A spatial dynamics approach to three-dimensional gravity-capillary steady water waves

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1 Introduction

A new approach to the question of the existence of small-amplitude, uniformly translating, two-dimensional capillary-gravity water waves was proposed by Kirchgässner (1988), who suggested writing the governing equations as an infinite-dimensional, quasilinear dynamical system in which the horizontal coordinate is the time-like variable. The centre-manifold reduction theorem of Mielke (1988) may then be used to show that this dynamical system is locally equivalent to a system of ordinary differential equations whose solution set can, in theory, be analysed. This method has become known as 'spatial dynamics' and is the basis for several existence theories for two-dimensional capillary-gravity water waves (see Iooss (1995) for a review). It was later noticed by Mielke (1991, ch. 9) and Baesens & MacKay (1992) that the two-dimensional steady water-wave problem can be written as a Hamiltonian system, and their observations were placed upon a secure functional-analytic foundation by Groves & Toland (1997). Mielke (1991) demonstrated that the reduced system of ordinary differential equations inherits the Hamiltonian structure, and this additional feature of the reduction process has since been exploited in recent existence theories for infinite families of multi-humped solitary waves (Buffoni, Groves & Toland 1996; Buffoni & Groves 1999).

This article is concerned with the application of spatial dynamics methods to a three-dimensional steady water-wave problem in which the waves are uniformly translating in one horizontal direction and periodic in the other. The issue was raised recently by Haragus-Courcelle & Illichev (1998), who studied a model equation for three-dimensional water waves of this type. The main difficulty in the full three-dimensional steady water-wave problem is the appropriate choice of variables for a correct formulation as a dynamical system. This matter is settled in the present article. Physically motivated arguments based upon a well-known formal variational formulation of the problem are used to select variables whose mathematical suitability is confirmed *a posteriori*. The dynamical system obtained in this fashion is Hamiltonian and amenable to the centre-manifold reduction procedure described above.

The first contribution to the study of variational formulations of the water-wave problem was made by Luke (1967), who published a formal variational principle that recovers the water-wave equations. A version of Luke's variational principle for the three-dimensional steady water-wave problem under consideration here is given in Section 2.2. It depends upon the velocity potential $\phi(x, y, z)$ and the free-surface elevation $\eta(x, z)$ and involves integrals over the unknown fluid domain $D_\eta = \{(x, y, z) : x \in \mathbb{R}, y \in (0, \eta(x, z)), z \in (0, P)\}$ (see equation (10)). A more workable variational principle is found by mapping D_η onto a strip using the transformation $y = Y\eta(x, z)$, where $Y \in (0, 1)$, and introducing a new variable $\Phi(x, Y, z) = \phi(x, y, z)$. The new variational functional can be interpreted as the action functional associated with a Lagrangian for the 'configuration variables' η and Φ and their 'velocities' η_x and Φ_x. Proceeding formally, one can carry out a Legendre transform to derive a Hamiltonian functional H which depends upon the canonical variables η, ω, Φ and ξ, where ω and ξ are 'momenta' conjugate to respectively η and Φ. 'Hamilton's equations' are written down by taking formal variational derivatives of this functional (equations (19)–(22)), and are accompanied by nonlinear boundary conditions (equations (17), (18)) which emerge from the necessary integration by parts.

The above procedure suggests strongly that a suitable formulation of the three-dimensional steady water-wave problem as a dynamical system is given by

$$u_x = f(u),$$

in which $u = (\eta, \omega, \Phi, \xi)$ and f is defined by the right-hand sides of 'Hamilton's equations', together with the nonlinear boundary conditions mentioned above. In Section 2.4 a Hilbert space X_s and a domain $\mathcal{D}(f)$ for f are identified with the properties that $\mathcal{D}(f)$ is an open subset of a dense subspace X_{s+1} of X_s, f maps $\mathcal{D}(f)$ smoothly into X_s and the nonlinear boundary conditions are well defined for $u \in \mathcal{D}(f)$. The spaces X_s and X_{s+1} are chosen as products of Hilbert spaces of fractional order, and results from interpolation theory are used to verify that $f : \mathcal{D}(f) \to X_s$ is well defined and smooth. One particular issue that arises here is that η, ω and Φ, ξ are defined upon respectively $S = \{z \in (0, P)\}$ and $\Sigma = \{(Y, z) \in (0, 1) \times (0, P)\}$, and it is necessary to relate these functions to each other. A key feature of the function f is that the trace operator $\Phi(Y, z) \mapsto \Phi(1, z)$, which maps $H^s(\Sigma)$ onto $H^{s-1/2}(S)$ for $s > 0$, does not appear: the operator $\Phi(Y, z) \mapsto \int_0^1 \Phi(Y, z)\, \mathrm{d}Y$, which maps $H^s(\Sigma)$ into $H^s(S)$, occurs in its place. The better mapping properties of the latter operator, which is selected by the physics of the problem, play an essential role in ensuring that f is well defined as an operator from its domain into X_s. It is also confirmed in Section 2.4 that solutions of the dynamical system under consideration define solutions of the steady water-wave problem (see Lemma 1) and its Hamiltonian structure is made rigorous. The remaining obstacle to its use in analysis, namely the nonlinear boundary conditions, is overcome in Section 3.2, where a diffeomorphism is constructed which transforms the system into an equivalent system with linear boundary conditions. Precise statements of the reduction theorems are given in Section 3, which also contains the material to verify that they are applicable to the present formulation of the three-dimensional steady water-wave problem.

The basic hydrodynamic problem depends upon two dimensionless parameters $\alpha = gh/c^2$ and $\beta = \sigma/hc^2$ where g, σ, h and c denote respectively the acceleration due to gravity, the coefficient of surface tension, the depth of the water in its undisturbed state and the velocity of the travelling wave; by assuming periodicity in the z-direction with period hP, a third dimensionless parameter $P > 0$ is introduced. The eigenvalues λ of the linearised problem satisfy the equation

$$(\alpha - \beta\sigma^2)\sigma \sin \sigma = |\lambda|^2 \cos \sigma, \qquad \sigma^2 = |\lambda|^2 - 4\pi^2 k^2/P^2 \tag{1}$$

for some $k \in \mathbb{Z}$. Figure 1 depicts the values of (β, α) at which the number of purely imaginary eigenvalues changes. Note that it incorporates the corresponding picture for two-dimensional water waves, which was first given by Kirchgässner (1988, Figure 1). 'Two-dimensional' eigenvalues (those associated with two-dimensional water waves) are shown as solid dots in Figure 1; they correspond to solutions of (1) with $k = 0$. 'Three-dimensional' eigenvalues (those which arise due to the z-dependent terms in the present problem) are shown as hollow dots and correspond to solutions of (1) with $k \neq 0$. In contrast to the geometrically simple 'two-dimensional' eigenvalues, the 'three-dimensional' eigenvalues are geometrically double: the corresponding eigenvectors occur in pairs with z-dependence of the form $\cos(2\pi k/P)$ and $\sin(2\pi k/P)$.

The three-dimensional steady water-wave problem is invariant under the transformation $z \mapsto -z$, and one may take advantage of this reflective symmetry to restrict to wave motions which are even in z and hence admit a representation as a Fourier cosine series. Its eigenvalues become geometrically simple, and for values of (β, α) away from the curves $C_0, C_1, \ldots$ and $\{\alpha = 1\}$ all purely imaginary eigenvalues are also algebraically simple. This fact, together with the Hamiltonian structure of the hydrodynamic problem, forms the basis of an existence theory for periodic

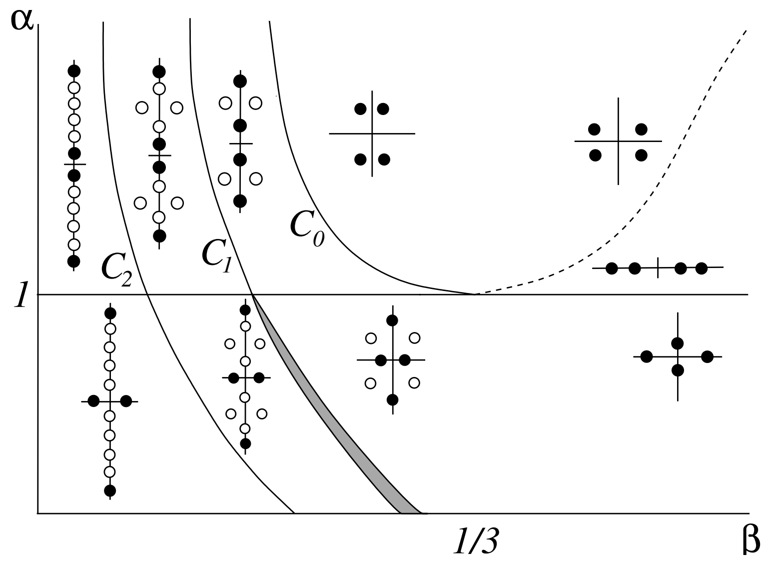

Figure 1: The solid and hollow dots represent respectively 'two-dimensional' and 'three-dimensional' eigenvalues. The curves C_0, C_1, C_2, ... consist of points in (β, α)-parameter space at which four purely imaginary eigenvalues become complex without passing through zero. At points along $\{\alpha = 1\}$ two purely imaginary 'two-dimensional' eigenvalues become real by passing though zero. The dashed curve consists of points at which four 'two-dimensional' real eigenvalues become complex without passing through zero. An existence theory for generalised solitary waves in the shaded region is given in Section 4.

or quasiperiodic waves. The centre-manifold reduction procedure preserves both the Hamiltonian structure and the configuration of the purely imaginary eigenvalues. Introducing canonical coordinates for the reduced equations using Darboux's theorem, one can complete the existence theory by a direct application of the Lyapunov centre theorem or an appropriate result from the KAM theory. The results which can be obtained in this way are stated precisely in Section 3.5.

The curves C_k, $k = 1, 2, \ldots$ in Figure 1 portray the loci of points in (β, α)-parameter space at which the number of purely imaginary 'three-dimensional' eigenvalues changes. More precisely, two purely imaginary eigenvalues $\pm \mathrm{i} s_k$ associated with the kth Fourier mode become complex without passing through zero at the curve C_k. Each curve C_k, $k = 1, 2, \ldots$ consists of two parts C_k^+ and C_k^- which lie respectively above and below the line $\{\alpha = 1\}$. The eigenvalue structure on C_k^+ and C_k^- is different because of the different nature of the 'two-dimensional' eigenvalues above and below this line. Codimension-one bifurcations of even water waves for parameter values near the curve C_1^- (at which the centre manifold is six dimensional) are considered in detail in this paper. Illichev & Kirchgässner (1998) recently derived an integrable normal form for six-dimensional reversible dynamical systems with an eigenvalue configuration of this type: as a bifurcation parameter μ is increased through zero, four of its six purely imaginary eigenvalues become complex without passing through zero. At each order their normal form admits a family of periodic solutions, each of which is connected to itself by a circle of homoclinic connections when $\mu > 0$. A Hamiltonian version of this theory is given in Section 4.1 of the present paper, and Sections 4.2 and 4.3 deal with the issue of how its solutions survive the re-introduction of higher-order remainder terms in the vector field which are neglected in the normal-form approx-

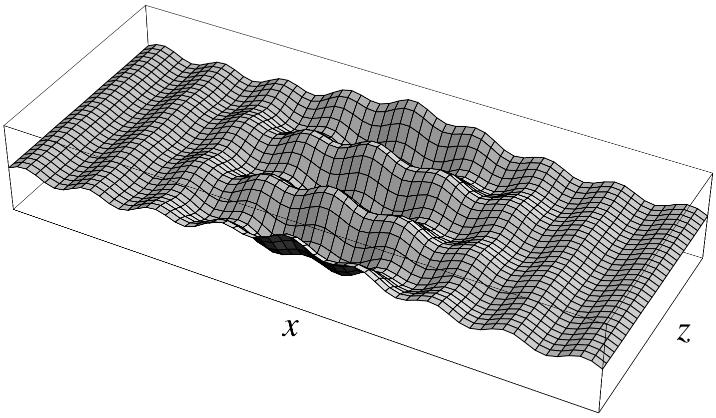

Figure 2: A sketch of a three-dimensional generalised solitary wave which decays to a two-dimensional periodic ripple at infinity.

imation. It is established that a family of periodic orbits persists, each of which is connected to itself by four reversible homoclinic connections. The method is quite general and complements the work of Illichev & Kirchgässner, who did not consider the persistence issue.

In the steady water-wave problem the periodic orbits in the above discussion are governed by 'two-dimensional' eigenvalues and therefore correspond to two-dimensional wave motions. On the other hand the homoclinic connections to these orbits are controlled by 'three-dimensional' complex eigenvalues. The corresponding solutions of the steady water-wave problem are generalised solitary waves which consist of a three-dimensional wave decaying to a two-dimensional periodic ripple far up- and downstream; a wave of this kind is sketched in Figure 2. The main result of the analysis in Section 4 may be stated mathematically as follows. Take $(\beta_0, \alpha_0) \in C_1^-$. There exists a positive number $\hat{I}_0$ such that the three-dimensional steady water-wave problem has a periodic orbit $\mathcal{P}_{\hat{I},\mu}$ at $(\beta, \alpha) = (\beta, \alpha + \mu)$ for each $\hat{I} \in (0, \hat{I}_0)$ and each $\mu \in (0, \mu_{\hat{I}})$, where $\mu_{\hat{I}}$ is a positive constant which depends upon $\hat{I}$. The orbit $\mathcal{P}_{\hat{I},\mu}$ is z-independent, has amplitude $O(\sqrt{\mu \hat{I}})$ and is connected to itself by four geometrically distinct, reversible homoclinic orbits which have nontrivial, periodic z-dependence.

The analysis in this paper was further motivated by previous work of Bridges (1994), who used perceptive physically-motivated arguments to suggest a Hamiltonian formulation for three-dimensional steady water waves which does not appear to be directly amenable to functional-analytic treatment. In his study he identifies the existence of the curves C_1, C_2, …in a different form (see his Figures 5.1 and 5.2) and argues that they should be associated with bifurcation phenomena. The only other rigorous treatments of three-dimensional gravity-capillary waves currently available are due to Reeder & Shinbrot (1981), who study periodic travelling waves whose fundamental domain is a 'symmetric diamond' and Craig & Nicholls (1999), who study periodic travelling waves with general fundamental domains.

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2 Formulation as an infinite-dimensional Hamiltonian system

2.1 The hydrodynamic problem

The physical problem in question concerns the three-dimensional flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. Let $(x, \tilde{y}, z)$ denote the usual Cartesian coordinates. The fluid occupies the domain $D_\eta = \{(x, \tilde{y}, z) : x, z \in \mathbb{R}, \tilde{y} \in (0, \eta(x, z, t))\}$, where $\eta > 0$ is a function of the spatial coordinates x, z and of time t. In terms of an Eulerian velocity potential ϕ, the mathematical problem is to solve Laplace's equation

$$\phi_{xx} + \phi_{\tilde{y}\tilde{y}} + \phi_{zz} = 0 \qquad \text{in } D_\eta, \tag{2}$$

with boundary conditions

$$\begin{aligned}
\phi_{\tilde{y}} &= 0 && \text{on } \tilde{y} = 0 && (3)\\
\eta_t &= \phi_{\tilde{y}} - \eta_x\phi_x - \eta_z\phi_z && \text{on } \tilde{y} = \eta, && (4)\\
\phi_t &= \frac{1}{2}(\phi_x^2 + \phi_{\tilde{y}}^2 + \phi_z^2) - g\eta && &&\\
&\quad + \sigma\left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_x + \sigma\left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_z + B && \text{on } \tilde{y} = \eta, && (5)
\end{aligned}$$

in which $\sigma > 0$ is the coefficient of surface tension and B is a constant called the Bernoulli constant (see, for example, Stoker (1957, §§1, 2.1)). Equation (3) is the kinematic condition that the water cannot permeate the rigid horizontal boundary at $\tilde{y} = 0$, while (4), (5) are respectively the kinematic and dynamic conditions at the free surface. Notice that equations (2)–(5) have the trivial solution $(\eta, \phi) = (h, 0)$, where the constant h represents the depth of the water in its undisturbed state.

This paper treats waves that are uniformly translating in the x-direction and periodic in the z-direction. These waves are described by solutions of (2)–(5) of the special form $\eta(x, z, t) = \eta'(x - ct, z)$, $\phi(x, \tilde{y}, z, t) = \phi'(x - ct, \tilde{y}, z)$. Substituting this form of η, ϕ into (2)–(5) and introducing the dimensionless variables

$$(x', \tilde{y}', z') = \frac{1}{h}(x, \tilde{y}, z), \quad \eta'(x', z') = \frac{1}{h}\eta(x, z), \quad \phi'(x', \tilde{y}', z') = \frac{1}{ch}\phi(x, \tilde{y}, z),$$

one obtains the equations

$$\begin{aligned}
&\phi_{xx} + \phi_{\tilde{y}\tilde{y}} + \phi_{zz} = 0 && 0 < \tilde{y} < \eta(x, z), && (6)\\
&\phi_{\tilde{y}} = 0 && \text{on } \tilde{y} = 0, && (7)\\
&\phi_{\tilde{y}} = \eta_x\phi_x + \eta_z\phi_z - \eta_x && \text{on } \tilde{y} = \eta(x, z) && (8)
\end{aligned}$$

and

$$\begin{aligned}
&-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_{\tilde{y}}^2 + \phi_z^2) + \alpha(\eta - 1)\\
&\quad - \beta\left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_x - \beta\left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_z = 0 \qquad \text{on } \tilde{y} = \eta(x, z) \qquad (9)
\end{aligned}$$

in which the primes have been dropped and x is now a shorthand for the variable $x - ct$. Here $\alpha = gh/c^2$, $\beta = \sigma/hc$ and the Bernoulli constant has been set to gh to allow for solitary waves, that is solutions of (6)–(9) with $\eta \to 1$, $(\phi_x, \phi_{\tilde{y}}, \phi_z) \to 0$ as $x \to \pm\infty$. The variables η, ϕ are assumed to be periodic in z; their period will henceforth be denoted by P.

The hydrodynamic problem (6)–(9) has certain symmetries. The equations are invariant under translations in the spatial coordinates x, z and in the velocity potential ϕ. There are also two discrete symmetries which play an important role in the following theory, namely $z \mapsto -z$ (*reflection*) and $(x, \eta, \phi) \mapsto (-x, \eta, -\phi)$ (*reversibility*).

2.2 *Physical considerations*

This section presents a formal, but physically motivated and direct argument which suggests the correct choice of Hamiltonian function for the three-dimensional steady water-wave problem. The starting point is the observation that equations (6)–(9) follow from the formal variational principle

$$\delta\left\{\int\int_0^P \left(\int_0^\eta \frac{1}{2}((\phi_x - 1)^2 + \phi_{\tilde{y}}^2 + \phi_z^2)\,\mathrm{d}\tilde{y} \right.\right.$$
$$\left.\left. + \frac{1}{2}\alpha(\eta-1)^2 - \frac{1}{2}(\eta - 1) + \beta(\sqrt{1+\eta_x^2+\eta_z^2} - 1)\right)\mathrm{d}z\,\mathrm{d}x\right\} = 0, \tag{10}$$

where the variation is taken in (η, ϕ). A more satisfactory version of this variational principle is obtained using the change of variable

$$\phi(x, \tilde{y}, z) = \Phi(x, y, z), \tag{11}$$

where $\tilde{y} = y\eta(x, z)$. One finds that

$$\delta\mathcal{L} = 0, \qquad \mathcal{L} = \int L(\eta, \Phi, \eta_x, \Phi_x)\,\mathrm{d}x,$$

in which

$$L(\eta, \Phi, \eta_x, \Phi_x) = \int_\Sigma \frac{1}{2\eta}[(\eta\Phi_x - \eta - \eta_x y \Phi_y)^2 + \Phi_y^2 + (\eta\Phi_z - \eta_z y\Phi_y)^2]\,\mathrm{d}y\,\mathrm{d}z$$
$$+ \int_S \left(\frac{1}{2}\alpha(\eta-1)^2 - \frac{1}{2}(\eta-1) + \beta[\sqrt{1+\eta_x^2+\eta_z^2} - 1]\right)\mathrm{d}z,$$

$S = (0, P)$ and $\Sigma = (0, 1) \times (0, P)$. The next step is to carry out a Legendre transform. Define new variables ω and ξ by the formulae

$$\omega = \frac{\delta\mathcal{L}}{\delta\eta_x} = -\int_0^1 (\eta\Phi_x - \eta - \eta_x y\Phi_y)\frac{y\Phi_y}{\eta}\,\mathrm{d}y + \frac{\beta\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}},$$
$$\xi = \frac{\delta\mathcal{L}}{\delta\Phi_x} = \eta\Phi_x - \eta - \eta_x y\Phi_y,$$

in which the variational derivatives are taken formally in respectively $L^2(S)$ and $L^2(\Sigma)$, and set

$$\begin{aligned} H(\eta,\omega,\Phi,\xi) &= \int_\Sigma \xi\Phi_x \,\mathrm{d}y\,\mathrm{d}z + \int_S \omega\eta_x \,\mathrm{d}z - L(\eta,\Phi,\eta_x,\Phi_x) \\ &= \int_\Sigma \left(\frac{1}{2\eta}(\xi^2-\Phi_y^2) - \frac{1}{2\eta}(\eta\Phi_z - \eta_z y\Phi_y)^2 + \xi\right) \mathrm{d}y\,\mathrm{d}z \\ &\quad + \int_S \left(\frac{1}{2}(\eta-1) - \frac{1}{2}\alpha(\eta-1)^2 + \beta - (\beta^2 - W^2)^{\frac{1}{2}}(1+\eta_z)^{\frac{1}{2}}\right) \mathrm{d}z, \end{aligned}$$

where

$$W = \omega + \frac{1}{\eta}\int_0^1 \xi y\Phi_y \,\mathrm{d}y.$$

The above procedure suggests that the equations

$$\eta_x = \frac{\delta H}{\delta\omega}, \quad \omega_x = -\frac{\delta H}{\delta\eta}, \qquad \Phi_x = \frac{\delta H}{\delta\Psi}, \quad \Psi_x = -\frac{\delta H}{\delta\Phi}$$

formally represent Hamilton's equations for a formulation of the above steady water-wave problem as a Hamiltonian system. Writing down these equations, one finds that $(\eta,\omega,\Phi,\xi) = (1,0,0,-1)$ is always a solution. Let us therefore introduce new variables $\eta = 1+\rho$, $\xi = \Psi - 1$ to arrive at the Hamiltonian

$$\begin{aligned} H(\rho,\omega,\Phi,\Psi) &= \int_\Sigma \left\{\frac{\Psi^2-\Phi_y^2}{2(\rho+1)} - \frac{\rho+1}{2}\left(\Phi_z - \frac{y\Phi_y\rho_z}{\rho+1}\right)^2 + \left(1-\frac{1}{\rho+1}\right)\Psi\right\} \mathrm{d}y\,\mathrm{d}z \\ &\quad + \int_S \left(\frac{1}{2(\rho+1)} + \frac{1}{2}\rho - \frac{1}{2}\alpha\rho^2 + \beta - (\beta^2-W^2)^{\frac{1}{2}}(1+\rho_z^2)^{\frac{1}{2}}\right) \mathrm{d}z, \end{aligned} \tag{12}$$

in which

$$W = \omega + \frac{1}{\rho+1}\int_0^1 (\Psi-1)y\Phi_y \,\mathrm{d}y. \tag{13}$$

In section 2.4 below it is confirmed rigorously that equation (12) defines a Hamiltonian function on a suitable symplectic manifold and that trajectories of Hamilton's equations result in solutions of the three-dimensional steady water-wave problem.

2.3 Nomenclature

Let us now briefly summarise the terminology used below in the formulation of the three-dimensional steady water-wave problem as a Hamiltonian system. Let M be a manifold modelled on a single coordinate chart U in a Hilbert space X with coordinate map $\phi : M \to U$. The tangent and cotangent spaces to M at m (Abraham, Marsden & Ratiu 1988, Definitions 3.3.3 and 5.2.9) are denoted by $TM|_m$ and $T^*M|_m$ and are identified with X and X^*. The *tangent* and *cotangent bundles* of M are denoted by TM and T^*M and treated as manifolds modelled on the single coordinate charts $U \times X$ and $U \times X^*$. A *manifold domain* N of M is a subset of M which is a manifold in its own right, modelled upon the single coordinate chart V in a Hilbert space Y which is densely and continuously embedded in X such that the coordinate map $\psi : N \to V$ is defined by $\psi = \phi|_N$.

A k-*form* α is an assignment of a bounded alternating k-linear map $\alpha|_m : (TM|_m)^k \to \mathbb{R}$ to each point m of its domain $\mathcal{D}(\alpha) \subseteq M$. The *differential* $\mathbf{d}f : \mathcal{D}(\mathbf{d}f) \subset M \to T^*M$ of a function $f \in C^1(N, \mathbb{R})$ is defined by

$$\mathcal{D}(\mathbf{d}f) = \{m \in M : \mathbf{d}f|_m \in T^*N|_m \text{ extends to } \mathbf{d}f|_m \in T^*M|_m\};$$

the extension is unique because Y is densely embedded in X. A 2-form Ω is *weakly nondegenerate* at a point $m \in M$ if the fact that $\Omega|_m(v_1, v_2) = 0$ for all $v_1 \in TM|_m$ implies that $v_2 = 0$, and is *closed* if its exterior derivative is zero. For the present purposes it suffices to note that the closure condition is always satisfied if $\Omega|_m$ is the same for each m (see Abraham, Marsden & Ratiu (1988, ch. 8).

A Hamiltonian system is a triple (M, Ω, H), where M is a manifold, $H \in C^1(N, \mathbb{R})$ is a function defined on a manifold domain N of M called the *Hamiltonian* and Ω is a symplectic 2-form (a 2-form on M that is closed and weakly nondegenerate at each point of M) whose domain contains N. A *vector field* v on M is an assignment of a tangent vector $v|_m \in TM|_m$ to each point m of its domain $\mathcal{D}(v) \subseteq M$. Each Hamiltonian system has an associated *Hamiltonian vector field* v_H, which is defined as follows. Let $\mathcal{D}(v_H) \subset M$ denote the subset of $\mathcal{D}(\mathbf{d}H)$ whose elements m have the property that there exists a tangent vector $v_H|_m \in TM|_m$ such that

$$\mathbf{d}H|_m(\tilde{v}|_m) = \Omega|_m(v_H|_m, \tilde{v}|_m)$$

for all $\tilde{v}|_m \in TM|_m$; the uniqueness of $v_H|_m$ is ensured by the weak nondegeneracy of Ω.

A *trajectory* $\gamma : (a, b) \to \mathcal{D}(v) \subseteq M$ of a vector field v on M is a continuously differentiable path whose tangent vector $\dot{\gamma}(x)$ coincides with $v|_{\gamma(x)}$ at each point $x \in (a, b)$. *Hamilton's equations* for the Hamiltonian system (M, Ω, H) are the differential equations

$$\dot{\gamma}(x) = v_H|_{\gamma(x)} \tag{14}$$

which determine the trajectories $\gamma : (a, b) \to \mathcal{D}(v_H) \subset M$ of the Hamiltonian vector field. A Hamiltonian system is said to be *reversible* if there exists a linear involution $S : X \to X$ (called the *reverser*) which anticommutes with the Hamiltonian vector field, so that $S\gamma(-x)$ is a trajectory of Hamilton's equations whenever $\gamma(x)$ is. A trajectory γ is *reversible* if $S\gamma(-x) = \gamma(x)$.

2.4 The Hamiltonian formulation

The theory below uses the Hilbert spaces

$$\begin{aligned} H^t_{\mathrm{per}}(S) &= \{u \in H^t_{\mathrm{loc}}(\mathbb{R}) : u(z + P) = u(z), z \in \mathbb{R}\}, \\ H^t_{\mathrm{per}}(\Sigma) &= \{u \in H^t_{\mathrm{loc}}((0,1) \times \mathbb{R}) : u(y, z + P) = u(y, z), y \in (0,1), z \in \mathbb{R}\}, \end{aligned}$$

in which $S = (0, P)$, $\Sigma = (0, 1) \times (0, P)$ and t is a nonnegative real number. Various results concerning the nature of these spaces, most of which are obtained by interpolation theory, are collected together in Appendix A. Take $s \in (0, 1/2)$, define $X_s = H^{s+1}_{\mathrm{per}}(S) \times H^s_{\mathrm{per}}(S) \times H^{s+1}_{\mathrm{per}}(\Sigma) \times H^s_{\mathrm{per}}(\Sigma)$ and consider the symplectic manifold (M, Ω), where $M = X_s$ and Ω is the position-independent 2-form on M given by

$$\Omega|_m((\rho_1, \omega_1, \Phi_1, \Psi_1), (\rho_2, \omega_2, \Phi_2, \Psi_2)) = \int_S (\omega_2 \rho_1 - \rho_2 \omega_1)\, \mathrm{d}z + \int_\Sigma (\Psi_2 \Phi_1 - \Phi_2 \Psi_1)\, \mathrm{d}y\, \mathrm{d}z \tag{15}$$

(the canonical 2-form with respect to the $L^2(S)\times L^2(S)\times L^2(\Sigma)\times L^2(\Sigma)$-inner product). Observe that the set

$$N = \{(\rho,\omega,\Phi,\Psi)\in X_{s+1} : |W(z)|<\beta, \rho(z)>-1 \text{ for each } z\in[0,1]\},$$

in which W is defined by (13), is a manifold domain of M (the pointwise constraints are valid since $\rho\in H^{s+1}_{\mathrm{per}}(S)\subset C(\overline{S})$ and it follows from the results in Appendix A that the same is true of W). These results also show that the function H given by (12) belongs to $C^\infty(N,\mathbb{R})$; a direct calculation shows that

$$\begin{aligned}
&\mathbf{d}H|_n(v_1|_n) = \\
&\int_S \omega_1 W\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}\mathrm{d}z + \int_\Sigma \rho_{1z}\left(\Phi_z - \frac{y\Phi_y\rho_z}{\rho+1}\right)\Phi_y y\,\mathrm{d}y\,\mathrm{d}z \\
&-\int_\Sigma \rho_1\left\{\frac{\Psi^2-\Phi_y^2}{2(\rho+1)^2}+\frac{1}{2}\left(\Phi_z^2-\frac{y^2\Phi_y^2\rho_z^2}{(\rho+1)^2}\right)-\frac{1}{(\rho+1)^2}\Psi\right\}\mathrm{d}y\,\mathrm{d}z \\
&-\int_S \rho_1\left(\frac{1}{2(\rho+1)^2}-\frac{1}{2}+\alpha\rho\right)\mathrm{d}z - \int_S \rho_{1z}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}\mathrm{d}z \\
&-\int_\Sigma \frac{\rho_1 W(\Psi-1)y\Phi_y}{(\rho+1)^2}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}\mathrm{d}y\,\mathrm{d}z \\
&+\int_\Sigma \Psi_1\left\{\frac{\Psi}{\rho+1}+1-\frac{1}{\rho+1}+\frac{Wy\Phi_y}{\rho+1}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}\right\}\mathrm{d}y\,\mathrm{d}z \\
&+\int_\Sigma \Phi_{1y}\left\{\left(\Phi_z-\frac{y\Phi_y\rho_z}{\rho+1}\right)y\rho_z+\frac{Wy(\Psi-1)}{\rho+1}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}-\frac{\Phi_y}{\rho+1}\right\}\mathrm{d}y\,\mathrm{d}z \\
&-\int_\Sigma \Phi_{1z}\left(\Phi_z-\frac{y\Phi_y\rho_z}{\rho+1}\right)(\rho+1)\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}\tag{16}$$

for $n=(\rho,\omega,\Phi,\Psi)\in N$ and $v_1|_n=(\rho_1,\omega_1,\Phi_1,\Psi_1)\in TN|_n\cong X_{s+1}$. The triple (M,Ω,H) is therefore a Hamiltonian system.

It follows from Lemma 19(ii) in Appendix A that $\mathbf{d}H|_n$ extends to an element of $T^*M|_n\cong X_s^*$ for each $n\in N$. Recall that the point $n\in N$ belongs to $\mathcal{D}(v_H)$ with $v_H|_n=\overline{v}|_n$ if and only if

$$\Omega|_n(\overline{v}|_n, v_1|_n) = \mathbf{d}H|_n(v_1|_n)$$

for all tangent vectors $v_1|_n\in TM|_n$. Using this criterion, equations (15), (16) and integrating by parts, one finds that $\mathcal{D}(v_H)$ is the set of functions $(\eta,\omega,\Phi,\Psi)\in N$ that satisfy the natural boundary conditions

$$\Phi_y = 0 \qquad \text{on } y=0, \tag{17}$$

$$\frac{\Phi_y}{\rho+1}-\left(\Phi_z-\frac{\Phi_y\rho_z}{\rho+1}\right)\rho_z-\frac{W(\Psi-1)}{\rho+1}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}=0 \qquad \text{on } y=1 \tag{18}$$

and that Hamilton's equations

$$\dot{u} = v_H(u)$$

are given explicitly by

$$
\begin{aligned}
\dot{\rho} &= W\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}, && (19)\\
\dot{\omega} &= \int_0^1 \left\{\frac{\Psi^2-\Phi_y^2}{2(\rho+1)^2}+\frac{1}{2}\left(\Phi_z^2-\frac{y^2\Phi_y^2\rho_z^2}{(\rho+1)^2}\right)+\left[\left(\Phi_z-\frac{y\Phi_y\rho_z}{\rho+1}\right)y\Phi_y\right]_z-\frac{\Psi}{(\rho+1)^2}\right\}\mathrm{d}y \\
&\quad +\frac{1}{2(\rho+1)^2}-\frac{1}{2}+\alpha\rho+\frac{W}{(\rho+1)^2}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}\int_0^1(\Psi-1)y\Phi_y\,\mathrm{d}y \\
&\quad -\left[\rho_z\left(\frac{\beta^2-W^2}{1+\rho_z^2}\right)^{\frac{1}{2}}\right]_z, && (20)\\
\dot{\Phi} &= \frac{\Psi}{(\rho+1)}+1-\frac{1}{\rho+1}+\frac{Wy\Phi_y}{\rho+1}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}, && (21)\\
\dot{\Psi} &= -\frac{\Phi_{yy}}{\rho+1}-\left[(\rho+1)\left(\Phi_z-\frac{y\Phi_y\rho_z}{\rho+1}\right)\right]_z+\left[\left(\Phi_z-\frac{y\Phi_y\rho_z}{\rho+1}\right)y\rho_z\right]_y \\
&\quad +\frac{W(y(\Psi-1))_y}{\rho+1}\left(\frac{1+\rho_z^2}{\beta^2-W^2}\right)^{\frac{1}{2}}. && (22)
\end{aligned}
$$

The results in Appendix A show that the right-hand sides of equations (19)–(22) define a smooth function $N \to X_s$ (the terms in square brackets may be expanded using the product rule in the usual fashion) and that the boundary conditions (17), (18) are also well defined.

The Hamiltonian system (M, Ω, H) has the conserved quantities

$$H(\rho,\omega,\Phi,\Psi), \qquad \int_S \omega\rho_z\,\mathrm{d}z+\int_\Sigma \Psi\Phi_z\,\mathrm{d}y\,\mathrm{d}z, \qquad \int_\Sigma \Psi\,\mathrm{d}y\,\mathrm{d}z,$$

which are associated with continuous symmetries, namely the invariance of Hamilton's equations under translations in x, z and Φ. Hamilton's equations also inherit the discrete symmetries of the hydrodynamic problem (6)–(9). They are invariant under the reflection $z \mapsto -z$ and are reversible; the reverser $S : X_s \to X_s$ is defined by $S(\rho,\omega,\Phi,\Psi) = (\rho,-\omega,-\Phi,\Psi)$.

It remains to confirm the connection between trajectories of Hamilton's equations with solutions of the three-dimensional steady water-wave problem.

Lemma 1 *Suppose that $\gamma = (\rho,\omega,\Phi,\Psi) : (a,b) \to \mathcal{D}(v_H) \subset M$ is a trajectory of Hamilton's equations for (M,Ω,H) which is continuous into N and continuously differentiable into M. Define functions η, ϕ by the formulae*

$$\eta(x,z) = 1+\rho(x)(z), \qquad (x,z) \in (a,b)\times\overline{S},$$
$$\phi(x,\tilde{y},z) = \Phi(x)(y,z), \qquad (x,y,z) \in (a,b)\times\overline{\Sigma},$$

where $\tilde{y} = y\eta(x,z)$, and let (x_1,x_2) be an interval with $a < x_1 < x_2 < b$. The functions $\tilde{\eta}$ and $\tilde{\phi}$ belong to respectively $C^1((x_1,x_2)\times\overline{S})$ and $C^1(D_\eta^1)\cap C^\infty(D_\eta^2)$, where

$$
\begin{aligned}
D_\eta^1 &= \{(x,\tilde{y},z) : x \in (x_1,x_2), z \in [0,P], \tilde{y} \in [0,\eta(x,z)]\},\\
D_\eta^2 &= \{(x,\tilde{y},z) : x \in (x_1,x_2), z \in [0,P], \tilde{y} \in (0,\eta(x,z))\},
\end{aligned}
$$

and solve equations (6)–(8) pointwise and equation (9) in the weak sense.

Proof Define functions $\tilde{\rho}$, $\tilde{\omega}$, $\tilde{\Phi}$, $\tilde{\Psi}$ by

$$\tilde{\rho}(x,z)=\rho(x)(z), \quad \tilde{\omega}(x,z)=\omega(x)(z), \qquad (x,z)\in(a,b)\times\overline{S},$$

$$\tilde{\Phi}(x,y,z)=\Phi(x)(y,z), \quad \tilde{\Psi}(x,y,z)=\Psi(x)(y,z), \qquad (x,y,z)\in(a,b)\times\overline{\Sigma}.$$

It follows from the hypotheses on γ that these functions are continuous and that $\tilde{\rho}$, $\tilde{\Phi}$ have continuous first derivatives on respectively $(a,b)\times\overline{S}$ and $(a,b)\times\overline{\Sigma}$. Straightforward arguments show that the first derivatives of $\tilde{\omega}$, $\tilde{\Psi}$ and all second derivatives of $\tilde{\rho}$, $\tilde{\Phi}$ except $\tilde{\rho}_{xx}$, $\tilde{\Phi}_{xx}$ exist in the weak sense on $(x_1,x_2)\times S$ and $(x_1,x_2)\times\Sigma$; these derivatives are related to those of ρ, Φ by statements of the following type: for almost all $x\in(x_1,x_2)$ the identity

$$\dot{\rho}_z(x)=\frac{\partial^2\tilde{\rho}}{\partial x\partial z}$$

holds for almost all $z\in(0,P)$. Equations (19), (21) show that the weak derivatives $\tilde{\rho}_{xx}$ and $\tilde{\Phi}_{xx}$ also exist in the weak sense on $(x_1,x_2)\times S$ and $(x_1,x_2)\times\Sigma$.

A straightforward but lengthy manipulation of equations (17), (18), (19)–(22) shows that $\eta=1+\tilde{\rho}$ and $\tilde{\Phi}$ satisfy

$$\begin{aligned}&\tilde{\Phi}_{xx}+\tilde{\Phi}_{zz}+\frac{\tilde{\Phi}_{yy}}{\eta^2}\\ &\quad+\frac{y^2\tilde{\Phi}_{yy}}{\eta^2}(\eta_x^2+\eta_z^2)-\frac{y\tilde{\Phi}_y}{\eta}(\eta_{xx}+\eta_{zz})+\frac{2y\tilde{\Phi}_y}{\eta^2}(\eta_x^2+\eta_z^2)-\frac{2y\tilde{\Phi}_{xy}\eta_x}{\eta}-\frac{2y\tilde{\Phi}_{yz}\eta_z}{\eta}=0\end{aligned}$$

in $L^2((x_1,x_2)\times\Sigma)$ and

$$\begin{aligned}\alpha(\eta-1)-\beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x-\beta\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z+\frac{1}{2}\left.\left(\tilde{\Phi}_x-\frac{\tilde{\Phi}_y\eta_x}{\eta}\right)^2\right|_{y=1}&\\ +\left.\frac{\tilde{\Phi}_y^2}{2\eta^2}\right|_{y=1}+\frac{1}{2}\left.\left(\tilde{\Phi}_z-\frac{\tilde{\Phi}_y\eta_z}{\eta}\right)^2\right|_{y=1}-\left.\left(\tilde{\Phi}_x-\frac{\tilde{\Phi}_y\eta_x}{\eta}\right)\right|_{y=1}&=0\end{aligned}$$

in $L^2((x_1,x_2)\times S)$; the boundary conditions

$$\begin{aligned}&\tilde{\Phi}_y=0 && \text{on } y=0,\\ &\frac{\tilde{\Phi}_y}{\eta}-\left(\tilde{\Phi}_z-\frac{\tilde{\Phi}_y\eta_z}{\eta}\right)\eta_z-\left(\tilde{\Phi}_x-\frac{\tilde{\Phi}_y\eta_x}{\eta}\right)\eta_x+\eta_x=0 && \text{on } y=1\end{aligned}$$

are satisfied pointwise. The function ϕ defined by $\phi(x,\tilde{y},z)=\tilde{\Phi}(x,y,z)$ clearly belongs to $C^1(D_1^\eta)$, and one finds from the above equations that η, ϕ solve (6), (9) in the weak sense and (7), (8) pointwise. Finally, note that ϕ belongs to $C^\infty(D_\eta^2)$ since it is a weak solution of Laplace's equation in D_η^2. □

3 Centre-manifold reduction

3.1 The reduction theorems

Theorems 2 and 3 below are the basic reduction results used in the sequel. Theorem 2 was obtained by Mielke (1988), while Theorem 3 is a parameterised version of a result given by Mielke (1991, Theorem 4.1) (see Buffoni, Groves & Toland (1996, Theorem 4.1)).

Theorem 2 *Consider the differential equation*

$$\dot{u} = Ku + n(u, \lambda), \tag{23}$$

in which u belongs to a Hilbert space X, $\lambda \in \mathbb{R}^n$ is a parameter and $K : \mathcal{D}(K) \subset X \to X$ is a densely defined, closed linear operator. Regarding $\mathcal{D}(K)$ as a Hilbert space equipped with the graph norm, suppose that 0 is an equilibrium point of (23) when $\lambda = 0$ and that

(H1) X admits a direct-sum decomposition $X = X_1 \oplus X_2$, where X_1, X_2 are closed, K-invariant subspaces, so that (23) can be written as the linearly-decoupled system

$$\dot{u}_1 = K_1 u_1 + n_1(u_1 + u_2, \lambda), \tag{24}$$
$$\dot{u}_2 = K_2 u_2 + n_2(u_1 + u_2, \lambda), \tag{25}$$

in which $K_i = K|_{\mathcal{D}(K) \cap X_i} : \mathcal{D}(K) \cap X_i \subset X_i \to X_i$, $i = 1, 2$;

(H2) X_1 is finite dimensional and the spectrum of K_1 lies on the imaginary axis;

(H3) The imaginary axis lies in the resolvent set of K_2 and

$$\|(K_2 - \mathrm{i}aI)^{-1}\| \leq \frac{C}{1 + |a|}, \qquad a \in \mathbb{R},$$

for some constant C that is independent of a;

(H4) There exists a natural number k and neighbourhoods $\Lambda \subset \mathbb{R}^n$ of 0 and $U \subset \mathcal{D}(K)$ of 0 such that n is $(k+1)$ times continuously differentiable on $U \times \Lambda$, its derivatives are bounded and uniformly continuous on $U \times \Lambda$ and

$$n(0,0) = 0, \qquad \mathrm{d}_1 n[0,0] = 0.$$

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ of 0 and $\tilde{U}_1 \subset U \cap X_1$, $\tilde{U}_2 \subset U \cap X_2$ of 0 and a reduction function $h : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2 \subset X_2$ with the following properties. The reduction function h is k times continuously differentiable on $\tilde{U}_1 \times \tilde{\Lambda}$, its derivatives are bounded and uniformly continuous on $\tilde{U}_1 \times \tilde{\Lambda}$ and

$$h(0,0) = 0, \qquad \mathrm{d}_1 h[0,0] = 0. \tag{26}$$

The graph

$$M_C^\lambda = \{u_1 + h(u_1, \lambda) \in \tilde{U}_1 \times \tilde{U}_2 : u_1 \in \tilde{U}_1\}$$

is a centre manifold for (23), so that

(1) M_C^λ is a locally invariant manifold of (23): through every point in M_C^λ there passes a unique solution of (23) that remains on M_C^λ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$;

(2) Every small bounded solution $u(t)$, $t \in \mathbb{R}$ of (23) that satisfies $(u_1(t), u_2(t)) \in \tilde{U}_1 \times \tilde{U}_2$ lies completely in M_C^λ;

(3) Every solution $\tilde{u} : (a, b) \to \tilde{U}_1$ of the reduced equation

$$\dot{u}_1 = K_1 u_1 + n_1(u_1 + h(u_1, \lambda), \lambda) \tag{27}$$

generates a solution

$$u(t) = u_1(t) + h(u_1(t), \lambda)$$

of the full equation (23).

Theorem 3 *Suppose that equation (23) represents Hamilton's equations for a Hamiltonian system $(M, \Omega^\lambda, H^\lambda)$. The centre manifold M_C^λ is a symplectic submanifold of M for each $\lambda \in \tilde{\Lambda}$. Moreover, the flow determined by the Hamiltonian system $(M_C^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, where the tilde denotes restriction to M_C^λ, coincides with the flow on M_C^λ determined by $(M, \Omega^\lambda, H^\lambda)$.*

In keeping with the notation in Theorems 2 and 3, let us write

$$\alpha = \alpha_0 + \lambda_1, \qquad \beta = \beta_0 + \lambda_2,$$

where α_0, β_0 are fixed and λ_1, λ_2 lie in neighbourhoods Λ_1, Λ_2 of the origin in $\mathbb{R}$, and consider trajectories of Hamilton's equations for (M, Ω, H) which lie in a neighbourhood V of the origin in X_{s+1}. Choosing V and Λ_2 small enough so that

$$\rho(z) > -\frac{1}{2} > -1, \quad |\lambda_2| < \frac{1}{4}\beta_0, \quad |W(z)| < \frac{\beta_0}{2} < \beta_0 + \lambda_2$$

for all $z \in [0, 1]$, one arrives at the equation

$$\dot{u} = f^\lambda(u), \tag{28}$$

where $f^\lambda : V \to X_s$ is the smooth function defined by the right-hand sides of (19)–(22) with (α, β) replaced by $(\alpha_0 + \lambda_1, \beta_0 + \lambda_2)$, together with the boundary conditions

$$\Phi_y = 0 \qquad \text{on } y = 0, \tag{29}$$

$$\frac{\Phi_y}{\rho+1} - \left(\Phi_z - \frac{\Phi_y \rho_z}{\rho+1}\right)\rho_z - \frac{W(\Psi-1)}{\rho+1}\left(\frac{1+\rho_z^2}{(\beta_0+\lambda_2)^2 - W^2}\right)^{\frac{1}{2}} = 0. \qquad \text{on } y = 1. \tag{30}$$

Clearly (28) represents Hamilton's equations for the Hamiltonian system (M, Ω, H^λ), where H^λ is the smooth functional defined upon the manifold domain V of M by

$$H^\lambda(\rho, \omega, \Phi, \Psi) =$$
$$\int_S \left(\frac{1}{2(\rho+1)} + \frac{1}{2}\rho - \frac{1}{2}(\alpha_0+\lambda_1)\rho^2 + \beta_0 + \lambda_2 - ((\beta_0+\lambda_2)^2 - W^2)^{\frac{1}{2}}(1+\rho_z^2)^{\frac{1}{2}}\right) \mathrm{d}z$$
$$+ \int_\Sigma \left\{\frac{\Psi^2 - \Phi_y^2}{2(\rho+1)} - \frac{\rho+1}{2}\left(\Phi_z - \frac{y\Phi_y\rho_z}{\rho+1}\right)^2 + \left(1 - \frac{1}{\rho+1}\right)\Psi\right\} \mathrm{d}y\,\mathrm{d}z;$$

the domain $\mathcal{D}(v_H^\lambda)$ of the Hamiltonian vector field v_H^λ is the subset of V specified by (29), (30) and $v_H^\lambda(u) = f^\lambda(u)$ for each $u \in \mathcal{D}(v_H^\lambda)$.

3.2 The nonlinear boundary condition

Theorem 2 cannot be applied directly to (28) because of the nonlinear boundary condition (30). It is therefore necessary to introduce a change of variable which leads to an equivalent problem in a linear space.

Define a function $F : V \times \Lambda_2 \subset X_{s+1} \times \mathbb{R} \to H^{s+1}_{\mathrm{per}}(\Sigma)$ by the formula

$$F(\rho, \omega, \Phi, \Psi, \lambda_2)(y) = [(1+\rho)\Phi_z - y\Phi_y \rho_z] y \rho_z + \left(\frac{1+\rho_z^2}{(\beta_0 + \lambda_2)^2 - W^2} \right)^{\frac{1}{2}} W y (\Psi - 1),$$

so that the boundary conditions (29), (30) are equivalent to

$$\Phi_y = F(\rho, \omega, \Phi, \Psi, \lambda_2) \qquad \text{on } y = 0, 1; \tag{31}$$

it follows from the results in Appendix A that F is smooth. The requisite change of variable $G^{\lambda_2} : V \to X_{s+1}$ is given by $G^{\lambda_2}(\rho, \omega, \Phi, \Psi) = (\rho, \nu, \Gamma, \Psi)$, where

$$\begin{aligned} \nu &= \omega - \int_0^1 y \Phi_y \, \mathrm{d}y, \\ \Gamma &= \Phi + \chi_y, \end{aligned}$$

in which $\chi \in H^{s+3}_{\mathrm{per}}(\Sigma)$ is the unique solution of the elliptic boundary-value problem

$$\begin{aligned} -\chi_{yy} - \chi_{zz} &= F(\rho, \omega, \Phi, \Psi, \lambda_2) && \text{in } \Sigma, \\ \chi &= 0 && \text{on } y = 0, 1. \end{aligned}$$

Lemma 4

(i) For each $\lambda_2 \in \Lambda_2$ the mapping G^{λ_2} is a smooth diffeomorphism from the neighbourhood V of 0 in X_{s+1} onto a neighbourhood $\tilde{V}$ of 0 in X_{s+1}. The mappings G^{λ_2}, $(G^{\lambda_2})^{-1}$ and their derivatives depend smoothly upon $\lambda_2 \in \Lambda_2$.

(ii) For each $(v, \lambda_2) \in V \times \Lambda_2$ the operator $\mathrm{d}G^{\lambda_2}[v] : X_{s+1} \to X_{s+1}$ extends to an isomorphism $\widetilde{\mathrm{d}G}^{\lambda_2}[v] : X_s \to X_s$. The operators $\widetilde{\mathrm{d}G}^{\lambda_2}[v]$, $\widetilde{\mathrm{d}G}^{\lambda_2}[v]^{-1} \in \mathcal{L}(X_s, X_s)$ depend smoothly upon $(v, \lambda_2) \in V \times \Lambda_2$.

Proof The mapping $G^{\lambda_2} : V \to X_{s+1} \times \mathbb{R}$ is smooth and depends smoothly upon λ_2 since it is a linear function of the smooth map $F : V \times \Lambda_2 \to H^{s+1}_{\mathrm{per}}(\Sigma)$. Clearly $G^{\lambda_2}(0) = 0$ and

$$\mathrm{d}G^{\lambda_2}[\rho, \omega, \Phi, \Psi](\rho_1, \omega_1, \Phi_1, \Psi_1) = \left(\rho_1, \omega_1 - \int_0^1 y \Phi_{1y} \, \mathrm{d}y, \Phi_1 + \xi_y, \Psi_1 \right),$$

in which $\xi \in H^{s+3}_{\mathrm{per}}(\Sigma)$ is the unique solution of the elliptic boundary-value problem

$$\begin{aligned} -\xi_{yy} - \xi_{zz} &= \mathrm{d}_1 F[\rho, \omega, \Phi, \Psi, \lambda_2](\rho_1, \omega_1, \Phi_1, \Psi_1) && \text{in } \Sigma, \\ \xi &= 0 && \text{on } y = 0, 1. \end{aligned}$$

A direct calculation shows that

$$
\begin{aligned}
&\mathrm{d}_1F[\rho,\omega,\Phi,\Psi,\lambda_2](\rho_1,\omega_1,\Phi_1,\Psi_1) = \\
&\quad y\Phi_z\rho_z\rho_1 + (\rho+1)y\Phi_{1z}\rho_z + (\rho+1)y\Phi_z\rho_{1z} - y^2\Phi_{1y}\rho_z^2 - 2y^2\Phi_y\rho_z\rho_{1z} \\
&\quad + Wy\Psi_1\left(\frac{1+\rho_z^2}{(\beta_0+\lambda_2)^2-W^2}\right)^{\frac{1}{2}} + \frac{Wy(\Psi-1)\rho_z\rho_{1z}}{((\beta_0+\lambda_2)^2-W^2)^{\frac{1}{2}}(1+\rho_z^2)^{\frac{1}{2}}} \\
&\quad + \left(\frac{1+\rho_z^2}{(\beta_0+\lambda_2)^2-W^2}\right)^{\frac{1}{2}} y(\Psi-1) \\
&\qquad \times\left[\omega_1 - \frac{\rho_1}{(\rho+1)^2}\int_0^1(\Psi-1)y\Phi_y\,\mathrm{d}y + \frac{1}{\rho+1}\int_0^1(\Psi_{1y}\Phi_y+(\Psi-1)y\Phi_{1y})\,\mathrm{d}y\right] \\
&\quad + \left(\frac{1+\rho_z^2}{((\beta_0+\lambda_2)^2-W^2)^3}\right)^{\frac{1}{2}} Wy(\Psi-1) \\
&\qquad \times\left[\omega_1 - \frac{\rho_1}{(\rho+1)^2}\int_0^1(\Psi-1)y\Phi_y\,\mathrm{d}y + \frac{1}{\rho+1}\int_0^1(\Psi_{1y}\Phi_y+(\Psi-1)y\Phi_{1y})\,\mathrm{d}y\right], \qquad (32)
\end{aligned}
$$

so that

$$\mathrm{d}_1F[0,\lambda_2](\rho_1,\omega_1,\Phi_1,\Psi_1) = -\frac{y}{\beta_0+\lambda_2}\left(\omega_1 - \int_0^1 y\Phi_{1y}\,\mathrm{d}y\right).$$

The map $\mathrm{d}G^{\lambda_2}[0] : X_{s+1}\times\mathbb{R}\to X_{s+1}\times\mathbb{R}$ is a bijection; its inverse is given by

$$\mathrm{d}G^{\lambda_2}[0]^{-1}(\rho_1,\nu_1,\Gamma_1,\Psi_1) = \left(\rho_1,\nu_1+\int_0^1 y\Phi_{1y}\,\mathrm{d}y,\Phi_1,\Psi_1\right),$$

where $\Phi_1=\Gamma_1-\zeta_y$ and ζ is the unique solution of the elliptic boundary-value problem

$$
\begin{aligned}
-\zeta_{yy}-\zeta_{zz} &= -\frac{y\nu_1}{\beta_0+\lambda_2} && \text{in } \Sigma,\\
\zeta &= 0 && \text{on } y=0,1.
\end{aligned}
$$

Replacing $\tilde{V}$ with a smaller neighbourhood of 0 in X_{s+1} if necessary, one obtains the result stated in the first part of the lemma using the inverse function theorem and the fact that in the present context all estimates used in its proof hold uniformly in $\lambda\in\Lambda_2$.

Turning to the second assertion, suppose that $(\rho_1,\omega_1,\Phi_1,\Psi_1)$ belongs to X_s and examine the right-hand side of (32). With the exception of the final term in each of the square brackets each term consists of several functions in $H^{s+1}_{\mathrm{per}}(\Sigma)$ multiplied by a single function in $H^s_{\mathrm{per}}(\Sigma)$. Using Lemma 19, one concludes that each such term represents a bounded linear mapping from X_s into $H^s_{\mathrm{per}}(\Sigma)$ which depends smoothly upon $(\rho,\omega,\Phi,\Psi,\lambda_2)\in\tilde{V}\times\Lambda_2$. Similar arguments involving also the other results in Appendix A show that the same is true of the remaining two terms. It follows that $\mathrm{d}_1F[v,\lambda_2] : X_{s+1}\to H^{s+1}_{\mathrm{per}}(\Sigma)$ extends to a bounded linear operator $\widetilde{\mathrm{d}_1F}[v,\lambda_2] : X_s\to H^s_{\mathrm{per}}(\Sigma)$ that depends smoothly upon $(v,\lambda_2)\in V\times\Lambda_2$. The corresponding result holds for $\mathrm{d}G^{\lambda_2}$ since G^{λ_2} depends linearly upon F; the extension $\widetilde{\mathrm{d}G}^{\lambda_2}$ is unique since X_{s+1} is a dense subspace of X_s.

Define $F_1,F_2 : \mathcal{L}(X_s,X_s)\times V\times\Lambda_2\to\mathcal{L}(X_s,X_s)$ by the formulae

$$F_1(T,v,\lambda_2) = \widetilde{\mathrm{d}G}^{\lambda_2}[v]T-I, \qquad F_2(T,v,\lambda_2) = T\widetilde{\mathrm{d}G}^{\lambda_2}[v]-I,$$

in which I denotes the identity operator $X_s \to X_s$. The above calculations show that $\widetilde{\mathrm{d}G}^{\lambda_2}[0] : X_s \to X_s$ is an isomorphism and clearly

$$F_j(\widetilde{\mathrm{d}G}^{\lambda_2}[0]^{-1}, 0, 0) = 0, \quad \mathrm{d}_1 F_j[\widetilde{\mathrm{d}G}^{\lambda_2}[0]^{-1}, 0, 0] = I, \qquad j = 1, 2.$$

Replacing V and Λ_2 with a smaller neighbourhoods of 0 in X_{s+1} and $\mathbb{R}$ if necessary, one finds from the implicit function theorem that the equations $F_1(T, v, \lambda_2) = 0$, $F_2(T, v, \lambda_2) = 0$ have unique solutions $T_1(v, \lambda_2)$, $T_2(v, \lambda_2)$ for each $(v, \lambda_2) \in V \times \Lambda_2$ which depend smoothly upon (v, λ_2). It follows that $\widetilde{\mathrm{d}G}^{\lambda_2}[v]$ is a bijection with $\widetilde{\mathrm{d}G}^{\lambda_2}[v]^{-1} = T_1 = T_2$ for each $(v, \lambda_2) \in V \times \Lambda_2$. □

The diffeomorphism G^{λ_2} transforms (28) into

$$\dot{u} = g^{\lambda}(u), \tag{33}$$

where $g^{\lambda} : \tilde{V} \to X_s$ is the smooth vector field defined by

$$g^{\lambda}(u) = \widetilde{\mathrm{d}G}^{\lambda_2}[(G^{\lambda_2})^{-1}(u)](f^{\lambda}((G^{\lambda_2})^{-1}(u))).$$

Notice that

$$\begin{aligned} \Gamma_y &= \Phi_y + \chi_{yy} \\ &= \Phi_y - \chi_{zz} - F(\rho, \omega, \Phi, \Psi, \lambda_2), \end{aligned}$$

and since $\chi_{zz} = 0$ on $y = 0, 1$ it follows from (31) that the boundary conditions (29), (30) are transformed into

$$\Gamma_y = 0 \qquad \text{on } y = 0, 1. \tag{34}$$

Formula (33) represents Hamilton's equations for the Hamiltonian system $(M, \Omega_2^{\lambda_2}, H_2^{\lambda})$, where $\Omega_2^{\lambda_2}$ and H_2^{λ} are defined on the manifold domain $\tilde{V}$ of M by

$$\Omega_2^{\lambda_2}|_m(v_1, v_2) = \Omega(\widetilde{\mathrm{d}G}^{\lambda_2}[(G^{\lambda_2})^{-1}(m)]^{-1}(v_1), \widetilde{\mathrm{d}G}^{\lambda_2}[(G^{\lambda_2})^{-1}(m)]^{-1}(v_2))$$

for $\lambda_2 \in \Lambda_2$, $v_1, v_2 \in TM|_m$ and

$$H_2^{\lambda}(m) = H^{\lambda}((G^{\lambda_2})^{-1}(m)).$$

The domain $\mathcal{D}(v_{H_2}^{\lambda})$ of the Hamiltonian vector field is the subset of $\tilde{V}$ given by the linear boundary conditions (34) and $v_{H_2}^{\lambda}(u) = g^{\lambda}(u)$ for each $u \in \mathcal{D}(v_{H_2}^{\lambda})$.

Equation (33) can be written as

$$\dot{u} = K_s u + n(u, \lambda), \tag{35}$$

where $K_s = \mathrm{d}g^0[0]$, $\mathcal{D}(K_s) = \{(\rho, \omega, \Gamma, \Psi) \in X_{s+1} : \Gamma_y|_{y=0} = \Gamma_y|_{y=1} = 0\}$ and $n(u, \lambda) = g^{\lambda}(u) - Ku$. This equation is in a form suitable for the application of Theorems 2 and 3. Writing $X = X_s$, $K = K_s$, $\Lambda = \Lambda_1 \times \Lambda_2$ and $U = \tilde{V}$, observe that K_s is densely defined and that (H4) is satisfied for any natural number r. The remaining hypotheses on K_s are verified in Section 3.3 below.

3.3 The linear operator

Let us introduce the operator $L_s : \mathcal{D}(L_s) \subset X_s \to X_s$ given by

$$L_s \begin{pmatrix} \rho \\ \omega \\ \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_0}\left(\omega - \int_0^1 y\Phi_y \,\mathrm{d}y\right) \\ -\int_0^1 \Psi \,\mathrm{d}y - \rho + \alpha_0\rho - \beta_0\rho_{zz} \\ \Psi + \rho \\ -\Phi_{yy} - \Phi_{zz} - \frac{1}{\beta_0}\left(\omega - \int_0^1 y\Phi_y \,\mathrm{d}y\right) \end{pmatrix}; \tag{36}$$

its domain $\mathcal{D}(L_s)$ is the subspace of elements of X_{s+1} that satisfy the boundary conditions

$$\Phi_y = 0 \qquad \text{on } y = 0, \tag{37}$$

$$\Phi_y + \frac{1}{\beta}\left(\omega - \int_0^1 y\Phi_y \,\mathrm{d}y\right) = 0 \qquad \text{on } y = 1. \tag{38}$$

(The operator L_s is the formal linearisation of $f^0(\rho, \omega, \Phi, \Psi)$ at 0.) Observe that $\mathrm{d}G^0[0]$ is a homeomorphism of $\mathcal{D}(L_s)$ onto $\mathcal{D}(K_s)$ and that

$$K_s = \widetilde{\mathrm{d}G^0}[0] L_s (\mathrm{d}G^0[0])^{-1} \tag{39}$$

for $s \in (0, 1/2)$, where $K_s : \mathcal{D}(K_s) \subset X_s \to X_s$ is the linear operator in equation (35). Equation (39) shows that the topological properties of L_s and K_s are identical; in particular their spectra coincide and L_s is densely defined. Let us also define densely defined operators $L_0 : \mathcal{D}(L_0) \subset X_0 \to X_0$ and $K_0 : \mathcal{D}(K_0) \subset X_0 \to X_0$ using the formulae (36)–(39); this procedure is valid since $dG^0[0]^{-1}$ is an isomorphism of $\mathcal{D}(K_0) = \{(\rho, \nu, \Gamma, \Psi) \in X_1 : \Gamma_y|_{y=0} = \Gamma_y|_{y=1} = 0\}$ onto $\mathcal{D}(L_0)$ and $\widetilde{\mathrm{d}G^0}[0]$ is an isomorphism of X_0 onto itself.

Lemma 5 *There exist real constants C, $\xi_0 > 0$ such that each solution $v \in \mathcal{D}(L_0)$ of the resolvent equation*

$$(L_0 - \mathrm{i}\xi I)v = f^\star, \tag{40}$$

in which $f^\star$ belongs to X_0 and ξ is a real number with $|\xi| > \xi_0$, satisfies the estimates

$$\|v\|_{X_1} \leq C\|f^\star\|_{X_0}, \tag{41}$$

$$\|v\|_{X_0} \leq \frac{C}{|\xi|}\|f^\star\|_{X_0}. \tag{42}$$

Proof Write (40) as

$$\frac{1}{\beta_0}(w - \chi) - \mathrm{i}\xi p = \rho^\star, \tag{43}$$

$$-\int_0^1 \Psi \,\mathrm{d}y + (\alpha_0 - 1)\rho - \beta_0\rho_{zz} - \mathrm{i}\xi\omega = \omega^\star, \tag{44}$$

$$\Psi + \rho - \mathrm{i}\xi\Phi = \Phi^\star, \tag{45}$$

$$-\Delta\Phi - \frac{1}{\beta_0}(\omega - \chi) - \mathrm{i}\xi\Psi = \Psi^\star, \tag{46}$$

where

$$\Phi_y = 0 \qquad \text{on } y = 0, \tag{47}$$

$$\Phi_y + \frac{1}{\beta_0}(\omega - \chi) = 0 \qquad \text{on } y = 1 \tag{48}$$

and

$$\chi = \int_0^1 y\Phi_y \,\mathrm{d}y.$$

It follows from equations (43), (44) that

$$\begin{aligned}
&\left\|\omega^\star + \int_0^1 \Psi \,\mathrm{d}y - (\alpha_0 - 1)\rho\right\|_0^2 + \|\beta_0\rho^\star + \chi\|_1^2 \\
&\qquad = \beta_0^2\|\rho_{zz}\|_0^2 + \|\omega\|_1^2 + \xi^2(\|\omega\|_0^2 + \beta_0^2\|\rho\|_1^2) + 2\xi\beta_0 \mathrm{Im}\int_S (\rho_{zz}\overline{\omega} - \omega\overline{\rho} - \omega_z\overline{\rho}_z)\,\mathrm{d}z,
\end{aligned}$$

in which the identity

$$|a + \mathrm{i}\xi b|^2 = |a|^2 + |b|^2 + 2\xi \mathrm{Im}(a\overline{b}), \qquad a, b \in \mathbb{R}$$

has been used. Using the integration by parts formula

$$\mathrm{Im}\int_S (\rho_{zz}\overline{\omega} - \omega_z\overline{\rho}_z)\,\mathrm{d}z = 0$$

and the estimate

$$\left|2\mathrm{Im}\int_S \omega\overline{\rho}\,\mathrm{d}z\right| \leq \|\omega\|_0^2 + \|\rho\|_0^2,$$

one finds that

$$\begin{aligned}
&\beta_0^2\|\rho_{zz}\|_0^2 + \|\omega\|_1^2 + \xi^2(\|\omega\|_0^2 + \beta_0^2\|\rho\|_1^2) \\
&\qquad \leq \|\omega^\star\|_0^2 + \|\Psi\|_0^2 + |(\alpha_0 - 1)|\|\rho\|_0^2 + \beta_0^2\|\rho^\star\|_1^2 + \|\chi\|_1^2 + |\xi|\|\omega\|_0^2 + \beta_0^2|\xi|\|\rho\|_0^2
\end{aligned}$$

so that

$$\|\rho_{zz}\|_0^2 + \|\omega\|_1^2 + \xi^2(\|\rho\|_1^2 + \|\omega\|_0^2) \leq c_1(\|\rho^\star\|_1^2 + \|\omega^\star\|_0^2 + \|\Psi\|_0^2 + \|\chi\|_1^2) \tag{49}$$

for sufficiently large $|\xi|$. (Each coefficient c_j appearing in this analysis is a constant which is independent of ξ, v and f.) Because ρ solves the equation

$$\beta_0\rho_{zz} = \omega^\star + \mathrm{i}\xi\omega + (1 - \alpha_0)\rho + \int_0^1 \Psi \,\mathrm{d}y$$

it is estimated by

$$\|\rho\|_2^2 \leq c_2(\|\omega^\star\|_0^2 + |\xi|^2\|\omega\|_0^2 + \|\rho\|_0^2 + \|\Psi\|_0^2), \tag{50}$$

and (49), (50) imply that

$$\|\rho\|_2^2 + \|\omega\|_1^2 + \xi^2(\|\rho\|_1^2 + \|\omega\|_0^2) \leq c_3(\|\rho^\star\|_1^2 + \|\omega^\star\|_0^2 + \|\Psi\|_0^2 + \|\chi\|_1^2) \tag{51}$$

for sufficiently large $|\xi|$.

Equations (45), (46) show that

$$
\begin{aligned}
&\|\Phi^\star - \rho\|_1^2 + \left\|\Psi^\star + \frac{1}{\beta_0}(\omega - \chi)\right\|_0^2 \\
&\qquad = \|\Delta\Phi\|_0^2 + \|\Psi\|_1^2 + \xi^2(\|\Psi\|_0^2 + \|\Phi\|_1^2) + 2\xi \mathrm{Im}\int_\Sigma (\Delta\Phi\overline{\Psi} - \Psi\overline{\Phi} - \nabla\Psi\nabla\overline{\Phi})\,\mathrm{d}y\,\mathrm{d}z. \qquad (52)
\end{aligned}
$$

Using the integration by parts formula

$$\int_\Sigma (\Delta\Phi\overline{\Psi} - \nabla\Psi\nabla\overline{\Phi})\,\mathrm{d}y\,\mathrm{d}z = \mathrm{Im}\frac{1}{\beta_0}\int_S \Psi|_{y=1}(\chi - \omega)\,\mathrm{d}z$$

and the estimate

$$
\begin{aligned}
\left|\frac{2}{\beta_0}\mathrm{Im}\int_0^1 \overline{\Psi}|_{y=1}(\chi - \omega)\,\mathrm{d}z\right| &\leq \delta\|\Psi|_{y=1}\|_0^2 + \frac{1}{\delta\beta_0^2}\|\chi - \omega\|_0^2 \\
&\leq \delta c_4\|\Psi\|_1\|\Psi\|_0 + \frac{1}{\delta\beta_0^2}(\|\chi\|_0^2 + \|\omega\|_0^2)
\end{aligned}
$$

one finds from (52) that

$$
\begin{aligned}
&\|\Delta\Phi\|_0^2 + \|\Psi\|_1^2 + \xi^2(\|\Psi\|_0^2 + \|\Phi\|_1^2) \\
&\quad \leq \|\Phi^\star\|_1^2 + \|\rho\|_1^2 + \|\Psi^\star\|_0^2 + \frac{1}{\beta_0^2}\|\omega\|_0^2 + \frac{1}{\beta_0^2}\|\chi\|_0^2 + c_4\delta|\xi|\|\Psi\|_1\|\Psi\|_0 \\
&\quad + \frac{|\xi|}{\delta\beta_0^2}\|\chi\|_0^2 + \frac{|\xi|}{\delta\beta_0^2}\|\omega\|_0^2 + |\xi|\|\Psi\|_0^2 + |\xi|\|\Phi\|_1.
\end{aligned}
$$

This estimate may be rewritten as

$$
\begin{aligned}
&\|\Delta\Phi\|_0^2 + \frac{1}{2}(|\xi|\|\Psi\|_0 - c_4\delta\|\Psi\|_1)^2 + \left(1 - \frac{c_4^2\delta^2}{2}\right)\|\Psi\|_1^2 + \frac{1}{2}\xi^2\|\Psi\|_0^2 + \xi^2\|\Phi\|_1^2 \\
&\quad \leq \|\Phi^\star\|_1^2 + \|\rho\|_1^2 + \|\Psi^\star\|_0^2 + \frac{1}{\beta_0^2}\|\omega\|_0^2 + \frac{1}{\beta_0^2}\|\chi\|_0^2 \\
&\quad + \frac{|\xi|}{\delta\beta_0^2}\|\chi\|_0^2 + \frac{|\xi|}{\delta\beta_0^2}\|\omega\|_0^2 + |\xi|\|\Psi\|_0^2 + |\xi|\|\Phi\|_1,
\end{aligned}
$$

and with $\delta < \sqrt{2}/c_4$ it follows that

$$\|\Delta\Phi\|_0^2 + \|\Psi\|_1^2 + \xi^2(\|\Psi\|_0^2 + \|\Phi\|_1^2) \leq c_5(\|\Phi^\star\|_1^2 + \|\Psi^\star\|_0^2 + \|\rho\|_1^2 + |\xi|\|\omega\|_0^2 + |\xi|\|\chi\|_0^2) \qquad (53)$$

for sufficiently large $|\xi|$.

Observe that Φ satisfies the equation

$$-\Delta\Phi = \Psi^\star + \mathrm{i}\xi\Phi + \frac{1}{\beta_0}(\omega - \chi) \qquad \text{in } \Sigma$$

subject to the boundary conditions (47), (48); standard results in elliptic regularity theory assert that

$$\|\Phi\|_2^2 \leq c_6(\|\Psi^\star\|_0^2 + \|\omega\|_{1/2}^2 + \xi^2\|\Phi\|_0^2 + \|\chi\|_{1/2}^2), \qquad (54)$$

and combining (53), (54), one finds that

$$\begin{aligned}
&\|\Phi\|_2^2 + \|\Psi\|_1^2 + \xi^2(\|\Psi\|_0^2 + \|\Phi\|_1^2) \\
&\qquad\qquad \leq c_7(\|\Phi^\star\|_1^2 + \|\Psi^\star\|_0^2 + \|\rho\|_1^2 + |\xi|\|\omega\|_0^2 + |\xi|\|\chi\|_0^2 + \|\chi\|_{1/2}^2 + \|\omega\|_{1/2}^2)
\end{aligned} \tag{55}$$

for sufficiently large $|\xi|$.

The formula

$$\chi(z) = \Phi(1,z) + \int_0^1 \Phi(y,z)\,\mathrm{d}y$$

shows that

$$\|\chi\|_t^2 \leq \|\Phi|_{y=1}\|_t^2 + \|\Phi\|_t^2 \leq c_8(t)\|\Phi\|_{t+1/2}^2$$

for $t \in [0,\infty)$; in particular one has that

$$\|\chi\|_0^2 \leq c_9\|\Phi\|_{1/2}^2 \leq c_{10}\|\Phi\|_1^2, \tag{56}$$

$$\|\chi\|_{1/2}^2 \leq c_{11}\|\Phi\|_1^2 \tag{57}$$

$$\|\chi\|_1^2 \leq c_{12}\|\Phi\|_{3/2}^2 \leq c_{13}\|\Phi\|_2^2. \tag{58}$$

Inequalities (55), (56). (57) show that

$$\|\Phi\|_2^2 + \|\Psi\|_1^2 + \xi^2(\|\Psi\|_0^2 + \|\Phi\|_1^2) \leq c_{14}(\|\Phi^\star\|_1^2 + \|\Psi^\star\|_0^2 + \|\rho\|_1^2 + |\xi|\|\omega\|_0^2 + \|\omega\|_0\|\omega\|_1), \tag{59}$$

where the estimate $\|\omega\|_{1/2}^2 \leq \|\omega\|_0\|\omega\|_1$ has also been used, and it follows from (58), (59) that

$$\|\chi\|_1^2 \leq c_{15}(\|\Phi^\star\|_1^2 + \|\Psi^\star\|_0^2 + \|\rho\|_1^2 + |\xi|\|\omega\|_0^2 + \|\omega\|_0\|\omega\|_1) \tag{60}$$

for sufficiently large $|\xi|$.

The next step is to combine (51) and (60) to conclude that

$$\|\rho\|_2^2 + \|\omega\|_1^2 + \xi^2(\|\rho\|_1^2 + \|\omega\|_0^2) \leq c_{16}(\|\rho^\star\|_1^2 + \|\omega^\star\|_0^2 + \|\Phi^\star\|_1^2 + \|\Psi^\star\|_0^2 + \|\Psi\|_0^2) \tag{61}$$

for sufficiently large $|\xi|$. Using (59) and (61), one finds that

$$\|\Phi\|_2^2 + \|\Psi\|_1^2 + \xi^2(\|\Phi\|_1^2 + \|\Psi\|_0^2) \leq c_{17}(\|\rho^\star\|_1^2 + \|\omega^\star\|_0^2 + \|\Phi^\star\|_1^2 + \|\Psi^\star\|_0^2), \tag{62}$$

and estimates (61), (62) show that

$$\|\rho\|_2^2 + \|\omega\|_1^2 + \xi^2(\|\rho\|_1^2 + \|\omega\|_0^2) \leq c_{18}(\|\rho^\star\|_1^2 + \|\omega^\star\|_0^2 + \|\Phi^\star\|_1^2 + \|\Psi^\star\|_0^2) \tag{63}$$

for sufficiently large $|\xi|$.

The stated result is a direct consequence of (62) and (63). □

Proposition 6 *A non-zero complex number λ is an eigenvalue of L_s, $s \in [0,1/2)$ if and only if there exists an integer $k \in \mathbb{Z}$ such that*

$$(\alpha_0 - \beta_0\sigma^2)\sigma\sin\sigma = |\lambda|^2\cos\sigma \qquad \textit{with } \sigma^2 = |\lambda|^2 - 4\pi^2k^2/P^2. \tag{64}$$

Zero is always an eigenvalue of L_s, $s \in [0,1/2)$. The spectrum of L_0 consists entirely of isolated eigenvalues of finite algebraic multiplicity and $\sigma(L_0) \cap \mathrm{i}\mathbb{R}$ is a finite set.

Proof Consider the resolvent equation

$$(L_s - \lambda I)v = f^\star, \tag{65}$$

where $v = (\rho, \omega, \Phi, \Psi)$ and $f^\star = (\rho^\star, \omega^\star, \Phi^\star, \Psi^\star)$ belong to respectively X_s and $\mathcal{D}(L_s)$. Let us write

$$\rho(z) = \sqrt{P}\sum_{k\in\mathbb{Z}} \rho_k \mathrm{e}^{2\pi \mathrm{i} kz/P}, \qquad \omega(z) = \sqrt{P}\sum_{k\in\mathbb{Z}} \omega_k \mathrm{e}^{2\pi \mathrm{i} kz/P},$$

$$\Phi(y,z) = \sqrt{P}\sum_{k\in\mathbb{Z}} \Phi_k(y) \mathrm{e}^{2\pi \mathrm{i} kz/P}, \qquad \Psi(y,z) = \sqrt{P}\sum_{k\in\mathbb{Z}} \Psi_k(y) \mathrm{e}^{2\pi \mathrm{i} kz/P}$$

with similar expansions for the components $(\rho^\star, \omega^\star, \Phi^\star, \Psi^\star)$ of $f^\star$ and examine the ordinary differential equations

$$\begin{aligned}
\frac{1}{\beta_0}\left(\omega_k - \int_0^1 y\Phi_{ky}\,\mathrm{d}y\right) - \lambda\rho_k &= \rho_k^\star, && (66)\\
-\int_0^1 \Psi_k\,\mathrm{d}y - (\alpha_0 - 1)\rho_k + \frac{4\beta_0\pi^2k^2}{P^2}\rho_k - \lambda\omega_k &= \omega_k^\star, && (67)\\
\Psi_k + \rho_k - \lambda\Phi_k &= \Phi_k^\star, && (68)\\
-\Phi_{kyy} - \frac{1}{\beta_0}\left(\omega_k - \int_0^1 y\Phi_{ky}\,\mathrm{d}y\right) + \frac{4\pi^2k^2}{P^2}\Phi_k - \lambda\Psi_k &= \Psi_k^\star, && (69)
\end{aligned}$$

where

$$\begin{aligned}
&\Phi_{ky} = 0 && \text{on } y = 0, && (70)\\
&\Phi_{ky} + \frac{1}{\beta_0}\left(\omega_k - \int_0^1 y\Phi_{ky}\,\mathrm{d}y\right) = 0 && \text{on } y = 1. && (71)
\end{aligned}$$

Choosing $(\rho_k^\star, \omega_k^\star, \Phi_k^\star, \Psi_k^\star) = (0,0,0,0)$ and $\lambda \neq 0$, one finds by elementary calculations that equations (66)–(71) have only the zero solution unless k satisfies (64); in this case they have a one-dimensional family $\mathrm{span}\{w_k\}$ of smooth solutions, and it follows that λ is an eigenvalue of L_s with 2-dimensional eigenspace $\mathrm{span}\{w_k \mathrm{e}^{2\pi\mathrm{i}kz/P}, w_k\mathrm{e}^{-2\pi\mathrm{i}kz/P}\}$ if $k \neq 0$ and 1-dimensional eigenspace $\mathrm{span}\{w_0\}$ if $k = 0$. When $\lambda = 0$ the equations have a one-dimensional family $\mathrm{span}\{w_0\}$ of smooth solutions, so that $\ker L_s = \mathrm{span}\{w_0\}$.

Now take $s = 0$ and $\lambda \neq 0$. Straightforward arguments show that equations (66)–(71) have a unique solution $v_k = (\rho_k, \omega_k, \Phi_k, \Psi_k) \in \mathbb{R}\times\mathbb{R}\times H^2(0,1)\times H^1(0,1)$ for each $f_k^\star = (\rho_k^\star, \omega_k^\star, \Phi_k^\star, \Psi_k^\star)$ if and only if k does not satisfy (64) (cf. Buffoni, Groves & Toland (1996, pp. 589–590), Buffoni & Groves (1999, Appendix A)). Note that (64) has no solutions of the form $\lambda = \mathrm{i}\xi$, $\xi > \xi_0$; such a solution would imply the existence of an eigenvector of L_0 with eigenvalue $i\xi$ (see above), and this is ruled out by Lemma 5. Choosing $\lambda = \mathrm{i}\xi$ with $\xi > \xi_0$, observe that

$$(L_0 - \mathrm{i}\xi I)v_k \mathrm{e}^{2\pi\mathrm{i}kz/P} = f_k^\star \mathrm{e}^{2\pi\mathrm{i}kz/P},$$

and Lemma 5 therefore shows that

$$\|v_k \mathrm{e}^{2\pi\mathrm{i}kz/P}\|_{X_1} \le C\|f_k^\star \mathrm{e}^{2\pi\mathrm{i}kz/P}\|_{X_0}.$$

Summing this inequality over all $k \in \mathbb{Z}$, one finds that $v \in X_1$ and that $(L_0 - \mathrm{i}\xi I)^{-1} : X_0 \to X_1$ is continuous. It follows that (40) has a unique solution $v \in \mathcal{D}(L_0)$ for each $f^\star \in X_0$, and inequality (42) indicates that $\mathrm{i}\xi$ belongs to the resolvent set of L_0. The resolvent operator $(L_0 - \mathrm{i}\xi I)^{-1} : X_0 \to X_0$ is compact since $(L_0 - \mathrm{i}\xi I)^{-1} : X_0 \to X_1$ is continuous and X_1 is compactly embedded in X_0; the spectrum of L_0 therefore consists of isolated eigenvalues of finite algebraic multiplicity (Kato 1976, Theorem III.6.29). □

Recall that the spectra of K_s and L_s are identical. The results of Proposition 6 therefore also apply to K_s, and it follows from Lemma 5 that there is a constant C_1 such that

$$\|(K_0 - \mathrm{i}\xi I)^{-1}\|_{X_0 \to X_0} \leq \frac{C_1}{|\xi|}$$

for each real number ξ with $|\xi| > \xi_0$. It remains to describe the spectrum of K_s for $s \in (0, 1/2)$.

Theorem 7 *For each $s \in (0, 1/2)$ the spectrum of K_s consists of isolated eigenvalues of finite algebraic multiplicity; these eigenvalues are precisely the solutions of (64). Moreover, there exists a constant C_2 (which depends upon s) such that*

$$\|(K_s - \mathrm{i}\xi I)^{-1}\|_{X_s \to X_s} \leq \frac{C_2}{|\xi|} \tag{72}$$

for each real number ξ with $|\xi| > \xi_0$.

Proof In this proof $\mathcal{D}(K_s)$ and $\mathcal{D}(L_s)$ are regarded as closed subsets of X_{s+1} and are equipped with the X_{s+1}-norm. First note that $(K_0 - \mathrm{i}\xi I)^{-1} : \mathcal{D}(K_0) \to \mathcal{D}(K_0)$ is bounded since X_1 is continuously embedded in X_0. Using the symbols c_j to denote constants which are independent of ξ, one has that

$$\begin{aligned}
\|(K_0 - \mathrm{i}\xi I)^{-1} v\|_{X_1} &\leq c_1 \|(K_0 - \mathrm{i}\xi_1 I)(K_0 - \mathrm{i}\xi I)^{-1} v\|_{X_0} \\
&= c_1 \|(K_0 - \mathrm{i}\xi I)^{-1}(K_0 - \mathrm{i}\xi_1 I) v\|_{X_0} \\
&\leq c_2 \|(K_0 - \mathrm{i}\xi I)^{-1}\|_{X_0 \to X_0} \|(K_0 - \mathrm{i}\xi_1 I) v\|_{X_0} \\
&\leq \frac{c_3}{|\xi|} \|v\|_{X_1}
\end{aligned}$$

for each $v \in \mathcal{D}(K_0)$; here ξ_1 is a fixed real number with $\xi_1 > \xi_0$. It follows by interpolation that $(K_0 - \mathrm{i}\xi I)^{-1}$ is a bounded operator from $[\mathcal{D}(K_0), X_0]_s$ into itself which satisfies the estimate

$$\|(K_0 - \mathrm{i}\xi I)^{-1} v\|_{[\mathcal{D}(K_0), X_0]_s} \leq \frac{c_4}{|\xi|} \|v\|_{[\mathcal{D}(K_0), X_0]_s}$$

for each $v \in [\mathcal{D}(K_0), X_0]_s$.

Recall that $\mathcal{D}(K_0)$ is the closed subset of X_1 consisting of those functions $(\rho, \omega, \Gamma, \Psi)$ which satisfy $\Gamma_y|_{y=0} = \Gamma_y|_{y=1} = 0$. These boundary conditions are *normal* in the sense of Grisvard (1967), and the interpolation results in this reference assert that $[\mathcal{D}(K_0), X_0]_s = X_s$ for $s \in (0, 1/2)$ (see also Lions & Magenes (1972, vol. I, ch. 1, §18.5 and vol. II, ch. 4, §14.5)). One may therefore identify $(K_0 - \mathrm{i}\xi I)^{-1} : X_s \to X_s$ with $(K_s - \mathrm{i}\xi I)^{-1} : X_s \to X_s$ to find that the latter operator satisfies (72) and that $(K_s - \mathrm{i}\xi I) : \mathcal{D}(K_s) \to X_s$ is a bijection. An examination of equation (36) shows that $L_s : \mathcal{D}(L_s) \to X_s$ and hence $(K_s - \mathrm{i}\xi I) : \mathcal{D}(K_s) \to X_s$ is continuous. It follows that $(K_s - \mathrm{i}\xi I)^{-1} : X_s \to \mathcal{D}(K_s)$ is continuous and, since X_{s+1} is compactly embedded in X_s, that $(K_s - \mathrm{i}\xi I)^{-1} : X_s \to X_s$ is compact. The spectrum of K_s therefore consists of isolated eigenvalues of finite algebraic multiplicity; it has already been established that these eigenvalues are precisely the solutions of (64). $\square$

Let us now verify the remaining hypotheses in Theorem 2 which ensure that it is applicable to equation (35). First note that K_s is closed because it has a non-empty resolvent set. Define the spectral projection P corresponding to the subset $\sigma(K_s) \cap \mathrm{i}\mathbb{R}$ of $\sigma(K_s)$ by

$$Px = \frac{1}{2\pi \mathrm{i}} \int_C (K_s - \lambda I)^{-1} x \, \mathrm{d}\lambda,$$

where C is a closed curve in the resolvent set of K_s which contains $\sigma(K_s) \cap \mathrm{i}\mathbb{R}$ and no other point of $\sigma(K_s)$. Writing $X_1 = P(X_s)$, $X_2 = (I-P)(X_s)$, one finds that hypotheses (H1) and (H2) of Theorem 2 are satisfied (see Kato (1976, Theorem III.6.17)), and Theorem 7 shows that (H3) is also satisfied.

Theorem 8 *For any natural number r equation (35) has a finite-dimensional centre manifold of class C^r in the sense of Theorem 2. The flow on this centre manifold is Hamiltonian and generates solutions of the steady water-wave problem (6)–(9).*

3.4 Coordinates on the centre manifold

According to Theorem 3 equation (27) represents Hamilton's equations for the Hamiltonian system $(M_C^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, where M_C^λ is equipped with the single coordinate chart $\tilde{U}_1 \subset X_1$ and coordinate map $\chi : M_C^\lambda \to \tilde{U}_1$ defined by

$$\chi^{-1}(u_1) = u_1 + h(u_1, \lambda).$$

In the following calculations it is more convenient to change to the coordinate chart $\tilde{W}_1 = (\mathrm{d}G^0[0])^{-1}(\tilde{U}_1)$, so that $\tilde{W}_1$ is the centre subspace of the linear operator L_s. The coordinate map $\zeta : M_C^\lambda \to \tilde{W}_1$ can be written as

$$\zeta^{-1}(w_1) = w_1 + k(w_1, \lambda),$$

where $k : \tilde{W}_1 \times \tilde{\Lambda} \to V$ is given by

$$k(w_1, \lambda) = (G^{\lambda_2})^{-1}(\mathrm{d}G^0[0](w_1) + h(\mathrm{d}G^0[0](w_1, \lambda)) - w_1.$$

In this coordinate system M_C^λ is $\{w_1 + k(w_1, \mu) : w_1 \in \tilde{W}_1\}$; it lies in $\mathcal{D}(v_H^\lambda)$ and defines a centre manifold for the Hamiltonian system (M, Ω, H^λ). The reduced Hamiltonian function $\tilde{H}^\lambda$ and reduced 2-form $\tilde{\Omega}^\lambda$ are respectively C^r- and C^{r-1}-functions of w_1 and λ given by the formulae

$$\begin{aligned}
\tilde{H}^\lambda(w_1) &= H_2^\lambda(w_1 + k(w_1, \lambda)), \\
\tilde{\Omega}^\lambda|_{w_1}(v^1, v^2) &= \Omega|_{w_1 + k(w_1, \lambda)}(v^1 + \mathrm{d}_1 k[w_1, \lambda](v^1), v^2 + \mathrm{d}_1 k[w_1, \lambda](v^2)) \\
&= \Omega(v^1, v^2) + O(|(\lambda, w_1)|)
\end{aligned}$$

as $(w_1, \lambda) \to 0$.

The next step is to choose a symplectic basis for X_1 consisting of generalised eigenvectors of L_s. The operator always has a zero eigenvalue with eigenvector $v_0 = (0,0,1,0)^{\mathrm{T}}$. Direct calculations show that this eigenvector has a Jordan chain of length 2 if $\alpha_0 \neq 1$, of length 4 if $\alpha_0 = 1$, $\beta_0 \neq 1/3$ and of length 6 if $\alpha_0 = 1$, $\beta_0 = 1/3$. Starting from this Jordan chain, one can find a basis $\{e_1,\dots,e_n,f_1,\dots,f_n\}$ for E_0 consisting of generalised eigenvectors of L_s such that $\Omega(e_i,f_i) = 1$ and the symplectic product of any other combination of basis vectors is zero. Here E_0 is the generalised eigenspace corresponding to the zero eigenvalue and n is 1, 2 or 3 as appropriate. Notice that it is always possible to choose either e_1 or f_1 to be a scalar multiple of v_0. The real coordinates q_i, p_i in the e_i and f_i directions are canonical coordinates for E_0 and the action of the reverser S on this generalised eigenspace is given by either $(q,p) \mapsto (-q,p)$ or $(q,p) \mapsto (q,-p)$.

Any other 'two-dimensional' purely imaginary eigenvalues are also geometrically simple. They occur in pairs $\pm\mathrm{i}q$ and there is an eigenvector e such that $L_s e = \mathrm{i}qe$, $L_s\bar{e} = -\mathrm{i}q\bar{e}$. A straightforward calculation shows that there is a Jordan chain (of length 2) if and only if α_0, β_0 and q are related by the equations

$$\beta_0 = \frac{-1}{2\sinh^2 q} + \frac{1}{2q\tanh q}, \qquad \alpha_0 = \frac{q^2}{2\sinh^2 q} + \frac{q}{2\tanh q}.$$

This relationship defines a curve C_0 in (β,α) parameter space which is parameterised by q (see Figure 1). When $(\beta_0,\alpha_0) \notin C_0$ one can normalise e in such a way that $\Omega(e,\bar{e}) = \pm\mathrm{i}$, so that $\{e,\bar{e}\}$ is a symplectic basis for $E_{\mathrm{i}q} \oplus E_{-\mathrm{i}q}$. Canonical coordinates for $E_{\mathrm{i}q} \oplus E_{-\mathrm{i}q}$ are given by the complex coordinates C, $\bar{C}$ in the e, $\bar{e}$ directions, and the action of the reverser S on $E_{\mathrm{i}q} \oplus E_{-\mathrm{i}q}$ is $C \mapsto \bar{C}$. When $(\beta_0,\alpha_0) \in C_0$ there is a generalised eigenvector f such that $(L_s - \mathrm{i}qI)f = e$, $(L_s + \mathrm{i}qI)\bar{f} = \bar{e}$. After normalising e and modifying f by the addition of a suitable multiple of e if necessary, one finds that $\{e,f,\bar{e},\bar{f}\}$ is a basis for $E_{\mathrm{i}q} \oplus E_{-\mathrm{i}q}$ such that $\Omega(e,\bar{f}) = 1$, $\Omega(f,\bar{e}) = -1$ and the symplectic product of any other combination of basis vectors is zero. The complex coordinates A, B in the e and f directions are canonical coordinates for $E_{\mathrm{i}q} \oplus E_{-\mathrm{i}q}$, and the action of the reverser S on this space is $(A,B) \mapsto (\bar{A},-\bar{B})$.

All 'three-dimensional' purely imaginary eigenvalues are geometrically double. They occur in pairs $\pm\mathrm{i}s_k$ with $L_s e^k_{\cos} = \mathrm{i}s_k e^k_{\cos}$, $L_s e^k_{\sin} = \mathrm{i}s_k e^k_{\sin}$, $L_s \bar{e}^k_{\cos} = -\mathrm{i}s_k \bar{e}^k_{\cos}$, $L_s \bar{e}^k_{\sin} = -\mathrm{i}s_k \bar{e}^k_{\sin}$, where $e^k_{\cos} = v_k(y)\cos(2k\pi z/P)$, $e^k_{\sin} = v_k(y)\sin(2k\pi z/P)$ for some smooth vector $v_k(y)$ which does not depend upon z. A straightforward calculation shows that there are no Jordan chains unless β_0, α_0 and s_k are related by the equations

$$\beta_0 = \frac{-s_k^2 + 2p_k^2}{2p_k^3\tanh p_k} - \frac{s_k^2}{2p_k^2\sinh^2 p_k}, \qquad \alpha_0 = \frac{3s_k^2 - 2p_k^2}{2p_k\tanh p_k} + \frac{s_k^2}{2\sinh^2 p_k}. \tag{73}$$

This relationship defines a curve C_k in (β,α) parameter space which is parameterised by s_k (see Figure 1). When $(\beta_0,\alpha_0) \notin C_k$ one can normalise the eigenvectors in such a way that $\Omega(e^k_{\cos},\bar{e}^k_{\cos}) = \pm\mathrm{i}$, $\Omega(e^k_{\sin},\bar{e}^k_{\sin}) = \pm\mathrm{i}$, so that $\{e^k_{\cos},\bar{e}^k_{\cos},e^k_{\sin},\bar{e}^k_{\sin}\}$ is a symplectic basis for $E_{\mathrm{i}s_k} \oplus E_{-\mathrm{i}s_k}$. Canonical coordinates for $E_{\mathrm{i}s_k} \oplus E_{-\mathrm{i}s_k}$ are given by the complex coordinates $C^k_{\cos}$, $C^k_{\sin}$ in the $e^k_{\cos}$, $e^k_{\sin}$ directions, and the action of the reverser S on $E_{\mathrm{i}s_k} \oplus E_{-\mathrm{i}s_k}$ is $(C^k_{\cos},C^k_{\sin}) \mapsto (\bar{C}^k_{\cos},\bar{C}^k_{\sin})$. When $(\beta_0,\alpha_0) \in C_k$ each eigenvector lies in a Jordan chain of length 2: there are generalised eigenvectors $f^k_{\cos}$ and $f^k_{\sin}$ such that $(L_s - \mathrm{i}s_kI)f^k_{\cos} = e^k_{\cos}$, $(L_s - \mathrm{i}s_kI)f^k_{\sin} = e^k_{\sin}$, $(L_s + \mathrm{i}s_kI)\bar{f}^k_{\cos} = \bar{e}^k_{\cos}$, $(L_s + \mathrm{i}s_kI)\bar{f}^k_{\sin} = \bar{e}^k_{\sin}$ and these vectors take the form $f^k_{\cos} =$

$w_k(y)\cos(2k\pi z/P)$, $f^k_{\mathrm{sin}} = w_k(y)\sin(2k\pi z/P)$ for some smooth vector $w_k(y)$ which does not depend upon z. Normalising e^k_{cos}, e^k_{sin} and adding suitable multiples of e^k_{cos}, e^k_{sin} to respectively f^k_{cos}, f^k_{sin}, one obtains a basis $\{e^k_{\mathrm{cos}}, \bar{e}^k_{\mathrm{cos}}, f^k_{\mathrm{cos}}, \bar{f}^k_{\mathrm{cos}}, e^k_{\mathrm{sin}}, \bar{e}^k_{\mathrm{sin}}, f^k_{\mathrm{sin}}, \bar{f}^k_{\mathrm{sin}}\}$ for $E_{\mathrm{i}s_k} \oplus E_{-\mathrm{i}s_k}$ with the property that $\Omega(e^k_{\mathrm{sin}}, \bar{f}^k_{\mathrm{sin}}) = 1$, $\Omega(f^k_{\mathrm{sin}}, \bar{e}^k_{\mathrm{sin}}) = -1$, $\Omega(e^k_{\mathrm{cos}}, \bar{f}^k_{\mathrm{cos}}) = 1$, $\Omega(f^k_{\mathrm{cos}}, \bar{e}^k_{\mathrm{cos}}) = -1$ and the symplectic product of any other combination of basis vectors is zero. The complex coordinates A^k_{cos}, A^k_{sin}, B^k_{cos}, B^k_{sin} in the e^k_{cos}, e^k_{sin}, f^k_{cos}, f^k_{sin} directions are canonical coordinates for $E_{\mathrm{i}q} \oplus E_{-\mathrm{i}q}$, and the action of the reverser S on this space is $(A^k_{\mathrm{cos}}, A^k_{\mathrm{sin}}, B^k_{\mathrm{cos}}, B^k_{\mathrm{sin}}) \mapsto (\bar{A}^k_{\mathrm{cos}}, \bar{A}^k_{\mathrm{sin}}, -\bar{B}^k_{\mathrm{cos}}, -\bar{B}^k_{\mathrm{sin}})$.

Applying a standard result from linear Hamiltonian systems theory (see e. g. Mielke (1991, Theorem 3.4)) to $(M, \Omega, \mathcal{H})$, where $\mathcal{H} = \mathrm{d}^2 H^0[0](\cdot,\cdot)/2$, one finds that E_λ and E_μ are orthogonal with respect to Ω whenever $\lambda + \mu \neq 0$. It follows that Ω is the canonical symplectic 2-form Υ on X_1 in the coordinate system introduced above. The subspaces

$$\mathcal{T}_1 = \{A^k_{\mathrm{cos}}, A^k_{\mathrm{sin}}, B^k_{\mathrm{cos}}, B^k_{\mathrm{sin}}, C^k_{\mathrm{cos}}, C^k_{\mathrm{sin}} = 0 \text{ for all } k\}$$

and

$$\mathcal{T}_2 = \{A^k_{\mathrm{sin}}, B^k_{\mathrm{sin}}, C^k_{\mathrm{sin}} = 0 \text{ for all } k\}$$

of X_1 are important in the following theory. They consist of respectively two-dimensional vectors (which do not depend upon z) and even vectors (which are invariant under the reflection $z \mapsto -z$). For later use note also that the action $\Phi \mapsto \Phi + c$ of the translation group on X_1 is given by either $q_1 \mapsto q_1 + c$ or $p_1 \mapsto p_1 + c$.

The general theory developed by Mielke (1991) shows that the centre-manifold reduction procedure preserves reversibility and symmetries. One finds from this theory that the action of the reverser S upon the coordinates for $\tilde{W}_1$ as described above correctly defines the reversibility of $(M^\lambda_C, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$. Similarly, the invariance of the flow of (M, Ω, H^λ) under the restriction to solutions which are independent of z or even in z is inherited by the reduced system $(M^\lambda_C, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, for which $\mathcal{T}_1$ and $\mathcal{T}_2$ are invariant subspaces. Finally, the translation invariance in Φ is reflected in the fact that the reduced system is invariant under translations in either q_1 or p_1.

The following result is a parameter-dependent, equivariant version of Darboux's theorem (see Mielke (1991, §5.4) and Buffoni & Groves (1999, Theorem 3) for the modifications to incorporate the parameter-dependency).

Lemma 9 *Consider the Hamiltonian system $(M^\lambda_C, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, in which M^λ_C is equipped with the coordinate system described above. There is a near identity change of coordinates of class C^{k-1} which transforms $\tilde{\Omega}^\lambda$ into Υ. It preserves reversibility, the invariance of $\mathcal{T}_1$, $\mathcal{T}_2$ and the invariance under translations in q_1 or p_1 as appropriate.*

Lemma 9 indicates a convenient choice of coordinates in which to discuss Hamilton's equations for $(M^\lambda_C, \Upsilon, \tilde{H}^\lambda)$. In particular, notice that either p_1 or q_1 is always cyclic. The conjugate variable is therefore a conserved quantity (which will be set to zero in the following analysis), and the dimension of the system of equations can always be reduced by two.

3.5 *Existence theories for periodic and quasiperiodic waves*

This section presents some existence results for small-amplitude solutions of the steady water-wave problem which can be obtained by direct applications of standard results in Hamiltonian-systems theory to the reduced equations on the centre manifold. They concern fixed values

(β_0, α_0) of (β, α) which do not lie on any of the curves shown in Figure 1, so that the bifurcation parameters λ_1 and λ_2 are both set to zero. Let us proceed by introducing coordinates for the centre manifold using the procedure explained in Section 3.4 and restricting attention to the invariant subspace $\mathcal{T}_2$ which corresponds to solutions of the water-wave problem which are even in z. The zero eigenvalue, which has a Jordan chain of length 2 in the present setting, can be removed using the procedure explained at the end of Section 3.4, and all other eigenvalues $\pm\mathrm{i}\omega_1$, ..., $\pm\mathrm{i}\omega_n$ of $\sigma(L_s) \cap \mathrm{i}\mathbb{R}$ are simple.

The following result is obtained from the Lyapunov centre theorem (e.g. see Ambrosetti & Prodi (1995, ch. 7, Theorem 3.4)).

Theorem 10 *Assume that $\omega_i/\omega_j \notin \mathbb{Z}$ for all $i \neq j$.*

(i) Suppose that $\alpha_0 > 1$ and (β_0, α_0) lies between C_j^+ and C_{j+1}^+ for some $j \geq 0$. The reduced equations on the centre manifold have $2(j+1)$ geometrically distinct periodic orbits on the energy surface $\{\tilde{H}^0 = \epsilon\}$ for each sufficiently small value of $\epsilon > 0$. Two of these periodic orbits correspond to solutions of the steady water-wave problem which are independent of z.

(ii) Suppose that $\alpha_0 < 1$ and (β_0, α_0) lies between C_j^- and C_{j+1}^- for some $j \geq 1$. The reduced equations on the centre manifold have $2j+1$ geometrically distinct periodic orbits on the surface energy surface $\{\tilde{H}^0 = \epsilon\}$ for each sufficiently small value of $\epsilon > 0$. One of these periodic orbits corresponds to a solution of the steady water-wave problem which is independent of z.

(iii) Suppose that $\alpha_0 < 1$ and (β_0, α_0) lies to the right of C_1^-. The reduced equations on the centre manifold have a periodic orbit on the energy surface $\{\tilde{H}^0 = \epsilon\}$ for each sufficiently small value of $\epsilon > 0$. This orbit corresponds to a two-dimensional solution of the steady water-wave problem.

Now suppose that ω_1, ..., ω_n satisfy no resonance relation of order three or four, that is any triple or quartet of frequencies is linearly independent over the field $\mathbb{Q}$. According to classical dynamical-systems theory the reduced Hamiltonian system given by the centre-manifold reduction procedure can be put into *fourth-order Birkhoff normal form* by a near identity symplectic transformation (e.g. see Arnold (1989, Appendix 7A)). The transformed Hamiltonian function takes the form

$$\tilde{H}^0(w_1) = \frac{1}{2}\omega_1 I_1 + \cdots + \frac{1}{2}\omega_n I_n + \frac{1}{8}(I_1, \ldots, I_n)Q(I_1, \ldots, I_n)^T + O(\|w_1\|^5)$$

as $w_1 \to 0$, in which I_1, ..., I_n are action variables associated with the eigenvalues $\pm\mathrm{i}\omega_1$, $\ldots \pm\mathrm{i}\omega_n$ and Q is an $n \times n$ matrix. Theorem 11 below follows from a result in KAM theory given by Pöschel (1980, Theorem 2.1).

Theorem 11 *Suppose that $\omega_1, \ldots, \omega_n$ satisfy no resonance relation of order three or four and that* $\det Q \neq 0$.

(i) The reduced equations on the centre manifold have $2(j+1)$-dimensional invariant Kronecker system whenever $\alpha_0 > 1$ and (β_0, α_0) lies between C_j^+ and C_{j+1}^+ for some $j \geq 0$.

(ii) The reduced equations on the centre manifold have a $(2j+1)$-dimensional invariant Kronecker system in each neighbourhood of the origin whenever $\alpha_0 > 1$ and (β_0, α_0) lies between C_j^- and C_{j+1}^- for some $j \geq 1$.

(An n-dimensional Kronecker system (T^n, ω) consists of an n-torus T^n and a frequency vector ω which describes linear quasiperiodic motion upon it. A $2n$-dimensional system of ordinary differential equations is said to have an invariant Kronecker system (T^n, ω) if there exists an embedding $\phi \in C^1(T^n, \mathbb{R}^{2n})$ such that $\phi(\omega t + x_0)$ solves the system for each $x_0 \in T^n$.)

4 Bifurcation of generalised solitary waves

4.1 Normal-form analysis

This section is concerned with a codimension-one, small-amplitude bifurcation phenomenon which occurs at the curve C_1^-. At each point (β_0, α_0) of C_1^- the purely imaginary part of $\sigma(L_s)$ consists of the 'two-dimensional' eigenvalues 0, $\pm \mathrm{i}q$ and the 'three-dimensional' eigenvalues $\pm \mathrm{i}s_1$ (see Figure 1). Let us introduce a bifurcation parameter μ by writing

$$(\beta, \alpha) = (\beta_0, \alpha_0 + \mu)$$

(so that $\lambda_1 = \mu$, $\lambda_2 = 0$ in the notation of Section 3) and carry out the centre-manifold reduction procedure using the coordinates described in Section 3.4.

The zero eigenvalue is geometrically simple and has a Jordan chain of length 2. Its generalised eigenvectors

$$e_1 = \sqrt{\frac{\alpha_0}{P(1-\alpha_0)}} \begin{pmatrix} \frac{1}{\alpha_0} \\ 0 \\ 0 \\ 1 - \frac{1}{\alpha_0} \end{pmatrix}, \qquad f_1 = \sqrt{\frac{\alpha_0}{P(1-\alpha_0)}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

which satisfy $L_s f_1 = 0$, $L_s e_1 = f_1$, have been chosen so that $\Omega(e_1, f_1) = 1$. The other 'two-dimensional' eigenvalues $\pm \mathrm{i}q$ are geometrically and algebraically simple. The vector

$$e = \frac{1}{\sqrt{\gamma_1}} \begin{pmatrix} \mathrm{i} \sinh q \\ -\left(\beta_0 q + \dfrac{1}{q}\right) \sinh q + \cosh q \\ \cosh qy \\ -\mathrm{i} \sinh q + \mathrm{i} q \cosh qy \end{pmatrix},$$

where

$$\gamma_1 = \frac{P}{2q}(-4\alpha \sinh^2 q + q(2q + \sinh 2q)),$$

is an eigenvector corresponding to the eigenvalue $\mathrm{i}q$ which is normalised so that $\Omega(\bar{e}, e) = \mathrm{i}$.

The 'three-dimensional' eigenvalues $\pm\mathrm{i}s_1$ are geometrically double, and in addition each eigenvector has a Jordan chain of length 2. Observe that

$$v^1_{\text{cos}} = \begin{pmatrix} \dfrac{\mathrm{i}p_1}{s_1}\sinh s_1 \\ -\left(\beta_0 p_1 + \dfrac{1}{p_1}\right)\sinh p_1 + \cosh p_1 \\ \cosh p_1 y \\ -\dfrac{\mathrm{i}p_1}{s_1}\sinh p_1 + \mathrm{i}s_1\cosh p_1 y \end{pmatrix} \cos\left(\frac{2\pi z}{P}\right),$$

$$w^1_{\text{cos}} = \begin{pmatrix} \left(\dfrac{1}{p_1} - \dfrac{p_1}{s_1^2}\right)\sinh p_1 + \cosh p_1 \\ \dfrac{\mathrm{i}s_1}{p_1}\left(\beta_0 - 1 - \dfrac{1}{p_1^2}\right)\sinh p_1 + \mathrm{i}s_1\left(\beta_0 + \dfrac{1}{p_1^2}\right)\cosh p_1 \\ -\dfrac{\mathrm{i}s_1}{p_1} y \sinh p_1 y \\ \left(\dfrac{p_1}{s_1^2} - \dfrac{1}{p_1}\right)\sinh p_1 - \cosh p_1 + \dfrac{s_1^2}{p_1} y \sinh p_1 y + \cosh p_1 y \end{pmatrix} \cos\left(\frac{2\pi z}{P}\right),$$

in which $p_1^2 = s_1^2 + 4\pi^2/P^2$, satisfy $(L_s - \mathrm{i}s_1 I)v^1_{\text{cos}} = 0$, $(L_s + \mathrm{i}s_1 I)w^1_{\text{cos}} = v^1_{\text{cos}}$, and defining v^1_{sin}, w^1_{sin} by the above formulae with $\cos(2\pi z/P)$ replaced by $\sin(2\pi z/P)$, one finds that $(L_s - \mathrm{i}s_1 I)v^1_{\text{sin}} = 0$, $(L_s + \mathrm{i}s_1 I)w^1_{\text{sin}} = v^1_{\text{sin}}$. Direct calculations show that $\Omega(v^1_{\text{cos}}, \bar{w}^1_{\text{cos}}) = -\Omega(w^1_{\text{cos}}, \bar{v}^1_{\text{cos}}) = \gamma_2$, $\Omega(w^1_{\text{cos}}, \bar{w}^1_{\text{cos}}) = \mathrm{i}\gamma_3$, $\Omega(v^1_{\text{sin}}, \bar{w}^1_{\text{sin}}) = -\Omega(w^1_{\text{sin}}, \bar{v}^1_{\text{sin}}) = \gamma_2$, $\Omega(w^1_{\text{sin}}, \bar{w}^1_{\text{sin}}) = \mathrm{i}\gamma_3$, where

$$\gamma_2 = \frac{P}{8p_1^3}\Big(- 4p_1^2 s_1^2 \coth p_1 + (4p_1^2 - 3s_1^2)(2p_1 + \sinh 2p_1)\Big) > 0, \tag{74}$$

$$\begin{aligned} \gamma_3 = \frac{1}{12p_1^5 P^3 s_1}\Big(& 192 p_1 \pi^4 + p_1^3 P^4 s_1^2(-3 + 8s_1^2) \\ & - 3p_1 P^2 s_1^2((16\pi^2 + P^2 s_1^2)\cosh 2p_1 + 2p_1(12\pi^2\coth p_1 - p_1 P^2 s_1^2 \,\text{cosech}^2 p_1)) \\ & + 6\pi^2(16\pi^2 + 3P^2 s_1^2)\sinh 2p_1\Big), \end{aligned}$$

and the symplectic product of any other combination of these vectors is zero. The generalised eigenvectors given by

$$e^1_{\text{cos}} = \frac{1}{\sqrt{\gamma_2}} v^1_{\text{cos}}, \qquad f^1_{\text{cos}} = \frac{1}{\sqrt{\gamma_2}}\left(w^1_{\text{cos}} - \frac{\mathrm{i}\gamma_3}{2\gamma_2} v^1_{\text{cos}}\right),$$

$$e^1_{\text{sin}} = \frac{1}{\sqrt{\gamma_2}} v^1_{\text{sin}}, \qquad f^1_{\text{sin}} = \frac{1}{\sqrt{\gamma_2}}\left(w^1_{\text{sin}} - \frac{\mathrm{i}\gamma_3}{2\gamma_2} v^1_{\text{sin}}\right)$$

are therefore normalised such that $\Omega(e^1_{\text{cos}}, \bar{f}^1_{\text{cos}}) = 1$, $\Omega(f^1_{\text{cos}}, \bar{e}^1_{\text{cos}}) = -1$, $\Omega(e^1_{\text{sin}}, \bar{f}^1_{\text{sin}}) = 1$, $\Omega(f^1_{\text{sin}}, \bar{e}^1_{\text{sin}}) = -1$ and the symplectic product of any other combination is zero.

The centre-manifold reduction procedure yields a 12-dimensional centre manifold M_C^μ for $\mu \in \tilde{\Lambda}$. Following the theory developed in Section 3.4, let us model M_C^μ upon the single coordinate chart $\tilde{W}_1$ and write

$$
\begin{aligned}
w_1 = {} & q_1 e_1 + p_1 f_1 + Ce + \bar{C}\bar{e} \\
& + A_{\cos}^1 e_{\cos}^1 + B_{\cos}^1 f_{\cos}^1 + \bar{A}_{\cos}^1 \bar{e}_{\cos}^1 + \bar{B}_{\cos}^1 \bar{f}_{\cos}^1 + A_{\sin}^1 e_{\sin}^1 + B_{\sin}^1 f_{\sin}^1 + \bar{A}_{\sin}^1 \bar{e}_{\sin}^1 + \bar{B}_{\sin}^1 \bar{f}_{\sin}^1,
\end{aligned}
$$

in which q_1, p_1 and $A_{\cos}^1$, $A_{\sin}^1$, $B_{\cos}^1$, $B_{\sin}^1$ are respectively real and complex. Suitable canonical coordinates for the reduced Hamiltonian system $(M_C^\mu, \tilde{\Omega}^\mu, \tilde{H}^\mu)$ are then found by a near-identity symplectic transformation (Lemma 9). Because p_1 is cyclic, the conjugate variable variable q_1 is a conserved quantity and it is convenient to reduce the order of Hamilton's equations by setting $q_1 = 0$. Furthermore, recall that

$$
\mathcal{T}_1 = \{A_{\cos}^1 = A_{\sin}^1 = B_{\cos}^1 = B_{\sin}^1 = 0\}, \qquad \mathcal{T}_2 = \{A_{\sin}^1 = B_{\sin}^1 = 0\}
$$

are invariant subspaces for Hamilton's equations; they correspond to solutions of the water-wave problem which are respectively independent of z and even in z. Let us now restrict to the subspace $\mathcal{T}_2$ to obtain a 6-dimensional Hamiltonian system $(\mathcal{T}, \Upsilon, \tilde{H}^\mu)$, where

$$
\mathcal{T} = \{Ae^1 + Bf^1 + \bar{A}\bar{e}^1 + \bar{B}\bar{f}^1 + Ce + \bar{C}\bar{e} : A, B, C \in \mathbb{C}\} \cap \tilde{W}_1,
$$

in which $A_{\cos}^1$, $B_{\cos}^1$ have been replaced by A, B for notational simplicity, and

$$
\begin{aligned}
& \Upsilon((A^1, B^1, \bar{A}^1, \bar{B}^1, C^1, \bar{C}^1), (A^2, B^2, \bar{A}^2, \bar{B}^2, C^2, \bar{C}^2)) \\
& \qquad = A^1\bar{B}^2 - A^2\bar{B}^1 + \bar{A}^1 B^2 - \bar{A}^2 B^1 + \mathrm{i}(C^2\bar{C}^1 - C^1\bar{C}^2).
\end{aligned}
$$

Note that solutions of Hamilton's equations for $(\mathcal{T}, \Upsilon, \tilde{H}^\mu)$ correspond to water waves which are even in z, and solutions which lie in the invariant subspace $\{A = B = 0\}$ correspond to two-dimensional (z-independent) water waves. Recall also that $(\mathcal{T}, \Upsilon, \tilde{H}^\mu)$ is a reversible Hamiltonian system; the reverser S maps (A, B, C) to $(\bar{A}, -\bar{B}, \bar{C})$.

Lemma 12 *There is a near-identity, analytic, symplectic change of coordinates with the property that*

$$
\tilde{H}^\mu = \mathrm{i}s_1(A\bar{B} - \bar{A}B) + |B|^2 + q|C|^2 + H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), |C|^2, \mu) + O(|(A, B, C)|^{N+2})
$$

as $(A, B, C) \to 0$ in the new coordinates. The function H_{NF} is a polynomial of order $N + 1$ in its first three arguments with coefficients which depend analytically upon μ; it satisfies

$$
H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), |C|^2, \mu) = O(|(A, B, C)|^2 |(\mu, A, B, C)|).
$$

The change of coordinates preserves reversibility and the invariance of the subspace $\{A = B = 0\}$.

Proof Expand the Hamiltonian function in a power series in $(A, B, C, \bar{A}, \bar{B}, \bar{C})$ with μ-dependent coefficients, so that

$$
\tilde{H}^\mu = \mathrm{i}s_1(A\bar{B} - \bar{A}B) + |B|^2 + q|C|^2 + \sum_{i=2}^{N+1} \tilde{H}_i^\mu(A, B, C, \bar{A}, \bar{B}, \bar{C}) + O(|(A, B, C)|^{N+2})
$$

as $(A, B, C) \to 0$, in which the terms H_i^μ are homogeneous of degree i in $(A, B, C, \bar{A}, \bar{B}, \bar{C})$ and $\tilde{H}_2^0$ is identically zero. The Taylor polynomials of a reduction function defining a centre manifold for an analytic system of equations always depend analytically upon any parameters; it follows that $\tilde{H}_i^\mu$ depends analytically upon μ. The next step is to apply the normal-form theory developed by Elphick *et al.* (1987) and extended to Hamiltonian systems by Meyer & Hall (1992, ch. VII). There is a near identity, analytic, symplectic changes of variable which modifies $\tilde{H}_i^\mu$, $i = 2, \ldots, N+1$ in such a way that in the new coordinates

$$\tilde{H}_i^\mu(e^{L^* x}(A, B, C, \bar{A}, \bar{B}, \bar{C})) = \tilde{H}_i^\mu(A, B, C, \bar{A}, \bar{B}, \bar{C}) \tag{75}$$

for all $\mu \in \tilde{\Lambda}$ and $x \in \mathbb{R}$; here the operator L is defined by

$$L \begin{pmatrix} A \\ B \\ C \\ \bar{A} \\ \bar{B} \\ \bar{C} \end{pmatrix} = \begin{pmatrix} \mathrm{i}s_1 A + B \\ \mathrm{i}s_1 B \\ \mathrm{i}qC \\ -\mathrm{i}s_1 \bar{A} + \bar{B} \\ -\mathrm{i}s_1 \bar{B} \\ -\mathrm{i}q\bar{C} \end{pmatrix}$$

and L^* is its adjoint. Equation (75) states that $\tilde{H}_i^\mu$ is an invariant of the linear system

$$\frac{d}{dx} \begin{pmatrix} A \\ B \\ C \\ \bar{A} \\ \bar{B} \\ \bar{C} \end{pmatrix} = L^* \begin{pmatrix} A \\ B \\ C \\ \bar{A} \\ \bar{B} \\ \bar{C} \end{pmatrix}.$$

Making the assumption that $q/s_1 \notin \mathbb{Q}$, one finds that $\tilde{H}_i^\mu$ is a real polynomial function of $|A|^2$, $\mathrm{i}(A\bar{B} - \bar{A}B)$, $|C|^2$ (cf. Illichev & Kirchgässner (1998), who study the corresponding normal form for general reversible vector fields). $\square$

Let us henceforth work with the coordinates A, B, C obtained in Lemma 12. Hamilton's equations for the Hamiltonian system $(\mathcal{T}, \Upsilon, \tilde{H}^\mu)$ are given by

$$\begin{aligned}
\dot{A} &= \mathrm{i}s_1 A + B + \mathrm{i}A\partial_2 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), |C|^2, \mu) + O(|(A, B, C)|^{N+1}), \\
\dot{B} &= \mathrm{i}s_1 B + \mathrm{i}B\partial_2 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), |C|^2, \mu) \\
&\quad - A\partial_1 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), |C|^2, \mu) + O(|(A, B, C)|^{N+1}), \\
\dot{C} &= \mathrm{i}qC + \mathrm{i}C\partial_3 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), |C|^2, \mu) + O(|(A, B, C)|^{N+1})
\end{aligned}$$

together with their complex conjugates, as $(A, B, C, \mu) \to 0$. This system of equations can be written schematically as

$$\dot{X} = F(X, \mu) + \mathcal{R}(X, \mu), \tag{76}$$

in which $X = (A, B, C, \bar{A}, \bar{B}, \bar{C})$ and $\mathcal{R}$ represents the terms of $O(|(A, B, C)|^{N+1})$. By choosing k sufficiently large, one may take N arbitrarily large and assume that $\mathcal{R}$ has derivatives of arbitrarily high order in X and μ.

Consider the Hamiltonian system $(\mathcal{T}, \Upsilon, \tilde{H}_0^\mu)$ obtained by omitting the remainder terms in $\tilde{H}^\mu$, so that Hamilton's equations are given in the above coordinates by

$$\dot{X} = F(X, \mu). \tag{77}$$

This Hamiltonian system is completely integrable: the quantities $\tilde{H}_0^\mu$, $\mathrm{i}(A\bar{B} - \bar{A}B)$ and $|C|^2$ are first integrals. It is reversible in the same sense as (76) and also has the two-dimensional symmetry

$$R_{\phi,\theta} F(X, \mu) = F(R_{\phi,\theta} X, \mu),$$

where $R_{\phi,\theta} : (A, B, C) \mapsto (A\mathrm{e}^{\mathrm{i}\phi}, B\mathrm{e}^{\mathrm{i}\phi}, C\mathrm{e}^{\mathrm{i}\theta})$. Solutions to equation (77) are easily found by introducing symplectic polar coordinates (I, α) given by

$$C = \sqrt{I}\mathrm{e}^{\mathrm{i}\alpha},$$

so that Hamilton's equations become

$$\begin{aligned} \dot{A} &= \mathrm{i}s_1 A + B + \mathrm{i}A\partial_2 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), I, \mu), && (78)\\ \dot{B} &= \mathrm{i}s_1 B + \mathrm{i}B\partial_2 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), I, \mu) - A\partial_1 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), I, \mu) && (79) \end{aligned}$$

together with their complex conjugates and

$$\begin{aligned} \dot{I} &= 0,\\ \dot{\alpha} &= q + \partial_3 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), I, \mu). \end{aligned}$$

The subspace $\{A = B = 0\}$ is clearly invariant under the flow generated by the above equations. It contains a family of periodic solutions which trace out the level sets of I with frequency $\Omega = q + \partial_3 H_{\mathrm{NF}}(0, 0, I, \mu)$. Note that the action-frequency map is nondegenerate whenever $\partial_3^2 H_{\mathrm{NF}}(0, 0, 0, \mu)$ is nonzero. This construction yields a foliation of the 6-dimensional phase space into a family of 4-dimensional reduced phase spaces parameterised by I: for each fixed value of this integral equations (78), (79) describe a completely integrable 4-dimensional system of the type studied by Iooss & Pérouème (1993), and the angle α is recovered by quadrature.

In certain cases this decomposition predicts the presence of homoclinic connections to each periodic orbit in $\{A = B = 0\}$. Write

$$\begin{aligned} \tilde{H}_{\mathrm{NF}} &= \mu c_1 |A|^2 + \mu c_2 \mathrm{i}(A\bar{B} - \bar{A}B) + \mu c_3 I + c_4 |A|^4 - c_5 (A\bar{B} - \bar{A}B)^2\\ &\quad + c_6 I^2 + c_7 \mathrm{i}|A|^2 (A\bar{B} - \bar{A}B) + c_8 |A|^2 I + c_9 \mathrm{i} I (A\bar{B} - \bar{A}B) + \cdots \end{aligned} \tag{80}$$

and suppose that $c_4 > 0$, $c_1 < 0$. (These coefficients are computed for the water-wave problem in Appendix B, where it is confirmed that they have the required signs.) Elementary methods show that equations (78), (79) have a circle $\{R_{\Omega_0 x + \phi} \mathcal{H}_0(x)\}_{\phi \in S^1}$ of homoclinic orbits for each positive value of μ and each value of I in $(0, \infty)$ if $c_8 < 0$ or in $(0, -c_1/c_8)$ if $c_8 > 0$. (Note, however, that only small values μ and I are relevant in the context of the centre-manifold reduction, so that the sign of c_8 causes no real restriction on I.) Here $\Omega_0 = s_1 + \partial_2 H_{\mathrm{NF}}(0, 0, I, \mu)$ and $\mathcal{H}_0 = (\mathcal{H}_A, \mathcal{H}_B, \overline{\mathcal{H}}_A, \overline{\mathcal{H}}_B)^T$ is a solution of the equations

$$\begin{aligned} \dot{A} &= B + \mathrm{i}A[\partial_2 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), I, \mu) - \partial_2 H_{\mathrm{NF}}(0, 0, I, \mu)], && (81)\\ \dot{B} &= \mathrm{i}B[\partial_2 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), I, \mu) - \partial_2 H_{\mathrm{NF}}(0, 0, I, \mu)]\\ &\quad - A\partial_1 H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), I, \mu) && (82) \end{aligned}$$

and their complex conjugates which is homoclinic to the origin and satisfies $\overline{\mathcal{H}}_A(x) = \mathcal{H}_A(-x)$, $\overline{\mathcal{H}}_B(x) = -\mathcal{H}_B(-x)$. Its components are given by

$$\begin{aligned}\mathcal{H}_A(x) &= \frac{\mu^{1/2}(-c_1 - c_8\hat{I})^{1/2}}{c_4^{1/2}}\mathrm{sech}((-c_1 - c_8\hat{I})^{1/2}\mu^{1/2}x) + O(\mu),\\ \mathcal{H}_B(x) &= -\frac{\mu(-c_1 - c_8\hat{I})}{c_4^{1/2}}\mathrm{sech}((-c_1 - c_8\hat{I})^{1/2}\mu^{1/2}x)\tanh((-c_1 - c_8\hat{I})^{1/2}\mu^{1/2}x) + O(\mu^{3/2}),\end{aligned}$$

where $I = \sqrt{\mu}\hat{I}$, and one has that

$$|\mathcal{H}_0(x)| \leq M\sqrt{\mu}\mathrm{e}^{-|\nu|x}, \qquad x \in \mathbb{R} \tag{83}$$

for every $\nu < \lambda = \sqrt{-\partial_1 H_{\mathrm{NF}}(0, 0, \mu\hat{I}, \mu)}$ and M denotes a constant which depends only upon $\hat{I}$. The above homoclinic orbits correspond to solutions of (77) which are homoclinic to the periodic orbit of amplitude I in $\{A = B = 0\}$. There is a two-parameter family $\{R_{\Omega_0 x+\phi,\Omega x+\theta}\mathcal{H}(x)\}$ of such solutions, where $\mathcal{H} = (\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \overline{\mathcal{H}}_A, \overline{\mathcal{H}}_B, \overline{\mathcal{H}}_C)^T$, $\mathcal{H}_C = \sqrt{I}\mathrm{e}^{\mathrm{i}\psi(x)}$ and ψ is the odd solution of the equation

$$\dot{\psi} = \partial_3 H_{\mathrm{NF}}(|\mathcal{H}_A(x)|^2, 0, I, \mu) - \partial_3 H_{\mathrm{NF}}(0, 0, I, \mu). \tag{84}$$

Precisely four of these homoclinic orbits are reversible, namely those for which $(\phi, \theta) \in \{0, \pi\}^2$. Observe that

$$\mathcal{H}(x) \to R_{0,\pm\delta}(0, 0, \sqrt{I}, 0, 0, \sqrt{I})^T \tag{85}$$

as $x \to \pm\infty$, in which the phase shift δ is defined by

$$2\delta = \int_{-\infty}^{\infty} (\partial_3 H_{\mathrm{NF}}(|\mathcal{H}_A(x)|^2, 0, I, \mu) - \partial_3 H_{\mathrm{NF}}(0, 0, I, \mu))\,\mathrm{d}x.$$

Furthermore, the function $\mathcal{H}$ is differentiable with respect to $\hat{I}$ and its derivative satisfies the estimate

$$\left|\frac{d\mathcal{H}}{d\hat{I}}(x)\right| \leq M\sqrt{\mu}, \qquad x \in \mathbb{R}. \tag{86}$$

It remains to obtain an analogous statement for equation (76), for which $\{A = B = 0\}$ is also an invariant subspace. Solutions which are homoclinic to a periodic solution in $\{A = B = 0\}$ correspond to *generalised solitary-wave solutions* of the steady water-wave problem; they consist of a three-dimensional disturbance which decays to a two-dimensional periodic wave at infinity. The rest of this paper is dedicated to a proof of the existence of a family of periodic solutions of (76) in $\{A = B = 0\}$, each of which is connected to itself by four reversible homoclinic solutions. These solutions are close (in a sense to be made precise) to the corresponding solutions of equation (77) found above.

4.2 *Persistence of periodic orbits in* $\{A = B = 0\}$

The change of variable $X = \sqrt{\mu}\hat{X}$ transforms (76) into

$$\dot{\hat{X}} = \hat{F}(\hat{X}, \mu) + \epsilon\hat{\mathcal{R}}(\hat{X}, \mu) \tag{87}$$

and (77) into the same equation with $\epsilon = 0$; here

$$\hat{F}(\hat{X}, \mu) = \frac{1}{\sqrt{\mu}} F(X, \mu), \quad \hat{\mathcal{R}}(\hat{X}, \mu) = \frac{1}{\sqrt{\mu^N}} \mathcal{R}(X, \mu), \quad \epsilon = \sqrt{\mu^{N-1}}.$$

In the following analysis ϵ is used to activate or deactivate the remainder terms which were neglected in the normal-form analysis in Section 4.1. It is treated as a parameter which is independent of μ, so that $\mu \in [0, \mu_0]$ and $\epsilon \in [0, \epsilon_0]$ for some positive constants μ_0 and ϵ_0; the compatibility of the results thus obtained with the relationship $\epsilon = \sqrt{\mu^{N-1}}$ is checked at the end of each section.

Solutions of (87) of the form $\hat{X} = (0, 0, \hat{C}, 0, 0, \overline{\hat{C}})^T$ satisfy the equation

$$\dot{\hat{C}} = \mathrm{i}\hat{C}[q + \partial_3 H_{\mathrm{NF}}(0, 0, \mu|\hat{C}|^2, \mu)] + \epsilon \hat{\mathcal{R}}_0(\hat{C}, \mu) \tag{88}$$

and its complex conjugate, where $\hat{\mathcal{R}}_0(\hat{C}, \mu) = O(\sqrt{\mu}|\hat{C}|^{N+1})$ as $(\hat{C}, \mu) \to 0$. Formula (88) represents Hamilton's equations for a two-dimensional canonical Hamiltonian system with Hamiltonian given by

$$\tilde{H}_0^{\mu,\epsilon}(\hat{C}, \overline{\hat{C}}) = q|\hat{C}|^2 + \frac{1}{\mu} H_{\mathrm{NF}}(0, 0, \mu|\hat{C}|^2, \mu) + O(\epsilon\sqrt{\mu}|\hat{C}|^{N+2})$$

as $(\hat{C}, \mu) \to 0$. Elementary arguments show that equation (88) has a family of small-amplitude periodic solutions which can be parameterised by $\mu \in [0, \mu_0]$, $\epsilon \in [0, \epsilon_0]$ and small values $\hat{J} \in [0, \hat{J}_0]$ of the Hamiltonian. Let us denote an orbit in this family by $u(\hat{J}, \mu, \epsilon)$ and write its frequency as $\omega(\hat{J}, \mu, \epsilon)$; by choosing k large enough one may assume that for each $x \in \mathbb{R}$ the functions $u(\hat{J}, \mu, \epsilon)(x)$, $\dot{u}(\hat{J}, \mu, \epsilon)(x)$ and $\omega(\hat{J}, \mu, \epsilon)$ are at least twice continuously differentiable in $\hat{J}$, μ and ϵ. Note that the invariance of the subspace $\{A = B = 0\}$ is not essential here; the same conclusion is reached in the general case by means of the Lyapunov centre theorem. (In fact even the Hamiltonian structure of the system is not crucial: there is a Lyapunov centre theorem for reversible systems (Devaney 1976, Theorem 8.1).)

Explicit formulae for $u(\hat{J}, \mu, 0)$ and $\omega(\hat{J}, \mu, 0)$ were found in Section 4.1, where they were parameterised by small values $\hat{I} \in [0, \hat{I}_0]$ of the action variable and $\mu \in [0, \mu_0]$ to give the periodic solutions

$$\hat{C}(x) = \sqrt{\hat{I}}\mathrm{e}^{\mathrm{i}\Omega x}, \qquad \Omega(\hat{I}, \mu) = q + \partial_3 H_{\mathrm{NF}}(0, 0, \mu\hat{I}, \mu)$$

of equations (88). For the purposes of the existence theory in Section 4.3 below it is convenient to reparameterise the periodic orbits $u(\hat{J}, \mu, \epsilon)$ for $\epsilon > 0$ in a similar fashion.

Proposition 13 *For each $\hat{I} \in [0, \hat{I}_0]$, $\mu \in (0, \mu_0]$ and $\epsilon \in [0, \mu^{5/2}]$ there exists a real number $\hat{J}_{\hat{I},\mu,\epsilon} \in [0, \hat{J}_0]$ such that $\omega(\hat{J}_{\hat{I},\mu,\epsilon}, \mu, \epsilon) = \Omega(\hat{I}, \mu)$ and*

$$\left\| u(\hat{J}_{\hat{I},\mu,\epsilon}, \mu, \epsilon) - \sqrt{\hat{I}}\mathrm{e}^{\mathrm{i}(\cdot)\Omega} \right\|_{C^1_{\mathrm{B}}(\mathbb{R})} \leq \frac{c\epsilon}{\mu},$$

where c is a constant which is independent of $\hat{I}$, μ and ϵ.

Proof Consider the equation

$$\omega(J,\mu,\epsilon) = \Omega(\hat{I},\mu) \tag{89}$$

for $\hat{J}$. The solution for $\epsilon = 0$ is clearly $\hat{J}_{\hat{I},\mu,0} = \tilde{H}_0^{\mu,0}(\sqrt{\hat{I}},\sqrt{\hat{I}})$. Suppose that the coefficient c_6 in the expansion of $H^{\mu}_{\rm NF}$ (see equation (80)) is nonzero, so that

$$|\partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu,0)^{-1}| = |\partial_1\Omega(\hat{I},\mu)^{-1}| = |(\mu\partial_3^2 H_{\rm NF}(0,\mu\hat{I},\mu))^{-1}| \leq \frac{c}{\mu}.$$

(The coefficient c_6 is computed in Appendix B and shown there to be nonzero.) Observe that $\hat{J}$ is a solution of (89) if and only if it solves the equation $B(\hat{J},\epsilon) = \hat{J}$, where

$$B(\hat{J},\epsilon) = \hat{J} - \partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu,0)^{-1}(\omega(\hat{J},\mu,\epsilon) - \omega(\hat{J}_{\hat{I},\mu,0},\mu,0)).$$

Standard arguments in nonlinear analysis show that $B(\cdot,\epsilon)$ is a contraction on $\overline{B}_\delta(\hat{J}_{\hat{I},\mu,0})$ provided that $|B(\hat{J}_{\hat{I},\mu,0},\epsilon) - \hat{J}_{\hat{I},\mu,0}| \leq \delta/2$, $|\partial_1 B(\hat{J},\epsilon)| \leq 1/2$ for each $\hat{J} \in \overline{B}_\delta(\hat{J}_{\hat{I},\mu,0})$. The estimates

$$\begin{aligned}
|B(\hat{J}_{\hat{I},\mu,0},\epsilon) - \hat{J}_{\hat{I},\mu,0}| &\leq |\partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu,0)^{-1}||\omega(\hat{J}_{\hat{I},\mu,0},\mu,\epsilon) - \omega(\hat{J}_{\hat{I},\mu,0},\mu,0)| \\
&\leq \frac{c\epsilon}{\mu}
\end{aligned}$$

and

$$\begin{aligned}
|\partial_1 B(\hat{J},\epsilon)| &\leq |\partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu_0)^{-1}||\partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu,0) - \partial_1\omega(\hat{J},\mu,\epsilon)| \\
&\leq |\partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu_0)^{-1}||\partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu,0) - \partial_1\omega(\hat{J},\mu,0)| \\
&\quad + |\partial_1\omega(\hat{J}_{\hat{I},\mu,0},\mu_0)^{-1}||\partial_1\omega(\hat{J},\mu,0) - \partial_1\omega(\hat{J},\mu,\epsilon)| \\
&\leq \frac{c\delta}{\mu} + \frac{c\epsilon}{\mu}
\end{aligned}$$

for $\hat{J} \in \overline{B}_\delta(\hat{J}_{\hat{I},\mu,0})$ follow from the fact that ω is a twice-continuously differentiable function of its arguments. Replacing μ_0 with a smaller constant if necessary, one concludes that $B(\cdot,\epsilon)$ is a contraction on $\overline{B}_\delta(\hat{J}_{\hat{I},\mu,0})$ for $\delta = c\epsilon/\mu$ whenever $\epsilon \leq \mu^{2+\tau}$ for some $\tau > 0$; in keeping with the relationship $\epsilon = \sqrt{\mu^{N-1}}$, let us choose $\tau = 1/2$ and $N \geq 6$. For these values of δ and ϵ equation (89) has a unique solution $\hat{J}_{\hat{I},\mu,\epsilon} \in \overline{B}_\delta(\hat{J}_{\hat{I},\mu,0})$, so that

$$|\hat{J}_{\hat{I},\mu,\epsilon} - \hat{J}_{\hat{I},\mu,0}| \leq \frac{c\epsilon}{\mu}.$$

It follows from the C^2-dependence of $u(\hat{J},\mu,\epsilon)(x)$ and $\dot{u}(\hat{J},\mu,\epsilon)(x)$ on the parameters that

$$|u(\hat{J}_{\hat{I},\mu,\epsilon},\mu,\epsilon)(x) - u(\hat{J}_{\hat{I},\mu,0},\mu,0)(x)|, |\dot{u}(\hat{J}_{\hat{I},\mu,\epsilon},\mu,\epsilon)(x) - \dot{u}(\hat{J}_{\hat{I},\mu,0},\mu,0)(x)| \leq \frac{c\epsilon}{\mu},$$

and these estimates hold uniformly in every compact subset of $\mathbb{R}$. Because $u(\hat{J}_{\hat{I},\mu,\epsilon},\mu,\epsilon)$ and $u(\hat{J}_{\hat{I},\mu,0},\mu,0)$ are periodic and have the same period, the estimates hold uniformly in $\mathbb{R}$. □

It remains to introduce suitable notation for the periodic orbits to (87) found by the above procedure. Define

$$\hat{X}_{\hat{I},\epsilon}(x) = (0, 0, u(\hat{J}, \mu, \epsilon), 0, 0, \overline{u(\hat{J}, \mu, \epsilon)}),$$

so that $\hat{X}_{\hat{I},\epsilon}$ is a periodic solution of (87) with frequency Ω. Clearly $\hat{X}_{\hat{I},0}$ is given by the explicit formula

$$\hat{X}_{\hat{I},0}(x) = R_{0,\Omega x}\hat{Y}_{\hat{I}}, \qquad \hat{Y}_{\hat{I}} = (0, 0, \sqrt{\hat{I}}, 0, 0, \sqrt{\hat{I}})^T$$

and one has that

$$\|\hat{X}_{\hat{I},\epsilon} - R_{0,\Omega x}\hat{Y}_{\hat{I}}\|_{C^1_{\mathrm{B}}(\mathbb{R})} \leq \frac{c\epsilon}{\mu}. \tag{90}$$

This notation is used in Section 4.3 below, where the estimate (90) plays a pivotal role.

4.3 Persistence of homoclinic connections to periodic orbits

This section is dedicated to the proof of the following persistence result.

Theorem 14 *Take $\hat{I} \in (0, \hat{I}_0]$. There exists a real number $\mu_{\hat{I}}$ such that equation (87) has four geometrically distinct reversible homoclinic connections to the periodic orbit $\hat{X}_{\hat{I},\epsilon}$ for each $\mu \in (0, \mu_{\hat{I}}]$ and each $\epsilon \in [0, \mu^{9/2}]$.*

The method employed to establish this result is motivated by the strategy devised by Iooss & Pérouème (1993, §IV) in a persistence proof for a simpler normal form. The first step is to fix $\hat{I} \in (0, \hat{I}_0]$, take $\mu \in (0, \mu_0]$, $\epsilon \in [0, \mu^{5/2}]$ and look for a solution to (87) of the form

$$\hat{\mathcal{S}}(x; \phi, \theta, \hat{I}, \mu, \epsilon) = R_{\Omega_0 x+\phi, \Omega x+\theta}(\hat{\mathcal{H}}(x) - R_{0,\delta}\hat{Y}_{\hat{I}} + \hat{Z}(x)) + \hat{X}_{\hat{I},\epsilon}(x + (\delta+\theta)/\Omega), \tag{91}$$

for $x \in [0, \infty)$, in which $\mathcal{H}(x) = \sqrt{\mu}\hat{\mathcal{H}}(x)$, $\Omega_0 = q + \partial_2 H_{\mathrm{NF}}(0, 0, \mu\hat{I}, \mu)$ and $\hat{Z}$ is a function which tends exponentially to 0 as $x \to \infty$, so that $\hat{\mathcal{S}}(\cdot; \phi, \theta, \hat{I}, \mu, \epsilon)$ is a solution to (87) which approaches the periodic orbit $\hat{X}_{\hat{I},\epsilon}$ as $x \to \infty$. (Notice that $\hat{Z} = 0$ is the appropriate choice for $\epsilon = 0$.) Since $\hat{X}_{\hat{I},\epsilon}(x + (\delta+\theta)/\Omega)$ solves equation (87) and $R_{\Omega_0 x+\phi, \Omega x+\theta}\hat{\mathcal{H}}(x)$, $R_{0,\Omega x+\theta}\hat{Y}_{\hat{I}}$ solve equation (87) with $\epsilon = 0$, it follows that

$$\frac{d\hat{Z}}{dx} + L_{\Omega_0,\Omega}\hat{Z} - \mathrm{d}_1\hat{F}[\hat{\mathcal{H}}, \mu](\hat{Z}) = \mathcal{N}_1(\hat{Z}, x, \phi, \theta, \mu) + \epsilon\mathcal{N}_2(\hat{Z}, x, \phi, \theta, \mu), \tag{92}$$

where $L_{\alpha,\beta} : (A, B, C) \mapsto (\mathrm{i}\alpha A, \mathrm{i}\alpha B, \mathrm{i}\beta C)$,

$$\begin{aligned} \mathcal{N}_1(\hat{Z}, x, \phi, \theta, \mu) \;=\; & \hat{F}(\hat{\mathcal{H}} - R_{0,\delta}\hat{Y}_{\hat{I}} + \hat{Z} + R_{-(\Omega_0 x+\phi), -(\Omega x+\theta)}\hat{X}_{\hat{I},\epsilon}(x + (\delta+\theta)/\Omega), \mu) \\ & - \hat{F}(\hat{\mathcal{H}}, \mu) + \hat{F}(R_{0,\delta}\hat{Y}_{\hat{I}}, \mu) - \mathrm{d}_1\hat{F}[\hat{\mathcal{H}}, \mu](\hat{Z}) \\ & - \hat{F}(R_{-(\Omega_0 x+\phi), -(\Omega x+\theta)}\hat{X}_{\hat{I},\epsilon}(x + (\delta+\theta)/\Omega), \mu) \end{aligned}$$

and

$$\begin{aligned} & \mathcal{N}_2(\hat{Z}, x, \phi, \theta, \mu) = \\ & \quad R_{-(\Omega_0 x+\phi), -(\Omega x+\theta)}\hat{\mathcal{R}}(R_{\Omega_0 x+\phi, \Omega x+\theta}(\hat{\mathcal{H}} - R_{0,\delta}\hat{Y}_{\hat{I}} + \hat{Z}) + \hat{X}_{\hat{I},\epsilon}(x + (\delta+\theta)/\Omega), \mu) \\ & \quad - R_{-(\Omega_0 x+\phi), -(\Omega x+\theta)}\hat{\mathcal{R}}(\hat{X}_{\hat{I},\epsilon}(x + (\delta+\theta)/\Omega), \mu). \end{aligned}$$

Choose ν such that ν/λ is a fixed real number between $1/2$ and 1, and let us use the symbol M to denote a constant which depends only upon $\hat{I}$. The following result is a straightforward consequence of formulae (83)–(85) in Section 4.1.

Proposition 15 *The function $\hat{\mathcal{H}}$ satisfies the estimate*

$$|\hat{\mathcal{H}}(x) - R_{0,\delta}\hat{Y}_{\hat{I}}| \leq M\mathrm{e}^{-\nu x}, \qquad x \in [0,\infty). \tag{93}$$

Using the estimate (93) and the fact that

$$|\hat{X}_{\hat{I},\epsilon}(x + (\delta+\theta)/\Omega) - R_{\Omega_0 x+\phi,\Omega x+\theta+\delta}\hat{Y}_{\hat{I}}| \leq \frac{M\epsilon}{\mu}, \qquad x \in [0,\infty), \tag{94}$$

one finds that

$$\begin{aligned} \mathcal{N}_1(\hat{Z},x,\phi,\theta,\mu) &\leq M\sqrt{\mu}\left(\frac{\epsilon}{\mu}(\mathrm{e}^{-\nu x} + |\hat{Z}|) + |\hat{Z}|^2\right), & (95)\\ \mathcal{N}_2(\hat{Z},x,\phi,\theta,\mu) &\leq M\sqrt{\mu}(\mathrm{e}^{-\nu x} + |\hat{Z}|) & (96) \end{aligned}$$

and

$$\begin{aligned} |\mathcal{N}_1(\hat{Z}_1,x,\phi,\theta,\mu) - \mathcal{N}_1(\hat{Z}_2,x,\phi,\theta,\mu)| &\leq M\sqrt{\mu}\left(\frac{\epsilon}{\mu} + |\hat{Z}_1| + |\hat{Z}_2|\right)|\hat{Z}_1 - \hat{Z}_2|,\\ |\mathcal{N}_2(\hat{Z}_1,x,\phi,\theta,\mu) - \mathcal{N}_2(\hat{Z}_2,x,\phi,\theta,\mu)| &\leq M\sqrt{\mu}|\hat{Z}_1 - \hat{Z}_2| \end{aligned}$$

for each $x \in [0,\infty)$.

To construct appropriate solutions of equation (92) it is necessary to obtain some information about the associated linear equation

$$\frac{d\hat{Z}}{dx} = \mathcal{L}(x)(\hat{Z}),$$

where $\mathcal{L}(x) = \mathrm{d}_1\hat{F}[\hat{\mathcal{H}}(x),\mu] - L_{\Omega_0,\Omega}$. This equation may be written as

$$\frac{d\hat{Z}}{dx} = \mathcal{L}_\infty\hat{Z} + (\mathcal{L}(x) - \mathcal{L}_\infty)\hat{Z}, \tag{97}$$

in which $\mathcal{L}_\infty = \lim_{x\to\infty}\mathcal{L}(x) = \mathrm{d}_1\hat{F}[R_{0,\delta}\hat{Y}_{\hat{I}}] - L_{\Omega_0,\Omega}$. The spectrum of $\mathcal{L}_\infty$ can be calculated directly. It consists of two geometrically double eigenvalues $\pm\lambda = \pm\sqrt{-\partial_1 H_{\mathrm{NF}}(0,0,\mu\hat{I},\mu)}$ and a zero eigenvalue with a Jordan chain of length 2. Using real coordinates $(\hat{q}_1,\hat{q}_2,\hat{q}_3,\hat{p}_1,\hat{p}_2,\hat{p}_3)^T$ defined by

$$\hat{A} = \frac{1}{\sqrt{2}}(\hat{q}_1 + \mathrm{i}\hat{q}_2), \qquad \hat{B} = \frac{1}{\sqrt{2}}(\hat{p}_1 + \mathrm{i}\hat{p}_2), \qquad C = \frac{1}{\sqrt{2}}(\hat{p}_3 + \mathrm{i}\hat{q}_3),$$

note that $s_1^\infty = (1,0,0,0,\lambda,0)$, $s_2^\infty = (0,1,0,\lambda,0,0)$ and $u_1^\infty = (1,0,0,0,-\lambda,0)$, $u_2^\infty = (0,1,0,-\lambda,0,0)$ are linearly independent eigenvectors corresponding to respectively the negative and positive eigenvalue. Differentiating the identity

$$F(R_{0,\delta}\hat{Y}_{\hat{I}},\mu) = L_{0,\Omega}R_{0,\delta}\hat{Y}_{\hat{I}\star}$$

with respect to δ and $\hat{I}$, one obtains the relations

$$\mathcal{L}_\infty R_{0,\delta} L_{0,1} \hat{Y}_{\hat{I}} = 0, \qquad \mathcal{L}_\infty R_{0,\delta} \frac{d\hat{Y}_{\hat{I}}}{d\hat{I}} = R_{0,\delta} L_{0,\Omega'} \hat{Y}_{\hat{I}},$$

where the prime denotes differentiation with respect to $\hat{I}$, which show that $c_1^\infty = R_{0,\delta} L_{0,1} \hat{Y}_{\hat{I}}$ is an eigenvector of $\mathcal{L}_\infty$ corresponding to the zero eigenvalue and $c_2^\infty = R_{0,\delta} d\hat{Y}_{\hat{I}}/d\hat{I}$ is an associated generalised eigenvector.

Since $\mathcal{L}(x)$ tends exponentially to $\mathcal{L}_\infty$ as $x \to \infty$ uniformly in μ, one may apply the theory given by Coddington & Levinson (1955, ch. 3, problems 29 and 35) to find that equation (97) has solutions $s_1(x)$, $s_2(x)$, $u_1(x)$, $u_2(x)$ such that

$$s_j(x)\mathrm{e}^{\lambda x} \to s_j^\infty \text{ as } x \to \infty, \quad |s_j(x)| \le M\mathrm{e}^{-\lambda x} \text{ for } x \in [0,\infty), \qquad j = 1,2, \tag{98}$$

$$u_j(x)\mathrm{e}^{-\lambda x} \to u_j^\infty \text{ as } x \to \infty, \quad |u_j(x)| \le M\mathrm{e}^{\lambda x} \text{ for } x \in [0,\infty), \qquad j = 1,2. \tag{99}$$

Differentiating the identity

$$R_{\phi,\phi} \frac{d\hat{\mathcal{H}}}{dx}(x) + L_{\Omega_0,\Omega} R_{\phi,\phi} \hat{\mathcal{H}}(x) = \hat{F}(R_{\phi,\phi}\hat{\mathcal{H}}(x), \mu) \tag{100}$$

with respect to ϕ and $\hat{I}$ and putting $\phi = 0$, one finds that $c_1(x)$ and $c_2(x) + xM_{\Omega_0',\Omega'} c_1(x)$ are solutions of (97), where $c_1(x) = L_{1,1}\hat{\mathcal{H}}(x)$, $c_2(x) = d\hat{\mathcal{H}}(x)/d\hat{I}$, $M_{\alpha,\beta}(A,B,C) \mapsto (\alpha A, \alpha B, \beta C)$. Observe that $c_1(x) \to c_1^\infty$, $c_2(x) \to c_2^\infty$ as $x \to \infty$ and the estimate (86) in Section 4.1 shows that

$$|c_1(x)|, |c_2(x)| \le M \tag{101}$$

for each $x \in [0,\infty)$. The matrix $(s_1(x)|s_2(x)|u_1(x)|u_2(x)|c_1(x)|c_2(x) + xM_{\Omega_0',\Omega'}c_1(x))$ is a fundamental matrix for (97) on $[0,\infty)$ and since this equation is reversible in the same sense as (76) its Wronskian is independent of x. It follows that

$$\begin{aligned}
\det(s_1(x)|s_2(x)|u_1(x)|c_1(x)|c_2(x) + xM_{\Omega_0',\Omega'}c_1(x)) \\
&= \det(s_1(x)|s_2(x)|u_1(x)|u_2(x)|c_1(x)|c_2(x)) \\
&= \det(s_1(x)\mathrm{e}^{\lambda x}|s_2(x)\mathrm{e}^{\lambda x}|u_1(x)\mathrm{e}^{-\lambda x}|u_2(x)\mathrm{e}^{-\lambda x}|c_1(x)|c_2(x)) \\
&= \det(s_1^\infty|s_2^\infty|u_1^\infty|u_2^\infty|c_1^\infty|c_2^\infty) \\
&= M\lambda^2 \\
&= M\mu.
\end{aligned}$$

Let $\{s_1^*(x), s_2^*(x), u_1^*(x), u_2^*(x), c_1^*(x), c_2^*(x)\}$ denote the dual basis to $\{s_1(x), s_2(x), u_1(x), u_2(x), c_1(x), c_2(x)\}$, so that $\{s_1^*(x), s_2^*(x), u_1^*(x), u_2^*(x), c_2^*(x), c_1^*(x) - xM_{\Omega_0',\Omega'}c_2^*(x)\}$ is a fundamental solution set for the adjoint equation to (97). The dual basis to

$$\mathcal{B} = \{s_1(x)\mathrm{e}^{\lambda x}, s_2(x)\mathrm{e}^{\lambda x}, u_1(x)\mathrm{e}^{-\lambda x}, u_2(x)\mathrm{e}^{-\lambda x}, c_1(x), c_2(x)\}$$

is clearly given by

$$\mathcal{B}^* = \{s_1^*(x)\mathrm{e}^{-\lambda x}, s_2^*(x)\mathrm{e}^{-\lambda x}, u_1^*(x)\mathrm{e}^{\lambda x}, u_2^*(x)\mathrm{e}^{\lambda x}, c_1^*(x), c_2^*(x)\},$$

and the elements of $\mathcal{B}^*$ can be computed from those of $\mathcal{B}$ using Cramer's rule; each entry in each vector is given by a fraction whose denominator is the above determinant and whose numerator can be estimated using (98), (99), (101). It follows that

$$|u_1^*(x)|, |u_2^*(x)| \leq \frac{M}{\mu}\mathrm{e}^{-\lambda x}, \qquad |s_1^*(x)|, |s_2^*(x)| \leq \frac{M}{\mu}\mathrm{e}^{\lambda x}, \qquad |c_1^*(x)|, |c_2^*(x)| \leq \frac{M}{\mu} \tag{102}$$

for each $x \in [0, \infty)$.

A straightforward calculation shows that any solution of the functional equation

$$\begin{aligned} \hat{Z} \;=\; & \int_0^x \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon\mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), s_1^*(s)\rangle \,\mathrm{d}s\; s_1(x) \\ & + \int_0^x \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon\mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), s_2^*(s)\rangle \,\mathrm{d}s\; s_2(x) \\ & - \int_x^\infty \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon\mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), u_1^*(s)\rangle \,\mathrm{d}s\; u_1(x) \\ & - \int_x^\infty \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon\mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), u_2^*(s)\rangle \,\mathrm{d}s\; u_2(x) \\ & - \int_x^\infty \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon\mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), c_1^*(s) - (s-x)M_{\Omega_0', \Omega'}c_2^*(s)\rangle \,\mathrm{d}s\; c_1(x) \\ & - \int_x^\infty \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon\mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), c_2^*(s)\rangle \,\mathrm{d}s\; c_2(x) \end{aligned} \tag{103}$$

solves equation (92); here $\langle\cdot\,,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^6$. Let us write the above equation as

$$\hat{Z} = \mathcal{F}(\hat{Z}, \epsilon)$$

and consider this fixed-point problem in the Banach space

$$E_\nu = \{\hat{Z} \in C([0,\infty) \times S^1 \times S^1) : \sup\{|\hat{Z}(x,\phi,\theta)|\mathrm{e}^{\nu x} : x \in [0,\infty), \theta, \phi \in S^1\} < \infty\}$$

equipped with the norm

$$\|Z\|_\nu = \sup\{|\hat{Z}(x,\phi,\theta)|\mathrm{e}^{\nu x} : x \in [0,\infty), \theta, \phi \in S^1\}.$$

Proposition 16 *The function $\mathcal{F}$ satisfies the estimates*

$$\begin{aligned} \|\mathcal{F}(\hat{Z}, \epsilon)\|_\nu \;&\leq\; \frac{M}{\mu}\left[\|\hat{Z}\|_\nu^2 + \frac{\epsilon}{\mu}(1 + \|\hat{Z}\|_\nu)\right], \\ \|\mathcal{F}(\hat{Z}_1, \epsilon) - \mathcal{F}(\hat{Z}_2, \epsilon)\|_\nu \;&\leq\; \frac{M}{\mu}\left[\frac{\epsilon}{\mu} + \|\hat{Z}_1\|_\nu + \|\hat{Z}_2\|_\nu\right]\|\hat{Z}_1 - \hat{Z}_2\|_\nu \end{aligned}$$

for each $\hat{Z}$, $\hat{Z}_1$, $\hat{Z}_2 \in E_\nu$.

Proof Using the estimates (95), (96) and (98), (99), (101), one finds that

$$\begin{aligned}
&\left|\int_0^x \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon \mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), s_1^*(s)\rangle \,\mathrm{d}s \; s_1(x)\right| \mathrm{e}^{\nu x} \\
&\qquad \leq \int_0^x \frac{M}{\mu}\sqrt{\mu}\left[\frac{\epsilon}{\mu}(1+\|\hat{Z}\|_\nu)\mathrm{e}^{-\nu s} + \|\hat{Z}\|_\nu^2 \mathrm{e}^{-2\nu s}\right] \mathrm{e}^{\lambda s}\,\mathrm{d}s\; \mathrm{e}^{-(\lambda-\nu)x} \\
&\qquad = \frac{M}{\mu}\sqrt{\mu}\left[\frac{\epsilon}{\mu}(1+\|\hat{Z}\|_\nu)\left(\frac{1-\mathrm{e}^{-(\lambda-\nu)x}}{\lambda-\nu}\right) + \|\hat{Z}\|_\nu^2\left(\frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-(\lambda-\nu)x}}{\lambda-2\nu}\right)\right] \\
&\qquad \leq \frac{M\sqrt{\mu}}{\mu\lambda}\left[\frac{\epsilon(1+\|\hat{Z}\|_\nu)}{\mu(1-\nu/\lambda)}) + \frac{\|\hat{Z}\|_\nu^2}{1-2\nu/\lambda}\right] \\
&\qquad \leq \frac{M}{\mu}\left[\|\hat{Z}\|_\nu^2 + \frac{\epsilon}{\mu}(1+\|Z\|_\nu)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_x^\infty \langle \mathcal{N}_1(\hat{Z}, s, \phi, \theta, \mu) + \epsilon \mathcal{N}_2(\hat{Z}, s, \phi, \theta, \mu), (s-x)M_{\Omega_0',\Omega'} c_2^*(s)\rangle \,\mathrm{d}s \; c_1(s)\right| \mathrm{e}^{\nu x} \\
&\qquad \leq \int_x^\infty \frac{M}{\mu}\sqrt{\mu}\left[\frac{\epsilon}{\mu}(1+\|\hat{Z}\|_\nu)\mathrm{e}^{-\nu s} + \|\hat{Z}\|_\nu^2 \mathrm{e}^{-2\nu s}\right](s-t)\mu\,\mathrm{d}s\;\mathrm{e}^{\nu x} \\
&\qquad = \frac{M}{\mu}\mu^{3/2}\left[\frac{1}{\nu^2}\left(1+\frac{\epsilon}{\mu}\right)\|\hat{Z}\|_\nu + \frac{1}{2\nu}\|\hat{Z}\|_\nu^2 \mathrm{e}^{-\nu x}\right] \\
&\qquad \leq \frac{M}{\mu}\left[\left(1+\frac{\epsilon}{\mu}\right)\|\hat{Z}\|_\nu + \|\hat{Z}\|_\nu^2\right],
\end{aligned}$$

in which the estimates Ω_0', $\Omega' \leq M\mu$ and $\sqrt{\mu}/\lambda \leq M$ (which follows because the coefficient c_1 in the expansion of H^μ_{NF} is nonzero) and the identity

$$\int_x^\infty (s-x)\mathrm{e}^{-\nu(s-x)}\,\mathrm{d}s = \frac{1}{\nu^2}$$

have also been used. The remaining integrals on the right-hand side of (103) are estimated in a similar fashion, and this procedure yields the stated estimate for $\|\mathcal{F}(\hat{Z})\|_\nu$. The second estimate is obtained by the same method. □

According to standard arguments in nonlinear analysis, $\mathcal{F}(\cdot,\epsilon)$ is a contraction on $\overline{B}_\delta(0) \subset E_\nu$ provided that $\|\mathcal{F}(0,\epsilon)\|_\nu \leq \delta/2$ and that $\mathcal{F}(\cdot,\epsilon)$ is Lipschitz continuous on $\overline{B}_\delta(0)$ with Lipschitz constant at most $1/2$. Replacing μ_0 with a smaller constant (which depends upon $\hat{I}$) if necessary, one finds from Proposition 16 that $\mathcal{F}(\cdot,\epsilon)$ is a contraction on $\overline{B}_{\delta_1}(0) \subset E_\nu$, $\delta_1 = M\epsilon/\mu^2$ whenever $\epsilon \leq \mu^{7/2}$. This result shows that equation (92) has a unique solution $\hat{Z}(x;\phi,\theta,\hat{I},\mu,\epsilon)$ for each $\phi,\theta \in S^1$ which satisfies the estimate

$$|\hat{Z}(x;\phi,\theta,\hat{I},\mu,\epsilon)| \leq \frac{M\epsilon}{\mu^2}, \qquad x \in [0,\infty).$$

The next step is to show that Z_ϕ, Z_θ exist and also satisfy estimates of this type.

Let us formally differentiate the functional equation (103) with respect to θ and replace $\hat{Z}_\theta$ by $\hat{Y}$. The resulting equation, which may be written schematically as

$$\hat{Y} = \mathcal{G}(\hat{Y}, \hat{Z}, \epsilon),$$

is given by (103) with $\mathcal{N}_1(\hat{Z}, x, \phi, \theta, \mu)$ and $\mathcal{N}_2(\hat{Z}, x, \phi, \theta, \mu)$ replaced by respectively

$$\begin{aligned}
&\mathcal{N}_3(\hat{Z}, \hat{Y}, s, \phi, \theta, \mu) = \\
&\quad \mathrm{d}_1 \hat{F}[\hat{\mathcal{H}} - R_{0,\delta}\hat{Y}_{\hat{I}} + \hat{Z} + R_{-(\Omega_0 x+\phi),-(\Omega x+\theta)}\hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega), \mu] \\
&\qquad \left(\hat{Y} + L_{-1,0}R_{-(\Omega_0 x+\phi),-(\Omega x+\theta)}\hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega) + \frac{1}{\Omega}R_{\Omega_0 x+\phi,\Omega x+\theta}\frac{d\hat{X}_{\hat{I},\epsilon}}{dx}(x+(\delta+\theta)/\Omega)\right) \\
&\quad - \mathrm{d}_1 \hat{F}[R_{-(\Omega_0 x+\phi),-(\Omega x+\theta)}\hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega), \mu] \\
&\qquad \left(L_{-1,0}R_{-(\Omega_0 x+\phi),-(\Omega x+\theta)}\hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega) + \frac{1}{\Omega}R_{\Omega_0 x+\phi,\Omega x+\theta}\frac{d\hat{X}_{\hat{I},\epsilon}}{dx}(x+(\delta+\theta)/\Omega)\right) \\
&\quad - \mathrm{d}_1 \hat{F}[\mathcal{H}, \mu](\hat{Y})
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{N}_4(\hat{Z}, \hat{Y}, s, \phi, \theta, \mu) = \\
&\quad L_{0,-1}R_{-(\Omega_0 x+\phi),-(qx+\theta)}\hat{\mathcal{R}}(\mu, R_{\Omega_0 x+\phi,qx+\theta}(\hat{\mathcal{H}} - R_{0,\delta}\hat{Y}_{\hat{I}} + \hat{Z}) + \hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega), \mu) \\
&\quad + R_{-(\Omega_0 x+\phi),-(qx+\theta)}\mathrm{d}_1\hat{\mathcal{R}}[R_{\Omega_0 x+\phi,\Omega x+\theta}(\hat{\mathcal{H}} - R_{0,\delta}\hat{Y}_{\hat{I}} + \hat{Z}) + \hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega), \mu] \\
&\qquad \left(L_{0,1}R_{\Omega_0 x+\phi,\Omega x+\theta}(\hat{\mathcal{H}} - R_{0,\delta}\hat{Y}_{\hat{I}} + \hat{Z}) + R_{\Omega_0 x+\phi,\Omega x+\theta}\hat{Y} + \frac{1}{\Omega}\frac{d\hat{X}_{\hat{I},\epsilon}}{dx}(x+(\delta+\theta)/\Omega)\right) \\
&\quad - L_{0,-1}R_{-(\Omega_0 x+\phi),-(\Omega x+\theta)}\hat{\mathcal{R}}(\hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega), \mu) \\
&\quad - R_{-(\Omega_0 x+\phi),-(\Omega x+\theta)}\mathrm{d}_1\hat{\mathcal{R}}[\hat{X}_{\hat{I},\epsilon}(x+(\delta+\theta)/\Omega), \mu]\left(\frac{1}{\Omega}\frac{d\hat{X}_{\hat{I},\epsilon}}{dx}(x+(\delta+\theta)/\Omega)\right).
\end{aligned}$$

Using the estimates (93), (94) and the fact that

$$\left|\frac{1}{\Omega}\frac{d\hat{X}_{\hat{I},\epsilon}}{dx}(x+(\delta+\theta)/\Omega) - L_{0,1}R_{\Omega_0 x+\phi,\Omega x+\theta+\delta}\hat{Y}_{\hat{I},\epsilon}\right| \le \frac{M\epsilon}{\mu}, \qquad x \in [0,\infty),$$

one finds that

$$\begin{aligned}
\mathcal{N}_3(\hat{Z}, \hat{Y}, x, \phi, \theta, \mu) &\le \frac{M}{\mu}\sqrt{\mu}\left(\frac{\epsilon}{\mu}|\hat{Z}| + \frac{\epsilon}{\mu}|\hat{Y}| + \frac{\epsilon}{\mu}\mathrm{e}^{-\nu x} + |\hat{Z}||\hat{Y}|\right), \\
\mathcal{N}_4(\hat{Z}, \hat{Y}, x, \phi, \theta, \mu) &\le \frac{M}{\mu}\sqrt{\mu}(e^{-\nu x} + |\hat{Z}| + |\hat{Y}| + |\hat{Z}||\hat{Y}|)
\end{aligned}$$

and

$$|\mathcal{N}_3(\hat{Z}, \hat{Y}_1, x, \phi, \theta, \mu) - \mathcal{N}_3(\hat{Z}, \hat{Y}_2, x, \phi, \theta, \mu)| \le \frac{M}{\mu}\sqrt{\mu}\left(\frac{\epsilon}{\mu}|\hat{Y}_1 - \hat{Y}_2| + |\hat{Z}|\right)|\hat{Y}_1 - \hat{Y}_2|,$$

$$
\begin{aligned}
|\mathcal{N}_3(\hat{Z}_1,\hat{Y},x,\phi,\theta,\mu)-\mathcal{N}_3(\hat{Z}_2,\hat{Y},x,\phi,\theta,\mu)| &\leq \frac{M}{\mu}\sqrt{\mu}\left(\frac{\epsilon}{\mu}|\hat{Z}_1-\hat{Z}_2|+|\hat{Y}|\right)|\hat{Z}_1-\hat{Z}_2|,\\
|\mathcal{N}_4(\hat{Z},\hat{Y}_1,x,\phi,\theta,\mu)-\mathcal{N}_4(\hat{Z},\hat{Y}_2,x,\phi,\theta,\mu)| &\leq \frac{M}{\mu}\sqrt{\mu}(1+|\hat{Z}|+|\hat{Y}_1|+|\hat{Y}_2|)|\hat{Y}_1-\hat{Y}_2|,\\
|\mathcal{N}_4(\hat{Z}_1,\hat{Y},x,\phi,\theta,\mu)-\mathcal{N}_4(\hat{Z}_2,\hat{Y},x,\phi,\theta,\mu)| &\leq \frac{M}{\mu}\sqrt{\mu}(1+|\hat{Y}|+|\hat{Z}_1|+|\hat{Z}_2|)|\hat{Z}_1-\hat{Z}_2|
\end{aligned}
$$

for each $x\in[0,\infty)$. The following result is proved in the same way as Proposition 16.

Proposition 17 *The function $\mathcal{G}$ satisfies the estimates*

$$
\begin{aligned}
\|\mathcal{G}(\hat{Z},\hat{Y},\epsilon)\|_\nu &\leq \frac{M}{\mu}\left[\|\hat{Z}\|_\nu\|\hat{Y}\|_\nu+\frac{\epsilon}{\mu}(1+\|\hat{Z}\|_\nu+\|\hat{Y}\|_\nu)\right],\\
\|\mathcal{G}(\hat{Z},\hat{Y}_1,\epsilon)-\mathcal{G}(\hat{Z},\hat{Y}_2,\epsilon)\|_\nu &\leq \frac{M}{\mu}\left[\frac{\epsilon}{\mu}+\|\hat{Z}\|_\nu\right]\|\hat{Y}_1-\hat{Y}_2\|_\nu,\\
\|\mathcal{G}(\hat{Z}_1,\hat{Y},\epsilon)-\mathcal{G}(\hat{Z}_2,\hat{Y},\epsilon)\|_\nu &\leq \frac{M}{\mu}\left[\frac{\epsilon}{\mu}+\|\hat{Z}_1\|_\nu+\|\hat{Z}_2\|_\nu+\|\hat{Y}\|_\nu\right]\|\hat{Z}_1-\hat{Z}_2\|_\nu
\end{aligned}
$$

for each $\hat{Z}$, $\hat{Y}$, $\hat{Z}_1$, $\hat{Z}_2$, $\hat{Y}_1$, $\hat{Z}_2\in E_\nu$.

The argument used above to show that $\mathcal{F}(\cdot,\epsilon)$ is a contraction on $\overline{B}_{\delta_1}(0)$ is also applicable here; it shows that for each $\hat{Z}\in\overline{B}_{\delta_1}(0)$ the function $\mathcal{G}(\hat{Z},\cdot,\epsilon):\overline{B}_{\delta_2}(0)\to\overline{B}_{\delta_2}(0)$, $\delta_2=M\epsilon/\mu^2$ is a contraction whose Lipschitz constant is independent of $\hat{Z}$ whenever $\epsilon\leq\mu^{7/2}$. Let us denote its fixed point by $\hat{Y}(\cdot;\cdot,\cdot,\hat{I},\mu,\epsilon)$. Note that $\mathcal{G}(\cdot,\hat{Y},\epsilon):\overline{B}_{\delta_1}(0)\to\overline{B}_{\delta_2}(0)$ is continuous for each $\hat{Y}\in\overline{B}_{\delta_2}(0)$, so that the map $\mathcal{H}(\cdot,\cdot,\epsilon):\overline{B}_{\delta_1}(0)\times\overline{B}_{\delta_2}(0)\to\overline{B}_{\delta_1}(0)\times\overline{B}_{\delta_2}(0)$ defined by $\mathcal{H}(\hat{Z},\hat{Y},\epsilon)=(\mathcal{F}(\hat{Z},\epsilon),\mathcal{G}(\hat{Z},\hat{Y},\epsilon))$ has the fixed-point property. Take $\hat{Z}\in\overline{B}_{\delta_1}(0)$ and suppose that $\hat{Z}_\theta$ exists and belongs to $\overline{B}_{\delta_2}(0)$ ($\hat{Z}=0$ is a trivial example); by construction $\mathcal{F}(\hat{Z},\epsilon)_\phi$ exists and equals $\mathcal{G}(\hat{Z},\hat{Z}_\phi,\epsilon)$. Because $\mathcal{H}(\cdot,\cdot,\epsilon)$ has the fixed-point property the sequence $\{(\hat{Z}_n,\hat{Y}_n)\}\subset\overline{B}_{\delta_1(0)}\times\overline{B}_{\delta_2(0)}$ given by $(\hat{Z}_0,\hat{Y}_0)=(0,0)$, $(\hat{Z}_{n+1},\hat{Y}_{n+1})=(\mathcal{F}(\hat{Z}_n,\epsilon),\mathcal{G}(\hat{Z}_n,\hat{Y}_n,\epsilon))$, $n=0,1,2,\ldots$ converges to the unique fixed point $(\mathcal{Z}(\cdot;\cdot,\cdot,\hat{I},\mu,\epsilon)$, $\mathcal{Y}(\cdot;\cdot,\cdot,\hat{I},\mu,\epsilon))$ of $\mathcal{H}$ in $\overline{B}_{\delta_1}(0)\times\overline{B}_{\delta_2(0)}$ and the uniformity of the convergence in $(t,\phi,\theta)\in[0,\infty)\times S^1\times S^1$ implies that $\hat{Y}(\cdot;\cdot,\cdot,\hat{I},\mu,\epsilon)=\hat{Z}_\theta(\cdot;\cdot,\cdot,\hat{I},\mu,\epsilon)$. In particular, one has that

$$
|\hat{Z}_\theta(x;\phi,\theta,\hat{I},\mu,\epsilon)\leq\frac{M\epsilon}{\mu^2},\qquad x\in[0,\infty).
$$

The same method shows that $\hat{Z}_\phi(x;\phi,\theta,\hat{I},\mu,\epsilon)$ exists and satisfies

$$
|\hat{Z}_\phi(x;\phi,\theta,\hat{I},\mu,\epsilon)|\leq\frac{M\epsilon}{\mu^2},\qquad x\in[0,\infty).
$$

Clearly any solution of (92) which lies in E_ν satisfies

$$
\begin{aligned}
|\hat{Z}_x| &\leq |L_{\Omega_0,\Omega}||\hat{Z}|+|\mathrm{d}_1\hat{F}[\hat{\mathcal{H}},\mu]||\hat{Z}|+|\mathcal{N}_1(\hat{Z},x,\phi,\theta,\mu)|+\epsilon|\mathcal{N}_2(\hat{Z},x,\phi,\theta,\mu)|\\
&\leq M|\hat{Z}|+\frac{M}{\mu}\sqrt{\mu}\left(\frac{\epsilon}{\mu}(1+|\hat{Z}|)+|\hat{Z}|^2\right),
\end{aligned}
$$

so that

$$|\hat{Z}_x(x;\phi,\theta,\hat{I},\mu,\epsilon)| \leq \frac{M\epsilon}{\mu^2}, \qquad x\in[0,\infty).$$

One concludes that equation (91) defines a solution of (92) which satisfies the estimates

$$|\hat{\mathcal{S}}(x;\phi,\theta,\hat{I},\mu,\epsilon)-\hat{\mathcal{S}}(x;\phi,\theta,\hat{I},\mu,0)| \leq \frac{M\epsilon}{\mu^2}, \tag{104}$$

$$|\hat{\mathcal{S}}_\phi(x;\phi,\theta,\hat{I},\mu,\epsilon)-\hat{\mathcal{S}}_\phi(x;\phi,\theta,\hat{I},\mu,0)| \leq \frac{M\epsilon}{\mu^2}, \tag{105}$$

$$|\hat{\mathcal{S}}_\theta(x;\phi,\theta,\hat{I},\mu,\epsilon)-\hat{\mathcal{S}}_\theta(x;\phi,\theta,\hat{I},\mu,0)| \leq \frac{M\epsilon}{\mu^2}, \tag{106}$$

$$|\hat{\mathcal{S}}_x(x;\phi,\theta,\hat{I},\mu,\epsilon)-\hat{\mathcal{S}}_x(x;\phi,\theta,\hat{I},\mu,0)| \leq \frac{M\epsilon}{\mu^2}. \tag{107}$$

Define a function $F:\mathbb{R}\times S^1\times S^1\times[0,\mu^{7/2}]\to \mathrm{Im}(I-S)$ by the formula

$$F(\gamma,\epsilon)=(I-S)\hat{\mathcal{S}}(x;\phi,\theta,\hat{I},\mu,\epsilon),$$

where $\gamma=(x,\phi,\theta)$, so that a solution $\gamma=(x,\phi,\theta)$ of the equation $F(\gamma,\epsilon)=0$ corresponds to a point $\hat{\mathcal{S}}(x;\phi,\theta,\hat{I},\mu,\epsilon)\in \mathrm{Fix}\, S$ on a solution to (92) which approaches the periodic orbit $\hat{X}_{\hat{I},\epsilon}$ as $x\to\infty$. Bearing in mind that $\gamma=(0,0,0)$ is a solution of the equation $F(\gamma,0)=0$ (see the remarks above equation (92)), note that γ is a solution of $F(\gamma,\epsilon)=0$ if and only if it solves the equation $B(\gamma,\epsilon)=\gamma$, where $B:\mathbb{R}\times S^1\times S^1\times[0,\epsilon]$ is given by $B(\gamma,\epsilon)=\gamma-\mathrm{d}_1F[0,0]^{-1}F(\gamma,\epsilon)$. The operator $\mathrm{d}_1F[0,0]:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathrm{Im}(I-S)$ may be computed explicitly. Differentiating the expression

$$\hat{S}(x;\phi,\theta,\hat{I},\mu,0)=R_{\Omega_0x+\phi,qx+\theta}\hat{\mathcal{H}}(x), \tag{108}$$

one finds that

$$\begin{aligned}
\hat{\mathcal{S}}_x(0;0,0,\hat{I},\mu,0) &= L_{\Omega_0,\Omega}\hat{\mathcal{H}}(0)+\frac{d\hat{\mathcal{H}}}{dx}(0),\\
\hat{\mathcal{S}}_\phi(0;0,0,\hat{I},\mu,0) &= L_{\Omega_0,0}\hat{\mathcal{H}}(0),\\
\hat{\mathcal{S}}_\theta(0;0,0,\hat{I},\mu,0) &= L_{0,\Omega}\hat{\mathcal{H}}(0).
\end{aligned}$$

It follows directly from the definition of S that $\dim \mathrm{Im}(I-S)=\dim\ker(I-S)=3$, and since $SL_{\Omega_0,0}\hat{\mathcal{H}}(0)=-L_{\Omega_0,0}\hat{\mathcal{H}}(0)$, $SL_{0,\Omega}\hat{\mathcal{H}}(0)=-L_{0,\Omega}\hat{\mathcal{H}}(0)$, $Sd\hat{\mathcal{H}}/dx=-d\hat{\mathcal{H}}/dx$, the set

$$\mathcal{B}_1=\left\{\frac{d\hat{\mathcal{H}}}{dx}(0), L_{\Omega_0,0}\hat{\mathcal{H}}(0), L_{0,\Omega}\hat{\mathcal{H}}(0)\right\}$$

is a basis for $\mathrm{Im}(I-S)$. The matrix of $\mathrm{d}_1F[0,0]$ with respect to this basis is

$$\begin{pmatrix} 2 & 0 & 0\\ 2 & 2 & 0\\ 2 & 0 & 2\end{pmatrix}.$$

Formula (100) and the remarks below (82) show that

$$\mathcal{B}_2 = \left\{ \frac{1}{\mu}\frac{d\hat{\mathcal{H}}}{dx}(0), L_{\Omega_0,0}\hat{\mathcal{H}}(0), L_{0,\Omega}\hat{\mathcal{H}}(0) \right\}$$

is a basis for $\mathrm{Im}(I - S)$ with the property that the norms of its elements are bounded above and below by a constant which is independent of μ. The same is therefore true of the dual basis to $\mathcal{B}_2$, so that the dual basis $\{v_1, v_2, v_3\}$ to $\mathcal{B}_1$ satisfies $\|v_1\| \leq M/\mu$, $\|v_1\|, \|v_2\| \leq M$. Since

$$\mathrm{d}_1 F[0,0]^{-1}(v) = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} \langle v, v_1 \rangle \\ \langle v, v_2 \rangle \\ \langle v, v_3 \rangle \end{pmatrix}$$

it follows that $\|\mathrm{d}_1 F[0,0]^{-1}\| \leq M/\mu$.

Notice that

$$\begin{aligned} \|B(0,\epsilon)\| &\leq \|\mathrm{d}_1 F[0,0]^{-1}\| \|F(0,\epsilon)\| \\ &\leq \|\mathrm{d}_1 F[0,0]^{-1}\| \|I - S\| \|\hat{\mathcal{S}}(0; 0, 0, \hat{I}, \mu, \epsilon)\| \\ &\leq M\frac{\epsilon}{\mu^3}, \end{aligned}$$

where the last line follows from (104) and the fact that $\hat{\mathcal{S}}(0; 0, 0, \hat{I}, \mu, 0) = 0$. Moreover

$$\begin{aligned} \|\mathrm{d}_1 B(\gamma,\epsilon)\| &\leq \|\mathrm{d}_1 F[0,0]^{-1}\| \|\mathrm{d}_1 F[\gamma, 0] - \mathrm{d}_1 F[0,0]\| \\ &\quad + \|\mathrm{d}_1 F[0,0]^{-1}\| \|\mathrm{d}_1 F[\gamma,\epsilon] - \mathrm{d}_1 F[\gamma, 0]\| \\ &\leq \frac{M}{\mu}\left(|\gamma| + \frac{\epsilon}{\mu^2}\right) \end{aligned}$$

because

$$\begin{aligned} &\|\mathrm{d}_1 F[\gamma, 0] - \mathrm{d}_1 F[0,0]\| \leq M|\gamma|, \\ &\|\mathrm{d}_1 F[\gamma,\epsilon] - \mathrm{d}_1 F[\gamma, 0]\| \leq \frac{M\epsilon}{\mu^2}; \end{aligned}$$

here the first estimate follows from the explicit formula (108) for $\hat{\mathcal{S}}(\cdot; \cdot, \cdot, \hat{I}, \mu, 0)$ and the second is a consequence of (105)–(107). Replacing μ_0 with a smaller constant (which depends upon $\hat{I}$) if necessary, one concludes that $B_2(\cdot, \epsilon)$ has a unique fixed point $(x_1, \phi_1, \theta_1) \in \overline{B}_\delta(0)$, $\delta = M\epsilon/\mu^3$ whenever $\epsilon \leq \mu^{9/2}$. The solution $\mathcal{S}(\cdot; \phi_1, \theta_1, \hat{I}, \mu, \epsilon)$ to equation (87) is therefore a reversible homoclinic connection to the periodic orbit $\hat{X}_{\hat{I},\epsilon}$.

Recall that $F(\gamma, 0) = 0$ has three additional solutions $\gamma = (0, \pi, 0)$, $\gamma = (0, 0, \pi)$ and $\gamma = (0, \pi, \pi)$. Repeating the above analysis, one therefore obtains three further reversible homoclinic solutions $\mathcal{S}(\cdot; \phi_i, \theta_i, \hat{I}, \mu, \epsilon)$, $i = 1$, 2, 3 to equation (87) whenever $\epsilon \leq \mu^{9/2}$. These remarks complete the proof of Theorem 14, and returning to the relationship $\epsilon = \sqrt{\mu^{N-1}}$, note that it is necessary to take $N \geq 10$.

Appendix A Results in interpolation theory

In this Appendix various properties of the Hilbert spaces

$$H^t_{\mathrm{per}}(S) = \{u \in H^t_{\mathrm{loc}}(\mathbb{R}) : u(z+P) = u(z), z \in \mathbb{R}\}$$

and

$$H^t_{\mathrm{per}}(\Sigma) = \{u \in H^t_{\mathrm{loc}}((0,1)\times\mathbb{R}) : u(y,z+P) = u(y,z), y \in (0,1), z \in \mathbb{R}\}$$

are collected together. The first result concerns the relationship between elements in these spaces.

Lemma 18 *Let t be a non-negative real number.*

(i) The mapping

$$u \mapsto \int_0^1 u(\cdot, y)\,\mathrm{d}y$$

defines a bounded linear operator $H^t_{\mathrm{per}}(\Sigma) \to H^t_{\mathrm{per}}(S)$.

(ii) The natural extension of $u \in H^t_{\mathrm{per}}(S)$ to Σ defines a bounded linear operator $H^t_{\mathrm{per}}(S) \to H^t_{\mathrm{per}}(\Sigma)$.

(iii) The trace mapping $u \mapsto u|_S$ is a bounded operator $H^{t+1/2}_{\mathrm{per}}(\Sigma) \to H^t_{\mathrm{per}}(S)$.

Proof The first two assertions are clearly true for $t \in \mathbb{N}_0$ and hence, by interpolation, for each $t \in [0,\infty)$. The proof of the third assertion is given by Lions & Magenes (1972, vol. I, ch. 1, Theorem 9.4). □

Lemma 19 indicates when it is possible to multiply functions in H^s_{per} and H^t_{per} together.

Lemma 19

(i) The spaces $H^s_{\mathrm{per}}(S)$ and $H^t_{\mathrm{per}}(\Sigma)$ are Banach algebras for $s > 1/2$, $t > 1$.

(ii) Take $u_1 \in H^{s+1}_{\mathrm{per}}(\Sigma)$, $u_2 \in H^s_{\mathrm{per}}(\Sigma)$ for some $s \in (0, 1/2)$. The function u_1u_2 belongs to $H^s_{\mathrm{per}}(\Sigma)$ and satisfies the estimate

$$\|u_1u_2\|_s \le c\|u_1\|_{1+s}\|u_2\|_s,$$

in which c is a positive constant. The same result holds when Σ is replaced by S.

Proof The proof of the first part of this lemma is given by Lions & Magenes (1972, vol. I, ch. 1, Theorem 9.8). Turning to the second part, fix $u_1 \in H^{s+1}_{\mathrm{per}}(\Sigma)$ and set $Tu_2 = u_1u_2$. For each $u_2 \in L^2_{\mathrm{per}}(\Sigma)$ one has that

$$\|Tu_2\|_0 = \|u_1u_2\|_0 \le \sup_{(x,z)\in\overline{S}} |u_1(x,z)|\|u_2\|_0 \le c_1\|u_1\|_{1+s}\|u_2\|_0, \tag{109}$$

in which the continuity of the embedding $H^{s+1}_{\mathrm{per}}(\Sigma) \subset C(\overline{\Sigma})$ has been used. It follows that T is a bounded linear operator $L^2_{\mathrm{per}}(\Sigma) \to L^2_{\mathrm{per}}(\Sigma)$.

Now take $u_2 \in H^1_{\mathrm{per}}(\Sigma)$ and note that

$$\|\nabla T u_2\|_0^2 \leq \frac{1}{2}\|u_1 \nabla u_2\|_0^2 + \frac{1}{2}\|u_2 \nabla u_1\|_0^2. \tag{110}$$

The above procedure shows that

$$\|u_1 \nabla u_2\|_0^2 \leq c_2 \|u_1\|_{1+s}^2 \|u_2\|_0^2, \tag{111}$$

and using Hölder's inequality and the continuity of the embeddings $H^1_{\mathrm{per}}(\Sigma) \subset L^{2p}_{\mathrm{per}}(\Sigma)$ for $2p \in [1, \infty)$ and $H^s_{\mathrm{per}}(\Sigma) \subset L^{2q}_{\mathrm{per}}(\Sigma)$ for $2q \in [1, 2/(1-s)]$, one finds that

$$\|u_2 \nabla u_1\|_0^2 \leq \|u_2\|_{0,2p}^2 \|\nabla u_1\|_{0,2q}^2 \leq c_2 \|u_2\|_1^2 \|\nabla u_1\|_s^2. \tag{112}$$

Equations (109)–(112) show that

$$\|T u_2\|_1 \leq c_3 \|u_1\|_{1+s} \|u_2\|_1, \tag{113}$$

so that T is a bounded linear operator $H^1_{\mathrm{per}}(\Sigma) \to H^1_{\mathrm{per}}(\Sigma)$.

The desired result for Σ now follows from (109), (113) by interpolation and an identical procedure gives the same conclusion for S. □

Appendix B Results of the normal-form computations

Let us write the Hamiltonian vector field for $(M_C^\mu, \tilde{\Omega}, \tilde{H}^\mu)$ as $Z \mapsto LZ + P(Z, \mu)$, where

$$P(Z, \mu) = \sum_{r+i+j+k+\ell+m+n=2}^{3} P_{r,ijk\ell mn} \mu^r A^i B^j C^k \bar{A}^\ell \bar{B}^m \bar{C}^n + O(|(A, B, C)||(\mu, A, B, C)|^3)$$

as $(Z, \mu) \to 0$. In similar notation, the reduction function $k : \tilde{W}_1 \times \tilde{\Lambda} \to \tilde{W}_2$ defining the centre manifold for (M, Ω, H^λ) takes the form

$$k(Z, \mu) = \sum_{r+i+j+k+\ell+m+n=2}^{3} k_{r,ijk\ell mn} \mu^r A^i B^j C^k \bar{A}^\ell \bar{B}^m \bar{C}^n + O(|(A, B, C)||(\mu, A, B, C)|^3)$$

as $(Z, \mu) \to 0$, where each $k_{r,ijk\ell mn}$ is an element of V. Recall that Hamilton's equations for (M, Ω, H^μ) take the form

$$\dot{u} = f^\mu(u)$$

with boundary conditions

$$\Phi_y^u = F(u) \qquad \text{on } y = 0, 1.$$

Substituting $u = Z + k(Z, \mu)$ and $\dot{Z} = LZ + P(Z, \mu)$ into these equations, one finds that

$$f_1^0 k_{0,2}(Z) - \mathrm{d}k_{0,2}[Z](f_1^0(Z)) = -f_2^0(Z, Z) + P_{0,2}(Z) \tag{114}$$

with boundary conditions

$$\Phi_y^{k_{0,2}} + \frac{1}{\beta}\left(\omega^{k_{0,2}} - \int_0^1 y \Phi_y^{k_{0,2}} \, \mathrm{d}y\right) = F_2(Z, Z) \qquad \text{on } y = 1, \tag{115}$$

$$\Phi_y^{k_{0,2}} = 0 \qquad \text{on } y = 0, \tag{116}$$

in which

$$k_{p,q}(Z) = \sum_{i+j+k+\ell+m+n=q} \mu^p A^i B^j C^k \bar{A}^\ell \bar{B}^m \bar{C}^n, \qquad P_{p,q}(Z) = \sum_{i+j+k+\ell+m+n=q} \mu^p A^i B^j C^k \bar{A}^\ell \bar{B}^m \bar{C}^n,$$

$f_n^0 = \frac{1}{n!}\mathrm{d}^n f^0[0]$ and $F_n = \frac{1}{n!}\mathrm{d}^n F[0]$, $n = 1, 2, 3$. Since $P_{0,2} = 0$, equations (114)–(116) consist of 21 boundary-value problems which uniquely determine $k_{0,2}$; note in particular that

$$f_1^0 k_{0,100100} = -2f_2^0(e^1, \bar{e}^1)$$

with

$$\begin{aligned} &\Phi_y^{k_{0,100100}} + \frac{1}{\beta_0}\left(\omega^{k_{0,100100}} - \int_0^1 y\Phi_y^{k_{0,100100}}\,\mathrm{d}y\right) = 2F_2(e^1, \bar{e}^1) && \text{on } y = 1, \\ &\Phi_y^{k_{0,100100}} = 0 && \text{on } y = 0 \end{aligned}$$

and

$$(f_1^0 - 2\mathrm{i}sI)k_{0,200000} = -f_2^0(e^1, e^1)$$

with

$$\begin{aligned} &\Phi_y^{k_{0,200000}} + \frac{1}{\beta_0}\left(\omega^{k_{0,200000}} - \int_0^1 y\Phi_y^{k_{0,200000}}\,\mathrm{d}y\right) = F_2(e^1, e^1) && \text{on } y = 1, \\ &\Phi_y^{k_{0,200000}} = 0 && \text{on } y = 0. \end{aligned}$$

Observe that the inhomogeneous terms in these equations have nonzero components in only the zeroth and second Fourier modes. Examining the contributions to the equations from these modes, one arrives at four systems of linear ordinary differential equations whose unique solutions can be found by straightforward calculations. This procedure yields expressions for $k_{0,100100}$, $k_{0,200000}$, and $k_{0,001001}$, $k_{0,002000}$ may be computed in the same manner (here it is necessary only to examine the zeroth Fourier mode).

Next note that $k_{0,3}$ satisfies the equation

$$f_1^0 k_{0,3}(Z) - \mathrm{d}k_{0,3}[Z](f_1^0 Z) = \mathrm{d}k_{0,2}[Z]((P_{0,2}(Z)) - 2f_2^0(Z, k_{0,2}(Z)) - f_3^0(Z, Z, Z) + P_{0,3}(Z) \tag{117}$$

with boundary conditions

$$\begin{aligned} &\Phi_y^{k_{0,3}} + \frac{1}{\beta_0}\left(\omega^{k_{0,3}} - \int_0^1 y\Phi_y^{k_{0,3}}\,\mathrm{d}y\right) = 2F_2(Z, k_{0,2}(Z)) + F_3(Z, Z, Z) && \text{on } y = 1, \\ &\Phi_y^{k_{0,2}} = 0 && \text{on } y = 0. \end{aligned}$$

The component of equation (117) which contains the term $P_{0,200100} = c_7 \mathrm{i}e^1 - 2c_4 f_2$ is

$$(f_1^0 - \mathrm{i}s_1 I)k_{0,200100} = -2f_2^0(e^1, k_{0,100100}) - 2f_2^0(\bar{e}^1, k_{0,200000}) - 3f_3^0(e^1, e^1, \bar{e}^1) + c_7 \mathrm{i}e^1 - 2c_4 f_2,$$

so that

$$\begin{aligned} 2c_4 = \Omega((f_1^0 - \mathrm{i}sI)k_{0,200100}, \bar{e}^1) \\ + 2\Omega(f_2^0(e^1, k_{0,100100}), \bar{e}^1) + 2\Omega(f_2^0(\bar{e}^1, k_{0,200000}), \bar{e}^1) + 3\Omega(f_3^0(e^1, e^1, \bar{e}^1), \bar{e}^1). \end{aligned} \tag{118}$$

Equation (16) shows that

$$\Omega(f^\mu(u), v) + [G_1(u)\Phi^v]_{y=0}^{y=1} = \mathrm{d}H^\mu[u](v)$$

for each $u, v \in \tilde{W}_1$, whereupon

$$\begin{aligned}
\Omega(f_1^0(u), v) + [G_1(u)\Phi^v]_{y=0}^{y=1} &= 2H_2^0(u, v),\\
\Omega(f_2^0(u_1, u_2), v) + [G_2(u_1, u_2)\Phi^v]_{y=0}^{y=1} &= 3H_3^0(u_1, u_2, v),\\
\Omega(f_3^0(u_1, u_2, u_3), v) + [G_3(u_1, u_2, u_3)\Phi^v]_{y=0}^{y=1} &= 4H_4^0(u_1, u_2, u_3, v),
\end{aligned}$$

in which $H_n^0 = \frac{1}{n!}\mathrm{d}^n H^0[0]$, $G_n = \frac{1}{n!}\mathrm{d}^n G$, $n = 1, 2, 3, 4$. These results indicate that

$$\begin{aligned}
\Omega(f_1^0 k_{0,200100}, \bar{e}^1) + \left[G_1(k_{0,200100})\Phi^{\bar{e}^1}\right]_{y=0}^{y=1} &= 2H_2^0(k_{0,200100}, \bar{e}^1)\\
&= \Omega(f_1^0 \bar{e}^1, k_{0,200100}) + \left[G_1(\bar{e}^1)\Phi^{k_{0,200100}}\right]_{y=0}^{y=1}\\
&= -\mathrm{i}s_1\Omega(\bar{e}^1, k_{0,200100}),
\end{aligned}$$

in which the second line follows because of the symmetry of H_2^0 and the third because $G_1(\bar{e}^1) = 0$ on $y = 0, 1$. It follows that

$$\Omega((f_1^0 - \mathrm{i}s_1 I)k_{0,200100}, \bar{e}^1) = -\left[G_1(k_{0,200100})\Phi^{\bar{e}^1}\right]_{y=0}^{y=1}$$

and similar arguments show that

$$\begin{aligned}
\Omega(f_2^0(e^1, k_{0,100100}), \bar{e}^1) + \left[G_2(e^1, k_{0,100100})\Phi^{\bar{e}^1}\right]_{y=0}^{y=1} &= 3H_3^0(e^1, k_{0,100100}), \bar{e}^1),\\
\Omega(f_2^0(\bar{e}^1, k_{0,200000}, \bar{e}^1) + \left[G_2(\bar{e}^1, k_{0,200000})\Phi^{\bar{e}^1}\right]_{y=0}^{y=1} &= 3H_3^0(\bar{e}^1, k_{0,200000}), \bar{e}^1),\\
\Omega(f_3^0(e^1, e^1, \bar{e}^1), \bar{e}^1) + \left[G_3(e^1, e^1, \bar{e}^1)\Phi^{\bar{e}^1}\right]_{y=0}^{y=1} &= 4H_4^0(e^1, e^1, \bar{e}^1, \bar{e}^1).
\end{aligned}$$

Inserting the above expressions into (118), one finds that

$$\begin{aligned}
2c_4 = &-\left[G_1(k_{0,200100})\Phi^{\bar{e}^1}\right]_{y=0}^{y=1} - 2\left[G_2(e^1, k_{0,100100})\Phi^{\bar{e}^1}\right]_{y=0}^{y=1}\\
&- 2\left[G_2(\bar{e}^1, k_{0,200000})\Phi^{\bar{e}^1}\right]_{y=0}^{y=1} - 3\left[G_3(e^1, e^1, \bar{e}^1)\Phi^{\bar{e}^1}\right]_{y=0}^{y=1}\\
&+ 6H_3^0(e^1, k_{0,100100}, \bar{e}^1) + 6H_3^0(\bar{e}^1, \bar{e}^1, k_{0,200000}) + 12H_4^0(e^1, e^1, \bar{e}^1, \bar{e}^1). \quad (119)
\end{aligned}$$

The fact that

$$G(Z + k(Z, \mu)) = 0 \qquad \text{on } y = 0, 1,$$

implies that

$$G_1(k_{0,3}) + 2G_2(u, k_{0,2}) + G_3(u, u, u) = 0$$

and hence that

$$G_1(k_{0,200100}) + 2G_2(e^1, k_{100100}) + 2G_2(\bar{e}^1, k_{0,200000}) + 3G_3(e^1, e^1, \bar{e}^1) = 0.$$

Equation (119) therefore reduces to

$$c_4 = 3H_3(e^1, k_{0,100100}, \bar{e}^1) + 3H_3(\bar{e}^1, \bar{e}^1, k_{0,200000}) + 6H_4(e^1, e^1, \bar{e}^1, \bar{e}^1)$$

and the formula

$$c_6 = 3H_3(e, k_{0,001001}, \bar{e}) + 3H_3(\bar{e}, \bar{e}, k_{0,002000}) + 6H_4(e, e, \bar{e}, \bar{e})$$

is obtained by similar arguments.

The results of the above calculations are

$$\begin{aligned}
\gamma_2^2 c_4 = {} & \frac{1}{128P^5 s_1^4}\, p_1 \sinh(p_1) \Big(- 2P^2 s_1^2(64\pi^2 + P^2 s_1^2(-8\pi^2 + P^2(3 + 6s_1^2))) \cosh(p_1) \\
& - 2P^4 s_1^4(8\pi^2 + P^2(-3 + 6s_1^2)) \cosh(3p_1) - P^6 p_1 s_1^4(-36 + s_1^2(16 + 27\beta_0)) \sinh(p_1) \\
& - p_1(1728\pi^6\beta_0 + 528P^2\pi^4 s_1^2\beta_0 + 4P^4\pi^2 s_1^4(16 + 33\beta_0)) \sinh(p_1) \\
& + p_1(576\pi^6\beta_0 + 176P^2\pi^4 s_1^2\beta_0 + 44P^2\pi^2 s_1^4\beta_0 + 3P^6 s_1^4(4 + 3s_1^2\beta_0)) \sinh(3p_1)\Big) \\
& + (4\pi^2 - P^2(s_1^2\cosh(2p_1) + p_1(-4p_1\sinh^2(p_1) + (\alpha_0 + 4s_1^2\beta_0)\sinh(2p_1)))) \\
& \quad \times \Big(- 2p_1 P^2 s_1 \cosh(2s_1)\sinh(2p_1) \\
& \qquad + (-2(2\pi^2 + P^2 s_1^2) + (8\pi^2 + P^2 s_1^2)\cosh(2p_1))\sinh(2s_1)\Big) \\
& \quad \times (16P^3(-2s_1\cosh(2s_1) + (\alpha_0 + 4s_1^2\beta_0)\sinh(2s_1)))^{-1} \\
& + p_1^5 P(5\cosh(p_1) + \cosh(3p_1))\sinh(p_1) \\
& \quad \times(-P^2 s_1^2(-2 + \cosh(2p_1)) + p_1(16\pi^2\beta_0 + P^2(\alpha_0 + 4s_1^2\beta_0))\sinh(2p_1)) \\
& \quad \times(32s_1^2(-2P^2 s_1^2\cosh(2p_1) + p_1(16\pi^2\beta_0 + P^2(\alpha_0 + 4s_1^2\beta_0))\sinh(2p_1))^{-1} \\
& - \frac{Pp_1}{24}\sinh^2(p_1)(6p_1 - 2p_1^3 + 3\cosh(p_1)\sinh(p_1) + 3p_1(\cosh(2p_1) + p_1\sinh(2p_1))) \\
& + \frac{p_1^4 P}{8\alpha_0} + \frac{3(P^2 s_1^2 - 4\pi^2\cosh(2p_1))^2}{48P(P^2\alpha_0 + 16\pi^2\beta_0)} + \frac{p_1^2 P}{48}(-6 + 2p_1^2 - 3\cosh(2p_1))\sinh^2(p_1) \\
& - \frac{3p_1}{48P}(8\pi^2 + P^2(1 + 2s_1^2))\cosh(p_1)\sinh^3(p_1)
\end{aligned}$$

and

$$\begin{aligned}
c_6 = {} & \frac{q^4}{16P\alpha_0}\Big(128q^3 + 6q\alpha_0^2 + q(64q^2 + 51\alpha_0^2)\cosh(2q) - 54q\alpha_0^2\cosh(4q) \\
& - 3q\alpha_0^2\cosh(6q) + 3\alpha_0(38q^2 + 15\alpha_0^2)\sinh(2q) \\
& + 12\alpha_0(5q^2 - 3\alpha_0^2)\sinh(4q) + (2q^2 + 9\alpha_0^2)\alpha_0\sinh(6q)\Big) \\
& \times(2(q^2 + \alpha_0) - 2\alpha_0\cosh(2q) + q\sinh(2q))^{-2}(2q(2 + \cosh(2q)) - 3\alpha_0\sinh(2q))^{-1},
\end{aligned}$$

in which α_0, β_0 and γ_2 are given by (73) with $k = 1$ and (74). It is apparent from the formula that c_4 depends only upon s_1 and P; the same is also true of c_6 since $\lambda = \pm iq$ are the purely imaginary solutions of

$$(\alpha_0 - \beta_0\sigma^2)\sigma\sin\sigma = |\lambda|^2\cos\sigma,$$

in which $\sigma^2 = |\lambda|^2 - 4\pi^2/P^2$.

It remains to compute the coefficient c_1. The 'three-dimensional' eigenvalues λ of the operator obtained by replacing α_0 by $\alpha_0 + \mu$, $\mu \in \tilde{\Lambda}$ in the definition of L satisfy the equation

$$(\alpha_0 + \mu - \beta_0\sigma^2)\sigma \sin\sigma = |\lambda|^2 \cos\sigma. \tag{120}$$

On the other hand it follows from the reduced equations (76) that these eigenvalues satisfy

$$\begin{aligned}\lambda &= \mathrm{i}(s_1 + \partial_2 H_{\mathrm{NF}}(0,0,0,\mu)) \pm \sqrt{-\partial_1 H_{\mathrm{NF}}(0,0,0,\mu)} \\ &= \mathrm{i}(s_1 + c_2\mu) \pm \sqrt{-c_1}\sqrt{\mu} + O(\mu^{\frac{3}{2}})\end{aligned} \tag{121}$$

as $\mu \to 0$. The value of c_1 may therefore be determined by substituting the expression (121) for λ into (120). One finds that

$$c_1 = -8p_1^5 \sinh^3 p_1 [s_1^2((3s_1^2 - 4p_1^2(1+2s_1^2))\cosh p_1 + (4p_1^2 - 3s_1^2)(\cosh 3p_1 + 4p_1 \sinh p_1))]^{-1}.$$

It can be verified that $c_1 < 0$, $c_4 > 0$ and $c_6 \neq 0$.

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