cdot))$.

Now choose x_0 such that $H^{(1)}(\varepsilon^{(1)}(\widehat{\eta}), \widehat{\eta}, x_0) \in \widehat{K}_{\text{out}}$ and take $x_j = x_j^{\text{out}} - x_0$ and $\beta_j = \psi_j^{\text{out}}$. Then, for $j = 1, \ldots, n$ we obtain

$$|H_r^{(n)}(\widehat{\varepsilon}, \widehat{\eta}, x) - T_{\beta_j} H^{(1)}(\varepsilon^{(1)}(\widehat{\eta}), \widehat{\eta}, x-x_j)| < \widetilde{C}\sqrt{r} \quad \text{for} \quad x \in I_j$$

with $I_j = ((x_j^{\text{out}} + x_{j-1}^{\text{in}})/2, (x_{j+1}^{\text{out}} + x_j^{\text{in}})/2)$, where $x_0^{\text{in}} = -\infty$ and $x_{n+1}^{\text{out}} = \infty$. Here we have used continuity arguments for $x \in (x_j^{\text{out}}, x_j^{\text{in}})$ and straightforward estimates using the hyperbolic structure of the fixed point $Y = 0$ for $x \in I_j \setminus (x_j^{\text{out}}, x_j^{\text{in}})$. Since $x_{j+1}^{\text{out}} - x_j^{\text{in}} \geq C \log(1/r)$ we obtain $|H^{(1)}(\varepsilon^{(1)}(\widehat{\eta}), \widehat{\eta}, x-x_j)| < \widehat{C}\, r$ for $x \in \mathbb{R} \setminus I_j$. Choosing the appropriate constant C we arrive at the required statement. QED

Theorem 2.3 easily follows from this statement and as a direct consequence of this result we obtain Theorem 2.5.

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